

POSITIVITY OF PARTIAL SUMS OF A RANDOM MULTIPLICATIVE FUNCTION AND CORRESPONDING PROBLEMS FOR THE LEGENDRE SYMBOL

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ABSTRACT. Let $f(n)$ be a random completely multiplicative function such that $f(p) = \pm 1$ with probabilities $1/2$ independently at each prime. We study the conditional probability, given that $f(p) = 1$ for all $p < y$, that all partial sums of $f(n)$ up to x are nonnegative. We prove that for $y \geq C \frac{(\log x)^2 \log_2 x}{\log_3 x}$ this probability equals $1 - o(1)$. We also study the probability P'_x that $\sum_{n \leq x} \frac{f(n)}{n}$ is negative. We prove that $P'_x \ll \exp\left(-\exp\left(\frac{\log x \log_4 x}{(1+o(1)) \log_3 x}\right)\right)$, which improves a bound given by Kerr and Klurman. Under a conjecture closely related to Halász's theorem, we prove that $P'_x \ll \exp(-x^\alpha)$ for some $\alpha > 0$. Let $\chi_p(n) = \left(\frac{n}{p}\right)$ be the Legendre symbol modulo p . For a prime p chosen uniformly at random from $(x, 2x]$, we express the probability that all partial sums of $\frac{\chi_p(n)}{n}$ are nonnegative in terms of the probability that partial sums of $\frac{f(n)}{n}$ are nonnegative.

1. INTRODUCTION

1.1. Partial sums of $f(n)$. Let $\chi_p(n) = \left(\frac{n}{p}\right)$ be a Legendre symbol $(\text{mod } p)$. Let \mathcal{L}^+ denote the set of primes p such that the partial sums of $\chi_p(n)$ are all nonnegative. The motivational problem for us is whether \mathcal{L}^+ is infinite. Primes in \mathcal{L}^+ are also remarkable, because the corresponding Fekete polynomial has no zeros in $(0, 1)$ and hence the L -function $L(s, \chi_p)$ has no real zeros.

Let x be a large number and let us choose a prime $p \in (x, 2x]$ uniformly at random. Kalmynin [21] proved that $\mathbb{P}(p \in \mathcal{L}^+) \ll (\log \log x)^{-c}$, where $c \approx 0.0368$. As indicated in [23, p. 5], this bound can be substantially improved conditionally on the non-existence of Siegel zeros. In this paper by $\log_k x$ we denote $\underbrace{\log \log \dots \log x}_k$.

Let us define a random completely multiplicative function f by taking $f(p) = \pm 1$ with probabilities $1/2$ independently at each prime. Denote by \mathcal{F} the probability space of such functions. Such a random function $f : [1, N] \cap \mathbb{N} \rightarrow \{1, -1\}$ is a good model for $\chi_p : [1, N] \cap \mathbb{N} \rightarrow \{1, -1\}$ if N is small enough in comparison with x . For large N the model becomes inadequate; for instance $\sum_{n=1}^{p-1} \chi_p(n) = 0$, a property not generally satisfied by an arbitrary $f \in \mathcal{F}$. See [20], where a better model on a large scale is defined.

One can try to find $p \in \mathcal{L}^+$ among those primes for which the least quadratic non-residue n_p is large. Then one can expect that $\sum_{n \leq y} \chi_p(n)$ is dominated by the contribution of the $(n_p - 1)$ -smooth part. With that in mind, we formulate the following problem. Let us denote by \mathcal{L}_x^+ the set of completely multiplicative functions f taking values ± 1 such that all partial sums of $f(n)$ up to x are nonnegative. What should be $y = y(x)$ so that $\mathbb{P}(f \in \mathcal{L}_x^+ \mid f(p) = 1 (p \leq y)) = 1 - o(1)$?

Theorem 1. *There exist $C > 0, x_0 > 0$ such that for any $x > x_0$ and any*

$$y \geq C \frac{(\log x)^2 \log_2 x}{\log_3 x}$$

we have $\mathbb{P}(f \in \mathcal{L}_x^+ \mid f(p) = 1 (p \leq y)) = 1 - o(1)$ as x tends to infinity.

It is worth mentioning that the best known lower bound $n_p = \Omega(\log p \log_3 p)$ proved by Graham and Ringrose [10] is much smaller than y in Theorem 1 if we set $p = x$. In the setting of Theorem 1 the value $y = (\log x)^{2+o(1)}$ seems crucial, because then the square root of the variation of $\sum_{n \leq x} f(n)$ surpasses $\Psi(x, y) = x^{1/2+o(1)}$, where by $\Psi(x, y)$ we denote the number of y -smooth numbers up to x . Let us state this as a conjecture.

Conjecture 1. *For any $\varepsilon > 0, x > x_0(\varepsilon)$ and $y \leq (\log x)^{2-\varepsilon}$ we have*

$$\mathbb{P}(f \in \mathcal{L}_x^+ \mid f(p) = 1 (p \leq y)) = o(1).$$

Since

$$\mathbb{P}(f \in \mathcal{L}_x^+) \geq \mathbb{P}(f(p) = 1 (p \leq y)) \mathbb{P}(f \in \mathcal{L}_x^+ \mid f(p) = 1 (p \leq y)) = 2^{-\pi(y)}(1 - o(1)),$$

Theorem 1 gives us the following corollary.

Corollary 1.

$$\mathbb{P}(f \in \mathcal{L}_x^+) \geq \exp\left(-C' \frac{(\log x)^2}{\log_3 x}\right).$$

The upper bound $\mathbb{P}(f \in \mathcal{L}_x^+) \ll (\log x)^{-c+o(1)}$ was proved in [21]. It is plausible that Corollary 1 can be substantially improved, since we used a very special construction to detect f in \mathcal{L}_x^+ .

1.2. Partial sums of $\frac{f(n)}{n}$. Now let us ask: What is the probability that the sums $\sum_{n \leq y} \frac{\chi_p(n)}{n}$ are positive for all $y \geq 1$? This problem seems to be much more approachable than the problem about \mathcal{L}^+ . First of all, the $\sum_n \frac{\chi_p(n)}{n}$ converges to $L(1, \chi_p)$, which is positive due to Dirichlet's class number formula. This shows that the partial sums $\sum_{n \leq y} \frac{\chi_p(n)}{n}$ are all strictly positive from a certain point. Second, the values of $\chi_p(q)$ for large primes q have an insignificant influence on the size of $\sum_{n \leq y} \frac{\chi_p(n)}{n}$. Hence, it is easier to prove that a random function $f \in \mathcal{F}$ is a good model for χ_p in this problem.

Let us start by discussing the analogous problem for $f \in \mathcal{F}$. First, what can be said about an arbitrary fixed $f \in \mathcal{F}$? The question of how negative the sum $\sum_{n \leq x} \frac{f(n)}{n}$ can be was discussed by Granville and Soundararajan [12]. They showed among other things that for x sufficiently large $\sum_{n \leq x} \frac{f(n)}{n} \geq -(\log \log x)^{-3/5}$ and constructed f such that $\sum_{n \leq x} \frac{f(n)}{n} < -\frac{c}{\log x}$. Kerr and Klurman [22] proved that $\sum_{n \leq x} \frac{f(n)}{n} \geq -(\log \log x)^{-1+\varepsilon}$ for any $\varepsilon > 0$ and x large enough. Of course these results can be applied to $f(n) = \chi_p(n)$.

Now let us denote by P the probability that

$$\sum_{n \leq y} \frac{f(n)}{n} > 0$$

for every $y \geq 1$. Angelo and Xu [2] proved that $1 - 10^{-45} < P$. Let $\lambda(n)$ be the Liouville function. Borwein, Ferguson, and Mossinghoff [6] showed that the minimal

N_0 such that

$$\sum_{n \leq N_0} \frac{\lambda(n)}{n} < 0,$$

is $N_0 = 72, 185, 376, 951, 205$. Hence

$$P \leq 1 - 2^{-\pi(N_0)} < 1 - 10^{-704 \times 10^9}.$$

Let us denote by P'_x the probability that

$$\sum_{n \leq x} \frac{f(n)}{n} < 0.$$

It turns out that P'_x tends to 0 very rapidly.

Angelo and Xu proved [2, Theorem 1.2] that

$$P'_x \ll \exp\left(-\exp\left(\frac{\log x}{C \log_2 x}\right)\right).$$

Kerr and Klurman [22, Theorem 1.2] improved this to

$$P'_x \ll \exp\left(-\exp\left(\frac{\log x \log_3 x}{C \log_2 x}\right)\right),$$

for some constant C . Although the authors do not state it explicitly, one can derive from the proof that $C = 1 + o(1)$ is admissible.

Theorem 2.

$$P'_x \ll \exp\left(-\exp\left(\frac{\log x \log_4 x}{(1 + o(1)) \log_3 x}\right)\right),$$

as $x \rightarrow +\infty$.

This theorem can be improved if the following conjecture is true.

Conjecture 2. *There exists $\varepsilon > 0$ such that for all $x \geq 3$ and all real-valued completely multiplicative functions f such that $|f(n)| \leq 1$ for all n and $f(p) = 0$ for all $p > x^\varepsilon$ we have*

$$\sum_{n \leq x} f(n) \ll \frac{x}{\log \log x} \exp\left(\sum_{p \leq x} \frac{f(p)}{p}\right).$$

Theorem 3. *If Conjecture 2 is true, then there exists $\alpha > 0$ such that*

$$P'_x \ll \exp(-x^\alpha).$$

We now comment on Conjecture 2. Proposition 4 is a weaker result that we use instead of Conjecture 2 to prove Theorem 2. To prove Proposition 4 we use a version of Halász's theorem for multiplicative functions with support on smooth numbers, which was proved by Granville, Harper, and Soundararajan in [11] (see Lemma 3.5). If we take $f(p) = -1$ for $p < x/2$, and $f(p) = 0$ otherwise, then

$$\sum_{n \leq x} f(n) \asymp \frac{x}{\log x} \asymp x \exp\left(\sum_{p \leq x} \frac{f(p)}{p}\right).$$

Hence the condition that f is supported on x^ε -smooth numbers cannot be dropped. But if we let $f(p) = -1$ for $p \leq x^{0.99}$ and $f(p) = 0$ otherwise, then, as follows from the result by Alladi [1, Theorem 2],

$$\sum_{n \leq x} f(n) \ll \frac{x}{(\log x)^2}.$$

Therefore, this does not produce a counterexample to Conjecture 2.

