ON TORSION IN THE HOMOLOGY OF THE TORELLI GROUP

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ABSTRACT. Let S_g be a closed, oriented surface of genus g, and let $\mathrm{Mod}(S_g)$ denote its mapping class group. The *Torelli group* $\mathcal{I}_g$ is the subgroup of $\mathrm{Mod}(S_g)$ consisting of mapping classes that act trivially on $H_1(S_g)$. For any collection of pairwise disjoint, separating simple closed curves on S_g, the corresponding Dehn twists pairwise commute and determine a homology class in $H_k(\mathcal{I}_g)$, which is called an *abelian cycle*. We prove that the subgroup of $H_k(\mathcal{I}_g)$ generated by such abelian cycles is a $\mathbb{Z}/2\mathbb{Z}$-vector space for all k, and that it is finite-dimensional for $k = 2$ and $g \geq 4$.

1. INTRODUCTION

Let S_g^b be a compact oriented surface of genus g with b boundary components. We omit b if it vanishes. The *mapping class group* of S_g^b is the group

$$\mathrm{Mod}(S_g^b) = \pi_0(\mathrm{Homeo}^+(S_g^b, \partial S_g^b)),$$

where $\mathrm{Homeo}^+(S_g^b, \partial S_g^b)$ is the group of orientation-preserving homeomorphisms of S_g^b that fix each boundary component pointwise. There is a natural action of $\mathrm{Mod}(S_g)$ on $H_1(S_g; \mathbb{Z})$. This action preserves the *intersection form*, yielding a surjection

$$\Psi\colon\ \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}).$$

The *Torelli group* of S_g is defined to be the kernel of Ψ and is denoted by $\mathcal{I}_g$. Similarly, one can define the Torelli group $\mathcal{I}_g^1$ for a surface with one boundary component. For $b \in \{0, 1\}$, we will denote by $\mathcal{I}_g^b = \mathcal{I}(S_g^b)$ the Torelli group of S_g^b.

It is a well-known fact that for $g = 1$, the homomorphism Ψ is actually an isomorphism, so $\mathrm{Mod}(S_1) \cong \mathrm{SL}(2, \mathbb{Z})$ and $\mathcal{I}_1$ is trivial. Mess showed [12] that the group $\mathcal{I}_2$ is an infinitely generated free group. Johnson proved [9] that the Torelli groups $\mathcal{I}_g$ and $\mathcal{I}_g^1$ are finitely generated for $g \geq 3$, and explicitly determined a generating set.

It is an important and interesting problem to study the homology groups of the Torelli groups $\mathcal{I}_g$ for $g \geq 3$. We denote the integral homology $H_k(G; \mathbb{Z})$ by $H_k(G)$.

Let us recall some known facts about homology groups $H_k(\mathcal{I}_g)$:

- Johnson [10] computed the abelianization of the Torelli group $\mathcal{I}_g$ for $g \geq 3$:

$$H_1(\mathcal{I}_g) \cong \mathbb{Z}^{\binom{2g}{3}-2g} \oplus (\mathbb{Z}/2\mathbb{Z})^{\binom{2g}{2}+2g}.$$

- Bestvina–Bux–Margalit [1] proved that for $g \geq 3$, the cohomological dimension of $\mathcal{I}_g$ is $3g-5$, and that the top homology group $H_{3g-5}(\mathcal{I}_g)$ is not finitely generated.
- Gaifullin [5] proved that $H_k(\mathcal{I}_g)$ contains a free abelian group of infinite rank for $2g-3 \leq k < 3g-5$, which implies that these homology groups are not finitely generated.

Studying $H_2(\mathcal{I}_g)$ is especially important, as it is closely related to the question of whether $\mathcal{I}_g$ is finitely presented. Let us list some known facts about $H_2(\mathcal{I}_g)$:

- In 2018, Kassabov and Putman [11] proved that for $g \geq 3$, the group $H_2(\mathcal{I}_g)$ is a finitely generated $\mathrm{Sp}(2g, \mathbb{Z})$-module. However, this result provides no information about the finite generation of $H_2(\mathcal{I}_g)$ as a group.

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- In 2023, Minahan [13] proved that for $g \geq 51$, the vector space $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite-dimensional.
- In 2025, Minahan and Putman [14] proved that for $g \geq 5$, the vector space $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite-dimensional. They also explicitly computed the second rational cohomology group of the Torelli groups for $g \geq 6$.

The torsion in $H_k(\mathcal{I}_g)$ remains largely mysterious for $k \geq 2$. In this paper, we study this torsion and, to state our main result precisely, we first introduce some necessary definitions.

Let $h_1, \dots, h_k$ be pairwise commuting elements of a group G. Consider the homomorphism $\phi\colon \mathbb{Z}^k \to G$ that sends the generator of the i^{th} factor to h_i. The *abelian cycle* $\mathcal{A}(h_1, \dots, h_k)$ is defined as the image of the standard generator $\mu_k \in H_k(\mathbb{Z}^k) \cong \mathbb{Z}$ under the pushforward map $\phi_*\colon H_k(\mathbb{Z}^k) \to H_k(G)$.

Throughout the paper, we will often use the following standard properties of abelian cycles:

(1) For commuting elements $h_1, h_1', \dots, h_k \in G$,
$$\mathcal{A}(h_1h_1', h_2, \dots, h_k) = \mathcal{A}(h_1, h_2, \dots, h_k) + \mathcal{A}(h_1', h_2, \dots, h_k),$$
whenever all three abelian cycles are defined.

(2) For any permutation π of $\{1, \dots, k\}$,
$$\mathcal{A}(h_{\pi(1)}, \dots, h_{\pi(k)}) = \operatorname{sgn}(\pi)\, \mathcal{A}(h_1, \dots, h_k).$$

We will frequently use the following immediate consequence of property (1): if $x, y \in G$ commute with $z_1, \dots, z_{n-1} \in G$, then
$$\mathcal{A}([x,y], z_1, \dots, z_{n-1}) = 0,$$
where $[x,y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y.

We denote by $H_k^{\text{ab,sep}}(\mathcal{I}_g)$ the subgroup of $H_k(\mathcal{I}_g)$ generated by all abelian cycles of the form $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$, where $\delta_1, \dots, \delta_k$ are pairwise disjoint, nonisotopic separating simple closed curves on S_g, and T_δ denotes the left Dehn twist about δ. In this paper, we consider this subgroup for $k \geq 2$, since $H_1^{\text{ab,sep}}(\mathcal{I}_g)$ was fully described by Johnson (see [10]).

The *genus* of a separating simple closed curve γ is defined as the smaller of the genera of the two surfaces bounded by γ and is denoted by $g(\gamma)$. One of the main results of this paper is the following theorem.

Theorem 1.1. *For $g \geq 3$ and $k \geq 2$, the following hold:*

- *If $k < g$, the group $H_k^{\text{ab,sep}}(\mathcal{I}_g)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space, generated by abelian cycles $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$ with $\delta_1, \dots, \delta_k$, pairwise disjoint, pairwise nonisotopic, separating simple closed curves of genus* 1.
- *If $k \geq g$, the group $H_k^{\text{ab,sep}}(\mathcal{I}_g)$ is trivial.*

Remark 1.2. If $k > 2g-3$, the group $H_k^{\text{ab,sep}}(\mathcal{I}_g)$ is trivial, since there are no abelian cycles of the form $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$. Indeed, a simple Euler characteristic argument shows that S_g admits at most $2g-3$ pairwise disjoint, pairwise nonisotopic separating simple closed curves.

Remark 1.3. Vautaw [15] showed that any abelian subgroup of $\mathcal{I}_g$ has rank at most $2g-3$. Hence, the subgroup of $H_k(\mathcal{I}_g)$ generated by abelian cycles is trivial for $2g-3 < k \leq 3g-5$, and therefore is not of interest.

Corollary 1.4. *The subgroup $H_k^{\text{ab,sep}}(\mathcal{I}_g) \subset H_k(\mathcal{I}_g)$ is finitely generated if and only if there are only finitely many distinct abelian cycles of the form $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$. In particular, if there are infinitely many such abelian cycles, then $H_k(\mathcal{I}_g)$ is not finitely generated.*

The main result of this paper is the following theorem.

Theorem 1.5. *The $\mathbb{Z}/2\mathbb{Z}$-vector space $H_2^{\text{ab,sep}}(\mathcal{I}_g)$ is finite-dimensional for $g \geq 4$.*

Remark 1.6. The question of whether $H_2^{\mathrm{ab,sep}}(\mathcal{I}_3)$ is finite-dimensional remains open. In [4], Gaifullin studied the spectral sequence associated with the action of $\mathcal{I}_3$ on the complex of cycles introduced by Bestvina–Bux–Margalit [1]. His work provides evidence suggesting that $H_2^{\mathrm{ab,sep}}(\mathcal{I}_3)$ may not be finitely generated.

Brendle and Farb [2] showed that $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ is nontrivial in $H_2(\mathcal{I}_g)$ for two disjoint, nonisotopic, separating simple closed curves δ_1 and δ_2. However, the analogous statement does not always hold in $H_k(\mathcal{I}_g)$ for $k > 2$. We now give a criterion for when $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$ vanishes in $H_k(\mathcal{I}_g)$.

Theorem 1.7. *Let $\delta_1, \dots, \delta_k$ be pairwise disjoint, pairwise nonisotopic, separating simple closed curves on S_g with $k < g$. Let $\Sigma_1, \dots, \Sigma_{k+1}$ be the surfaces obtained by cutting S_g along $\delta_1, \dots, \delta_k$. Then $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$ in $H_k(\mathcal{I}_g)$ if and only if at least one of the surfaces $\Sigma_1, \dots, \Sigma_{k+1}$ has genus 0.*

Let us introduce some notation and conventions used throughout the paper. A closed curve is called *simple* if it has no self-intersections, and a simple closed curve is called *essential* if it is not nullhomotopic. Unless stated otherwise, by a *curve* we mean an essential simple closed curve. For a curve α on S_g^b, we denote by $[\alpha]$ its homology class in $H_1(S_g^b)$. For homology classes $a, b \in H_1(S_g^b)$, we denote by $a \cdot b$ their algebraic intersection number.

For integral homology classes $x, y, \dots$, we denote their reductions modulo 2 by bold letters $\mathbf{x}, \mathbf{y}, \dots$. Similarly, for symplectic subgroups $V, U, \dots$ of the integral homology groups, their reductions modulo 2 are denoted by $\mathbf{V}, \mathbf{U}, \dots$.

We denote by $g(S)$ the genus of a surface S. For an element $h \in G$, we denote by $[h]$ its homology class in $H_1(G)$.

This paper is organized as follows. In §2, we review known relations in the mapping class group and the Torelli group. In §3, we prove that abelian cycles of the form $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ have order 2 in $H_2(\mathcal{I}_g)$ for $g \geq 4$ (Proposition 3.1). In §4, we establish an auxiliary relation (Proposition 4.2) and use it to prove the main relations (Propositions 5.1 and 5.6) in §5. There, we also prove Theorem 1.1 for $k = 2$ and $g \geq 3$ (Propositions 5.3 and 5.5). In §6, we derive a key relation between abelian cycles (Proposition 6.1) using the Birman–Craggs–Johnson homomorphism. In §7, we prove Theorem 1.5. Finally, in §8, we complete the proof of Theorem 1.1 by treating the remaining cases. There, we also prove Theorem 1.7.

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2. Basic relations in $\mathrm{Mod}(S)$ and $\mathcal{I}(S)$

In this section, we recall some known relations in the groups $\mathrm{Mod}(S)$ and $\mathcal{I}(S)$. We begin by restating the following result of Johnson (see [10, Corollary from Lemma 3]):

$$2[T_\delta] = 0 \quad \text{in } H_1(\mathcal{I}_g^b), \quad b \in \{0, 1\}. \tag{2.1}$$

Proposition 2.1. *Let δ be a separating simple closed curve on S_g^b with $g \geq 3$ and $b \in \{0, 1\}$. Then there exist elements $h_1, f_1, \dots, h_m, f_m \in \mathcal{I}_g^b$ such that*

$$T_\delta^2 = [h_1, f_1] \cdots [h_m, f_m].$$

Corollary 2.2. *Let δ be a simple closed curve on S_g bounding a subsurface Σ of genus at least 3. Then there exist elements $h_1, f_1, \dots, h_m, f_m \in \mathcal{I}_g$ that are all supported on Σ such that*

$$T_\delta^2 = [h_1, f_1] \cdots [h_m, f_m].$$

One of the most important relations in Torelli group theory is the *lantern relation* in $\mathrm{Mod}(S_g)$ (see [3, Proposition 5.1]).

