

HADWIGER'S CONJECTURE FOR CAP BODIES

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ABSTRACT. Hadwiger's covering conjecture is that every n -dimensional convex body can be covered by at most 2^n of its smaller positive homothetic copies, with 2^n copies required only for affine images of n -cube. Convex hull of a ball and an external point is called a spike. The union of finitely many spikes of a ball is a cap body if it is a convex set.

In this note, we confirm the Hadwiger's conjecture for the class of cap bodies in all dimensions, bridging recently established cases of $n = 3$ and large n . The proof uses probabilistic techniques, and additionally, for moderate dimensions $4 \leq n \leq 15$, integer linear programming performed with computer assistance.

1. INTRODUCTION

Hadwiger [10] asked the following question in 1957: For $n \geq 3$, what is the smallest number $H(n)$ such that every n -dimensional convex body can be covered by the union at most $H(n)$ of translates of the body's interior? An equivalent formulation in terms of illumination was offered by Boltyanski [6] in 1960. For a convex body K , a direction (unit vector) v illuminates a point x on the boundary ∂K of K , if the ray $\{x + vt : t \geq 0\}$ has nonempty intersection with interior of K . The set of directions $\{v_i\}_{i=1}^k$ is said to illuminate K if every point of ∂K is illuminated by some v_i . The illumination number $I(K)$ of K is the smallest k for which K can be illuminated by k directions. For a collection \mathcal{K} of convex bodies, we define the illumination number as $I(\mathcal{K}) := \max\{I(K) : K \in \mathcal{K}\}$. The Hadwiger's covering number $H(n)$ is then the illumination number of the class of all n -dimensional convex bodies.

The Hadwiger Covering Conjecture, the Levi-Hadwiger(-Gohberg-Markus) Conjecture, or the Hadwiger-Boltyanski Illumination Conjecture is that $H(n) = 2^n$ with $I(K) = 2^n$ if and only if K is an affine copy of an n -cube. The conjecture attracted many mathematicians, but remains unsolved for all $n \geq 3$. For further details and various partial cases, an interested reader is referred to the survey is [5], and more recent works [13], [1].

The goal of this paper is to complete confirmation of the Hadwiger's conjecture for the class of cap bodies in all dimensions. Following the notation from [4], convex hull of a ball and an external point is called a spike (or cap). The union of finitely many spikes of a ball is a cap body if it is a convex set. Let \mathcal{K}_c^n denote the class of all n -dimensional cap bodies, while $\mathcal{K}_{c,s}^n$ and $\mathcal{K}_{c,us}^n$ denote the subclasses of \mathcal{K}_c^n of all symmetric and unconditionally symmetric (about every coordinate hyperplane) cap bodies, respectively.

The problem of illumination of cap bodies was first studied by Ivanov and Strachan [11], where $I(\mathcal{K}_{c,s}^3) = 6$ and $I(\mathcal{K}_{c,us}^4) = 8$ were established. Then, Bezdek, Ivanov and Strachan [4] proved that $I(\mathcal{K}_{c,s}^n) < 2^n$ for $n = 3, 4, 9$ and all $n \geq 19$, thus confirmed the conjecture for centrally symmetric cap bodies in these dimensions. They used a reduction to symmetric coverings of sphere with spherical caps of radius $\pi/4$. For large n , the estimate obtained in [4] is actually $I(\mathcal{K}_{c,s}^n) < (\sqrt{2} + o(1))^n$. This was improved in [2] using a different method combining spherical coverings and packings, which also

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allowed to get rid of symmetry assumption. Namely, [2, Th. 5] implies $I(\mathcal{K}_c^n) < 1.19851^n$ for some implicit $n \geq n_0$ confirming the conjecture for all (not necessarily centrally symmetric) cap bodies in high dimensions. In the three-dimensional case, we recently proved [3] that $I(\mathcal{K}_c^3) = 6$. The proof is based on choosing 4 of the illuminating directions as vertices of a randomly rotated regular simplex, and showing, by reduction to an integer linear programming and computer assistance, that at most 2 caps remain unilluminated, so an additional 2 directions bring the total to 6.