Now let us return to the problem about χ_p . Denote by \tilde{P}_x the probability that for all $y \geq 1$

$$\sum_{n \leq y} \frac{\chi_p(n)}{n} > 0.$$

By $\text{Cov}(X, Y)$ we define the covariance of random variables X and Y . If A is an event, then $\mathbb{1}_A$ denotes the indicator function of A .

Theorem 4. *Let p be a random prime in $(x, 2x]$ chosen uniformly. Let A be the event in \mathcal{F} that partial sums of the sequence $\frac{f(n)}{n}$ are all positive. Let $k = 8 \prod_{2 < q \leq c_1 \sqrt{\log x}} q$. Let $E_0 = 0$ if no character $(\text{mod } k)$ has a Siegel zero. If there is a character $\chi_1 \pmod{k}$ with Siegel zero β_1 , then χ_1 can be written in the form $\chi_1(n) = \left(\frac{d}{n}\right)$, where $d|k$. In this case, we set $E_0 = 0$ if $d < 0$ and $E_0 = 1$ otherwise. Then*

$$\tilde{P}_x = P - E_0 \text{Cov}(\mathbb{1}_A, f(d)) \frac{\int_x^{2x} \frac{u^{\beta_1-1}}{\log u} du}{\text{Li}(2x) - \text{Li}(x)} + O\left(\exp\left(-\exp\left(\frac{\log_2 x \log_4 x}{(2+o(1)) \log_3 x}\right)\right)\right).$$

Corollary 2.

$$\tilde{P}_x - P \ll \exp\left(-\exp\left(\frac{\log_3 x \log_6 x}{(1+o(1)) \log_5 x}\right)\right).$$

We studied in this paper the positivity of partial sums $\sum_{n \leq y} f(n)$ and $\sum_{n \leq y} \frac{f(n)}{n}$. The analogous problems for $\sum_{n \leq y} \frac{f(n)}{n^\sigma}$, where $0 < \sigma < 1$, were investigated in recent years. See, for example, [4], [3]; the latter includes a survey of results on random multiplicative functions.

2. PROOF OF THEOREM 1

Denote by $\alpha = \alpha(x, y)$ the solution to the equation

$$\sum_{p \leq y} \frac{\log p}{p^\alpha - 1} = \log x.$$

Lemma 2.1. *There exist $K > 0$ and $y_0 > 0$, such that for any $y_0 < y < x$ that satisfy $y = (\log x)^{2+\varepsilon}$, $\varepsilon > (\log x)^{-1/4}$ and*

$$(1) \quad (2\alpha(x, y) - 1)y^{2\alpha(x, y)-1} \log y \geq K \log \log x$$

we have $\mathbb{P}(f \in \mathcal{L}_x^+ \mid f(p) = 1 (p \leq y)) = 1 - o(1)$.

Deduction of Theorem 1 from Lemma 2.1.

$$\log x = \sum_{p < y} \frac{\log p}{p^\alpha - 1} \geq \frac{\theta(y)}{y^\alpha - 1},$$

where $\theta(y) = \sum_{p \leq y} \log p$. Hence for $y \gg (\log x)^{1.01}$

$$\alpha \geq \frac{\log\left(1 + \frac{\theta(y)}{\log x}\right)}{\log y} = \frac{\log\left(\frac{y}{\log x}\right)}{\log y} + o\left(\frac{1}{\log y}\right).$$

If $y = (\log x)^{2+\varepsilon}$, then

$$2\alpha(x, y) - 1 \geq \frac{\varepsilon}{2 + \varepsilon} + o(1/\log y).$$

Hence

$$(2) \quad (2\alpha(x, y) - 1)y^{2\alpha(x, y)-1} \log y \gg \varepsilon y^{\frac{\varepsilon}{2+\varepsilon}} \log y.$$

We want the right-hand side to be $\gg \log \log x \asymp \log y$. This is satisfied if

$$\varepsilon = \frac{\log_3 x}{\log_2 x} - \frac{\log_4 x}{\log_2 x} + \frac{R}{\log_2 x},$$

where R is a sufficiently large constant.

Thus we can take

$$y = (\log x)^{2+\varepsilon} \gg \frac{(\log x)^2 \log_2 x}{\log_3 x}.$$

□

2.1. Proof of Lemma 2.1. We first require some preliminary results.

Lemma 2.2. *We have uniformly in $1 \leq t \leq x$, $2 \leq y \leq x$*

$$\Psi\left(\frac{x}{t}, y\right) \ll \frac{\Psi(x, y)}{t^{\alpha(x, y)}}.$$

Proof. This is proved by Bretèche and Tenenbaum [8, Theorem 2.4], using a formula for $\Psi(x, y)$ due to Hildebrand and Tenenbaum [17, Theorem 1] that was obtained via Perron's formula and the saddle point method. □

Lemma 2.3. *Uniformly in $x \geq y \geq 2$ we have*

$$\alpha(x, y) = \frac{\log(1 + y/\log x)}{\log y} \left(1 + O\left(\frac{\log_2(1 + y)}{\log y}\right)\right).$$

Proof. See [17, Theorem 2]. □

Let $p(n)$ be the least prime divisor of n . We also define $p(1) := +\infty$. Denote

$$\Psi^*(x, y) := \sum_{p(m) > y} \Psi\left(\frac{x}{m^2}, y\right).$$

Then

$$\sum_{n \leq x} f(n) = \Psi^*(x, y) + \sum_{\substack{p(n) > y \\ n \neq 1}} {}^b f(n) \Psi^*\left(\frac{x}{n}, y\right),$$

where the flat (b) indicates that the sum is over square-free integers.

Now let $y = (\log x)^{2+\varepsilon}$, $\varepsilon > 0$. Then, by Lemma 2.3

$$\alpha(x, y) = \frac{1 + \varepsilon}{2 + \varepsilon} \left(1 + O\left(\frac{\log_2 y}{\log y}\right)\right).$$

Note that $\frac{1+\varepsilon}{2+\varepsilon} = \frac{1}{2} + \frac{\varepsilon}{2(2+\varepsilon)}$.

By Lemma 2.2

$$(3) \quad \frac{\Psi^*(x, y) - \Psi(x, y)}{\Psi(x, y)} \ll \sum_{\substack{p(m) > y \\ m \neq 1}} m^{-2\alpha(x, y)} = \prod_{p > y} (1 - p^{-2\alpha(x, y)})^{-1} - 1.$$

We have

$$\sum_{p > y} p^{-2\alpha(x, y)} \ll \frac{y^{1-2\alpha}}{(\log y)(2\alpha - 1)}.$$

Let us assume that y is such that $\frac{y^{1-2\alpha}}{(\log y)^{(2\alpha-1)}} = o(1)$. Then (3) gives us

$$\frac{\Psi^*(x, y) - \Psi(x, y)}{\Psi(x, y)} \ll \sum_{p>y} p^{-2\alpha(x, y)} = o(1).$$

Hence, under this condition, $\Psi^*(u, y) \sim \Psi(u, y)$ for any $u \leq x$ since $\alpha(x, y)$ is monotonically decreasing in x .

Lemma 2.4 (Bonami-Halász's inequality). *Let $f(n)$ be a Rademacher random variable and let $b_j(n) \in \mathbb{C}$ be fixed coefficients. Then*

$$\left| \mathbb{E} \left(\prod_{1 \leq j \leq m} \sum_{n \geq 1} {}^b b_j(n) f(n) \right) \right| \leq \left(\prod_{1 \leq j \leq m} \sum_{n \geq 1} {}^b |b_j(n)|^2 (m-1)^{\omega(n)} \right)^{1/2}.$$

Proof. This statement was proved by Bonami in [5]. See [13, Lemma 2] for an alternative proof. \square

Lemma 2.5. *Let $\delta > 0$, $x > y$ and*

$$R_x := \mathbb{P} \left(\left| \sum_{\substack{p(n)>y \\ n \neq 1}} {}^b f(n) \Psi^* \left(\frac{x}{n}, y \right) \right| > \delta \Psi^*(x, y) \right).$$

Suppose that $y = y(x)$ satisfies $\frac{y^{1-2\alpha}}{(\log y)^{(2\alpha-1)}} = o(1)$. Then there exists $c_0 > 0$ such that

$$R_x \ll \exp(-c_0 \delta^2 (2\alpha - 1) y^{2\alpha-1} \log y).$$

Proof. By Lemma 2.2 we have

$$\Psi^* \left(\frac{x}{n}, y \right) \leq C_1 n^{-\alpha(x, y)} \Psi^*(x, y)$$

for some absolute constant C_1

Thus Lemma 2.4 and the moment inequality give us

$$R_x \ll C_1^{2m} \delta^{-2m} \left(\sum_{\substack{p(n)>y \\ n \neq 1}} {}^b n^{-2\alpha(x, y)} (2m-1)^{\omega(n)} \right)^m.$$

Hence

$$R_x \ll C_1^{2m} \delta^{-2m} \left(\prod_{p>y} \left(1 + \frac{2m-1}{p^{2\alpha}} \right) - 1 \right)^m.$$

Suppose that $\sum_{p>y} \frac{m}{p^{2\alpha}} < 1/2$. Let $C_2 = \max(2C_1^2, 2)$, then

$$R_x \ll C_1^{2m} \delta^{-2m} \left(\exp \left(2 \sum_{p>y} \frac{m}{p^{2\alpha}} \right) - 1 \right)^m \ll C_2^m \delta^{-2m} \left(\sum_{p>y} \frac{m}{p^{2\alpha}} \right)^m.$$

The bound still holds if $\sum_{p>y} \frac{m}{p^{2\alpha}} \geq 1/2$ since $R_x \leq 1$.