Proposition 2.3 (Lantern relation). *Let b_1, b_2, b_3, b_4 be the boundary components of S_0^4, and let x, y, z be the simple closed curves on S_0^4 as shown in Figure 1. In* $\mathrm{Mod}(S_0^4)$*, the following relation holds*

$$T_{b_1}T_{b_2}T_{b_3}T_{b_4} = T_xT_yT_z.$$

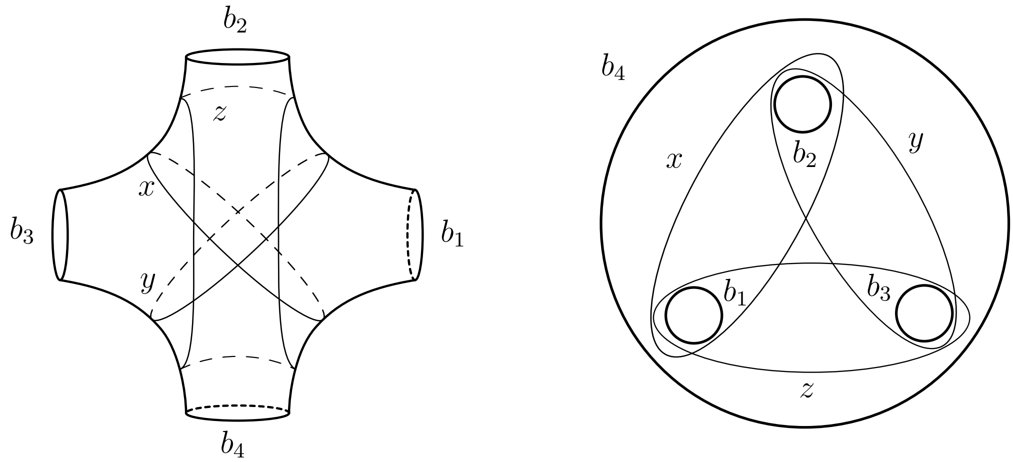

Figure 1. Two views of the lantern relation.

For any orientation-preserving embedding $S_0^4 \hookrightarrow S$, the corresponding curves in S satisfy the same relation in $\mathrm{Mod}(S)$.

3. Abelian cycles have order 2 for $k = 2$ and $g \geq 4$

In this section, we use Proposition 2.1 and Corollary 2.2 to extend (2.1) to $H_2(\mathcal{I}_g)$ for $g \geq 4$. For the case $g = 3$, we use certain auxiliary relations in $H_2(\mathcal{I}_g)$ (see §4 and §5).

Proposition 3.1. *Let δ_1, δ_2 be disjoint, nonisotopic, separating simple closed curves on S_g with $g \geq 4$. Then we have*

$$2\,\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = 0.$$

Proof. Since $g \geq 4$ and the separating curves δ_1, δ_2 are disjoint, one of them bounds a subsurface Σ of genus at least 3. For concreteness, we will assume that δ_1 bounds such a subsurface. If δ_2 lies in Σ, then by Proposition 2.1 we have

$$T_{\delta_2}^2 = [h_1, f_1] \cdots [h_m, f_m],$$

where h_j and f_j are supported on Σ for all $j = 1, \ldots, m$. It follows that

$$2\,\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = \mathcal{A}(T_{\delta_1}, T_{\delta_2}^2) = \mathcal{A}(T_{\delta_1}, [h_1, f_1]) + \cdots + \mathcal{A}(T_{\delta_1}, [h_m, f_m]) = 0,$$

since each h_j and f_j commutes with T_{δ_1}.

If δ_2 does not lie in Σ, then by Corollary 2.2 we have

$$T_{\delta_1}^2 = [\widetilde{h}_1, \widetilde{f}_1] \cdots [\widetilde{h}_n, \widetilde{f}_n],$$

where $\widetilde{h}_j$ and $\widetilde{f}_j$ are supported on Σ for all $j = 1, \ldots, n$. We then have

$$2\,\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = \mathcal{A}(T_{\delta_1}^2, T_{\delta_2}) = \mathcal{A}([\widetilde{h}_1, \widetilde{f}_1], T_{\delta_2}) + \cdots + \mathcal{A}([\widetilde{h}_n, \widetilde{f}_n], T_{\delta_2}) = 0.$$

$\square$

Corollary 3.2. *For $g \geq 4$, the group $H_2^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space.*

4. The BP-relation in $H_k(\mathcal{I}_g)$

In this section, we derive an auxiliary relation in $H_k(\mathcal{I}_g)$. We begin by introducing the necessary terminology.

A pair of simple closed curves (α, α') is called a *bounding pair* if α and α' are disjoint, nonisotopic, nonseparating simple closed curves on S_g such that $\alpha \cup \alpha'$ separates S_g. We say that a bounding pair (α, α') is *oriented* if we fix one of the two possible orientations. By definition, a bounding pair satisfies $[\alpha] = \pm[\alpha']$, and choosing an orientation means simultaneously orienting α and α' so that $[\alpha] = [\alpha']$.

Cutting S_g along $\alpha \cup \alpha'$ yields two surfaces R_1 and R_2, with R_1 lying *to the left* of α. Let $\mathcal{H}_1 = H_1(R_1)$ and $\mathcal{H}_2 = H_1(R_2)$ denote the subspaces of $H_1(S_g)$ induced by the inclusions $R_1 \hookrightarrow S_g$ and $R_2 \hookrightarrow S_g$. We set $\mathcal{H}_{\alpha,\alpha'} = (\mathcal{H}_1, \mathcal{H}_2)$.

Remark 4.1. For an oriented bounding pair (α, α'), it is *not true* that $\mathcal{H}_{\alpha,\alpha'} = \mathcal{H}_{\alpha',\alpha}$. In fact, $\mathcal{H}_{\alpha,\alpha'} = \mathcal{H}_{\overline{\alpha}',\overline{\alpha}}$, where $\overline{\alpha}$ denotes α with the reversed orientation.

Proposition 4.2 (BP-relation)**.** *Let (α, α') and (β, β') be oriented bounding pairs such that $\mathcal{H}_{\alpha,\alpha'} = \mathcal{H}_{\beta,\beta'}$. Let $\delta_1, \dots, \delta_{k-1}$ be pairwise disjoint, pairwise nonisotopic separating simple closed curves on S_g, each disjoint from $\alpha, \alpha', \beta, \beta'$. Then in $H_k(\mathcal{I}_g)$ we have*

$$\mathcal{A}(T_\alpha T_{\alpha'}^{-1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) = \mathcal{A}(T_\beta T_{\beta'}^{-1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}).$$

To prove this proposition, we first state the following lemma, which immediately implies the relation.

Lemma 4.3. *Let $\alpha, \alpha', \beta, \beta'$ be oriented, nonseparating simple closed curves on S_g representing the same homology class in $H_1(S_g)$, such that α is disjoint from α' and β is disjoint from β'.*

1. *If the oriented bounding pairs (α, α') and (β, β') satisfy $\mathcal{H}_{\alpha,\alpha'} = \mathcal{H}_{\beta,\beta'}$, then there exists $f \in \mathcal{I}_g$ such that $f(\alpha) = \beta$ and $f(\alpha') = \beta'$.*
2. *Moreover, if $\delta_1, \dots, \delta_n$ are pairwise disjoint, pairwise nonisotopic, separating simple closed curves disjoint from $\alpha, \alpha', \beta, \beta'$, then f can be chosen so that $f(\delta_j) = \delta_j$ for all $j = 1, \dots, n$.*

Proof. Let $\Sigma_1, \dots, \Sigma_{n+1}$ be the subsurfaces of S_g obtained by cutting along $\delta_1, \dots, \delta_n$ (where $n = 0$ and $\Sigma_1 = S_g$ if there are no δ_i's). Since $\mathcal{H}_{\alpha,\alpha'} = \mathcal{H}_{\beta,\beta'}$, all the four curves lie in some $\Sigma \in \{\Sigma_1, \dots, \Sigma_{n+1}\}$. Hence, it suffices to construct a homeomorphism $f \in \mathrm{Homeo}(\Sigma, \partial\Sigma)$ such that:

- it takes (α, α') to (β, β'); and
- its extension by the identity on S_g acts trivially on $H_1(S_g)$.

Let $N_1, N_2 \subset \Sigma$ be the subsurfaces obtained by cutting Σ along $\alpha \cup \alpha'$, with N_1 lying to the left of α. Similarly, let $M_1, M_2 \subset \Sigma$ be the subsurfaces obtained by cutting Σ along $\beta \cup \beta'$, with M_1 lying to the left of β. Since $\mathcal{H}_{\alpha,\alpha'} = \mathcal{H}_{\beta,\beta'}$, it follows that N_1 is homeomorphic to M_1 and N_2 is homeomorphic to M_2.

We can choose a homeomorphism $\phi_1 \colon N_1 \to M_1$ fixing each δ_j and taking (α, α') to (β, β'). Moreover, there exists a homeomorphism $\psi_1 \in \mathrm{Homeo}(M_1, \partial M_1)$ such that the extension of $\psi_1 \circ \phi_1$ to S_g by the identity acts trivially on $H_1(S_g)$.

Similarly, we choose $\phi_2 \colon N_2 \to M_2$ and $\psi_2 \in \mathrm{Homeo}(M_2, \partial M_2)$ so that $\psi_2 \circ \phi_2$ takes (α, α') to (β, β') and its extension to S_g by the identity acts trivially on $H_1(S_g)$.

Then $\psi_1 \circ \phi_1 \cup \psi_2 \circ \phi_2 \in \mathrm{Homeo}(\Sigma, \partial\Sigma)$ is the desired homeomorphism, and the lemma follows. □

Proof of Proposition 4.2. By Lemma 4.3, there exists $f \in \mathcal{I}_g$ taking (α, α') to (β, β') and fixing each δ_j, $j = 1, \dots, k-1$. Applying f_* to the abelian cycle, we obtain

$$\mathcal{A}(T_\alpha T_{\alpha'}^{-1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) = f_*(\mathcal{A}(T_\alpha T_{\alpha'}^{-1}, T_{\delta_1}, \dots, T_{\delta_{k-1}})) = \mathcal{A}(T_\beta T_{\beta'}^{-1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}),$$

which proves the proposition. □

5. Main relations in $H_k^{\mathrm{ab,sep}}(\mathcal{I}_g)$

In this section, we prove two auxiliary relations in $H_k(\mathcal{I}_g)$. Recall that a surface homeomorphic to a sphere with three boundary components is called a *pair of pants.*

Lemma 5.1. *Consider separating simple closed curves $\gamma_1, \gamma_2, \gamma_3$ and $\delta_1, \ldots, \delta_{k-1}$ on S_g such that:*

(1) the curves $\gamma_1, \gamma_2, \gamma_3$ decompose S_g into a pair of pants and three subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$, where $\partial\Sigma_j = \gamma_j$ for $j = 1, 2, 3$;
(2) Σ_2 has genus 1;
(3) the curves $\delta_1, \ldots, \delta_{k-1}$ are pairwise disjoint, pairwise nonisotopic, and lie in $\Sigma_1, \Sigma_2, \Sigma_3$.

Then in $H_k(\mathcal{I}_g)$ we have

$$\mathcal{A}(T_{\gamma_1}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) + \mathcal{A}(T_{\gamma_3}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) = \mathcal{A}(T_{\gamma_2}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}).$$

Remark 5.2. In Lemmas 5.1 and 5.6, some of the curves $\delta_1, \ldots, \delta_{k-1}$ may be isotopic to one of $\gamma_1, \gamma_2, \gamma_3$.