In this paper, we extend and generalize the techniques of [3] to show that $I(\mathcal{K}^n) < 2^n$ for all n , and thus, we confirm the Hadwiger's conjecture for cap bodies in all dimensions. More precisely, for small/moderate dimensions, we illuminate by the directions of several randomly rotated regular simplexes and/or several randomly rotated cross-polytopes, and estimate, by integer linear programming and computer assistance, the expected number of unilluminated caps. We obtain:

Theorem 1. $I(\mathcal{K}_c^n) < 2^n$ for all $4 \leq n \leq 15$, moreover, $I(\mathcal{K}_c^4) \leq 11$, $I(\mathcal{K}_c^5) \leq 17$, $I(\mathcal{K}_c^6) \leq 29$, $I(\mathcal{K}_c^7) \leq 44$, $I(\mathcal{K}_c^8) \leq 69$. (See Table 1 in Section 3 for more details.)

One of the necessary ingredients for estimates of expectations is a bound on the intersection of two equal hyperspherical caps, given in Theorem 5.

For large dimensions, we use an appropriate number of random rotations of the cross-polytope and direct estimates (no computer assistance required) to prove the following:

Theorem 2. For all $n \geq 9$

$$(1) \quad I(\mathcal{K}_c^n) < n + 7 + (\sqrt{2})^n \sqrt{\pi n} \left(1 + \frac{3}{n} + \frac{1}{2}n \ln \frac{n}{2} + \ln \frac{2}{\sqrt{n-1}}\right).$$

In particular, $I(\mathcal{K}_c^n) < 2^n$ for all $n \geq 13$.

While asymptotically (1) gives $I(\mathcal{K}_c^n) < (\sqrt{2} + o(1))^n$, which is weaker than in [2, Th. 5], our bound here is explicit and provides the desired bound for all $n \geq 13$.

Now Theorems 1 and 2 and $I(\mathcal{K}_c^3) = 6$ ([3, Th. 2]) imply that the Hadwiger conjecture is valid for all cap bodies:

Corollary 3. $I(\mathcal{K}_c^n) < 2^n$ for $n \geq 3$.

Note that $I(\mathcal{K}_c^2) = 4$, as the upper bound is valid for all convex bodies [12], while \mathcal{K}_c^2 contains a square, which has illumination number 4. On the other hand, \mathcal{K}_c^n for $n \geq 3$ does not contain any affine copy of n -cube.

We gather all preliminaries in Section 2. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

2. PRELIMINARIES

2.1. Cap bodies and illumination. Let \mathbb{R}^n denote the n -dimensional Euclidean space with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. The unit sphere and the unit ball centred at the origin are denoted as $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and $\mathbb{B}^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, respectively. For any points $x, y \in \mathbb{S}^{n-1}$, the geodesic distance between them is defined by $\theta(x, y) := \arccos \langle x, y \rangle$. For $\xi \in \mathbb{S}^{n-1}$, define the open and closed spherical caps on \mathbb{S}^{n-1} centred at ξ of radius φ by $C(\xi, \varphi) := \{y \in \mathbb{S}^{n-1} : \langle \xi, y \rangle > \cos \varphi\}$, $C[\xi, \varphi] := \{y \in \mathbb{S}^{n-1} : \langle \xi, y \rangle \geq \cos \varphi\}$.

Let $\text{conv}(X)$ be the convex hull of the set X . K is a convex body in \mathbb{R}^n if it is convex compact set with non-empty interior. A convex body K is called a cap body if and only if

$$K = \bigcup_{i=1}^m \text{conv}(\{x_i\} \cup \mathbb{B}^n)$$

for some points $x_i \in \mathbb{R}^n \setminus \mathbb{B}^n$, $1 \leq i \leq m$, which are called vertices of K . For a given vertex x_i , the corresponding base cap (or simply cap, with slight abuse of earlier terminology) is defined to

be the set

$$S_i := \text{cl}(\text{conv}(\{x_i\} \cup \mathbb{B}^n) \setminus \mathbb{B}^n) \cap \mathbb{S}^{n-1},$$

where $\text{cl}(\cdot)$ denotes the closure of the set. Note that $S_i = C[\hat{x}_i, \varphi_i]$, where $\hat{x}_i := \frac{x_i}{\|x_i\|}$ is the centre of the cap, and $\varphi_i = \arccos \frac{1}{\|x_i\|}$ is the radius of the cap. It is an easy observation that the base caps always have acute radius, i.e., $\varphi_i < \pi/2$ for $1 \leq i \leq m$. Furthermore, observe that the convexity of K implies that

$$(2) \quad C(\hat{x}_\alpha, \varphi_\alpha) \cap C(\hat{x}_\beta, \varphi_\beta) = \emptyset \quad \text{for any distinct } x_\alpha, x_\beta \in \{x_i\}_{i=1}^m.$$

We use the following proposition from [2, 4]:

Proposition 4. *A cap body K with vertices $\{x_i\}_{i=1}^m$ is illuminated by the directions $\{v_j\}_{j=1}^k \subset \mathbb{S}^{n-1}$ if:*

- i) $C(-\hat{x}_i, \frac{\pi}{2} - \varphi_i) \cap \{v_j\}_{j=1}^k \neq \emptyset$ for each i , $1 \leq i \leq m$,
- ii) positive hull of $\{v_i\}_{i=1}^k$ is \mathbb{R}^n , i.e. for all $x \in \mathbb{R}^n$ there are positive c_1, \dots, c_k such that $x = c_1 v_1 + \dots + c_k v_k$.

