

Walrasian equilibria are almost always finite in number

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Abstract

We show that in the context of exchange economies defined by aggregate excess demand functions on the full open price simplex, the generic economy has a finite number of equilibria. Genericity is proved also for critical economies and, in both cases, in the strong sense that it holds for an open dense subset of economies in the Whitney topology. We use the concept of finite singularity type from singularity theory. This concept ensures that the number of equilibria of a map appear only in finite number. We then show that maps of finite singularity type make up an open and dense subset of all smooth maps and translate the result to the set of aggregate excess demand functions of an exchange economy.

Along the way, we extend the classical results of Sonnenschein-Mantel-Debreu to aggregate excess demand functions defined on the full open price simplex, rather than just compact subsets of the simplex.

Keywords: Sonnenschein-Mantel-Debreu, general equilibrium, aggregate excess demand, determinacy, genericity

JEL classification: C02, D50, C62

1 Introduction

We show that the set of economies with a finite number of Walrasian equilibria is an open and dense subset of all economies. Our results apply to both regular and critical economies. This means that economies with an infinite number of equilibria are difficult to encounter.

Our results add to those in Debreu [10], Allen [2], Mas-Colell and Nachbar [18], and more recently Castro *et al.* [7]. In this latter work the authors prove that the set of all economies defined by an aggregate excess demand (AED) on the trimmed simplex (that is, such that every price remains bounded away from zero) and with a finite number of equilibria is residual (that is, a countable intersection of open and dense sets) in the space of all maps for the compact-open topology. In other words, having a finite number of equilibria is a generic property for an economy. Although suitable, the compact-open topology is not the finer topology on a space of functions and the same result for the Whitney topology was left unproved.

Several technical points prevented the authors of [7] from completing the argument for the Whitney topology as well as extending it to economies defined in the full open price simplex. These are overcome in the current article where the results hold for the Whitney topology and the full open simplex. We note that, in the present context of the open simplex, the Whitney topology is the more appropriate topology because, by its very definition, the compact-open topology is not sensitive to the behaviour of functions close to the boundary of the simplex. Besides, in the Whitney topology, we obtain a stronger genericity result on finiteness of equilibria, showing that it holds for an open and dense space of economies, rather than the weaker notion of residual. To illustrate why open and dense is a stronger notion of generic, recall that in the simpler context of measure spaces the complement to such a subset has measure zero, whereas this is not necessarily the case for the complement of a residual set.

Since there are examples of economies with an infinite number of equilibria the results we present here are the best possible. Thus we conclude the program initiated by Debreu [10].

Our approach rests on identifying the space of economies, via their aggregate excess demand functions, with the space of proper maps defined on the open price simplex. To this purpose we also prove a version of the Sonnenschein-Mantel-Debreu result that holds on the open simplex, rather than the standard versions which hold only on compact subsets of the open simplex.

Additionally, we show that any exchange economy stems from a finite number of Cobb–Douglas utility maximising consumers in a way to be made precise in Theorem 3.1. Related examples using Cobb–Douglas consumers appear in Aloqeili [3] and Chiappori and Ekeland [9].

It is important to stress that our results rely on the use of singularity theory, an area of mathematics that has not been commonly used in economics.

Moreover, we use singularity theory results that first appeared in the 1990's¹, well after economists had settled for an incomplete answer to the question of finiteness of equilibria.

The next section presents the technical concepts we use. These repeat some parts of [7] but we include them here for ease of reference. Section 3 presents a constructive proof of the results by Sonnenschein [20], Mantel [16], and Debreu [11] that holds in the open price simplex. Section 4 contains our main results and the final section concludes.

2 Preliminary results and definitions

We start by defining notation to be used throughout this article.

Let $\mathbb{R}_+$ denote the set of non-negative numbers and let $\mathbb{R}_{++}$ denote the set of strictly positive numbers.

Let $S_{++}^{\ell-1} = \{p \in \mathbb{R}^\ell : p_i > 0, i = 1, \dots, \ell \text{ and } \|p\| = 1\}$ denote the intersection of the unit sphere with the strictly positive orthant. This is the space of normalised positive prices.

Let $\Delta = \{p \in \mathbb{R}_+^\ell : \sum_{i=1}^\ell p_i = 1\}$ be the unit simplex in $\mathbb{R}^\ell$. This is the full price simplex for the economy. Its interior is denoted $\mathring{\Delta}$ and also referred to as the open price simplex.