Proof. Let $\varepsilon, \varepsilon', \varepsilon''$ be curves on S_g, arranged as shown in Figure 2. Applying the lantern relation, we obtain

$$T_\varepsilon^2 T_{\gamma_1} T_{\gamma_3} = T_{\gamma_2} T_{\varepsilon'} T_{\varepsilon''}.$$

It follows that

$$\begin{aligned}
\mathcal{A}(T_{\gamma_1}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) + \mathcal{A}(T_{\gamma_3}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) &= \mathcal{A}(T_{\gamma_1} T_{\gamma_3}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) \\
&= \mathcal{A}(T_{\gamma_2} T_{\varepsilon'} T_{\varepsilon''} T_\varepsilon^{-2}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) \\
&= \mathcal{A}(T_{\gamma_2}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) \\
&\qquad + \mathcal{A}(T_{\varepsilon'} T_\varepsilon^{-1}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) \\
&\qquad - \mathcal{A}(T_\varepsilon T_{\varepsilon''}^{-1}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}).
\end{aligned} \tag{5.1}$$

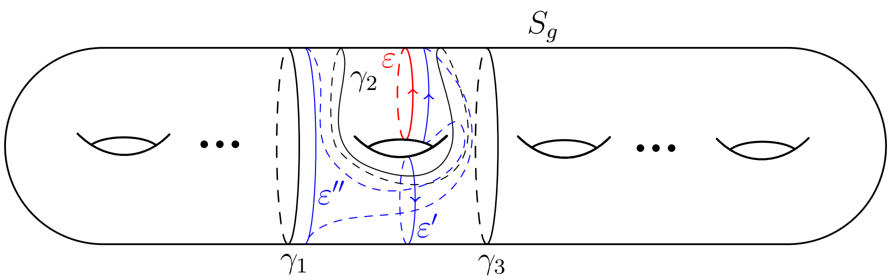

Figure 2. The curves $\varepsilon, \varepsilon', \varepsilon''$ on S_g.

We have $\mathcal{H}_{\varepsilon,\varepsilon''} = \mathcal{H}_{\varepsilon',\varepsilon} = (\mathcal{H}_1, \mathcal{H}_2)$, where

$$\mathcal{H}_1 = H_1(\Sigma_3) \oplus \langle x \rangle, \qquad \mathcal{H}_2 = H_1(\Sigma_1) \oplus \langle x \rangle,$$

and $x = [\varepsilon] = [\varepsilon'] = [\varepsilon'']$. It then follows from the BP-relation (Proposition 4.2) that

$$\mathcal{A}(T_{\varepsilon'} T_\varepsilon^{-1}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}) = \mathcal{A}(T_\varepsilon T_{\varepsilon''}^{-1}, T_{\delta_1}, \ldots, T_{\delta_{k-1}}).$$

The lemma now follows from (5.1). □

We now consider the following two important consequences of Lemma 5.1.

Proposition 5.3. *Let δ_1 and δ_2 be disjoint, nonisotopic, separating simple closed curves on S_3. Then in $H_2(\mathcal{I}_3)$ we have*

$$2\,\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = 0.$$

Proof. Choose a simple closed curve δ_3 on S_3 such that $\delta_1 \cup \delta_2 \cup \delta_3$ bounds a pair of pants. Applying Lemma 5.1 first to $\delta_1, \delta_2, \delta_3$ and then to $\delta_1, \delta_3, \delta_2$, we obtain

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}) + \mathcal{A}(T_{\delta_3}, T_{\delta_2}) = \mathcal{A}(T_{\delta_2}, T_{\delta_2}),$$
$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}) + \mathcal{A}(T_{\delta_2}, T_{\delta_2}) = \mathcal{A}(T_{\delta_3}, T_{\delta_2}).$$

Summing these two relations yields the desired equality. □

Remark 5.4. We have the following splitting:

$$H_*(\mathcal{I}_g; \mathbb{Z}[1/2]) = H_*(\mathcal{I}_g; \mathbb{Z}[1/2])^+ \oplus H_*(\mathcal{I}_g; \mathbb{Z}[1/2])^-,$$

where $H_*(\mathcal{I}_g; \mathbb{Z}[1/2])^\pm$ are the eigenspaces of the action of the hyperelliptic involution.

Hain [6] proved that $H_2(\mathcal{I}_3; \mathbb{Z}[1/2])^+ = 0$. Moreover, the images of all abelian cycles $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$, where δ_1 and δ_2 are disjoint separating simple closed curves, lie in $H_2(\mathcal{I}_3; \mathbb{Z}[1/2])^+$. It follows that, in $H_2(\mathcal{I}_3)$, each abelian cycle $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ has order 2^ℓ for some $\ell \in \mathbb{N}$.

From Corollary 3.2 and Proposition 5.3 it follows that $H_2^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space for $g \geq 3$.

Proposition 5.5. *For $g \geq 3$, the $\mathbb{Z}/2\mathbb{Z}$-vector space $H_2^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is generated by abelian cycles $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$, where δ_1 and δ_2 are disjoint, nonisotopic, separating simple closed curves of genus 1 on S_g.*

Proof. We show that for nonisotopic separating simple closed curves δ_1 and δ_2 on S_g, the abelian cycle $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ can be expressed as a sum of abelian cycles $\mathcal{A}(T_{\gamma_i}, T_{\theta_j})$, where γ_i and θ_j are disjoint, nonisotopic, separating simple closed curves of genus 1 for all i and j.

Suppose that the curves δ_1 and δ_2 bound subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$ of S_g such that $\partial\Sigma_1 = \delta_1$ and $\partial\Sigma_2 = \delta_2$. Choose separating simple closed curves γ_1 and η_1 on Σ_1 (see Figure 3) such that:

- γ_1 has genus 1; and
- $\gamma_1 \cup \eta_1 \cup \delta_1$ bounds a pair of pants.

By Lemma 5.1 we have

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = \mathcal{A}(T_{\gamma_1}, T_{\delta_2}) + \mathcal{A}(T_{\eta_1}, T_{\delta_2}).$$

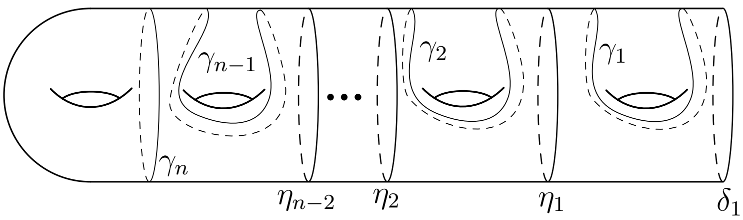

FIGURE 3. The surface Σ_1 and the curves on Σ_1.

Continuing similarly with $\mathcal{A}(T_{\eta_1}, T_{\delta_2})$ and repeating this process, we obtain a collection of separating simple closed curves $\gamma_1, \ldots, \gamma_n$ of genus 1 on Σ_1 such that

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = \mathcal{A}(T_{\gamma_1}, T_{\delta_2}) + \cdots + \mathcal{A}(T_{\gamma_n}, T_{\delta_2}).$$

Applying the same argument to each abelian cycle $\mathcal{A}(T_{\gamma_j}, T_{\delta_2})$, we obtain a collection of separating simple closed curves $\theta_1, \ldots, \theta_m$ of genus 1 on Σ_2 such that

$$\mathcal{A}(T_{\gamma_j}, T_{\delta_2}) = \mathcal{A}(T_{\gamma_j}, T_{\theta_1}) + \cdots + \mathcal{A}(T_{\gamma_j}, T_{\theta_m}).$$

Hence, we obtain the following decomposition, which proves the proposition:

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = \sum_{i=1}^{n} \mathcal{A}(T_{\gamma_i}, T_{\delta_2}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathcal{A}(T_{\gamma_i}, T_{\theta_j}).$$

□

We now state the second auxiliary relation in $H_k(\mathcal{I}_g)$.

Lemma 5.6. *Consider separating simple closed curves $\gamma_1, \gamma_2, \gamma_3$ and $\delta_1, \dots, \delta_{k-1}$ on S_g such that:*

(1) the curves $\gamma_1, \gamma_2, \gamma_3$ decompose S_g into a pair of pants and three subsurfaces $\Sigma_1, \Sigma_2, \Sigma_3$, where $\partial \Sigma_j = \gamma_j$ for $j = 1, 2, 3$;
(2) the curves $\delta_1, \dots, \delta_{k-1}$ are pairwise disjoint, pairwise nonisotopic, and lie in $\Sigma_1, \Sigma_2, \Sigma_3$;
(3) there exists a (possibly nonessential) separating simple closed curve θ such that $\gamma_3 \cup \theta$ bounds a subsurface Σ of genus 1, and the curves $\delta_1, \dots, \delta_{k-1}$ are disjoint from Σ.

Then in $H_k(\mathcal{I}_g)$ we have

$$\mathcal{A}(T_{\gamma_1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) + \mathcal{A}(T_{\gamma_2}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) = \mathcal{A}(T_{\gamma_3}, T_{\delta_1}, \dots, T_{\delta_{k-1}}).$$

Remark 5.7. Note the difference in the order of $\gamma_1, \gamma_2, \gamma_3$ in the relations of Lemmas 5.1 and 5.6. This difference is the key idea that will be used in the proof of Lemma 5.9 below.

Proof. Let $\alpha, \alpha', \beta, \beta'$ be curves on S_g, arranged as shown in Figure 4. Applying the lantern relation, we obtain

$$T_{\alpha'} T_\alpha T_{\gamma_2} T_{\gamma_1} = T_{\gamma_3} T_\beta T_{\beta'}.$$

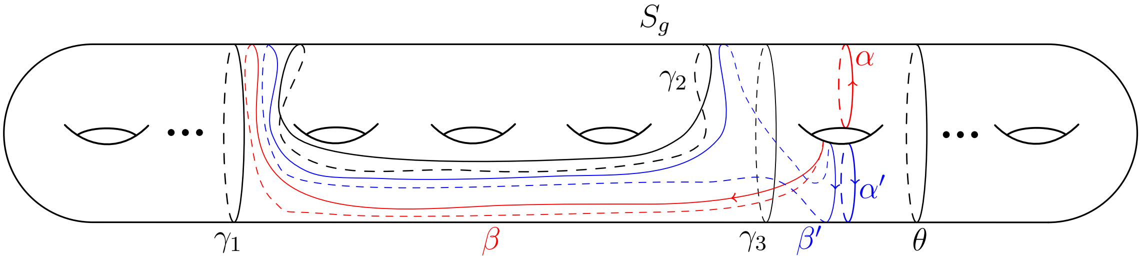

FIGURE 4. The curves $\alpha, \alpha', \beta, \beta'$ on S_g.

As in the proof of Lemma 5.1, we have $\mathcal{H}_{\beta,\alpha} = \mathcal{H}_{\alpha',\beta'} = (\mathcal{H}_1, \mathcal{H}_2)$, where

$$\mathcal{H}_1 = H_1(\Sigma_1) \oplus H_1(\Sigma_\theta) \oplus \langle x \rangle, \qquad \mathcal{H}_2 = H_1(\Sigma_2) \oplus \langle x \rangle.$$

Here, $x = [\alpha] = [\alpha'] = [\beta] = [\beta']$, and Σ_θ is the subsurface bounded by θ lying to the right of θ in Figure 4. By the BP-relation, we then obtain

$$\begin{aligned}
\mathcal{A}(T_{\gamma_1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) + \mathcal{A}(T_{\gamma_2}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) &= \mathcal{A}(T_{\gamma_1} T_{\gamma_2}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) \\
&= \mathcal{A}(T_{\gamma_3} T_\beta T_{\beta'} T_\alpha^{-1} T_{\alpha'}^{-1}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) \\
&= \mathcal{A}(T_{\gamma_3}, T_{\delta_1}, \dots, T_{\delta_{k-1}}) \\
&\qquad + \mathcal{A}(T_\beta T_\alpha^{-1}, T_{\delta_1} \dots, T_{\delta_{k-1}}) \\
&\qquad - \mathcal{A}(T_{\alpha'} T_{\beta'}^{-1}, T_{\delta_1} \dots, T_{\delta_{k-1}}) \\
&= \mathcal{A}(T_{\gamma_3}, T_{\delta_1}, \dots, T_{\delta_{k-1}}),
\end{aligned}$$

and the lemma follows. □

Remark 5.8. For $k = 2$, Lemmas 5.1 and 5.6 are special cases of the more general Proposition 6.1, which will be proved in Section 6.