Let C_3 be a constant such that

$$T := \frac{C_3 \delta^{-2} y^{1-2\alpha}}{(2\alpha-1) \log y} \geq C_2 \delta^{-2} \sum_{p>y} \frac{1}{p^{2\alpha}}.$$

We have $R_x \ll (Tm)^m$. Note that $T = o(1)$ by assumption. Let $m = \lceil T^{-1}/e \rceil$. We obtain

$$R_x \ll \exp(-c_0 \delta^2 (2\alpha - 1) y^{2\alpha-1} \log y)$$

for some $c_0 > 0$. □

Lemma 2.6. *Suppose that $(\log X)^3 \leq y$. Then $\mathbb{P}(f \in \mathcal{L}_X^+ \mid f(p) = 1 (p \leq y)) = 1 - o(1)$.*

Proof. Let us apply Lemma 2.5 with $\delta = 1/10$ at each integer in the interval $[y, X]$. We obtain

$$1 - \mathbb{P}(f \in \mathcal{L}_X^+ \mid f(p) = 1 (p \leq y)) \leq \sum_{y \leq n \leq X} R_n \ll X \exp\left(-\frac{c_0}{100} (2\alpha(X, y) - 1) y^{2\alpha(X, y)-1} \log y\right).$$

The right-hand side is $o(1)$ if

$$(4) \quad (2\alpha(X, y) - 1) y^{2\alpha(X, y)-1} \log y \geq K_0 \log X,$$

where K_0 is sufficiently large. Inequality (4) follows from the assumption $(\log X)^3 \leq y$. This can be shown in the same way as we deduced Theorem 1 from Lemma 2.1. □

Lemma 2.7. *There exists a constant $y_1 \geq 2$, such that for all $y \geq y_1$, $x, z > 0$ we have*

$$\Psi(x + z, y) - \Psi(x, y) \leq \Psi(z, y) + 1.$$

Proof. This was proved by Hildebrand [15, Theorem 4] without +1 on the right side but with an additional assumption that $x, z \geq y$. But if $z < y$, then

$$\Psi(x + z, y) - \Psi(x, y) \leq [z] + 1 = \Psi(z, y) + 1$$

still holds. The same argument works if $x < y$. □

Konyagin and Pomerance [24] showed the following lower bound.

Lemma 2.8. *If $x \geq 4$ and $2 \leq y \leq x$, then*

$$\Psi(x, y) \geq x^{1 - \frac{\log_2 x}{\log y}}.$$

Proof. See [24, Theorem 2.1]. □

From now on we assume that x is sufficiently large so that $y = (\log x)^{2+\varepsilon} > y_0$, $\varepsilon > 0$ and the conditions of Lemma 2.7 are satisfied. Let $\log_2 X = \frac{2}{3} \log_2 x$. Lemma 2.6 shows that with probability $1 - o(1)$ the partial sums of $f(n)$ are nonzero up to X .

Let $x_0 = X$, $x_{i+1} = x_i + \frac{x_i}{h(x_i)}$, where $(\log x_i)^{2.01} \leq h(x_i) \ll (\log x_i)^{100}$ is a monotonically increasing function that will be defined later. There are $O((\log x_i)^{102})$ points x_i . We apply Lemma 2.5 with $\delta = 1/100$ at each x_i and obtain that

$$\mathbb{P}\left(\exists x_i \sum_{n \leq x_i} f(n) \leq 0.99 \Psi^*(x_i, y)\right) \ll (\log x)^{102} \exp\left(-\frac{c_0}{10^4} (2\alpha - 1) y^{2\alpha-1} \log y\right).$$

This is $o(1)$ if (1) is satisfied with K sufficiently large.

We denote

$$R := \mathbb{P}\left(\exists i \exists u \in [x_i, x_{i+1}] : \sum_{x_i < n \leq u} f(n) \geq \frac{1}{10} \Psi^*(x_i, y)\right).$$

It is enough to prove that $R = o(1)$ if the assumptions of Lemma 2.1 hold.

First let us rewrite $\sum_{x_i < n \leq u} f(n)$ as

$$\sum_{x_i < n \leq u} f(n) = \Psi^*(x_i + u, y) - \Psi^*(x_i, y) + \sum_{\substack{p(n) > y \\ n \neq 1}} {}^b f(n) \left(\Psi^* \left(\frac{x_i + u}{n}, y \right) - \Psi^* \left(\frac{x_i}{n}, y \right) \right).$$

Since $x_{i+1} - x_i = o(x_i)$, Lemma 2.7 gives us

$$\Psi^*(x_i + u, y) - \Psi^*(x_i, y) \leq \Psi^*(u, y) + \sqrt{x_i} = o(\Psi^*(x_i, y)).$$

In the last equality we used Lemma 2.2, Lemma 2.8 and the assumption that $y = (\log x)^{2+\varepsilon}$, where $\varepsilon(\log x)$ tends to infinity.

Hence it is enough to give a good upper bound on

$$R'_i := \mathbb{P} \left(\exists u \in [x_i, x_{i+1}] : \sum_{x_i < n \leq u}^{\sim} f(n) \geq \frac{1}{11} \Psi^*(x_i, y) \right),$$

where \sum^{\sim} means that the sum is over integers that are not of the form m^2s , where s is y -smooth.

Let

$$R'_{k,l} := \mathbb{P} \left(\sum_{x_i + 2^k l < n \leq x_i + 2^k(l+1)}^{\sim} f(n) \geq \frac{(\log(x_{i+1} - x_i))^{-1}}{50} \Psi^*(x_i, y) \right).$$

Let $u \in [x_i, x_{i+1}]$ be such that $u - x_i$ is a natural number. Let $u - x_i = 2^{\alpha_1} + 2^{\alpha_2} + \dots + 2^{\alpha_j}$ be the binary expansion, where $\alpha_1 > \alpha_2 > \dots > \alpha_j$. Of course $j \leq \log(u - x_i)/\log 2$. This gives a partition of interval $(x_i, u]$ into subintervals $(x_i, x_i + 2^{\alpha_1}]$, $(x_i + 2^{\alpha_1}, x_i + 2^{\alpha_1} + 2^{\alpha_2}]$, \dots , $(x_i + \sum_{\tau \leq j-1} 2^{\alpha_\tau}, x_i + \sum_{\tau \leq j} 2^{\alpha_\tau}]$. All of them are of the form $(x_i + 2^k l, x_i + 2^k(l+1)]$.

Hence

$$(5) \quad R'_i \leq \sum_{2^k l \leq x_{i+1} - x_i} R'_{k,l}.$$

We have

$$\sum_{x_i + 2^k l < n \leq x_i + 2^k(l+1)}^{\sim} f(n) = \sum_{\substack{p(n) > y \\ n \neq 1}} {}^b f(n) \left(\Psi^* \left(\frac{x_i + 2^k(l+1)}{n}, y \right) - \Psi^* \left(\frac{x_i + 2^k l}{n}, y \right) \right).$$

Lemma 2.7 give us

$$\Psi^* \left(\frac{x_i + 2^k(l+1)}{n}, y \right) - \Psi^* \left(\frac{x_i + 2^k l}{n}, y \right) \leq \Psi^* \left(\frac{2^k}{n}, y \right) + \sqrt{\frac{x_{i+1}}{n}}.$$

Thus we can fix two sequences $b_{k,l,i}(n)$ and $d_{k,l,i}(n)$ such that

$$\Psi^* \left(\frac{x_i + 2^k(l+1)}{n}, y \right) - \Psi^* \left(\frac{x_i + 2^k l}{n}, y \right) = b_{k,l,i}(n) + d_{k,l,i}(n),$$

$$b_{k,l,i}(n) \leq \Psi^* \left(\frac{2^k}{n}, y \right), \quad d_{k,l,i}(n) \leq \sqrt{\frac{x_{i+1}}{n}}.$$

Then $R'_{k,l} \leq B'_{k,l} + D'_{k,l}$, where

$$B'_{k,l} = \mathbb{P} \left(\sum_{\substack{p(n) > y \\ n \neq 1}} {}^b b_{k,l,i}(n) f(n) \geq \frac{(\log(x_{i+1} - x_i))^{-1}}{100} \Psi^*(x_i, y) \right),$$

$$D'_{k,l} = \mathbb{P} \left(\sum_{\substack{p(n) > y \\ n \neq 1}} {}^b d_{k,l,i}(n) f(n) \geq \frac{(\log(x_{i+1} - x_i))^{-1}}{100} \Psi^*(x_i, y) \right).$$

We apply Lemma 2.4, Lemma 2.2 and the moment inequality to obtain

$$B'_{k,l} \ll C_4^m (\log(x_{i+1} - x_i))^{2m} \left(\left(\frac{2^k}{x_i} \right)^{2\alpha(x_{i+1}, y)} \sum_{\substack{p(n) > y \\ n \neq 1}} {}^b n^{-2\alpha(x_{i+1}, y)} (2m-1)^{w(n)} \right)^m.$$