2.2. Area of spherical cap. Let σ be the probabilistic spherical measure on \mathbb{S}^{n-1} . It is well-known that $\sigma(C[x, \theta]) = \frac{1}{2} I_{\sin^2 \theta}(\frac{n-1}{2}, \frac{1}{2})$ for $0 \leq \theta \leq \pi/2$, where $I_z(a, b) = \int_0^z u^a (a-u)^b du$ is the incomplete beta function. We will also use the estimate (see, e.g. [7, Cor. 3.2(i), (iii)])

$$(3) \quad \frac{\sin^{n-1} \theta}{\sqrt{2\pi n}} \leq \sigma(C[x, \theta]) \leq \frac{\sin^{n-1} \theta}{\sqrt{2\pi(n-1)} \cdot \cos \theta},$$

where the upper bound is valid for $\theta \leq \arccos \frac{1}{\sqrt{n}}$.

2.3. Area of intersection of two equal spherical caps. $|\mathbb{S}^{n-1}| = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ denotes the Lebesgue measure of the unit sphere. We need the following technical result.

Lemma 5. *Suppose $0 \leq \beta \leq \alpha \leq \pi/2$, and $x, y \in \mathbb{S}^{n-1}$ with $\theta(x, y) = 2\beta$. Then*

$$(4) \quad \sigma(C[x, \alpha] \cap C[y, \alpha]) = \frac{n-2}{\pi} \int_{\frac{\cos \alpha}{\cos \beta}}^1 (1-r^2)^{\frac{n-4}{2}} \left(\arccos\left(\frac{\cos \alpha}{r}\right) - \beta \right) r dr =: A_n(\alpha, \beta).$$

Proof. Suppose that $\mathcal{D} \subset \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ is a measurable subset of the unit disk. We wish to calculate the probabilistic measure of $C_{n, \mathcal{D}} = \{(x_1, x_2, \dots, x_n) \in \mathbb{S}^{n-1} : (x_1, x_2) \in \mathcal{D}\}$.

We use standard spherical coordinates

$$x_i = \cos \varphi_i \prod_{1 \leq j < i} \sin \varphi_j, \quad 1 \leq i < n, \quad x_n = \prod_{j=1}^n \sin \varphi_j, \quad \varphi_1, \dots, \varphi_{n-2} \in [0, \pi], \quad \varphi_{n-1} \in [0, 2\pi),$$

and by $C'_{n, \mathcal{D}}$ and \mathcal{D}' denote $C_{n, \mathcal{D}}$ and \mathcal{D} in $(\varphi_1, \dots, \varphi_{n-1})$ and (φ_1, φ_2) coordinates, respectively.

Then $dx_1 dx_2 = \sin^2 \varphi_1 \sin \varphi_2 d\varphi_1 d\varphi_2$ and $\sin \varphi_1 \sin \varphi_2 = \sqrt{1 - x_1^2 - x_2^2}$, so

$$(5) \quad \begin{aligned} |\mathbb{S}^{n-1}| \sigma(C_{n, \mathcal{D}}) &= \int_{C'_{n, \mathcal{D}}} \sin^{n-2} \varphi_1 \cdots \sin \varphi_{n-2} d\varphi_1 \cdots d\varphi_{n-1} \\ &= \iint_{\mathcal{D}'} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 d\varphi_1 d\varphi_2 |\mathbb{S}^{n-3}| \\ &= \iint_{\mathcal{D}} (1 - x_1^2 - x_2^2)^{\frac{n-4}{2}} dx_1 dx_2 |\mathbb{S}^{n-3}|. \end{aligned}$$

Now we apply this in our context. Clearly, $\sigma(C[x, \alpha] \cap C[y, \alpha]) = 2\sigma(C_{n, \mathcal{D}_{\alpha, \beta}})$, where

$$\mathcal{D}_{\alpha, \beta} = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1, x_2 \geq 0, (x_1, x_2) \cdot (\cos \beta, -\sin \beta) \geq \cos \alpha\}.$$