An important consequence of Lemmas 5.1 and 5.6 is the following lemma, which will play a key role in Section 8.

Lemma 5.9. *Let $\delta_1, \dots, \delta_k$ be pairwise disjoint, pairwise nonisotopic, separating simple closed curves on S_g. Assume that one of the following holds:*

(1) there exists a subsurface Σ of genus at least 2 among the subsurfaces $\Sigma_1, \dots, \Sigma_{k+1}$ obtained by cutting S_g along $\delta_1, \dots, \delta_k$;

(2) there exists a curve δ_j that bounds a subsurface Σ of genus at least 2 containing exactly one other curve δ_i, and all remaining curves δ_ℓ are disjoint from Σ.

Then we have

$$2\,\mathcal{A}(T_{\delta_1},\dots,T_{\delta_k})=0.$$

Proof. We first consider the case in which condition (2) holds. We may assume, without loss of generality, that

- δ_1 bounds a subsurface Σ of genus at least 2;
- δ_2 lies inside Σ; and
- the remaining curves $\delta_3,\dots,\delta_k$ are disjoint from Σ.

We begin with the case $g(\Sigma)\geq 3$. By Proposition 2.1, we have $T_{\delta_2}^2\in[\mathcal{I}(\Sigma),\mathcal{I}(\Sigma)]$. Since each T_{δ_j} commutes with all elements of $\mathcal{I}(\Sigma)$, it follows that

$$2\,\mathcal{A}(T_{\delta_1},\dots,T_{\delta_k})=0.$$

We now turn to the case $g(\Sigma)=2$. In this case, δ_2 bounds a subsurface of genus 1. Choose a separating simple closed curve γ of genus 1 on Σ, disjoint from δ_2. Apply Lemma 5.1 first to δ_2,γ,δ_1, and then to γ,δ_2,δ_1 to obtain

$$\mathcal{A}(T_{\delta_1},\dots,T_{\delta_k})+\mathcal{A}(T_{\gamma_1},T_{\delta_2},\dots,T_{\delta_k})=\mathcal{A}(T_{\gamma_2},T_{\delta_2},\dots,T_{\delta_k}),$$
$$\mathcal{A}(T_{\delta_1},\dots,T_{\delta_k})+\mathcal{A}(T_{\gamma_2},T_{\delta_2},\dots,T_{\delta_k})=\mathcal{A}(T_{\gamma_1},T_{\delta_2},\dots,T_{\delta_k}).$$

Summing these two relations yields the desired equality.

Next, we consider case (1). For simplicity, assume that $\delta_1,\dots,\delta_n$ bound a subsurface Σ of genus at least 2. Choose separating simple closed curves γ_1 and γ_2 on Σ such that

- γ_1 has genus 1; and
- γ_1,γ_2, and δ_1 bound a pair of pants and they are arranged as shown in Figure 5.

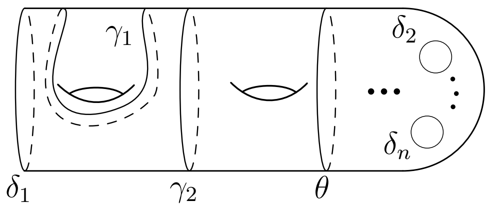

FIGURE 5. The surface Σ and the curves γ_1,γ_2,θ on Σ.

In Figure 5, the curves $\delta_2,\dots,\delta_n$ lie to the right of γ_2. We can, however, choose a separating simple closed curve θ such that $\delta_2,\dots,\delta_n$ lie to the right of θ, as illustrated in the same figure.

Apply Lemma 5.1 to $\delta_1,\gamma_1,\gamma_2$ to obtain

$$\mathcal{A}(T_{\delta_1},\dots,T_{\delta_k})+\mathcal{A}(T_{\gamma_2},T_{\delta_2},\dots,T_{\delta_k})=\mathcal{A}(T_{\gamma_1},T_{\delta_2},\dots,T_{\delta_k}).$$

The choice of θ allows us to apply Lemma 5.6 to $\delta_1,\gamma_1,\gamma_2$, giving

$$\mathcal{A}(T_{\delta_1},\dots,T_{\delta_k})+\mathcal{A}(T_{\gamma_1},T_{\delta_2},\dots,T_{\delta_k})=\mathcal{A}(T_{\gamma_2},T_{\delta_2},\dots,T_{\delta_k}).$$

Adding these two relations yields the desired equality. □

6. Birman–Craggs–Johnson homomorphism and relations in $H_2(\mathcal{I}_g)$

To state the main result of this section, we first recall the *Birman–Craggs–Johnson homomorphism* $\sigma\colon\mathcal{I}_g\to\mathbb{B}_3'$ (see [8] for details). We briefly review its construction.

A basis $a_1,\dots,a_g,b_1,\dots,b_g$ of the group $H_1(S_g)$ is called *symplectic* if

$$a_i\cdot b_j=\delta_{ij},\quad a_i\cdot a_j=b_i\cdot b_j=0,$$

where δ_{ij} denotes the Kronecker delta. A subgroup V of $H_1(S_g)$ is called *symplectic* if the restriction of the intersection form to V is unimodular. Equivalently, this means that

$H_1(S_g) = V \oplus V^{\perp}$, where the decomposition is orthogonal with respect to the intersection form. A symplectic basis of $H_1(S_g;\mathbb{Z}/2\mathbb{Z})$ and a symplectic subspace of $H_1(S_g;\mathbb{Z}/2\mathbb{Z})$ are defined in the same way.

Recall that for integral homology classes $x, y, \ldots$, their reductions modulo 2 are denoted by bold letters $\mathbf{x}, \mathbf{y}, \ldots$.

Let $S = S_g^b$, where $b \in \{0,1\}$. Let $\mathbb{B}(S)$ be the algebra over $\mathbb{Z}/2\mathbb{Z}$ generated by formal variables $\overline{\mathbf{x}}$ for all $\mathbf{x} \in H_1(S;\mathbb{Z}/2\mathbb{Z})$, subject to the following relations for all $\mathbf{x}, \mathbf{y} \in H_1(S;\mathbb{Z}/2\mathbb{Z})$:

(1) $\overline{\mathbf{x}+\mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}} + (\mathbf{x}\cdot\mathbf{y})$, where $\mathbf{x}\cdot\mathbf{y}$ denotes the mod 2 intersection number;
(2) $\overline{\mathbf{x}}^2 = \overline{\mathbf{x}}$.

We have the following description of $\mathbb{B}(S)$. From relations (1) and (2) it follows that, for any basis $\mathbf{e}_1, \ldots, \mathbf{e}_{2g}$ of $H_1(S;\mathbb{Z}/2\mathbb{Z})$, $\mathbb{B}(S)$ is the algebra of Boolean polynomials in the formal variables $\overline{\mathbf{e}}_1, \ldots, \overline{\mathbf{e}}_{2g}$.

The *Arf invariant* is the quadratic polynomial

$$\mathrm{Arf} = \sum_{j=1}^{g} \overline{\mathbf{a}}_j \overline{\mathbf{b}}_j,$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_g, \mathbf{b}_1, \ldots, \mathbf{b}_g$ is a symplectic basis of $H_1(S; Z/2Z)$. It is well known that Arf is independent of the choice of symplectic basis

Let $\mathbb{B}'(S)$ denote the quotient algebra $\mathbb{B}(S)/(\mathrm{Arf})$. For each k, let $\mathbb{B}_k(S) \subset \mathbb{B}(S)$ and $\mathbb{B}'_k(S) \subset \mathbb{B}'(S)$ denote the subspaces consisting of polynomials of degree at most k. When the surface is clear from the context, we will simply write $\mathbb{B}_k$ and $\mathbb{B}'_k$.

The Birman–Craggs–Johnson homomorphisms are given by:

$$\sigma\colon \mathcal{I}_g \to \mathbb{B}'_3(S_g),$$
$$\sigma\colon \mathcal{I}_g^1 \to \mathbb{B}_3(S_g^1).$$

For convenience, we denote all types of Birman–Craggs–Johnson homomorphisms by σ; this does not lead to any confusion.

Recall that the value of the Birman–Craggs–Johnson homomorphism on a Dehn twist about a separating simple closed curve δ is given by

$$\sigma(T_\delta) = \sum_{i=1}^{g'} \overline{\mathbf{a}}_i \overline{\mathbf{b}}_i, \tag{6.1}$$

where δ bounds a subsurface Σ of genus g', and $\mathbf{a}_1, \ldots, \mathbf{a}_{g'}, \mathbf{b}_1, \ldots, \mathbf{b}_{g'}$ form a symplectic basis of $H_1(\Sigma;\mathbb{Z}/2\mathbb{Z})$. This formula is well defined for a closed surface S_g, since $\mathrm{Arf} = 0$ in $\mathbb{B}'_3(S_g)$.

Now we are in a position to state the main result of this section.

Proposition 6.1. *Let δ be a separating simple closed curve on S_g with $g \geq 4$, bounding a subsurface Σ of genus at least 3. Then for separating simple closed curves $\gamma_1, \ldots, \gamma_n$ on Σ, the following are equivalent:*

- $\sigma(T_{\gamma_1}) + \cdots + \sigma(T_{\gamma_n}) = 0$ *or* $\sigma(T_{\gamma_1}) + \cdots + \sigma(T_{\gamma_n}) = \sigma(T_\delta)$;
- $\mathcal{A}(T_{\gamma_1}, T_\delta) + \cdots + \mathcal{A}(T_{\gamma_n}, T_\delta) = 0$ *in* $H_2(\mathcal{I}_g)$.

Remark 6.2. Proposition 6.1 also holds without the assumption that $g(\Sigma) \geq 3$. In particular, using Lemma 5.1, the case $g(\Sigma) = 2$ can be deduced from the case $g(\Sigma) \geq 3$. Moreover, the statement remains valid even if the curves $\gamma_1, \ldots, \gamma_n$ are not all required to lie on one side of δ. Since these extensions will not be needed, we omit the proof.

To prove Proposition 6.1 and to understand its nature, we require the following construction. Consider the pushforward homomorphism $\sigma_2\colon H_2(\mathcal{I}_g) \to H_2(\mathbb{B}'_3)$ induced by the Birman–Craggs–Johnson homomorphism $\sigma\colon \mathcal{I}_g \to \mathbb{B}'_3$.

Since $\mathbb{B}'_3$ is an abelian group, we have an isomorphism $H_2(\mathbb{B}'_3) \cong \wedge^2 \mathbb{B}'_3$, where $\wedge^2 \mathbb{B}'_3$ is defined as the quotient of $\mathbb{B}'_3 \otimes \mathbb{B}'_3$ by all relations of the form $x \otimes x = 0$.

It is easy to check, for example using the bar resolution, that on an abelian cycle $\mathcal{A}(h_1, h_2)$ with $h_1, h_2 \in \mathcal{I}_g$, the homomorphism σ_2 acts as

$$\sigma_2(\mathcal{A}(h_1, h_2)) = \sigma(h_1) \wedge \sigma(h_2). \tag{6.2}$$

By restricting σ_2 to the subgroup $H_2^{\mathrm{ab,sep}}(\mathcal{I}_g) \subset H_2(\mathcal{I}_g)$ and using formula (6.1), we obtain a homomorphism

$$\sigma_2\colon H_2^{\mathrm{ab,sep}}(\mathcal{I}_g) \to \wedge^2 \mathbb{B}_2'\,.$$

It is straightforward to check that this is indeed a homomorphism of $\mathrm{Sp}(2g, \mathbb{Z})$-modules, since σ itself is such a homomorphism. Consequently, we obtain an $\mathrm{Sp}(2g, \mathbb{Z})$-equivariant homomorphism of $\mathbb{Z}/2\mathbb{Z}$-vector spaces

$$\sigma_2\colon H_2^{\mathrm{ab,sep}}(\mathcal{I}_g) \to \wedge^2 \mathbb{B}_2',$$

and the value of σ_2 on an abelian cycle $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ is given by formula (6.2).