These sets are used to define an economy with a finite number of consumers and goods. The consumers are represented by continuous utility functions. From Proposition 3.3 in [6], there is a homeomorphism between the space of utilities and the space of preferences describing the economy. Since we work with the open price simplex we impose a boundary condition, as usual: in this context the utilities are proper maps².

An excess demand function $f(p)$ on $S_{++}^{\ell-1}$ with image in $\mathbb{R}^\ell$ satisfies Walras' Law $f(p) \cdot p = 0$ and can therefore be identified with a tangent vector field on $S_{++}^{\ell-1}$. Trivialising the tangent bundle such vector fields can be identified with maps taking values in $\mathbb{R}^{\ell-1}$. Moreover, by a standard renormalisation procedure, we can use the sphere $S_{++}^{\ell-1}$ and the open simplex $\mathring{\Delta}$ interchangeably. We then define the set of AEDs with domain the open price simplex as

$$\mathcal{Z}^* = \{z \in C^k(\mathring{\Delta}, \mathbb{R}^{\ell-1}) : z \text{ is proper}\}.$$

The results of [7] hold in the same way for the compact-open and Whitney topology because they are obtain in a compact set, the price simplex.

¹We became aware of the relevant results in singularity theory, namely the concept of finite singularity type, by chance while having a conversation with Andrew du Plessis, the first author of [12].

²A proper map is such that the inverse image of a compact set is itself compact.

Here, we need to distinguish between the two topologies and we recall them next. Let $C^k(N, P) = \{f : N \rightarrow P : f \text{ is of class } C^k\}$ for $1 \leq k \leq \infty$. Throughout the next section we state the results for arbitrary manifolds N as the source and P as the target for a space of functions. These results were obtained without reference to their application to general equilibrium in economics.

In the space of C^k maps between spaces N and P there are two natural topologies: the C^m compact-open and the C^m Whitney topology for $1 \leq m \leq \infty$, where $m \leq k$. The Whitney topology is finer than the compact-open in the sense that it has more open sets. Our definitions follow du Plessis and Wall [12]. We start with $m = 0$ and note that the C^0 topologies compare values of functions. The C^m topologies compare values of derivatives up to order m and therefore contain the topologies for all smaller degrees of differentiability. The basic open sets for the C^0 compact-open topology on $C^k(N, P)$ are subsets defined by

$$A(K, U) = \{f \mid f(K) \subset U\},$$

where $K \subset N$ is compact and $U \subset P$ is open. These open sets are used to define the basic open sets for the C^0 Whitney topology which are intersections of a collection of sets $A(K, U), \cap_\alpha A(K_\alpha, U_\alpha)$ where $\{K_\alpha\}$ is a locally finite collection of compact subsets of N and $\{U_\alpha\}$ is a collection of open subsets of P . The C^m topologies, for $m > 0$, are defined by using not just the values of f but also of its derivatives up to order m .

The concept of finite singularity type (FST) is presented in various forms in Section 2.4 of [12] to which we refer the interested reader. For our results it suffices to think that a map is FST if it is locally equivalent to its Taylor polynomial up to a finite order. By locally equivalent we mean that there are coordinate changes in the source and the target space, defined in a small neighbourhood of a point and its value by f , that transform f into its Taylor polynomial of finite order. This is a subtle concept in singularity theory and was used by the authors in [7] where more detail can be found.

3 “Anything goes” to the boundary

For the genericity results of Section 4 below it is important that the space of AED functions can be identified with the space $\mathcal{Z}^*$. Thus, we require that the Sonnenschein-Mantel-Debreu results hold for functions $z(p)$ defined on the open price simplex $\mathring{\Delta}$. Recall that up until now these results are shown to hold only on compact subsets of the open simplex. In this section we provide a proof in $\mathring{\Delta}$. The argument follows the idea outlined by Mas-Colell

[17, p. 193]. However, we show that one can in fact use a finite number of Cobb–Douglas consumers to span the space of AEDs, and we complete the argument by appealing to a theorem of Afriat stating that a positive multiple of an individual demand function is again the individual demand of a continuous consumer.

For the arguments we present in this section it is convenient normalise prices to belong to $S_{++}^{\ell-1}$, the positive part of the sphere since this is the natural mathematical setting for the proofs.