In polar coordinates $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, we have

$$\mathcal{D}_{\alpha, \beta} = \left\{ (r, \theta) : 0 \leq \theta \leq \arccos\left(\frac{\cos \alpha}{r}\right) - \beta, \frac{\cos \alpha}{\cos \beta} \leq r \leq 1 \right\}.$$

Now (4) readily follows from (5). \square

Clearly, the area of a single cap can be obtained as $\sigma(C[x, \alpha]) = A_n(\alpha, 0)$, so we will use the notation $A_n(\alpha) := A_n(\alpha, 0)$.

2.4. Union of equal caps centered at simplex or cross-polytope. Let S_n be the $n+1$ vertices of a regular simplex inscribed into \mathbb{S}^{n-1} , and let C_n be the vertices of the cross-polytope, i.e. points with coordinates $(\pm 1, 0, \dots, 0)$ and permutations thereof. For a discrete subset $X \subset \mathbb{S}^{n-1}$ and $\theta \in (0, \frac{\pi}{2}]$, denote $C[X, \theta] := \bigcup_{x \in X} C[x, \theta]$. We need values/estimates of $\sigma(C[X, \theta])$ when X is either S_n or C_n . We assume $n \geq 3$ below in this subsection.

Lemma 6. *We have:*

$$\sigma(C[S_n, \theta]) \begin{cases} = (n+1)A_n(\theta), & 0 < \theta \leq \frac{1}{2} \arccos(-\frac{1}{n}), \\ = (n+1)A_n(\theta) - \frac{n(n+1)}{2} A_n(\theta, \frac{1}{2} \arccos(-\frac{1}{n})), & \frac{1}{2} \arccos(-\frac{1}{n}) < \theta \leq \arccos \frac{1}{3}, \\ > \sigma(C[S_n, \arccos \frac{1}{3}]), & \arccos \frac{1}{3} < \theta \leq \arccos \frac{1}{n}, \\ = 1, & \arccos \frac{1}{n} < \theta \leq \frac{\pi}{2}. \end{cases}$$

Proof. The geodesic distance between any two distinct points in S_n is $\arccos(-\frac{1}{n})$. Therefore, when $0 < \theta \leq \frac{1}{2} \arccos(-\frac{1}{n})$ the set $C[S_n, \theta]$ consists of $n+1$ congruent caps of radius θ , and so has measure $(n+1)A_n(\theta)$. When $\frac{1}{2} \arccos(-\frac{1}{n}) < \theta \leq \arccos \frac{1}{3}$, each two caps in $C[S_n, \theta]$ intersect, but no three caps have common interior, so we compute the measure of the union by the inclusion/exclusion principle. The case $\arccos \frac{1}{3} < \theta \leq \arccos \frac{1}{n}$ follows from monotonicity of $\sigma(C[S_n, \theta])$ with respect to θ . Finally, it is straightforward that $C[S_n, \theta] = \mathbb{S}^{n-1}$ for $\arccos \frac{1}{n} \leq \theta \leq \frac{\pi}{2}$. \square

Lemma 7. *We have:*

$$\sigma(C[C_n, \theta]) \begin{cases} = 2nA_n(\theta), & 0 < \theta \leq \frac{\pi}{4}, \\ = 2nA_n(\theta) - n(n-1)A_n(\theta, \frac{\pi}{4}), & \frac{\pi}{4} < \theta \leq \arccos \frac{1}{\sqrt{3}}, \\ > \sigma(C[C_n, \arccos \frac{1}{\sqrt{3}}]), & \arccos \frac{1}{\sqrt{3}} < \theta \leq \arccos \frac{1}{\sqrt{n}}, \\ = 1, & \arccos \frac{1}{\sqrt{n}} < \theta \leq \frac{\pi}{2}. \end{cases}$$

We omit the proof as it is completely similar to that of Theorem 6.

2.5. A sphere packing bound. The next lemma shows that one cannot pack $n+2$ caps of radius $> \frac{\pi}{4}$ on \mathbb{S}^{n-1} . This is not new, but we include a proof for completeness.