Remark 6.3. From this perspective, we can reformulate Proposition 6.1 as follows: the relation

$$\mathcal{A}(T_{\gamma_1}, T_\delta) + \cdots + \mathcal{A}(T_{\gamma_n}, T_\delta) = 0$$

holds if and only if

$$\sigma_2(\mathcal{A}(T_{\gamma_1}, T_\delta) + \cdots + \mathcal{A}(T_{\gamma_n}, T_\delta)) = 0.$$

In the proof of Proposition 6.1, we will also require the *Johnson homomorphisms*, defined by

$$\tau\colon \mathcal{I}_g \to \wedge^3 H_1(S_g)/H_1(S_g),$$
$$\tau\colon \mathcal{I}_g^1 \to \wedge^3 H_1(S_g^1).$$

Here, the inclusion $H_1(S_g) \hookrightarrow \wedge^3 H_1(S_g)$ is given by $x \mapsto x \wedge \Omega$, where $\Omega \in \wedge^2 H_1(S_g)$ denotes the dual of the intersection form. For the construction of the Johnson homomorphism and its basic properties, see [7]. By [7, Lemma 4A], it follows that $\tau(T_\delta) = 0$ whenever δ is a separating simple closed curve.

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. Suppose $\mathcal{A}(T_{\gamma_1}, T_\delta) + \cdots + \mathcal{A}(T_{\gamma_n}, T_\delta) = 0$ in $H_2(\mathcal{I}_g)$. Then we have

$$\begin{aligned} 0 = \sigma_2(\mathcal{A}(T_{\gamma_1}, T_\delta)) + \cdots + \sigma_2(\mathcal{A}(T_{\gamma_n}, T_\delta)) &= \sigma_2(\mathcal{A}(T_{\gamma_1} \cdots T_{\gamma_n}, T_\delta)) \\ &= \sigma(T_{\gamma_1} \cdots T_{\gamma_n}) \wedge \sigma(T_\delta). \end{aligned}$$

Since $\sigma(T_\delta) \neq 0$ by formula 6.1, one of the following holds:

$$\sigma(T_{\gamma_1} \cdots T_{\gamma_n}) = 0 \quad \text{or} \quad \sigma(T_{\gamma_1} \cdots T_{\gamma_n}) = \sigma(T_\delta),$$

as required.

We now prove the converse statement. We may assume that $\sigma(T_{\gamma_1}) + \cdots + \sigma(T_{\gamma_n}) = 0$, since otherwise we could simply add the curve $\gamma_{n+1} = \delta$ to the set $\{\gamma_1, \ldots, \gamma_n\}$; this does not affect the statement to be proved, as $\mathcal{A}(T_\delta, T_\delta) = 0$ in $H_2(\mathcal{I}_g)$.

Let $f = T_{\gamma_1} \cdots T_{\gamma_n}$. We need to show that the equality $\sigma(f) = 0$ implies $\mathcal{A}(f, T_\delta) = 0$ in $H_2(\mathcal{I}_g)$. We consider the restrictions of the Birman–Craggs–Johnson and Johnson homomorphisms to the Torelli subgroups (see [8, Section 10] and [7, Section 6]):

$$\sigma|_\Sigma = \sigma_\Sigma\colon \mathcal{I}(\Sigma) \to \mathbb{B}_3(\Sigma),$$
$$\tau|_\Sigma = \tau_\Sigma\colon \mathcal{I}(\Sigma) \to \wedge^3 H_1(\Sigma).$$

Since $\sigma_\Sigma(f) = \tau_\Sigma(f) = 0$, it follows from [10, Theorem 3] that $f \in [\mathcal{I}(\Sigma), \mathcal{I}(\Sigma)]$; that is,

$$f = [g_1, h_1] \cdots [g_k, h_k],$$

where $g_i, h_i \in \mathcal{I}_g$ are supported on Σ for $i = 1, \ldots, k$.

It then follows immediately that $\mathcal{A}(f, T_\delta) = 0$, since each g_i and h_i commutes with T_δ for all $i = 1, \ldots, k$. ☐

7. Proof of Theorem 1.5

In this section, we prove that the vector space $H_2^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is finite-dimensional for $g \geq 4$. This follows from the fact that there are only finitely many pairwise distinct abelian cycles $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$, where δ_1 and δ_2 are separating simple closed curves of genus 1. We now state the precise proposition.

Proposition 7.1. *Let δ_1, δ_2 and γ_1, γ_2 be two pairs of separating simple closed curves of genus 1 on S_g with $g \geq 4$, such that the curves in each pair are disjoint and nonisotopic. If $\sigma(T_{\delta_1}) = \sigma(T_{\gamma_1})$ and $\sigma(T_{\delta_2}) = \sigma(T_{\gamma_2})$, then*

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}) = \mathcal{A}(T_{\gamma_1}, T_{\gamma_2}).$$

Proof of Theorem 1.5. Since $\mathbb{B}_2'$ is a finite set, there are only finitely many possibilities for $\sigma(T_{\delta_1})$ and $\sigma(T_{\delta_2})$, and the theorem follows. □

To prove Proposition 7.1, we first reduce the topological problem to a linear-algebraic one by describing the connection between the abelian cycles $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ and the corresponding splittings of $H_1(S_g)$.

Consider disjoint, nonisotopic, separating simple closed curves δ_1 and δ_2 on S_g. Suppose that $\delta_1 \cup \delta_2$ decomposes S_g into three subsurfaces $\Sigma_1, \Sigma, \Sigma_2$ such that

$$\partial\Sigma_1 = \delta_1, \qquad \partial\Sigma_2 = \delta_2, \qquad \partial\Sigma = \delta_1 \cup \delta_2.$$

This induces a splitting

$$H_1(S_g) = U_1 \oplus U \oplus U_2,$$

where $U_1 = H_1(\Sigma_1)$, $U_2 = H_1(\Sigma_2)$, and $U = H_1(\Sigma)$. The abelian cycle $\mathcal{A}(T_{\delta_1}, T_{\delta_2})$ is completely determined by the pair (U_1, U_2), and we will also denote it by $\mathcal{A}(U_1, U_2)$.

Throughout this paper, by a *splitting* of a symplectic group H we mean a decomposition

$$H = X_1 \oplus \cdots \oplus X_n,$$

where $X_1, \ldots, X_n \subset H$ are pairwise orthogonal symplectic subgroups with respect to the symplectic form on H.

From this perspective, we obtain the following consequence of the proof of Proposition 5.3.

Corollary 7.2. *Consider a splitting*

$$H_1(S_3) = U_1 \oplus U_2 \oplus U_3.$$

Then

$$\mathcal{A}(U_1, U_2) = \mathcal{A}(U_2, U_3) = \mathcal{A}(U_3, U_1).$$

Therefore, in $H_2(\mathcal{I}_3)$, the abelian cycle of this form is completely determined by the splitting and does not depend on the choice of summands.

We say that a separating simple closed curve γ *bounds* a symplectic subgroup $V \subset H_1(S_g)$ if it induces a decomposition $H_1(S_g) = V \oplus V^\perp$. The value $\sigma(T_\gamma)$ depends only on the unordered splitting $\{V, V'\}$. We define

$$\sigma(V) = \sigma(V') = \sigma(T_\gamma).$$

Moreover, $\sigma(V)$ depends only on $\mathbf{V}$ (the reduction of V modulo 2). We then set $\sigma(\mathbf{V}) = \sigma(V)$.

Consider two disjoint, nonisotopic, separating simple closed curves δ_1, δ_2 of genus 1 on S_g. Let $U_1, U_2 \subset H_1(S_g)$ be the symplectic subgroups bounded by δ_1 and δ_2, respectively. For $j = 1, 2$, let x_j, y_j form a symplectic basis of U_j. Then

$$\sigma_2(\mathcal{A}(U_1, U_2)) = \sigma(T_{\delta_1}) \wedge \sigma(T_{\delta_2}) = \overline{\mathbf{x}}_1\overline{\mathbf{y}}_1 \wedge \overline{\mathbf{x}}_2\overline{\mathbf{y}}_2.$$

We now reformulate Proposition 6.1 in linear algebraic terms.

Proposition 7.3. *Let $g \geq 4$, and let $U \subset H_1(S_g)$ be a symplectic subgroup. Let $V_1, \dots, V_n \subset H_1(S_g)$ be symplectic subgroups such that one of the following conditions holds:*

(1) $V_1, \dots, V_n \subset U$ and $\operatorname{rank} U \geqslant 6$; or
(2) $V_1, \dots, V_n \subset U^\perp$ and $\operatorname{rank} U^\perp \geqslant 6$.

Then the following statements are equivalent:

- *either $\sigma(V_1) + \cdots + \sigma(V_n) = 0$ or $\sigma(V_1) + \cdots + \sigma(V_n) = \sigma(U)$;*
- *$\mathcal{A}(V_1, U) + \cdots + \mathcal{A}(V_n, U) = 0$ in $H_2(\mathcal{I}_g)$.*

In the proof of Proposition 7.1, we will require the following lemmas.

Lemma 7.4. *For $g \geq 3$, let $\mathbf{V} \subset H_1(S_g; \mathbb{Z}/2\mathbb{Z})$ be a two-dimensional symplectic subspace with $\sigma(\mathbf{V}) = \overline{\mathbf{a}}\,\overline{\mathbf{b}}$ for some $\mathbf{a}, \mathbf{b} \in H_1(S_g; \mathbb{Z}/2\mathbb{Z})$ satisfying $\mathbf{a} \cdot \mathbf{b} = 1$. Then $\mathbf{V} = \langle \mathbf{a}, \mathbf{b} \rangle$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in H_1(S_g; \mathbb{Z}/2\mathbb{Z})$ form a symplectic basis for $\mathbf{V}$. Then $\overline{\mathbf{x}}\,\overline{\mathbf{y}} = \overline{\mathbf{a}}\,\overline{\mathbf{b}}$. We have the following decomposition

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_1, \qquad \text{with } \mathbf{x}_0 \in \langle \mathbf{a}, \mathbf{b} \rangle, \mathbf{x}_1 \in \langle \mathbf{a}, \mathbf{b} \rangle^\perp,$$
$$\mathbf{y} = \mathbf{y}_0 + \mathbf{y}_1, \qquad \text{with } \mathbf{y}_0 \in \langle \mathbf{a}, \mathbf{b} \rangle, \mathbf{y}_1 \in \langle \mathbf{a}, \mathbf{b} \rangle^\perp.$$

If $\mathbf{x}_0 = \mathbf{y}_0 = 0$, we obtain a contradiction with $\sigma(\mathbf{V}) = \overline{\mathbf{a}}\,\overline{\mathbf{b}}$. Assume instead that $\mathbf{x}_0 \neq 0$. Then in $\mathbb{B}'_3$ we have

$$\overline{\mathbf{a}}\,\overline{\mathbf{b}} = \overline{\mathbf{x}}\,\overline{\mathbf{y}} = \overline{\mathbf{x}}\,\overline{\mathbf{a}}\,\overline{\mathbf{b}} = (\overline{\mathbf{x}}_0 + \overline{\mathbf{x}}_1)\,\overline{\mathbf{a}}\,\overline{\mathbf{b}} = \overline{\mathbf{a}}\,\overline{\mathbf{b}} + \overline{\mathbf{x}}_1\,\overline{\mathbf{a}}\,\overline{\mathbf{b}}.$$

Hence $\overline{\mathbf{x}}_1\,\overline{\mathbf{a}}\,\overline{\mathbf{b}} = 0$ in $\mathbb{B}'_3$, which implies $\mathbf{x}_1 = 0$. Therefore, $\mathbf{x} \in \langle \mathbf{a}, \mathbf{b} \rangle$.