Theorem 3.1. *There are ℓ Cobb–Douglas utility maximising consumers with individual excess demand functions $z_i^{CD} : S_{++}^{\ell-1} \rightarrow \mathbb{R}^{\ell-1}$ satisfying:*

For any $z \in C^k(S_{++}^{\ell-1}, \mathbb{R}^{\ell-1})$, there are ℓ strictly positive functions $\mu_i \in C^k(S_{++}^{\ell-1}, \mathbb{R})$ such that

$$z(p) = \sum_{i=1}^{\ell} \mu_i(p) z_i^{CD}(p).$$

Proof. Consider the Cobb–Douglas utility function

$$u(x_1, \dots, x_\ell) = x_1^{\alpha_1} \dots x_\ell^{\alpha_\ell}$$

where the parameters α_i are strictly positive and satisfy $\alpha_1 + \dots + \alpha_\ell = 1$. The corresponding demand of a consumer with endowment $\omega = (\omega_1, \dots, \omega_\ell)$ is

$$x(p, \omega) = \left(\frac{\alpha_1}{p_1} w, \dots, \frac{\alpha_\ell}{p_\ell} w \right)$$

where $w = p \cdot \omega$ is wealth. Writing $A = \alpha_1^{\alpha_1} \dots \alpha_\ell^{\alpha_\ell}$ this translates into the indirect utility

$$V(p, \omega) = A p_1^{-\alpha_1} \dots p_\ell^{-\alpha_\ell} \cdot \left((1 - \alpha_1) \omega_1 - \frac{\alpha_1}{p_1} (w - p_1 \omega_1), \dots, (1 - \alpha_\ell) \omega_\ell - \frac{\alpha_\ell}{p_\ell} (w - p_\ell \omega_\ell) \right).$$

We define ℓ consumers by letting the endowment of consumer i be $\omega^i = (\omega_1^i, \dots, \omega_\ell^i)$, with $\omega_j^i = 0$ for $j \neq i$ and $\omega_i^i \neq 0$ for $i = 1, \dots, \ell$. With this choice, the ℓ gradient vectors (with respect to p)

$$\nabla_p V_i(p, \omega^i) = A p_1^{-\alpha_1} \dots p_\ell^{-\alpha_\ell} \left(-\alpha_1 \frac{p_i}{p_1} \omega_i^i, \dots, (1 - \alpha_i) \omega_i^i, \dots, -\alpha_\ell \frac{p_i}{p_\ell} \omega_i^i \right)$$

correspond to the individual excess demands

$$\begin{aligned} z_i^{CD}(p, \omega^i) &= -\frac{1}{A} p_1^{\alpha_1} \dots p_\ell^{\alpha_\ell} \nabla_p V_i(p, \omega^i) \\ &= \left(\alpha_1 \frac{p_i}{p_1} \omega_i^i, \dots, -(1 - \alpha_i) \omega_i^i, \dots, \alpha_\ell \frac{p_i}{p_\ell} \omega_i^i \right). \end{aligned}$$

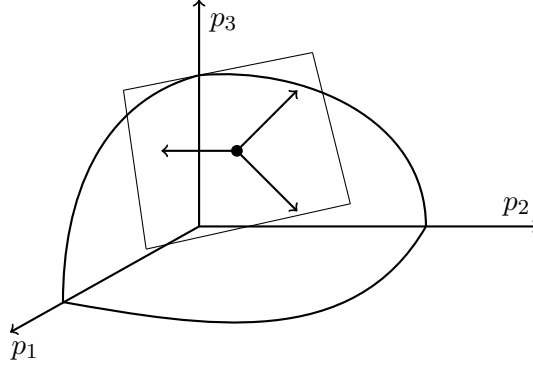


Figure 1: The tangent space is generated by positive linear combinations of three vectors.

Note that all coordinates are strictly positive, except for the i th one, which is strictly negative. Note also that, for each i there is a hyperplane in the tangent space $T_p S_{++}^{\ell-1}$ such that the tangent vector $z^i(p, \omega)$ and each of the remaining ones point to opposite sides of this hyperplane, as illustrated in the case $\ell = 3$ in Figure 1. Hence any vector in the $(\ell - 1)$ -dimensional tangent space to $S_{++}^{\ell-1}$ can be written as a linear combination of these with strictly positive coefficients. Therefore, given any $z \in C^k(S_{++}^{\ell-1}, \mathbb{R}^{\ell-1})$, one can find ℓ strictly positive functions $\mu_i \in C^k(S_{++}^{\ell-1}, \mathbb{R})$ such that

$$z(p) = \sum_{i=1}^{\ell} \mu_i(p) z_i^{CD}(p).$$

□

We remark that the individual excess demands $\mu_i(p) z_i^{CD}(p)$ used in the preceding proof may not be those of a Cobb–Douglas utility maximising consumer. That they are individual excess demands for some consumer is shown next.