Lemma 8. *For any $\{x_1, \dots, x_{n+2}\} \subset \mathbb{S}^{n-1}$ there exist $1 \leq i < j \leq n+2$ with $\theta(x_i, x_j) \leq \frac{\pi}{2}$.*

Proof. Assume to the contrary that $\langle x_i, x_j \rangle < 0$, $1 \leq i < j \leq n+2$. The points x_1, x_2, \dots, x_{n+2} are affinely dependent, so there are c_1, c_2, \dots, c_{n+2} (not all zero) such that

$$c_1 x_1 + \dots + c_{n+2} x_{n+2} = 0$$

and $\sum_{i=1}^{n+2} c_i = 0$. Suppose that $I_+ = \{c_i : c_i \geq 0\}$ and $I_- = \{c_j : c_j < 0\}$, now

$$\sum_{c_i \in I_+} c_i x_i = \sum_{c_j \in I_-} -c_j x_j.$$

Finally, multiplying by $\sum_{c_i \in I_+} c_i x_i$ on both sides we get

$$0 \leq \left\| \sum_{c_i \in I_+} c_i x_i \right\|^2 = \left\langle \sum_{c_i \in I_+} c_i x_i, \sum_{c_j \in I_-} -c_j x_j \right\rangle = \sum c_i (-c_j) \langle x_i, x_j \rangle < 0,$$

which is the desired contradiction. \square

3. PROOF OF THEOREM 1 (DIMENSIONS $4 \leq n \leq 15$)

Suppose K is a cap body in \mathbb{R}^n with vertices x_1, x_2, \dots, x_m , and the corresponding radii of the caps $\varphi_1, \varphi_2, \dots, \varphi_m$. Without loss of generality, assume $\varphi_1 \geq \varphi_2 \geq \dots \geq \varphi_m$.

To illuminate K , we begin by using a set of directions \mathcal{T} which is the union of s random rotations of S_n and l random rotations of C_n , where s and l satisfying $s + l \geq 1$ will be selected later. It is routine work to show that any rotation of either S_n or C_n satisfies Theorem 4 (ii), and thus we will only need to verify Theorem 4 (i).

Since $C[S_n, \theta] = \mathbb{S}^{n-1}$ when $\theta \geq \arccos\left(\frac{1}{n}\right)$, and $C[C_n, \theta] = \mathbb{S}^{n-1}$ when $\theta \geq \arccos\left(\frac{1}{\sqrt{n}}\right)$, by Theorem 4 (i), any vertex x_j of K with $\varphi_j < \frac{\pi}{2} - \arccos\left(\frac{1}{n}\right)$ is illuminated by one of the directions from \mathcal{T} . Thus, in what follows, we assume that $\varphi_m \geq \frac{\pi}{2} - \arccos\left(\frac{1}{n}\right)$.

Next, we will upper bound the expected number of caps not illuminated by \mathcal{T} . This will be done utilizing integer linear programming. We need to consider various cases depending on the number of caps of different sizes. Set $a_0 := \frac{\pi}{2} - \arccos\left(\frac{1}{n}\right) \leq \varphi_m$. For a suitable positive integer t that will be selected later, we define a discretization array $a = [a_0, a_1, \dots, a_t]$ and $a_i = a_0 + \frac{\pi}{180}i$ for $1 \leq i \leq t-1$, where $a_t := \frac{\pi}{2}$ and $a_i \leq a_t$. We have $\varphi_j \in (a_i, a_{i+1}]$ for any j . Let n_i , $0 \leq i \leq t-1$, denote the number of indices j such that $\varphi_j \in (a_i, a_{i+1}]$. The non-negative integers n_i , $0 \leq i \leq t-1$, will be the variables in the linear integer programming problem we are going to set up.

Note that by Theorem 4 (i) a vertex x_j with $\varphi_j \in (a_i, a_{i+1}]$ is illuminated, provided at least one of the points of \mathcal{T} is (strictly) within the geodesic distance $\frac{\pi}{2} - \varphi_j$ of \hat{x}_j . Therefore, the probability that x_j is not illuminated by \mathcal{T} is

$$(1 - \sigma(C[S_n, \frac{\pi}{2} - \varphi_j]))^s \cdot (1 - \sigma(C[C_n, \frac{\pi}{2} - \varphi_j]))^l,$$

and, overall, the expected number of caps which are not illuminated does not exceed

$$(6) \quad \sum_{i=0}^{t-1} n_i (1 - \sigma(C[S_n, \frac{\pi}{2} - a_{i+1}]))^s \cdot (1 - \sigma(C[C_n, \frac{\pi}{2} - a_{i+1}]))^l =: \sum_{i=0}^{t-1} n_i u_i^s v_i^l.$$

Evaluating precisely u_i and v_i may be difficult (e.g. for certain values of angles three or more caps may overlap), but for our purposes, it suffices to upper bound them. For this, we use Theorems 6 and 7, where $A_n(\theta)$ and $A_n(\alpha, \beta)$ are computed by numerical integration using (4). Denoting the corresponding upper bounds \bar{u}_i and \bar{v}_i , respectively we obtain that the expected number of caps not illuminated by \mathcal{T} does not exceed

$$(7) \quad \sum_{i=0}^{t-1} n_i (\bar{u}_i)^s (\bar{v}_i)^l,$$

which will be our target function in the integer programming problem.