Note that $(2^k/x_i)^{2\alpha(x_{i+1}, y)} \leq 2^k/x_i = \frac{2^k}{(x_{i+1}-x_i)} h(x_i)^{-1}$. Following the proof of Lemma 2.5 we obtain

$$(6) \quad B'_{k,l} \ll \exp \left(-c_1 \frac{(x_{i+1} - x_i)}{2^k} \frac{h(x_i)}{(\log x_i)^2} (2\alpha - 1) y^{2\alpha-1} \log y \right).$$

Now we provide an upper bound on $D'_{k,l}$. From Lemma 2.8 we deduce that

$$\Psi(x_i, y) \geq x_i^{1-\frac{1}{2+\varepsilon}} = x_i^{\frac{1}{2} + \frac{\varepsilon}{2(2+\varepsilon)}}.$$

This and Lemma 2.4 with the moment inequality imply that

$$D'_{k,l} \ll C_5^m (\log x_i)^{2m} x_i^{-\frac{m\varepsilon}{2+\varepsilon}} \left(\sum_{\substack{p(n) > y \\ 1 \neq n \leq 2x_i}} \frac{{}^b (2m-1)^{w(n)}}{n} \right)^m$$

A standard application of Rankin's trick shows that

$$\sum_{n \leq 2x_i} \frac{(2m-1)^{w(n)}}{n} \ll (C_6 \log x_i)^{2m-1}.$$

Thus

$$(7) \quad D'_{k,l} \ll C_7^m (\log x_i)^{2m^2+m} x_i^{-\frac{m\varepsilon}{2+\varepsilon}}.$$

Inequalities (5), (6) and (7) imply that

$$(8) \quad R'_i \ll C_7^m h(x_i)^{-1} (\log x_i)^{2m^2+m} x_i^{1-\frac{m\varepsilon}{2+\varepsilon}} + \sum_{2^k \leq x_{i+1}-x_i} \frac{|x_{i+1}-x_i|}{2^k} \exp \left(-c_1 \frac{(x_{i+1}-x_i)}{2^k} \frac{h(x_i)}{(\log x_i)^2} (2\alpha-1) y^{2\alpha-1} \log y \right) \ll C_7^m h(x_i)^{-1} (\log x_i)^{2m^2+m} x_i^{1-\frac{m\varepsilon}{2+\varepsilon}} + \exp(-c_2(2\alpha-1)y^{2\alpha-1} \log y).$$

There are no more than $O(h(x)(\log x))$ check points x_i . Hence

$$R \ll h(x)(\log x) \exp(-c_2(2\alpha-1)y^{2\alpha-1} \log y) + \sum_{x_i > y} C_7^m h(x_i)^{-1} (\log x_i)^{2m^2+m} x_i^{1-\frac{m\varepsilon}{2+\varepsilon}}.$$

Let us take $m = 10[\varepsilon^{-1}]$. Note that $(\log x_i)^{2m^2+m} = o(x_i^{1/10})$ is guaranteed by $\varepsilon \geq 1000 \sqrt{\frac{\log_2 X}{\log X}}$. This is satisfied, because $\varepsilon > 1000 \frac{(\log_2 x)^{1/2}}{(\log x)^{1/3}} \geq 1000 \sqrt{\frac{\log_2 X}{\log X}}$.

Hence

$$R \ll h(x)(\log x) \exp(-c_2(2\alpha-1)y^{2\alpha-1} \log y) + o(y^{-1}).$$

Now we take $h(x) = (\log x)^3$. We see that $R = o(1)$ if K in (1) is large enough. This finishes the proof of Lemma 2.1. \square

3. PROOF OF THEOREM 2 AND THEOREM 3

3.1. **Some discussions.** To provide an upper bound on P'_x Angelo and Xu [2, Theorem 1.2] used the identity

$$(9) \quad \sum_{n \leq x} \frac{f(n)}{n} = \prod_{p \leq x} \left(1 - \frac{f(p)}{p}\right)^{-1} - \sum_{\substack{n > x \\ P(n) \leq x}} \frac{f(n)}{n},$$

where $P(n)$ is the greatest prime divisor of n . The first term on the right-hand side is obviously positive and can be proved to be relatively large with high probability. After that one provides an upper bound on the absolute value of the second term which holds with high probability.

Let $g = f * 1$. Then g is multiplicative and nonnegative. The following equation is the analog of (9) and is the basis for the proof of Theorem 2 and Proposition 1.

$$(10) \quad \sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{x} \sum_{n \leq x} g(n) + \frac{1}{x} \sum_{n \leq x} f(n) \left\{ \frac{x}{n} \right\}.$$

The proof of [22, Theorem 1.2] by Kerr and Klurman goes as follows. The authors show that the two probabilities

$$(11) \quad P_0 = \mathbb{P} \left(\sum_{n \leq x} g(n) \ll \frac{x}{\log x} \right), \quad P_1 = \mathbb{P} \left(\sum_{n \leq x} f(n) \left\{ \frac{x}{n} \right\} \gg \frac{x}{\log x} \right)$$

are small.

Note that $\sum_{n \leq x} g(n) \geq \sum_{p \leq x} g(p) = \sum_{p \leq x} (1 + f(p))$ and the good upper bound on P_0 comes from a Chernoff-type bound. To bound P_1 the authors use [22, Proposition 5.2]: the moment inequality for a high moment as large as $\exp\left(\frac{\log x \log_3 x}{C \log_2 x}\right)$.

Let us state [22, Proposition 5.2] in a slightly generalized form.

Proposition 1 (Kerr and Klurman). *Let $\beta_0 > 0$ and $\beta(x) = \beta_0 + o(1)$ as $x \rightarrow \infty$. Let q be an even positive integer. Then there exists a function $o_\beta(1)$ such that*

$$\mathbb{E} \left[\left(\sum_{n \leq x} f(n) \right)^q \right]^{1/q} = o \left(\frac{x}{(\log x)^{\beta(x)}} \right)$$

uniformly for

$$q \leq \exp \left(\frac{\log x \log_3 x}{(\beta_0 + o_\beta(1)) \log_2 x} \right).$$

One can hope to improve the result of [22, Theorem 1.2] by providing a better lower bound on $\sum_{n \leq x} g(n)$. If we use Proposition 1, then we need a good upper bound on

$$P_\varepsilon := \mathbb{P} \left(\sum_{n \leq x} g(n) < \frac{x}{(\log x)^{1-\varepsilon}} \right)$$

for some $\varepsilon > 0$ to improve the constant $C = 1$. Unfortunately this approach does not work, as one can show that

$$P_\varepsilon \gg \exp \left(- \exp \left((\log x)^{1-\varepsilon} \right) \right).$$

Let us note, however, that in view of (10) we only need

$$(12) \quad \sum_{n \leq x} g(n) \geq \left| \sum_{n \leq x} f(n) \left\{ \frac{x}{n} \right\} \right|,$$

to be sure that $\sum_{n \leq x} \frac{f(n)}{n} \geq 0$. The two sides of (12) are strongly dependent, which we will utilize. First we will provide a lower bound for $\sum_{n \leq x} g(n)$ in terms of the function f which holds with high probability.

3.2. Lower bound for $\sum_{n \leq x} g(n)$.

Proposition 2. *Let f be a random completely multiplicative function such that for each prime p we have $\mathbb{P}(f(p) = 1) = \mathbb{P}(f(p) = -1) = 1/2$. Let $g = f * 1$.*

Then there exist $c > 0$ and $\beta > 0$ such that

$$(13) \quad 1 - \mathbb{P} \left(\sum_{n \leq x} g(n) \geq cx \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right) \right) \ll \exp(-x^\beta).$$

One can see from the proof, that any fixed $\beta < e^{-2}$ is admissible in Proposition 2.

Note that the lower bound $\sum_{n \leq x} g(n) \gg x \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right)$ is essentially the best, since for all f the upper bound

$$\sum_{n \leq x} g(n) \ll \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots \right) \asymp x \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right)$$

holds (see, for example, [14]).

Our proof of Proposition 2 is based on [22, Proposition 3.3] by Kerr and Klurman which in turn is based on theorem by Matomäki and Shao [25, Hypothesis P].

Lemma 3.1. *Let $\varepsilon > 0$ be sufficiently small, let f be a multiplicative function with $-1 \leq f(n) \leq 1$ for all n . Let $g = 1 * f$. For $0 < \delta < 1$, let $\mathcal{P}_\delta = \{p \text{ prime} : f(p) \geq -\delta\}$, and suppose for some*

$$\frac{40000}{\varepsilon^2} \leq v \leq \frac{\log x}{1000 \log_2 x}$$

we have

$$\sum_{\substack{p \in \mathcal{P}_\delta \\ x^{1/v} \leq p \leq x}} \frac{1}{p} \geq 1 + \varepsilon.$$

Then

$$(14) \quad \sum_{n \leq x} g(n) \gg \varepsilon^2 \left(\frac{(1-\delta)}{v} \right)^{v(1+o(1))/e} \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right) x.$$

Proof. See [22, Proposition 3.3]. □

Lemma 3.2 (Hoeffding's inequality). *Let X_1, X_2, \dots, X_n be independent random variables such that $a_i \leq X_i \leq b_i$ almost surely. Let*

$$S_n = X_1 + \dots + X_n.$$

Then

$$\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geq t) \leq 2 \exp \left(- \frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2} \right).$$

Proof. See [19, Theorem 2]. □

Proof of Proposition 2. In Lemma 3.1 we fix ε, δ and $v = \max(e^{2+4\varepsilon}, 40000\varepsilon^{-2})$. Let $v_0 = e^{2+4\varepsilon}$. Then for sufficiently large x

$$\mathbb{P} \left(\sum_{\substack{p \in \mathcal{P}_\delta \\ x^{1/v} \leq p \leq x}} \frac{1}{p} \leq 1 + \varepsilon \right) \leq \mathbb{P} \left(\sum_{\substack{p \in \mathcal{P}_\delta \\ x^{1/v_0} \leq p \leq x}} \frac{1}{p} \leq 1 + \varepsilon \right) \leq \\ \mathbb{P} \left(\left| \sum_{x^{1/v_0} \leq p \leq x} \frac{f(p)}{p} \right| \geq \varepsilon \right) \ll \exp(-x^{1/v_0}).$$

Here we used Hoeffding's inequality (Lemma 3.2). Hence the conditions of Lemma 3.1 are satisfied with probability $1 - O(\exp(-x^{1/v_0}))$. The equation (14) implies

$$\sum_{n \leq x} g(n) \gg x \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right),$$

since ε, δ and v are fixed. □

Remark 1. *Let us sketch an alternative proof. A result of Tenenbaum [26, Theorem 1.2] implies the following. Let g be a nonnegative multiplicative function such that $g(p^k) \leq k$ for all primes p , and let $\varrho > 0, \varepsilon > 0$, and $x > x_0$. Suppose that, uniformly in y ,*

$$(15) \quad \sum_{p \leq y} \frac{(g(p) - \varrho) \log p}{p} \ll \varepsilon \log y \quad (x^\varepsilon < y \leq x).$$

Then

$$(16) \quad \sum_{n \leq x} g(n) \gg \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{g(p)}{p} \right).$$

It remains to show that (15) holds with admissible probability. This can be done using Lemma 3.2.