Theorem 3.2. *Let $z(p)$ be an individual excess demand function obtained from utility maximisation and $\mu(p) > 0$ for all $p \in S_{++}^{\ell-1}$. Then $\mu(p)z(p)$ is the individual excess demand of a continuous consumer.*

Proof. Since $z(p)$ is obtained from utility maximisation it is bounded below and satisfies the Strong Axiom of Revealed Preferences. Moreover, the unique extension $\tilde{z}(p)$ of $z(p)$ to a homogeneous function of degree zero on the open orthant $\mathbb{R}_{++}^{\ell}$ satisfies Walras' Law because $z(p)$ is tangent to the unit sphere. Hence it follows from Afriat's Theorem [1] (cf. Jehle and Reny [14, p. 96]) that $\tilde{z}(p)$ is the individual excess demand of a continuous consumer. □

The previous result is attributed to Debreu by Chiappori and Ekeland [9] whose methods, however, require working on compact sets.

Recall that $\mathcal{Z}^* = \{z \in C^k(\mathring{\Delta}, \mathbb{R}^{\ell-1}) : z \text{ is proper}\}$. Any AED of class C^k belongs to this space. Conversely, we have the following version of the Sonnenschein-Mantel-Debreu results for the open simplex which follows immediately from Theorems 3.1 and 3.2 above.

Theorem 3.3. *For any $z \in \mathcal{Z}^*$, there is an economy with ℓ consumers such that z is the AED of this economy.*

Remark 3.4. It has been argued by Balasko [5] that there are natural topological constraints on AEDs defined on the open simplex: viewing an AED as a tangent vector field on $S_{++}^{\ell-1}$, this vector field should be inward pointing (corresponding to positive excess demand of a good whose price goes to zero). This means that the Brouwer degree of the restriction of the AED to the boundary of $S_{++}^{\ell-1}$ is ± 1 (the sign depends on the parity of ℓ). All our results continue to hold under this restriction, since it amounts to fixing a connected component of $\mathcal{Z}^*$ labelled by the Brouwer degree.

4 Finiteness of equilibria

In this section we extend the results of [7] to the Whitney topology which allows us to make a stronger genericity statement for finiteness of equilibria, even for critical economies.

Theorem 4.1. *The subspace of smooth FST maps is open and dense in $\mathcal{Z}^*$.*

Proof. This proof follows an analogous strategy to that of Theorem 5.1 in [7]. The key point is the following: the open simplex $\mathring{\Delta}$ has a locally finite covering by closed balls.

In view of the definition of the Whitney topology this means that the proof of Proposition 3.3 in [7] immediately adapts to show that the subspace $T_{W,pr}$ of that proposition is actually *open* in the Whitney topology. It follows, as in Theorem 5.1 and its Corollary 5.3 in [7], that FST maps are open and dense in the Whitney topology. $\square$

Theorem 4.2. *There is an open and dense set of AEDs in $\mathcal{Z}^*$ for which the number of equilibria is finite.*

Proof. Since any f of FST has a finite number of zeros, this follows from Theorem 4.1. $\square$

Let $\mathcal{Z}_{crit}^* \subset \mathcal{Z}^*$ denote the subspace of critical economies.

Theorem 4.3. *There is an open and dense subset of economies in $\mathcal{Z}_{\text{crit}}^*$ with a finite number of equilibria.*

Proof. It follows from Theorem 5.4 of [7] that the subset of $\mathcal{Z}_{\text{crit}}^*$ which are of FST is dense in $\mathcal{Z}_{\text{crit}}^*$. Since $\mathcal{Z}_{\text{crit}}^* \subset \mathcal{Z}^*$ has the subspace topology, openness follows from Theorem 4.1. $\square$

5 Final remarks

The results presented above close in a satisfactory way a question that persisted for over half a century. They also provide sound foundations for decision-making in economics based on the possibility of distinguishing between equilibrium states that are distinct, and in finite number.

Although the interest in general equilibrium theory is not at its peak, several aspects of existence and finiteness of Walrasian equilibria continue to receive attention from the scientific community. Recently the existence question has been treated with relaxed assumptions by Podczeck and Yannelis [19], Khan *et al.* [15], and Anderson and Duanmu [4]. The computation of equilibria has been addressed by Gauthier *et al.* [13] who provide an algorithm to construct examples with multiple Walrasian equilibria. Also, Cheung *et al.* [8] determine when a gradient descent method can be used to compute the equilibria.

Our results here and in [7] provide a solid foundation for the continued quest for methods to calculate and study Walrasian equilibria since we establish that equilibria are almost always in finite number.

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