Besides non-negativity of n_i , we will use two constraints. First, recalling (2), we get that the total measure of these caps is at most 1. In terms of n_i , this provides the following constraint:

$$(8) \quad \sum_{i=0}^{t-1} n_i A_n(a_i) \leq 1.$$

The second constraint directly follows from (2) and Theorem 8:

$$(9) \quad \sum_{0 \leq i < t: a_i \geq \frac{\pi}{4}} n_i \leq n + 1.$$

Let $M_{s,l,t}$ denote the solution of the integer linear programming problem

$$\text{maximize (7) subject to (8) and (9)}$$

with non-negative integer variables n_i , $0 \leq i \leq t-1$. We get $I(\mathcal{K}_c^n) \leq \lfloor M_{s,l,t} \rfloor + (n+1)s + 2nl$. For each dimension n , $4 \leq n \leq 15$, by computer search taking $50 \leq t \leq 500$, and suitable values of s, l we obtained the results in Table 1. The computations can be verified by running the corresponding SageMath ([8]) script provided in the appendix which takes about ten minutes on a modern personal computer.

n	$I(\mathcal{K}_c^n) \leq$	s (simplices)	l (cross-polytopes)
4	11	1	0
5	17	0	1
6	29	1	1
7	44	1	1
8	69	2	1
9	106	4	1
10	159	4	2
11	239	4	4
12	355	5	5
13	525	5	8
14	776	6	12
15	1138	6	17

TABLE 1. Bound for $4 \leq n \leq 15$.

4. PROOF OF THEOREM 2 (DIMENSIONS $n \geq 9$)

Assume that K is a cap body in \mathbb{R}^n for $n \geq 9$. We will begin by illuminating K with the union of $x \geq 1$ independent random rotations of the cross-polytope C_n . It is immediate that any rotation of C_n satisfies Theorem 4 (ii), so we will only need to worry about Theorem 4 (i). Also, it is a simple computation that $C[C_n, \theta] = \mathbb{S}^{n-1}$ when $\theta \geq \arccos \frac{1}{\sqrt{n}}$. Therefore, any cap with radius smaller than $\frac{\pi}{2} - \arccos \frac{1}{\sqrt{n}}$ will be illuminated. On the other hand, by Theorem 8, there are at most $n+1$ caps of radius $> \pi/4$.

Now let \mathcal{C} be the caps of K with radii in the interval $\left[\frac{\pi}{2} - \arccos \frac{1}{\sqrt{n}}, \frac{\pi}{4}\right]$, and M be the cardinality of \mathcal{C} . Define $p := A_n\left(\frac{\pi}{4}\right)$, and $q := A_n\left(\frac{\pi}{2} - \arccos \frac{1}{\sqrt{n}}\right)$. Note that the $2n$ caps in $C[C_n, \pi/4]$ have no common relative interior, so $2np < 1$. A cap from \mathcal{C} is not illuminated by any of x rotations of C_n with probability $(1 - 2np)^x$, so we can choose the rotations in a way that there are at most $(1 - 2np)^x M$ unilluminated caps from \mathcal{C} . All together, taking into account that $Mq \leq 1$ as base caps do not overlap, we see that K can be illuminated by at most

$$(10) \quad f(x) := (1 - 2np)^x \frac{1}{q} + 2nx + n + 1$$

directions. Now we want to minimize $f(x)$ over $x \in \mathbb{Z}^+$.