3.3. Applying Rankin's trick. In this section we follow the main steps of the proof of [22, Proposition 5.2].

Proposition 2 implies that if c_1 is sufficiently small, then we have

$$P'_x \leq \mathbb{P} \left(\left| \sum_{n \leq x} f(n) \left\{ \frac{x}{n} \right\} \right| > c_1 x \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right) \right) + O(\exp(-x^\beta)).$$

Let

$$S := \mathbb{E} \left[\left(x^{-1} \exp \left(- \sum_{p \leq x} \frac{f(p)}{p} \right) \sum_{n \leq x} f(n) \left\{ \frac{x}{n} \right\} \right)^q \right]^{1/q},$$

where $q > 0$ is an even integer.

Suppose that for $q = q(x)$ we have $S = o(1)$. The moment inequality implies

$$(17) \quad P'_x \leq (c_1^{-1} S)^q + O(\exp(-x^\beta)) \ll \exp(-q) + O(\exp(-x^\beta)).$$

That is why we want to prove that $S = o(1)$ for q as large as possible.

By $P(n), p(n)$ we denote the largest and the smallest prime divisors of n respectively. Let $\frac{1}{\log_2 x} \ll \varepsilon = \varepsilon(x) = o(1)$. Let us denote $S(x, y) := \{n \leq x : P(n) \leq y\}$,

$R(x, y) := \{n \leq x : p(n) > y\}$. Note that $1 \in R(x, y)$ for all $x, y > 0$ and if $y \geq x$, then $R(x, y) = \{1\}$.

We partition the summation over n as

$$\sum_{n \leq x} f(n) \left\{ \frac{x}{n} \right\} = \sum_{\substack{nl \leq x \\ n \in R(x, x^\varepsilon) \\ l \in S(x, x^\varepsilon)}} f(l)f(n) \left\{ \frac{x}{nl} \right\} = \sum_{j \leq \log x + 1} \sum_{\substack{nl \leq x \\ n \in R(x, x^\varepsilon) \\ l \in S(x, x^\varepsilon) \\ e^j \leq l < e^{j+1}}} f(l)f(n) \left\{ \frac{x}{nl} \right\}.$$

Let $h_1(x) = o(\varepsilon \log x)$ be a function to be chosen later.

Applying Minkowski's inequality we obtain $S \leq S_1 + S_2$, where

$$S_1 = x^{-1} \sum_{j \leq \log x - h_1(x)} \mathbb{E} \left[\left(\exp \left(- \sum_{p \leq x} \frac{f(p)}{p} \right) \sum_{\substack{nl \leq x \\ n \in R(x, x^\varepsilon) \\ l \in S(x, x^\varepsilon) \\ e^j \leq l < e^{j+1}}} f(l)f(n) \left\{ \frac{x}{nl} \right\} \right)^q \right]^{1/q},$$

$$S_2 = x^{-1} \sum_{\log x - h_1(x) < j \leq \log x + 1} \mathbb{E} \left[\left(\exp \left(- \sum_{p \leq x} \frac{f(p)}{p} \right) \sum_{\substack{nl \leq x \\ n \in R(x, x^\varepsilon) \\ l \in S(x, x^\varepsilon) \\ e^j \leq l < e^{j+1}}} f(l)f(n) \left\{ \frac{x}{nl} \right\} \right)^q \right]^{1/q}.$$

We first evaluate S_1 , following the method used to bound the analogous sum in the proof of [22, Proposition 5.2]. We use the majorant principle with Rankin's trick. By the majorant principle, we mean that when all coefficients $a_n \geq 0$, the moment

$$\mathbb{E} \left[\left(\sum_n a_n f(n) \right)^q \right] = \sum_{\substack{n_1, n_2, \dots, n_q \\ n_1 n_2 \dots n_q = \square}} a_{n_1} a_{n_2} \dots a_{n_q}$$

is monotone non-decreasing in each individual coefficient a_n . Since in the expansion $\exp \left(- \sum_{p \leq x} \frac{f(p)}{p} \right) = \sum_n a_n f(n)$ some of the coefficients a_n are negative, it prevents us from using the majorant principle immediately. That is why we use a trivial upper bound $\exp \left(- \sum_{p \leq x} \frac{f(p)}{p} \right) \ll \log x$. This provides

$$S_1 \ll \frac{\log x}{x} \sum_{j \leq \log x - h_1(x)} \mathbb{E} \left[\left(\sum_{\substack{nl \leq x \\ n \in R(x, x^\varepsilon) \\ l \in S(x, x^\varepsilon) \\ e^j \leq l < e^{j+1}}} f(l)f(n) \left\{ \frac{x}{nl} \right\} \right)^q \right]^{1/q}$$

Now we apply the majorant principle to obtain

$$(18) \quad S_1 \ll \frac{\log x}{x} \sum_{j \leq \log x - h_1(x)} \mathbb{E} \left[\left(\sum_{\substack{n \leq x/e^j \\ n \in R(x, x^\varepsilon)}} f(n) \sum_{\substack{l \leq e^{j+1} \\ l \in S(x, x^\varepsilon)}} f(l) \right)^q \right]^{1/q}$$

Note that if $n \leq x/e^j$ and $l \leq e^{j+1}$, then for $0 < \delta < 1$

$$\left(\frac{x}{nl}\right) \left(\frac{e^j}{x}\right)^\delta n^\delta \geq e^{-1} \left(\frac{x}{ne^j}\right) \left(\frac{ne^j}{x}\right)^\delta \geq e^{-1}.$$

This inequality, together with the majorant principle, gives us

$$S_1 \ll (\log x)x^{-\delta} \sum_{j \leq \log x - h_1(x)} e^{j\delta} \mathbb{E} \left[\left(\sum_{\substack{n \leq x/e^j \\ n \in R(x, x^\varepsilon)}} \frac{f(n)}{n^{1-\delta}} \sum_{\substack{l \leq e^{j+1} \\ l \in S(x, x^\varepsilon)}} \frac{f(l)}{l} \right)^q \right]^{1/q}.$$

Hence

$$(19) \quad S_1 \ll (\log x)x^{-\delta} \delta^{-1} e^{\delta(\log x - h_1(x))} \mathbb{E} \left[\left(\sum_{n \in R(x, x^\varepsilon)} \frac{f(n)}{n^{1-\delta}} \sum_{l \in S(x, x^\varepsilon)} \frac{f(l)}{l} \right)^q \right]^{1/q} \ll$$

$$(\log x)\delta^{-1} e^{-\delta h_1(x)} \mathbb{E} \left[\left(\prod_{p > x^\varepsilon} \left(1 - \frac{f(p)}{p^{1-\delta}}\right)^{-1} \right)^q \right]^{1/q} \mathbb{E} \left[\left(\prod_{p \leq x^\varepsilon} \left(1 - \frac{f(p)}{p}\right)^{-1} \right)^q \right]^{1/q}.$$

Let $|z| < 1/2$. For all such z we have

$$\log(1+z) \geq -z^2 + z.$$

Thus

$$(20) \quad (1+z)^{-q} \leq \exp(qz^2) \exp(-qz).$$

Suppose that $\delta \leq 1/3$. Applying inequality (20) twice for $z = p^{\delta-1}$ and $z = -p^{\delta-1}$, we obtain for $p \geq 3$

$$(21) \quad \mathbb{E} \left[\left(1 - \frac{f(p)}{p^{1-\delta}}\right)^{-q} \right] = \frac{1}{2} \left(\left(1 + \frac{1}{p^{1-\delta}}\right)^{-q} + \left(1 - \frac{1}{p^{1-\delta}}\right)^{-q} \right) \leq$$

$$\exp(qp^{-4/3}) \frac{\exp(q/p^{1-\delta}) + \exp(-q/p^{1-\delta})}{2} \leq \exp(qp^{-4/3}) \exp\left(\frac{q^2}{2p^{2-2\delta}}\right).$$

The last inequality follows from

$$\frac{e^z + e^{-z}}{2} \leq e^{\frac{z^2}{2}},$$

which holds for all $z \in \mathbb{R}$.