Similarly, if $\mathbf{y}_0 \neq 0$, then $\mathbf{y} \in \langle \mathbf{a}, \mathbf{b} \rangle$ and we are done. Otherwise, if $\mathbf{y}_0 = 0$, we obtain a contradiction with $\mathbf{x} \cdot \mathbf{y} = 1$. The lemma follows. □

Corollary 7.5. *For $g \geq 3$, let $V \subset H_1(S_g)$ be a symplectic subgroup of rank 2 with $\sigma(V) = \overline{\mathbf{a}}\,\overline{\mathbf{b}}$ for some $a, b \in H_1(S_g)$ satisfying $a \cdot b = 1$. Then there exists a symplectic basis x, y for V such that $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b}$.*

Lemma 7.6. *Let $g \geq 4$, and let $W \subset H_1(S_g)$ be a symplectic subgroup of rank 6. Suppose $X_1, X_2, Y_1, Y_2 \subset W$ are symplectic subgroups of rank 2 such that*

- *$X_1 \perp X_2, Y_1 \perp Y_2$ and $Y_2 \subset X_1 \oplus X_2$;*
- *$\sigma(X_1) = \sigma(Y_1)$ and $\sigma(X_2) = \sigma(Y_2)$.*

Then in $H_2(\mathcal{I}_g)$ we have

$$\mathcal{A}(X_1, X_2) = \mathcal{A}(Y_1, Y_2).$$

Proof. Let $X_3, Y_3 \subset W$ be symplectic subgroups of rank 2 such that

$$\begin{aligned} W &= X_1 \oplus X_2 \oplus X_3, \\ W &= Y_1 \oplus Y_2 \oplus Y_3. \end{aligned} \tag{7.1}$$

Choose symplectic bases a_1, b_1, a_2, b_2, and a_3, b_3 for X_1, X_2, and X_3, respectively. By Proposition 7.3, we have

$$\mathcal{A}(X_1, X_2) + \mathcal{A}(X_3, X_2) = \mathcal{A}(W, X_2),$$
$$\mathcal{A}(Y_1, Y_2) + \mathcal{A}(Y_3, Y_2) = \mathcal{A}(W, Y_2).$$

Since $\sigma(X_2) = \sigma(Y_2)$, Proposition 7.3 implies $\mathcal{A}(W, X_2) = \mathcal{A}(W, Y_2)$. Therefore, the desired equality is equivalent to

$$\mathcal{A}(X_3, X_2) = \mathcal{A}(Y_3, Y_2).$$

Since $Y_2 \subset X_1 \oplus X_2$, we have $Y_2 \perp X_3$, so $\mathcal{A}(X_3, Y_2)$ is well-defined. Proposition 7.3 then gives $\mathcal{A}(X_3, X_2) = \mathcal{A}(X_3, Y_2)$, as $\sigma(X_2) = \sigma(Y_2)$. Thus it remains to prove that

$$\mathcal{A}(X_3, Y_2) = \mathcal{A}(Y_3, Y_2).$$

From (7.1) we have $\sigma(X_3) = \sigma(Y_3)$, so the equality follows from Proposition 7.3. □

Now we are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Let $U_1, U_2, V_1, V_2 \subset H_1(S_g)$ be symplectic subgroups of rank 2, bounded by the curves $\delta_1, \delta_2, \gamma_1, \gamma_2$, respectively. Let a_1, b_1 be a symplectic basis for U_1, and a_2, b_2 a symplectic basis for U_2. Then we have

$$\sigma(V_1) = \sigma(U_1) = \bar{\mathbf{a}}_1\overline{\mathbf{b}}_1,$$
$$\sigma(V_2) = \sigma(U_2) = \bar{\mathbf{a}}_2\overline{\mathbf{b}}_2.$$

By Corollary 7.5 applied to V_2, there exists a symplectic basis x_2, y_2 for V_2 such that, for some $v, w \in H_1(S_g)$, we have

$$x_2 = a_2 + 2v,$$
$$y_2 = b_2 + 2w.$$

Thus, we can choose $a_3, b_3 \in \langle a_1, b_1, a_2, b_2\rangle^\perp$ such that $a_3 \cdot b_3 = 1$ and

$$x_2 = 2\zeta_1 a_1 + 2\eta_1 b_1 + (2\zeta_2 + 1)a_2 + 2\eta_2 b_2 + 2\zeta_3 a_3.$$

Let

$$a_2' = (2\alpha_1 + 1)a_2 + 2\alpha_2 b_2 + 2\alpha_3 a_3, \quad \text{with } (2\alpha_1 + 1, \alpha_2, \alpha_3) = 1,$$

be a primitive representative of $(2\zeta_2+1)a_2+2\eta_2 b_2+2\zeta_3 a_3$. Then $(2\alpha_1+1, 4\alpha_2, 4\alpha_3) = 1$, and there exist integers $\beta_1, \beta_2, \beta_3 \in \mathbb{Z}$ such that

$$(2\alpha_1 + 1)(2\beta_1 + 1) + 4\alpha_2\beta_2 + 4\alpha_3\beta_3 = 1.$$

Let

$$b_2' = (2\beta_1 + 1)b_2 - 2\beta_2 a_2 + 2\beta_3 b_3.$$

Then $a_2' \cdot b_2' = 1$, and $U_2' = \langle a_2', b_2'\rangle$ satisfies $U_2' \perp U_1$. From Proposition 7.3, it follows that $\mathcal{A}(U_1, U_2) = \mathcal{A}(U_1, U_2')$.

For some integer $\zeta_2' \in \mathbb{Z}$, we have

$$x_2 = 2\zeta_1 a_1 + 2\eta_1 b_1 + (2\zeta_2' + 1)a_2',$$

and we can choose $a_3' \in \langle a_1, b_1, a_2', b_2'\rangle^\perp$ such that

$$y_2 = 2\lambda_1 a_1 + 2\mu_1 b_1 + 2\lambda_2' a_2' + (2\mu_2' + 1)b_2' + 2\lambda_3' a_3'.$$

There exist $x_1', y_1' \in H_1(S_g)$ such that:

- $\mathbf{x}_1' = \mathbf{a}_1$ and $\mathbf{y}_1' = \mathbf{b}_1$;
- $\langle x_1', y_1'\rangle \perp V_2$;
- $x_1' \cdot y_1' = 1$.

Indeed, we may set

$$x_1' = (2\zeta_2' + 1)(2\mu_2' + 1)a_1 + (2\mu_2' + 1)2\eta_1 b_2' + (2\eta_1 \cdot 2\lambda_2' - (2\zeta_2' + 1) \cdot 2\mu_1)\, a_2' + a_4'$$
$$y_1' = (2\zeta_2' + 1)(2\mu_2' + 1)b_1 - (2\mu_2' + 1)2\zeta_1 b_2' + (-2\zeta_1 \cdot 2\lambda_2' + (2\zeta_2' + 1) \cdot 2\lambda_1)\, a_2' + 2\nu b_4',$$

where $a_4', b_4' \in \langle a_1, b_1, a_2', b_2', a_3', b_3'\rangle^\perp$ with $a_4' \cdot b_4' = 1$, and the integer constant ν is chosen so that $x_1' \cdot y_1' = 1$. This is possible because

$$(2\zeta_2' + 1)^2(2\mu_2' + 1)^2 \equiv 1 \pmod 4.$$

It follows that, for $V_1' = \langle \mathbf{x}_1', \mathbf{y}_1'\rangle$, we have

$$\sigma(V_1') = \sigma(V_1) = \bar{\mathbf{a}}_1\overline{\mathbf{b}}_1.$$

Proposition 7.3 then implies that $\mathcal{A}(V_1', V_2) = \mathcal{A}(V_1, V_2)$. Let $y_2' = y_2 - 2\lambda_3' a_3'$ and set $V_2' = \langle x_2, y_2'\rangle$. Then, again by Proposition 7.3, we obtain $\mathcal{A}(V_1', V_2) = \mathcal{A}(V_1', V_2')$.

The proposition now follows from Lemma 7.6 applied to $U_1, U_2', V_1', V_2' \subset W$, where $W = \langle a_1, b_1, a_2', b_2', a_4', b_4'\rangle$. $\square$

8. Abelian cycles in $H_k(\mathcal{I}_g)$

In this section, we extend several results on $H_2^{\mathrm{ab,sep}}(\mathcal{I}_g)$ to the higher homology groups $H_k(\mathcal{I}_g)$.

Proposition 8.1. *Let $\delta_1, \dots, \delta_k$ be pairwise disjoint, pairwise nonisotopic separating simple closed curves on S_g. Then, for $g \geq 3$ and $3 \leq k \leq 2g-3$, we have*

$$2\,\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0.$$

Remark 8.2. Combining Propositions 2.1,3.1, and 8.1, we obtain $2\,\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$ for $1 \leq k \leq 2g-3$.

We introduce some terminology needed for the proof of Proposition 8.1. Throughout this section, we assume that the surface S_g has genus at least 3.

We call a set of curves $\{\delta_1, \dots, \delta_k\}$ on S_g an *admissible partition* if the curves $\delta_1, \dots, \delta_k$ are pairwise disjoint, pairwise nonisotopic, simple closed separating curves. For convenience, we will also say that $\delta_1, \dots, \delta_k$ form an admissible partition, implicitly referring to the set $\{\delta_1, \dots, \delta_k\}$.

Let $\delta_1, \dots, \delta_k$ be an admissible partition of S_g, and fix this set of curves. Denote by $\mathcal{S}$ the set of surfaces obtained by cutting S_g along $\delta_1 \cup \dots \cup \delta_k$.

We call a curve δ_j *outermost* if there exists a surface $\Sigma \in \mathcal{S}$ such that $\partial\Sigma = \delta_j$. We call Σ a *cap over* δ_j and denote it by $\mathcal{C}ap(\delta_j)$. The set of outermost curves among $\delta_1, \dots, \delta_k$ is denoted by $\mathcal{O}(\{\delta_1, \dots, \delta_k\})$, or simply by $\mathcal{O}$ when the context makes clear which admissible partition is considered.

An admissible partition $\delta_1, \dots, \delta_k$ of S_g is called *simple* if, for each $\delta_i \in \mathcal{O}$, we have $g(\mathcal{C}ap(\delta_i)) = 1$.

We call a curve δ_i *grouping* if δ_i is not outermost and there exists a surface $\Sigma \in \mathcal{S}$ such that $\partial\Sigma \subset \delta_i \cup \mathcal{O}$; we call Σ a *cap base of* δ_i and denote it by $\mathcal{CB}(\delta_i)$. Equivalently, δ_i is grouping if it bounds a subsurface containing only outermost curves. The cap base is uniquely defined when there are at least two grouping curves; if there is exactly one grouping curve, then either of the two possible surfaces may be chosen.

For a grouping curve δ_i, let $\mathcal{O}_i$ denote the subset of $\mathcal{O}$ consisting of curves lying in $\mathcal{CB}(\delta_i)$. We then say that δ_i *groups the curves* $\delta_{i_1}, \dots, \delta_{i_k}$, where $\mathcal{O}_i = \{\delta_{i_1}, \dots, \delta_{i_k}\}$. For such a curve δ_i, we define

$$\mathcal{G}\mathcal{C}ap(\delta_i) = \mathcal{CB}(\delta_i) \cup \mathcal{C}ap(\delta_{i_1}) \cup \dots \cup \mathcal{C}ap(\delta_{i_k}).$$

The set of all grouping curves among $\delta_1, \dots, \delta_k$ is denoted by $\mathcal{G}(\{\delta_1, \dots, \delta_k\})$, or simply by $\mathcal{G}$ when the context is clear.

Lemma 8.3. *Let $\delta_1, \dots, \delta_k$ be a simple admissible partition of S_g. If there exists a grouping curve $\delta_i \in \mathcal{G}$ such that $g(\mathcal{CB}(\delta_i)) = 0$, then $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$ in $H_k(\mathcal{I}_g)$.*

Proof. Without loss of generality, assume that the grouping curve δ_1, which groups the curves $\delta_2, \dots, \delta_m$ (with $m \geq 3$), satisfies $g(\mathcal{CB}(\delta_1)) = 0$. The proof proceeds by induction on m.

Base of induction. For $m = 3$, the statement follows from Lemma 5.1 applied to $\delta_1, \delta_2, \delta_3$. Indeed, we have

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\delta_3}, T_{\delta_2}, T_{\delta_3}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta_2}, T_{\delta_2}, T_{\delta_3}, \dots, T_{\delta_k}),$$

which gives $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$.