Note that $f'(x) = (1 - 2np)^x \frac{1}{q} \ln(1 - 2np) + 2n$, and the only point $x_0 > 0$ satisfying $f'(x_0) = 0$ is

$$(11) \quad x_0 = \frac{1}{-\ln(1 - 2np)} (\ln(-\ln(1 - 2np)) - \ln(2nq)).$$

Thus, $\min_{x \in \mathbb{Z}^+} f(x)$ must be attained at $y_1 = \lfloor x_0 \rfloor$ or $y_2 = \lceil x_0 \rceil$. We get

$$(12) \quad \frac{f(y_i) - f(x_0)}{y_i - x_0} = f'(z_0),$$

where $|z_0 - x_0| < 1$. Now, suppose that $z_0 = x_0 + t$, then by $f'(x_0) = 0$

$$\begin{aligned} f'(z_0) &= f'(x_0 + t) \\ &= (1 - 2np)^t (1 - 2np)^{x_0} \frac{1}{q} \ln(1 - 2np) + 2n \\ &= 2n(1 - (1 - 2np)^t) =: g(t). \end{aligned}$$

Since g is a monotone function of $t \in [-1, 1]$, we have $\max_{t \in [-1, 1]} |g(t)|$ is achieved either at $t = 1$ or $t = -1$, and a direct verification gives $|g(-1)| > |g(1)|$. We conclude that $|f'(z_0)| \leq |g(-1)| = \frac{4n^2p}{1-2np}$. By (12), $|f(y_i) - f(x_0)| \leq |f'(z_0)|$, so with the help of (11), we get

$$(13) \quad \begin{aligned} f(y_i) &\leq \frac{4n^2p}{1-2np} + f(x_0) \\ &= \frac{4n^2p}{1-2np} + \frac{2n}{-\ln(1-2np)} + \frac{2n}{-\ln(1-2np)} (\ln(-\ln(1-2np)) - \ln(2nq)) + n + 1. \end{aligned}$$

Using (3) with angles $\theta = \pi/4, \pi/2 - \arccos(1/\sqrt{n})$ we obtain estimates on p, q

$$(14) \quad \left(\frac{1}{\sqrt{2}}\right)^n \frac{1}{\sqrt{\pi n}} \leq p \leq \left(\frac{1}{\sqrt{2}}\right)^n \sqrt{\frac{2}{\pi(n-1)}}, \quad \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{n}}\right)^n \leq q.$$

We will use the estimates $x/(1+x) \leq \ln(1+x) \leq x$ which are valid for $x > -1$. We have $-\ln(1-2np) \leq 2np/(1-2np) \leq 3/n$ for all $n \geq 9$, where the second inequality is verified by direct computation for initial values and standard calculus for larger values.

So, continuing (13),

$$\begin{aligned} f(y_i) &\leq 6 + \frac{2n}{2np} + \frac{2n}{2np} \left(\ln \left(-\frac{-2np}{1-2np} \right) - \ln(2nq) \right) + n + 1 \\ &= n + 7 + \frac{1}{p} (1 + \ln(p/q) - \ln(1-2np)) \\ &\leq n + 7 + \frac{1}{p} (1 + \ln(p/q) + 3/n) \\ &\leq n + 7 + (\sqrt{2})^n \sqrt{\pi n} \left(1 + 3/n + \ln \left((\sqrt{n}/2)^n \frac{2}{\sqrt{n-1}} \right) \right) \\ &\leq n + 7 + (\sqrt{2})^n \sqrt{\pi n} \left(1 + \frac{3}{n} + \frac{1}{2} n \ln \frac{n}{2} + \ln \frac{2}{\sqrt{n-1}} \right), \end{aligned}$$

which is the claimed bound for $n \geq 9$. It is routine to check that this bound does not exceed $2^n - 1$ for $n \geq 13$.

5. DISCUSSION AND REMARKS

For each dimension n and fixed s, l , the best bound one can obtain with our method is $I(\mathcal{K}_c^n) \leq \lim_{t \rightarrow \infty} [M_{s,l,t}] + (n+1)s + 2nl$, which is attained for some possibly large t . We varied the number of intervals t between 50 and 500 selecting a smaller value when possible which still yields the same upper bound on $I(\mathcal{K}_c^n)$. Values of s and l were chosen on a case-by-case basis to fit the dimension n . It is likely that the resulting bounds we obtained for $4 \leq n \leq 8$ are best possible by the method, in other words, taking larger t or other s and l will not yield any improvement. For other dimensions, one can further increase t at expense of longer computations and possibly improve the estimates in Table 1. We note, however, that the computation cost of integer linear programming may be non-linear in t and could increase rapidly.

If one is interested only in establishing $I(\mathcal{K}_c^n) < 2^n$ for $4 \leq n \leq 15$, then t can be chosen to be much smaller, between 4 and 7. The corresponding values of t, s, l are available in the last portion of the script.