Thus

$$(22) \quad \mathbb{E} \left[\left(\prod_{p > x^\varepsilon} \left(1 - \frac{f(p)}{p^{1-\delta}}\right)^{-1} \right)^q \right] \ll \exp\left(\frac{c_2 q^2}{\varepsilon(\log x)x^{(1-2\delta)\varepsilon}} + O(q)\right),$$

Also

$$(23) \quad \mathbb{E} \left[\left(\prod_{p \leq x^\varepsilon} \left(1 - \frac{f(p)}{p}\right)^{-1} \right)^q \right] \ll \exp\left(q \sum_{p \leq x^\varepsilon} \frac{1}{p} + O(q)\right) \ll$$

$$\exp(q \log \log x + q \log \varepsilon + O(q)).$$

Combining together (19), (22) and (23) we deduce

$$(24) \quad S_1 \ll (\log x) \delta^{-1} e^{-\delta h_1(x)} \exp\left(\frac{c_2 q}{\varepsilon (\log x) x^{(1-2\delta)\varepsilon}} + \log \log x + \log \varepsilon\right) \ll \frac{\varepsilon}{\delta} (\log x)^2 e^{-\delta h_1(x)} \exp\left(\frac{c_2 q}{\varepsilon (\log x) x^{(1-2\delta)\varepsilon}}\right)$$

Now let us estimate S_2 . There exists j_0 satisfying $\log x - h_1(x) < j_0 \leq \log x + 1$ such that

$$(25) \quad S_2 \ll x^{-1} h_1(x) \mathbb{E} \left[\left(\exp\left(-\sum_{p \leq x} \frac{f(p)}{p}\right) \sum_{\substack{nl \leq x \\ n \in R(x, x^\varepsilon) \\ l \in S(x, x^\varepsilon) \\ e^{j_0} \leq l < e^{j_0+1}}} f(l) f(n) \left\{ \frac{x}{nl} \right\} \right)^q \right]^{1/q}.$$

In (25) we have $n \ll e^{h_1(x)} = o(x^\varepsilon)$. But $n \in R(x, x^\varepsilon)$, thus $n = 1$.

We conclude that

$$S_2 \ll x^{-1} h_1(x) \mathbb{E} \left[\left(\exp\left(-\sum_{p \leq x} \frac{f(p)}{p}\right) \sum_{\substack{l \in S(x, x^\varepsilon) \\ e^{j_0} \leq l < e^{j_0+1}}} f(l) \left\{ \frac{x}{l} \right\} \right)^q \right]^{1/q}$$

and hence

$$(26) \quad S_2 \ll x^{-1} h_1(x) \sup_f \left| \exp\left(-\sum_{p \leq x} \frac{f(p)}{p}\right) \sum_{\substack{l \in S(x, x^\varepsilon) \\ e^{j_0} \leq l < e^{j_0+1}}} f(l) \left\{ \frac{x}{l} \right\} \right|,$$

where the supremum is over the set of completely multiplicative functions that take values in $\{1, -1\}$.

Let us denote

$$\Psi_f(x, y) := \sum_{n \in S(x, y)} f(n).$$

We will provide an upper bound on $\Psi_f(x, y)$.

3.4. Upper bound on $\Psi_f(x, y)$.

Proposition 3. *Let $f(n)$ be a completely multiplicative function such that $|f(n)| \leq 1$ for all n . Let $a \geq 0$ and $u_x := \frac{\log x}{\log y}$.*

Let $y \geq 2$ and suppose that uniformly for $y^a \leq t \leq y^{a+1}$ we have

$$(27) \quad |\Psi_f(t, y)| \leq c_f(y) \rho(u_t) t,$$

where $\rho(u)$ is the Dickman function which is defined by $u\rho'(u) + \rho(u-1) = 0$ and $\rho(u) = 1$ for $0 \leq u \leq 1$. Also suppose that $c_f(y) \gg y^{-1/7}$.

Then for any $\varepsilon > 0$ we have uniformly in the range $x \geq y^a$, $\log y \geq (\log_2 x)^{5/3+\varepsilon}$

$$(28) \quad |\Psi_f(x, y)| \leq c_f(y) x \rho(u_x) \left(1 + O_\varepsilon \left(\frac{u_x \log(u_x + 1)}{\log x} \right) \right).$$

In the following Lemma we collect the properties of the Dickman function that we will need.

Lemma 3.3. (i) $\rho(u)u = \int_{u-1}^u \rho(t) dt$ ($u \geq 1$),

(ii) Uniformly for $y \geq 1.5$ and $1 \leq u \leq \sqrt{y}$ we have

$$\int_0^u \rho(u-t)y^{-t} dt \ll \frac{\rho(u)}{\log y},$$

(iii) Uniformly for $y \geq 1.5$ and $1 \leq u \leq \sqrt{y}$ we have

$$\int_1^u \rho(u-t)y^{-t} dt \ll \frac{\rho(u)}{(\log y)y^{1/3}},$$

(iv) Uniformly for $y \geq 1.5$ and $1 \leq u \leq y^{1/4}$ we have

$$\sum_{\substack{y < p^m \leq y^u \\ p \leq y}} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) \ll \rho(u) \frac{\log y}{y^{1/6}}.$$

(v) For every fixed $\varepsilon > 0$ and uniformly for $y \geq 1.5$, $u \geq 1$ and $0 \leq \theta \leq 1$ we have

$$\sum_{p^m \leq y^\theta} \frac{\log p}{p^m} \rho\left(u - \frac{\log p^m}{\log y}\right) = (\log y) \int_{u-\theta}^u \rho(t) dt + O_\varepsilon(\rho(u) \{1 + u \log^2(u+1) \exp(-(\log y)^{3/5-\varepsilon})\}).$$

Proof. For (i) see, for example, [16, Lemma 1 (ii)]. (ii) is [16, Lemma 2] and (iii) easily follows from the proof of [16, Lemma 2]. (iv) follows from the proof of [16, Lemma 3], and (v) is [16, Lemma 4]. \square

Proof of Proposition 3. The formula

$$(29) \quad \Psi(x, y) = x\rho(u) \left(1 + O_\varepsilon\left(\frac{u \log(u+1)}{\log x}\right)\right),$$

where $\Psi(x, y) := |S(x, y)|$ was proved by Hildebrand [16, Theorem 1] in the range $\log y \geq (\log_2 x)^{5/3+\varepsilon}$.

The proof uses the identity

$$(30) \quad \Psi(x, y) \log x = \int_1^x \frac{\Psi(t, y)}{t} dt + \sum_{\substack{p^m \leq x \\ p \leq y}} \Psi\left(\frac{x}{p^m}, y\right) \log p.$$

The estimate is derived by an inductive argument provided by (30).

Let $S = \sum_{n \in S(x, y)} f(n) \log n$. Integrating by parts we obtain

$$S = \Psi_f(x, y) \log x - \int_1^x \frac{\Psi_f(t, y)}{y} dt.$$

On the other hand

$$S = \sum_{n \in S(x, y)} f(n) \sum_{p^m | n} \log p = \sum_{\substack{p^m \leq x \\ p \leq y}} f(p^m) \Psi_f\left(\frac{x}{p^m}, y\right) \log p.$$

Here we used that by assumption f is completely multiplicative.

Hence the analog of (30) is

$$\Psi_f(x, y) \log x = \int_1^x \frac{\Psi_f(t, y)}{t} dt + \sum_{\substack{p^m \leq x \\ p \leq y}} f(p^m) \Psi_f\left(\frac{x}{p^m}, y\right) \log p,$$

which implies

$$(31) \quad |\Psi_f(x, y)| \log x \leq \int_1^x \frac{|\Psi_f(t, y)|}{t} dt + \sum_{\substack{p^m \leq x \\ p \leq y}} \left| \Psi_f \left(\frac{x}{p^m}, y \right) \right| \log p.$$

For $u \geq a$ let $\Delta(y, u)$ be the minimal nonnegative real number such that the inequality

$$|\Psi_f(y^u, y)| \leq c_f(y) y^u \rho(u) (1 + \Delta(y, u))$$

holds. Also denote $\Delta^*(y, u) := \sup_{\max(a, u-1) \leq u' \leq u} \Delta(y, u')$, which is well defined for $u \geq a$. Finally let us denote $\Delta^{**}(y, u) := \sup_{a \leq u' \leq u} \Delta(y, u')$. We will prove by induction that $\Delta^{**}(y, u) \ll_\varepsilon \log(u+1)/\log y$.

By the assumption (27) we have $\Delta(y, u) = 0$ ($a \leq u \leq a+1$).

The inequality (31) and the trivial upper bound $|\Psi_f(t, y)| \leq \Psi(t, y)$ imply that for $a+1 \leq u \leq \exp((\log y)^{3/5-\varepsilon})$ we have

$$(32) \quad \begin{aligned} \frac{|\Psi_f(y^u, y)|}{\rho(u) y^u} &\leq \frac{1}{\rho(u) y^u \log y^u} \int_{y^{u-1}}^{y^u} \frac{|\Psi_f(t, y)|}{t} dt + \frac{1}{\rho(u) y^u \log y^u} \int_1^{y^{u-1}} \frac{\Psi(t, y)}{t} dt + \\ &\quad \frac{1}{\rho(u) y^u \log y^u} \sum_{\sqrt{y} < p^m \leq y} \left| \Psi_f \left(\frac{y^u}{p^m}, y \right) \right| \log p + \\ &\quad \frac{1}{\rho(u) y^u \log y^u} \sum_{p^m \leq \sqrt{y}} \left| \Psi_f \left(\frac{y^u}{p^m}, y \right) \right| \log p + \frac{1}{\rho(u) y^u \log y^u} \sum_{\substack{y < p^m \leq y^u \\ p \leq y}} \Psi \left(\frac{y^u}{p^m}, y \right) \log p, \end{aligned}$$