Induction step. Assume that the statement holds for all $m \leq \ell - 1$, and consider the case $m = \ell$. Choose a separating simple closed curve $\gamma \subset \mathcal{CB}(\delta_1)$ such that $\delta_1 \cup \delta_2 \cup \gamma$ bounds a pair of pants. Then, by Lemma 5.1, we have

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) + \mathcal{A}(T_\gamma, T_{\delta_2}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta_2}, \dots, T_{\delta_k}),$$

thus $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = -\,\mathcal{A}(T_\gamma, T_{\delta_2}, \dots, T_{\delta_k})$. Since γ groups $\delta_3, \dots, \delta_k$, the induction hypothesis implies $\mathcal{A}(T_\gamma, T_{\delta_2}, \dots, T_{\delta_k}) = 0$, yielding the desired result. $\square$

Lemma 8.4. *Let $\delta_1, \dots, \delta_k$ be a simple admissible partition of S_g such that all surfaces obtained by cutting S_g along $\delta_1, \dots, \delta_k$ have genus 1. Then there exists a simple admissible partition $\delta'_1, \dots, \delta'_k$ such that:*

(1) none of the curves $\delta'_1, \dots, \delta'_k$ is a grouping curve;
(2) $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta'_1}, \dots, T_{\delta'_k})$.

Proof. It suffices to show the following: for a curve δ_1 that groups the curves $\delta_2, \dots, \delta_m$, there exists an outermost separating simple closed curve δ'_1 on $\mathcal{CB}(\delta_1)$ that is disjoint from $\delta_1, \dots, \delta_k$ and such that

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta'_1}, T_{\delta_2}, \dots, T_{\delta_k}).$$

Since $g(\mathcal{CB}(\delta_1)) = 1$, the hypotheses of the lemma are satisfied for the curves $\delta'_1, \delta_2, \dots, \delta_k$.

Choose an outermost separating simple closed curve δ'_1 of genus 1 and a separating simple closed curve γ on $\mathcal{CB}(\delta_1)$, as illustrated in Figure 6.

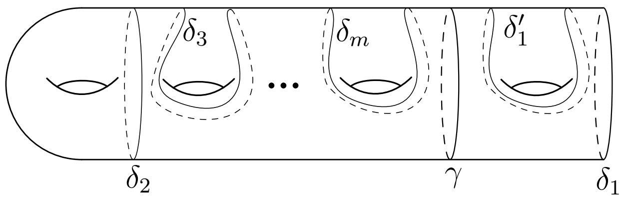

FIGURE 6. The curves δ'_1 and γ on the surface $\mathcal{GC}ap(\delta_1)$.

Apply Lemma 5.1 to the curves $\gamma, \delta'_1, \delta_1$:

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\gamma}, T_{\delta_2}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta'_1}, T_{\delta_2}, \dots, T_{\delta_k}).$$

By Lemma 8.3, we have $\mathcal{A}(T_{\gamma}, T_{\delta_2}, \dots, T_{\delta_k}) = 0$ in $H_k(\mathcal{I}_g)$. Hence,

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta'_1}, T_{\delta_2}, \dots, T_{\delta_k}),$$

and the lemma follows. □

Lemma 8.5. *Let $\delta_1, \dots, \delta_k$ be a simple admissible partition of S_g, and assume that none of the curves $\delta_1, \dots, \delta_k$ is a grouping curve. Then in $H_k(\mathcal{I}_g)$ we have*

$$2\,\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0.$$

Proof. Since there are no grouping curves, all curves $\delta_1, \dots, \delta_k$ are outermost. Choose a separating simple closed curve γ on S_g, disjoint from $\delta_1, \dots, \delta_k$, such that $\delta_1 \cup \delta_2 \cup \gamma$ bounds a pair of pants. Then, applying Lemma 5.1 to the curves $\gamma, \delta_1, \delta_2$, we obtain

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\gamma}, T_{\delta_2}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta_2}, T_{\delta_2}, \dots, T_{\delta_k}).$$

That is, $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\gamma}, T_{\delta_2}, \dots, T_{\delta_k})$. The curve γ is outermost and $g(\mathcal{C}ap(\gamma)) = 2$. Then by Lemma 5.9, we have $2\,\mathcal{A}(T_{\gamma}, T_{\delta_2}, \dots, T_{\delta_k}) = 0$ in $H_k(\mathcal{I}_g)$, which completes the proof. □

Proof of Proposition 8.1. It follows from Lemma 5.9 that $2\,\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$ in $H_k(\mathcal{I}_g)$ if any of the following conditions hold:

- for some outermost curve $\delta_i \in \mathcal{O}$, we have $g(\mathcal{C}ap(\delta_i)) \geq 2$;
- for some grouping curve $\delta_j \in \mathcal{G}$, we have $g(\mathcal{CB}(\delta_j)) \geq 2$.

We now assume that $g(\mathcal{C}ap(\delta_i)) = 1$ for all $\delta_i \in \mathcal{O}$ and $g(\mathcal{CB}(\delta_j)) \leq 1$ for all $\delta_j \in \mathcal{G}$. If $g(\mathcal{CB}(\delta)) = 0$ for some grouping curve δ, then the claim follows from Lemma 8.3. Otherwise, if $g(\mathcal{CB}(\delta)) = 1$ for all $\delta \in \mathcal{G}$, we first apply Lemma 8.4 to obtain an admissible partition $\delta'_1, \dots, \delta'_k$, and then apply Lemma 8.5 to this partition. □

Corollary 8.6. *For $g \geq 3$ and $3 \leq k \leq 2g-3$, the group $H_k^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space.*

As another consequence, we obtain the analogue of Proposition 5.5.

Proposition 8.7. *For $g \geq 3$, the following hold:*
- *if $k < g$, then the $\mathbb{Z}/2\mathbb{Z}$-vector space $H_k^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is generated by $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$, where $\delta_1, \dots, \delta_k$ are pairwise disjoint, pairwise nonisotopic, separating simple closed curves of genus 1;*
- *if $k \geq g$, then the $\mathbb{Z}/2\mathbb{Z}$-vector space $H_k^{\mathrm{ab,sep}}(\mathcal{I}_g)$ is trivial.*

Proof. First, we show that an arbitrary abelian cycle $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$ can be expressed as a sum of abelian cycles $\mathcal{A}(T_{\theta_{1,i}}, \dots, T_{\theta_{k,i}})$, each satisfying

$$\max\{g(\theta_{j,i}) \mid \theta_{j,i} \in \mathcal{O}(\theta_{1,i}, \dots, \theta_{k,i})\} = 1.$$

It suffices to show that, for the outermost curve (without loss of generality, δ_1), there exist separating simple closed curves $\varepsilon_1, \dots, \varepsilon_n$ of genus 1, disjoint from $\delta_2, \dots, \delta_k$, such that

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \sum_{i=1}^{n} \mathcal{A}(T_{\varepsilon_i}, T_{\delta_2}, \dots, T_{\delta_k}).$$

The same argument as in the proof of Proposition 5.5 applies here.

Thus, it remains to prove the statement for the abelian cycle $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$, where

$$\max\{g(\delta) \mid \delta \in \mathcal{O}\} = 1.$$

If none of the curves $\delta_1, \dots, \delta_k$ is grouping, the claim follows immediately. Otherwise, consider a grouping curve, which we may assume without loss of generality to be δ_1. If $g(\mathcal{CB}(\delta_1)) = 0$, then $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$ by Lemma 8.3. If $g(\mathcal{CB}(\delta_1)) > 0$, we can choose separating simple closed curves $\gamma_1, \dots, \gamma_n \subset \mathcal{CB}(\delta_1)$ of genus 1 such that in $H_k(\mathcal{I}_g)$ we have

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \sum_{i=1}^{n} \mathcal{A}(T_{\gamma_i}, T_{\delta_2}, \dots, T_{\delta_k}).$$

Indeed, we proceed in the same way as in the proof of Proposition 5.5. Choose separating simple closed curves $\gamma_1, \dots, \gamma_{n+1}$ and $\eta_1, \dots, \eta_{n-1}$ on $\mathcal{CB}(\delta_1)$, as shown in Figure 7.

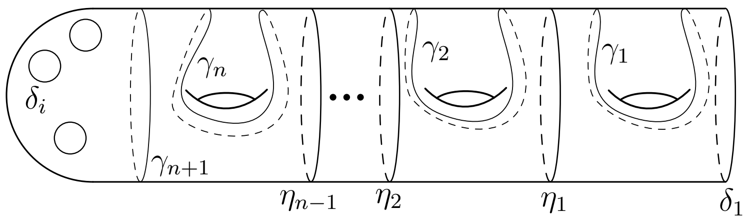

FIGURE 7. The curves on the surface $\mathcal{CB}(\delta_1)$.

Then, by Lemma 5.1, we have

$$\mathcal{A}(T_{\delta_1}, T_{\delta_2}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\gamma_1}, T_{\delta_2}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\eta_1}, T_{\delta_2}, \dots, T_{\delta_k}).$$

Applying the same procedure repeatedly to $\mathcal{A}(T_{\eta_1}, T_{\delta_2}, \dots, T_{\delta_k})$, we obtain a collection of separating simple closed curves $\gamma_1, \dots, \gamma_{n+1}$ on $\mathcal{CB}(\delta_1)$ such that $g(\gamma_1) = \cdots = g(\gamma_n) = 1$ and

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \sum_{i=1}^{n+1} \mathcal{A}(T_{\gamma_i}, T_{\delta_2}, \dots, T_{\delta_k}).$$

Since $\mathcal{A}(T_{\gamma_{n+1}}, T_{\delta_2}, \dots, T_{\delta_k}) = 0$ by Lemma 8.3, it follows that

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \sum_{i=1}^{n} \mathcal{A}(T_{\gamma_i}, T_{\delta_2}, \dots, T_{\delta_k}).$$

This yields the desired result.

Repeating the procedure described above for each grouping curve $\delta_i \in \mathcal{G}$, we obtain that the abelian cycle $\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k})$ can be expressed as a sum of abelian cycles $\mathcal{A}(T_{\gamma_{j_1}}, \ldots, T_{\gamma_{j_k}})$, where all outermost curves among $\gamma_{j_1}, \ldots, \gamma_{j_k}$ have genus 1.

Hence, if $k < g$, the claim follows. If $k \geq g$, there is necessarily at least one grouping curve among $\gamma_{j_1}, \ldots, \gamma_{j_k}$, and the resulting sum is empty, i.e., zero, as desired. □

Theorem 1.1 now follows from Propositions 3.1, 5.3, 5.5, 8.1, and 8.7.

We conclude the paper by proving a criterion for when an abelian cycle $\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k})$ vanishes in $H_k(\mathcal{I}_g)$ for $k < g$. We begin with the following lemma.

Lemma 8.8. *Let $\delta_1, \ldots, \delta_k$ be an admissible partition of S_g with $k < g$. Suppose that the curves $\delta_1, \ldots, \delta_m$ (with $m \geq 3$) bound a genus-0 subsurface Σ that does not contain any of the remaining curves $\delta_{m+1}, \ldots, \delta_k$.*

Then $\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k})$ can be expressed as a sum of abelian cycles $\mathcal{A}(T_{\varepsilon_{i,1}}, \ldots, T_{\varepsilon_{i,k}})$ such that, for each i, the following conditions hold:

(1) $\varepsilon_{i,1}, \ldots, \varepsilon_{i,k}$ form an admissible partition. Let $\mathcal{E}$ denote the set of subsurfaces obtained by cutting S_g along these curves.
(2) $\varepsilon_{i,1}, \ldots, \varepsilon_{i,m}$ bound a genus-0 subsurface $E \in \mathcal{E}$ that contains none of $\varepsilon_{i,m+1}, \ldots, \varepsilon_{i,k}$.
(3) For each $j = 1, 2$, the curve $\varepsilon_{i,j}$ bounds a subsurface $E_j \in \mathcal{E} \setminus \{E\}$, with $\varepsilon_{i,j} \subset \partial E_j$ and $g(E_j) \geq 1$.