It is interesting to note that for $n = 4$ using a (random rotation of) simplex is better, while for $n = 5$ using a cross-polytope is better. Confirming $I(\mathcal{K}_c^n) < 2^n$ could have been done using only simplexes or using only cross-polytopes, but we chose to obtain better bounds and include both. It is apparent from Table 1, where we found heuristically optimal choices of s and l , that both configurations are useful for obtaining better estimates. One can also try and utilize random rotations of other various “good” configurations of points available in specific dimensions.

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APPENDIX

```
def inter(n, alpha, beta, mult=0): #prob area of intersection of two caps centres 2*
                                #beta apart, radius alpha as per (4), plus
                                #error of numerical integration
                                #multiplied by mult
```

```

integ = numerical_integral(lambda r: (n-2)/pi*(1-r^2)^((n-4)/2)*(arccos(cos(alpha)/r)-beta)*r, cos(alpha)/cos(beta), 1)

return integ[0]+mult*integ[1]

def cross_cap_prob(n,theta): #probabilistic measure of the union of spherical caps
                             centered at cross polytopes, for theta>=
                             pi/4,
#when theta is too large and three caps overlap, this will return an upper
                             bound
return 2*n*inter(n,theta,0,-1)-2*n*(n-1)*inter(n,theta,pi/4,1)

def simplex(n,theta): #same for simplex
return (n+1)*inter(n,theta,0,-1)-1/2*n*(n+1)*inter(n,theta,arccos(-1/n)/2,1)

def min_illum(n,t,s,c): #given dimension, discretization, number of simplexes,
                        number of cross-polytopes, compute the
                        bound

A=pi/2-arccos(1/n)
B=pi/2
a=[A+(B-A)*i/t for i in range(t+1)]

probc=[]
probs=[]

pc_trip = cross_cap_prob(n,arccos(1/sqrt(3)))

for i in range(1,t+1): #note loop starts with 1 so probc and probs indexes one
                        lower
    if a[i]<=pi/2-arccos(1/sqrt(n)): #covering radius for cross polytope
        probc.append(1)
    elif a[i]<=pi/2-arccos(1/sqrt(3)): #there is triple intersection, simple
        bound
        probc.append(pc_trip)
    elif a[i]<pi/4: #double only
        probc.append(cross_cap_prob(n,pi/2-a[i]))
    else: #no intersections
        probc.append((2*n)*inter(n,pi/2-a[i],0))

ps_trip = simplex(n,arccos(1/3))

for i in range(1,t+1):
    if a[i]<pi/2-arccos(1/n): #never true, for homogeneity
        probs.append(1)
    elif a[i]<=pi/2-arccos(1/3): #triple, simple bound
        probs.append(ps_trip)
    elif a[i]<pi/2-1/2*arccos(-1/n): #double
        probs.append(min(1,simplex(n,pi/2-a[i])))
    else: #no intersections
        probs.append((n+1)*inter(n,pi/2-a[i],0))

p = MixedIntegerLinearProgram()
v = p.new_variable(integer=True, nonnegative=True)
p.set_objective(sum((v[i]*(1-probc[i])^c*(1-probs[i])^s) for i in range(t)))
p.add_constraint(sum(v[i]*inter(n,a[i],0,-1) for i in range(t))<= 1)
p.add_constraint(sum(v[i] for i in range(t) if a[i]>=pi/4)<=n+1) #no more than
n+1 caps at least pi/4
target=round(p.solve() + 2*n*c + s*(n+1), 4)

```

```
    return n, target

#larger t for better bound
print(min_illum(4, 50, 1, 0))
print(min_illum(5, 100, 0, 1))
print(min_illum(6, 300, 1, 1))
print(min_illum(7, 200, 1, 1))
print(min_illum(8, 250, 2, 1))
print(min_illum(9, 400, 4, 1))
print(min_illum(10, 500, 4, 2))
print(min_illum(11, 500, 4, 4))
print(min_illum(12, 500, 5, 5))
print(min_illum(13, 500, 5, 8))
print(min_illum(14, 500, 6, 12))
print(min_illum(15, 500, 6, 17))

#smaller t only to confirm conjecture
print(min_illum(4,7,0,1))
print(min_illum(5,6,0,1))
print(min_illum(6,6,0,2))
print(min_illum(7,5,2,3))
print(min_illum(8,5,2,5))
print(min_illum(9,5,4,6))
print(min_illum(10,4,0,40))
print(min_illum(11,4,0,63))
print(min_illum(12,4,0,97))
print(min_illum(13,4,0,148))
print(min_illum(14,4,0,224))
print(min_illum(15,4,0,337))
```

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