Now we use the definition of $\Delta(y, u)$ and formula (29) to obtain

$$(33) \quad \begin{aligned} \frac{|\Psi_f(y^u, y)|}{\rho(u) y^u} &\leq \frac{c_f(y)(1 + \Delta^*(y, u))}{\rho(u) y^u \log y^u} \int_{y^{u-1}}^{y^u} \rho \left(\frac{\log t}{\log y} \right) dt + \frac{O(1)}{\rho(u) y^u \log y^u} \int_1^{y^{u-1}} \rho \left(\frac{\log t}{\log y} \right) dt + \\ &\quad \frac{c_f(y)(1 + \Delta^*(y, u - 1/2))}{\rho(u) \log y^u} \sum_{\sqrt{y} < p^m \leq y} \frac{\log p}{p^m} \rho \left(u - \frac{\log p^m}{\log y} \right) + \\ &\quad \frac{c_f(y)(1 + \Delta^*(y, u))}{\rho(u) \log y^u} \sum_{p^m \leq \sqrt{y}} \frac{\log p}{p^m} \rho \left(u - \frac{\log p^m}{\log y} \right) + \frac{O(1)}{\rho(u) \log y^u} \sum_{\substack{y < p^m \leq y^u \\ p \leq y}} \frac{\log p}{p^m} \rho \left(u - \frac{\log p^m}{\log y} \right). \end{aligned}$$

By Lemma 3.3 (ii) we have

$$(34) \quad \int_{y^{u-1}}^{y^u} \rho \left(\frac{\log t}{\log y} \right) dt = (\log y) y^u \int_0^1 \rho(u - \tau) y^{-\tau} d\tau \ll y^u \rho(u).$$

Lemma 3.3 (iii) implies

$$(35) \quad \int_1^{y^{u-1}} \rho \left(\frac{\log t}{\log y} \right) dt = (\log y) y^u \int_1^u \rho(u - \tau) y^{-\tau} d\tau \ll \frac{y^u \rho(u)}{y^{1/3}}.$$

By part (i) of Lemma 3.3,

$$(36) \quad 1 = \frac{1}{\rho(u) u} \int_{u-1/2}^u \rho(t) dt + \frac{1}{\rho(u) u} \int_{u-1}^{u-1/2} \rho(t) dt =: \alpha(u) + (1 - \alpha(u)).$$

Part (v) of Lemma 3.3, equation (36) and the assumption $u \leq \exp((\log y)^{3/5-\varepsilon})$ imply

$$(37) \quad \sum_{\sqrt{y} < p^m \leq y} \frac{\log p}{p^m} \rho \left(u - \frac{\log p^m}{\log y} \right) = (\log y) \int_{u-1}^{u-\frac{1}{2}} \rho(t) dt + \\ O_\varepsilon \left(\rho(u) \{1 + u \log^2(u+1) \exp(-(\log y)^{3/5-\varepsilon/2})\} \right) = \\ \rho(u) \log y^u \left((1 - \alpha(u)) + O_\varepsilon \left(\frac{1}{\log y^u} \right) \right).$$

In exactly the same way we get

$$(38) \quad \sum_{\sqrt{y} < p^m \leq y} \frac{\log p}{p^m} \rho \left(u - \frac{\log p^m}{\log y} \right) = \rho(u) \log y^u \left(\alpha(u) + O_\varepsilon \left(\frac{1}{\log y^u} \right) \right).$$

Applying (34), (35), (37), (38) and Lemma 3.3 (iv) to the corresponding terms on the right-hand side of (33) we derive an estimate

$$(39) \quad \frac{|\Psi_f(y^u, y)|}{\rho(u)y^u} \leq \frac{c_f(y)O(1)(1 + \Delta^*(y, u))}{u \log y} + O \left(\frac{1}{uy^{1/3}} \right) + \\ c_f(y)(1 + \Delta^*(y, u - 1/2)) \left((1 - \alpha(u)) + O_\varepsilon \left(\frac{1}{u \log y} \right) \right) + \\ c_f(y)(1 + \Delta^*(y, u)) \left(\alpha(u) + O_\varepsilon \left(\frac{1}{u \log y} \right) \right) + O \left(\frac{1}{uy^{1/6}} \right).$$

Since $\rho(u)$ is a nonincreasing function of u , we have $\alpha(u) \leq (1 - \alpha(u))$ and hence $\alpha(u) \leq 1/2$.

We obtain

$$(40) \quad \frac{|\Psi_f(y^u, y)|}{\rho(u)y^u} \leq c_f(y) \left(1 + \frac{1}{2} \Delta^{**}(y, u) + \frac{1}{2} \Delta^{**}(y, u - 1/2) + O_\varepsilon \left(\frac{(1 + \Delta^{**}(y, u))}{u \log y} \right) \right).$$

By changing the constant in O_ε , we can assume, that the right-hand side of (40) is an upper bound for $\Delta^*(y, u)$.

We find that

$$\Delta^{**}(y, u) \left(1 + O_\varepsilon \left(\frac{1}{u \log y} \right) \right) \leq \Delta^{**}(y, u - 1/2) + O_\varepsilon \left(\frac{1}{u \log y} \right).$$

By induction

$$\Delta^{**}(y, u) \ll_\varepsilon \frac{\log(u+1)}{\log y}.$$

This finishes the proof of Proposition 3. □

3.5. Corollary from Halász's theorem.

Lemma 3.4.

(i) Let $|a_p| \leq 1$ for all p and $\alpha = 1 + \frac{1}{\log x}$. Then

$$\sum_{p \leq x} \frac{a_p}{p} = \sum_p \frac{a_p}{p^\alpha} + O(1).$$

(ii) Let $\operatorname{Re}(s) > 1$ and let $F(s) := \sum_n \frac{f(n)}{n^s}$, where f is a multiplicative function taking values in the unit disk and such that $f(2) = 0$. Then

$$\log F(s) = \sum_p \frac{f(p)}{p^s} + O(1).$$

(iii) Let $s = \sigma + it$. In the region $\sigma \geq 1, |t| \geq 2$

$$\frac{1}{\zeta(s)} \ll (\log |t|)^7.$$

Proof. (i) follows from Chebyshev upper bound on the prime counting function. For (iii) see [27, Section 3.6]. See [27, Section 6.19] for a better upper bound. To prove (ii) note that for $\operatorname{Re}(s) > 1$

$$\begin{aligned} \log F(s) &= \sum_p \log \left(1 + \sum_{k \geq 1} \frac{f(p^k)}{p^{ks}} \right) = \sum_p \frac{f(p)}{p^s} + \\ &\quad \sum_p \sum_{k \geq 2} \frac{f(p^k)}{p^k} + \sum_p O \left(\left(\sum_{k \geq 1} \frac{f(p^k)}{p^{ks}} \right)^2 \right) = \sum_p \frac{f(p)}{p^s} + O(1). \end{aligned}$$

Here we used that $\left| \sum_{k \geq 1} \frac{f(p^k)}{p^{ks}} \right| \leq 1/2$ for all p and $\log(1+x) = x + O(x^2)$ in the range $|x| \leq 1/2$. \square

The following lemma is a form of Halász's theorem by Granville, Harper and Soundararajan [11].

Lemma 3.5 (Halász's theorem). *Let f be a multiplicative function such that $|f(n)| \leq 1$ for all n . Let*

$$F_x(s) := \prod_{p \leq x} \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right).$$

Let

$$L(x) := \left(\sum_{|N| \leq (\log x)^{2+1}} \frac{1}{N^2 + 1} \sup_{|t-N| \leq 1/2} |F_x(1+it)|^2 \right)^{1/2}.$$

Then
(i)

$$\sum_{n \leq x} f(n) \ll x \frac{L(x)}{\log x} \log \left(100 \frac{\log x}{L(x)} \right) + x \frac{\log_2 x}{\log x}.$$

(ii) *If the multiplicative function $f(n)$ is supported only on numbers with all their prime factors less than $O(x^{0.99})$, then*

$$\sum_{n \leq x} f(n) \ll \frac{x}{\log x} (L(x) + 1).$$

Proof. Part (i) is [11, Theorem 1]. The proof of (ii) is sketched in [11, Remark 3.2]. Let us discuss the details.

Following the proof of [11, Theorem 1] one deduce

$$\sum_{n \leq x} f(n) = \frac{1}{\log x} \sum_{k=1}^{10} S_k(x) + O\left(\frac{x}{\log x} \sum_{\log^4 x < p \leq x^{0.99}} \frac{\log p}{p \log(x/p)}\right) + \frac{1}{\log x} \sum_{p \leq \log^4 x} f(p) \log p \sum_{m \leq x/p} f(m) + O\left(\frac{x}{\log x}\right),$$

where

$$S_k(x) = \sum_{\substack{pqn \leq x \\ x^{1-e^{1-k}} < p \leq x^{1-e^{-k}}}} \frac{f(p) \log p}{\log(x/p)} f(n) f(q) \log q.$$

It is proved in [11] that $S_k(x) \ll xL(x) + x$. Also note that

$$\frac{x}{\log x} \sum_{\log^4 x < p \leq x^{0.99}} \frac{\log p}{p \log(x/p)} \ll \frac{x}{\log x}.$$

Thus it is enough to prove that

$$\sum_{p \leq \log^4 x} f(p) \log p \sum_{m \leq x/p} f(m) \ll xL(x) + x.$$

For $p \leq \log^4 x$ we have $F_x(s) \asymp F_{x/p}(s)$ if $\operatorname{Re}(s) \geq 1$. Hence $L(x/p) \ll L(x)$ and Lemma 3.5(i) implies that

$$\sum_{p \leq \log^4 x} f(p) \log p \sum_{m \leq x/p} f(m) \ll \sum_{p \leq \log^4 x} \log p \frac{x(L(x) + 1)}{p \log x} \log_2(x) \ll x(L(x) + 1) \frac{(\log_2 x)^2}{\log x}.$$

This finishes the proof of part (ii). \square

Proposition 4. *Let $x \geq 1$ and let f be a real-values multiplicative function supported only on numbers with all their prime factors less than $O(x^{0.99})$ and such that $|f(n)| \leq 1$ for all n . Then*

$$\sum_{n \leq x} f(n) \ll x \exp\left(\sum_{p \leq x} \frac{f(p)}{p}\right).$$

Proof. By $\nu_2(n)$ we denote the largest integer v such that $2^v | n$. Since

$$\sum_{n \leq x} f(n) = \sum_{\substack{n \leq x \\ \nu_2(n) \neq 1}} f(n) + f(2) \sum_{\substack{n \leq x/2 \\ (n,2)=1}} f(n),$$

it suffices to prove Proposition 4 for f with $f(2) = 0$.