Proof. Denote by $\mathcal{S}$ the set of subsurfaces obtained by cutting S_g along $\delta_1, \ldots, \delta_k$. Let $\Sigma_1, \Sigma_2 \in \mathcal{S} \setminus \{\Sigma\}$ be such that $\delta_j \subset \partial\Sigma_j$ for $j = 1, 2$. For each $j = 1, 2$, let $\widehat{\Sigma}_j$ denote the subsurface of S_g bounded by δ_j that does not contain Σ.

We express the abelian cycle $\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k})$ as a sum of abelian cycles $\mathcal{A}(T_{\eta_{i,1}}, \ldots, T_{\eta_{i,k}})$ satisfying:

- $\eta_{i,1}, \ldots, \eta_{i,k}$ form an admissible partition; let $\mathcal{N}$ denote the subsurfaces obtained by cutting S_g along these curves.
- $\eta_{i,1}, \ldots, \eta_{i,m}$ bound a genus-0 subsurface $N \in \mathcal{N}$ that contains none of $\eta_{i,m+1}, \ldots, \eta_{i,k}$.
- $\eta_{i,1}$ bounds a subsurface $N_1 \in \mathcal{N} \setminus \{N\}$, with $g(N_1) \geq 1$.

We will refer to these as the $(\star)$ conditions.

Assume without loss of generality that $\delta_{m+1} \subset \widehat{\Sigma}_1$. There are two cases:

- $g(\delta_{m+1}) = 1$. Choose a separating simple closed curve $\gamma \subset \widehat{\Sigma}_1$ as shown in Figure 8, with the remaining curves δ_ℓ on $\widehat{\Sigma}_1$ lying to the left of γ.

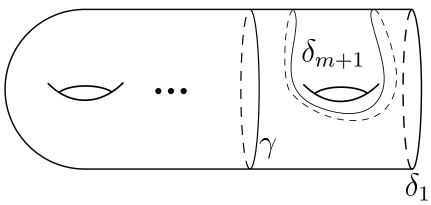

FIGURE 8. The curves γ and δ_{m+1} on $\widehat{\Sigma}_1$.

We then apply Lemma 5.1 to the curves $\delta_1, \delta_{m+1}, \gamma$:

$$\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_m}, T_{\delta_1}, T_{\delta_{m+2}}, \ldots, T_{\delta_k}) + \mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_m}, T_\gamma, T_{\delta_{m+2}}, \ldots, T_{\delta_k}) = \mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k}),$$

from which we obtain

$$\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k}) = \mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_m}, T_\gamma, T_{\delta_{m+2}}, \ldots, T_{\delta_k}).$$

For the curves $\delta_1, \ldots, \delta_m, \gamma, \delta_{m+2}, \ldots, \delta_k$, the $(\star)$ conditions are satisfied.

- $g(\delta_{m+1}) \geq 2$. Then choose separating simple closed curves $\gamma_1, \gamma_2 \subset \mathcal{C}ap(\delta_{m+1})$, as shown in Figure 9.

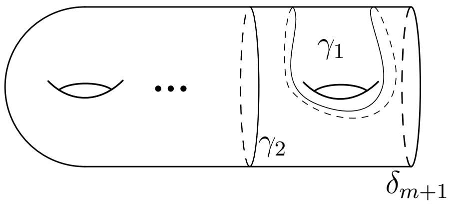

FIGURE 9. The curves γ_1, γ_2 on $\mathcal{C}ap(\delta_{m+1})$.

We then apply Lemma 5.1 to the curves $\gamma_1, \gamma_2, \delta_{m+1}$:

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_{m+1}}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\delta_1}, \dots, T_{\delta_m}, T_{\gamma_2}, T_{\delta_{m+2}}, \dots, T_{\delta_k}) \\ = \mathcal{A}(T_{\delta_1}, \dots, T_{\delta_m}, T_{\gamma_1}, T_{\delta_{m+2}}, \dots, T_{\delta_k}).$$

The $(\star)$ conditions are satisfied for the following two sets of curves:
- $\delta_1, \dots, \delta_m, \gamma_1, \delta_{m+2}, \dots, \delta_k$;
- $\delta_1, \dots, \delta_m, \gamma_2, \delta_{m+2}, \dots, \delta_k$.

By performing, if necessary, similar arguments for each admissible partition $\eta_{i,1}, \dots, \eta_{i,k}$ with respect to $\eta_{i,2}$, we obtain the desired expression of $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})$ as a sum of abelian cycles of the form specified in the lemma. □

For the proof of Theorem 1.7, we use the homomorphism $\sigma_k : H_k^{\mathrm{ab,sep}}(\mathcal{I}_g) \to \wedge^k \mathbb{B}_2'$, which is induced by the Birman–Craggs–Johnson homomorphism. As in the case $k = 2$, we have

$$\sigma_k(\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})) = \sigma(T_{\delta_1}) \wedge \cdots \wedge \sigma(T_{\delta_k}).$$

Proof of Theorem 1.7. Let the curves $\delta_1, \dots, \delta_m$ (with $m \geq 3$) bound a genus-0 subsurface $\Sigma \subset S_g$ that contains none of the remaining curves $\delta_{m+1}, \dots, \delta_k$. We will show that in this case $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$ in $H_k(\mathcal{I}_g)$.

Denote by $\mathcal{S}$ the set of subsurfaces obtained by cutting S_g along $\delta_1, \dots, \delta_k$, and consider the following condition:

$(\star\star)$ For $j = 1, 2$, the subsurface $\Sigma_j \in \mathcal{S} \setminus \{\Sigma\}$ with $\delta_j \subset \partial\Sigma_j$ satisfies $g(\Sigma_j) \geq 1$.

By Lemma 8.8 we may assume that, the curves $\delta_1, \dots, \delta_m$ satisfy the condition $(\star\star)$. The proof will proceed by induction on m.

Base of induction. Let $m = 3$. Then we can apply Lemma 5.6 to the curves $\delta_1, \delta_2, \delta_3$, obtaining

$$\mathcal{A}(T_{\delta_2}, T_{\delta_2}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\delta_3}, T_{\delta_2}, T_{\delta_3}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}),$$

from which it follows that $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$.

Inductive step. Suppose the statement holds for all $m \leq \ell$, and let us show that it holds for $m = \ell + 1$. Choose a curve γ such that $\delta_1, \delta_2, \gamma$ bound a genus-0 subsurface. Then, by Lemma 5.6, we have

$$\mathcal{A}(T_\gamma, T_{\delta_2}, \dots, T_{\delta_k}) + \mathcal{A}(T_{\delta_2}, T_{\delta_2}, \dots, T_{\delta_k}) = \mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}),$$

that is

$$\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = \mathcal{A}(T_\gamma, T_{\delta_2}, \dots, T_{\delta_k}).$$

By the induction hypothesis applied to the $m - 1$ curves $\gamma, \delta_3, \dots, \delta_m$, we conclude that $\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k}) = 0$.

To prove the statement in the reverse direction, it is sufficient to show that

$$\sigma_k(\mathcal{A}(T_{\delta_1}, \dots, T_{\delta_k})) \neq 0.$$

Using an argument similar to the proof of Lemma 8.4, one can show that there exists an admissible partition $\delta'_1, \ldots, \delta'_k$ such that:

- there are no grouping curves among $\delta'_1, \ldots, \delta'_k$; and
- $\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k}) = \mathcal{A}(T_{\delta'_1}, \ldots, T_{\delta'_k})$.

Therefore, we may assume that $\delta_1, \ldots, \delta_k$ all curves are outermost. Under this assumption, we can choose a symplectic basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ such that

$$\begin{aligned}
\sigma(T_{\delta_1}) &= \overline{\mathbf{a}}_1\overline{\mathbf{b}}_1 + \cdots + \overline{\mathbf{a}}_{i_1}\overline{\mathbf{b}}_{i_1},\\
\sigma(T_{\delta_2}) &= \overline{\mathbf{a}}_{i_1+1}\overline{\mathbf{b}}_{i_1+1} + \cdots + \overline{\mathbf{a}}_{i_2}\overline{\mathbf{b}}_{i_2},\\
&\vdots\\
\sigma(T_{\delta_k}) &= \overline{\mathbf{a}}_{i_{k-1}+1}\overline{\mathbf{b}}_{i_{k-1}+1} + \cdots + \overline{\mathbf{a}}_{i_k}\overline{\mathbf{b}}_{i_k},
\end{aligned}$$

where $1 < i_1 < \cdots < i_k < g$. Then

$$\begin{aligned}
\sigma_k(\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k})) &= \sigma(T_{\delta_1}) \wedge \cdots \wedge \sigma(T_{\delta_k})\\
&= (\overline{\mathbf{a}}_1\overline{\mathbf{b}}_1 + \cdots + \overline{\mathbf{a}}_{i_1}\overline{\mathbf{b}}_{i_1}) \wedge \cdots \wedge (\overline{\mathbf{a}}_{i_{k-1}+1}\overline{\mathbf{b}}_{i_{k-1}+1} + \cdots + \overline{\mathbf{a}}_{i_k}\overline{\mathbf{b}}_{i_k}).
\end{aligned}$$

Since $i_k < g$, the elements

$$\overline{\mathbf{a}}_1\overline{\mathbf{b}}_1, \ldots, \overline{\mathbf{a}}_{i_k}\overline{\mathbf{b}}_{i_k}$$

are linearly independent in $\mathbb{B}'_2$. It then follows that expanding the parentheses yields a sum of linearly independent vectors, so

$$\sigma_k(\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k})) \neq 0,$$

which implies $\mathcal{A}(T_{\delta_1}, \ldots, T_{\delta_k}) \neq 0$. □

References

[1] M. Bestvina, K.-U. Bux, and D. Margalit. "The dimension of the Torelli group". *Journal of the American Mathematical Society* 23.1 (2010), pp. 61–105.

[2] T. E. Brendle and B. Farb. "The Birman–Craggs–Johnson homomorphism and abelian cycles in the Torelli group". *Mathematische Annalen* 338.1 (2007), pp. 33–53.

[3] B. Farb and D. Margalit. *A Primer on Mapping Class Groups*. Princeton University Press, 2012.

[4] A. A. Gaifullin. "On a spectral sequence for the action of the Torelli group of genus 3 on the complex of cycles". *Izvestiya: Mathematics* 85.6 (2021), pp. 1060–1127.

[5] A. A. Gaifullin. "On infinitely generated homology of Torelli groups". *St. Petersburg Mathematical Journal* 35.6 (2024), pp. 959–993.

[6] R. Hain. "The Rational Cohomology Ring of the Moduli Space of Abelian 3-folds". *Mathematical Research Letters* 9.4 (2002), pp. 473–491.

[7] D. Johnson. "An abelian quotient of the mapping class group $\mathcal{I}_g$". *Math. Ann.* 249 (1980), pp. 225–242.

[8] D. Johnson. "Quadratic forms and the Birman–Craggs homomorphisms". *Transactions of the American Mathematical Society* 261.1 (1980), pp. 235–254.

[9] D. Johnson. "The structure of the Torelli group I: A finite set of generators for $\mathcal{I}$". *Ann. of Math.* 118.3 (1983), pp. 423–442.

[10] D. Johnson. "The structure of the Torelli group III: The abelianization of $\mathcal{I}$". *Topology* 24.2 (1985), pp. 127–144.

[11] M. Kassabov and A. Putman. "Equivariant group presentations and the second homology group of the Torelli group". *Mathematische Annalen* 376.1–2 (2019), pp. 227–241.

[12] G. Mess. "The Torelli groups for genus 2 and 3 surfaces". *Topology* 31.4 (1992), pp. 775–790.

[13] D. Minahan. “The second rational homology of the Torelli group is finitely generated”. *arXiv preprint arXiv:2307.07082* (2023).
[14] D. Minahan and A. Putman. “The second rational homology of the Torelli group”. *arXiv preprint arXiv:2504.00211* (2025).
[15] W. R. Vautaw. “Abelian subgroups of the Torelli group”. *Algebraic and Geometric Topology* 2.1 (2002), pp. 157–170.

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