Lemma 3.4 (ii) implies that $|F_x(s)| \asymp \exp\left(\sum_{p \leq x} \operatorname{Re} \frac{f(p)}{p^s}\right)$ in the region $\operatorname{Re}(s) \geq 1$. Then Lemma 3.5 implies

$$(41) \quad \exp\left(-\sum_{p \leq x} \frac{f(p)}{p}\right) \sum_{n \leq x} f(n) \ll \frac{x}{\log x} H(x) + x,$$

where

$$H(x) = \left(\sum_{|N| \leq (\log x)^{2+1}} \frac{1}{N^2 + 1} \sup_{|t-N| \leq 1/2} \exp\left(2 \sum_{p \leq x} \operatorname{Re} \frac{f(p)}{p} (p^{-it} - 1)\right) \right)^{1/2}.$$

It is clear that $\operatorname{Re} \frac{f(p)}{p} (p^{-it} - 1)$ is maximized when $f(p) = -1$. Hence

$$(42) \quad H(x) \ll \left(\sum_{|N| \leq (\log x)^2 + 1} \frac{1}{N^2 + 1} \sup_{|t-N| \leq 1/2} \exp \left(2 \sum_{p \leq x} \operatorname{Re} \frac{(1 - p^{-it})}{p} \right) \right)^{1/2} \ll (\log x) \left(\sum_{|N| \leq (\log x)^2 + 1} \frac{1}{N^2 + 1} \sup_{|t-N| \leq 1/2} \zeta^{-2} \left(1 + \frac{1}{\log x} + it \right) \right)^{1/2},$$

where we used Lemma 3.4 (i), (ii).

Lemma 3.4(iii) implies that

$$\sum_{|N| \leq (\log x)^2 + 1} \frac{1}{N^2 + 1} \sup_{|t-N| \leq 1/2} \zeta^{-2} \left(1 + \frac{1}{\log x} + it \right) = O(1).$$

Thus $H(x) \ll \log x$. This finishes the proof in view of (41). \square

3.6. The end of the proof of Theorem 2. Let $y = x^\varepsilon$ and $\frac{\log_4 x}{\log_3 x} \ll \varepsilon < 1/2$.

Proposition 4 implies that for $y^2 \leq t \leq y^3$ we have

$$(43) \quad \sum_{n \in S(t, y)} f(n) \ll \varepsilon^{-1} t \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right).$$

We see that the conditions of Proposition 3 are satisfied with $a = 2$ and

$$c_f(y) = O \left(\varepsilon^{-1} \exp \left(\sum_{p \leq x} \frac{f(p)}{p} \right) \right).$$

Hence

$$|\Psi_f(t, y)| \ll c_f(y) t \rho \left(\frac{\log t}{\log y} \right),$$

for $y^2 \leq t \leq x$, if $\log y \geq (\log_2 x)^{5/3+\varepsilon}$.

Let $\log x - h_1(x) < j \leq \log x + 1$. Integrating by parts we obtain

$$(44) \quad \sum_{\substack{l \in S(x, x^\varepsilon) \\ e^j \leq l < e^{j+1}}} f(l) \left\{ \frac{x}{l} \right\} = \int_{e^j}^{e^{j+1}} \left\{ \frac{x}{t} \right\} d\Psi_f(t, y) + O(1) \ll \frac{x}{e^j} \sup_{t \in [e^j, e^{j+1}]} |\Psi_f(t, y)| + O(1) \ll c_f(y) x \rho \left(\varepsilon^{-1} - \varepsilon^{-1} \frac{h_1(x)}{\log x} \right) + O(1).$$

We use (26), (44) and the upper bound $\rho(u) \ll \exp(-(1+o(1))u \log u)$ (see, for example, [18, Corollary 2.3]) to obtain

$$(45) \quad S_2 \ll h_1(x) \varepsilon^{-1} \exp \left(-(1+o(1)) \varepsilon^{-1} \log \varepsilon^{-1} \right).$$

Now recall (24). Let us choose $q = \varepsilon(\log x)x^{(1-2\delta)\varepsilon} + O(1)$, $\delta = (\log_3 x)^{-1}$, $h_1(x) = 10(\log_2 x)(\log_3 x)$. Finally let

$$\varepsilon = \frac{\log_4 x}{(1+o(1)) \log_3 x},$$

where $o(1)$ is chosen in such a way that $S_2 = o(1)$.

Hence for

$$q = \exp\left(\frac{\log x \log_4 x}{(1 + o(1)) \log_3 x}\right),$$

we have $S_1 = o(1)$, $S_2 = o(1)$ and thus $S = o(1)$.

This finishes the proof of Theorem 2 in view of (17). \square

3.7. Proof of Theorem 3. We do the same steps as in the proof of Theorem 2, but we use Conjecture 3 instead of Proposition 4.

In the notation of the proof of Theorem 2 this gives us

$$S_2 \ll \frac{h_1(x) \varepsilon^{-1} \exp(-(1 + o(1)) \varepsilon^{-1} \log \varepsilon^{-1})}{\log_2 x},$$

where ε is small enough.

Inequality (17) states that $P'_x \leq (c_1^{-1} S)^q + O(\exp(-x^\beta))$ for some fixed $c_1 > 0$, $\beta > 0$.

Let us take $q = \varepsilon(\log x)x^{(1-2\delta)\varepsilon} + O(1)$, $\delta = 1/10$, $h_1(x) = 100 \log_2 x$, and $\varepsilon > 0$ to be a fixed constant such that $S_2 < c_1/3$.

Our choice of variables implies that $S_1 = o(1)$, in view of (24). Therefore

$$P'_x \ll \exp(-q) + \exp(-x^\beta) \ll \exp(-x^\alpha),$$

where $\alpha = \min(\beta, (8/10)\varepsilon)$. \square

3.8. Proof of Proposition 1. Let

$$S' := \mathbb{E} \left[\left(\sum_{n \leq x} f(n) \right)^q \right]^{1/q}$$

Let $h_1(x) = o(\log x)$, $\varepsilon = o(\log x)$, $\delta > 0$. Following the steps of the proof of Theorem 2 (section 3.3), we obtain $S' \leq S'_1 + S'_2$, where

$$S'_1 \ll x \frac{\varepsilon}{\delta} (\log x) e^{-\delta h_1(x)} \exp\left(\frac{c_2 q}{\varepsilon (\log x) x^{(1-2\delta)\varepsilon}}\right), \quad S'_2 \ll h_1(x) \sup_f \left| \sum_{\substack{l \in S(x, x^\varepsilon) \\ e^{j_0} \leq l < e^{j_0+1}}} f(l) \left\{ \frac{x}{l} \right\} \right|.$$

Here $\log x - h_1(x) < j_0 \leq \log x + 1$. Hence $S'_2 \ll h_1(x) \Psi(x, x^\varepsilon)$.

For $q \leq \varepsilon(\log x)x^{(1-2\delta)\varepsilon}$ this gives us

$$(46) \quad S' \ll x \left(h_1(x) \exp(-(1 + o(1)) \varepsilon^{-1} \log \varepsilon^{-1}) + \frac{\varepsilon}{\delta} (\log x) e^{-\delta h_1(x)} \right).$$

Let us choose $\delta = (\log_3 x)^{-1}$, $h_1(x) = (10 + \beta_0)(\log_2 x)(\log_3 x)$. Finally let

$$\varepsilon = \frac{\log_3 x}{(\beta(x) + o(1)) \log_2 x},$$

where $o(1)$ is chosen in such a way that (46) gives $S' = o\left(\frac{x}{(\log x)^{\beta(x)}}\right)$.

This finishes the proof of Proposition 1. \square

4. PROOF OF THEOREM 4

Lemma 4.1 (Elliott). *Let b_1, b_2, \dots be a sequence of complex numbers such that $\forall i |b_i| \leq 1$. Then*

$$\mathbb{E} \left[\left(\sum_{n \leq y} b_n \chi_p(n) \right)^2 \right] \ll (\log x) y \log y + \frac{\log x}{x} y^3 \log y$$

Proof. This follows from [9, Lemma 10] if we note that

$$\sum_{\substack{m, n \leq y \\ mn = \square}} |b_n b_m| \ll y \log y.$$

□

Lemma 4.2. *Let b_1, b_2, \dots be a sequence of complex numbers such that $\forall i |b_i| \leq 1$. Let $h(y)$ be a function such that $0 < h(y) \ll (\log y)^{o(1)}$ and let q be an even positive integer. Then*

$$\mathbb{E} \left[\left(\sum_{n \leq y} b_n \chi_p(n) \right)^q \right] \ll (\log x) o \left(\frac{y}{h(y) \log y} \right)^q + \frac{\log x}{x} (\log y^q) (4y^{3/2})^q$$

for all

$$(47) \quad q \leq \exp \left(\frac{\log y \log_3 y}{(1 + o_h(1)) \log_2 y} \right).$$

Proof. We follow the proof of [9, Lemma 10]. We have

$$\mathbb{E} \left[\left(\sum_{n \leq y} b_n \chi_p(n) \right)^q \right] \ll \frac{\log x}{x} \sum_{x < p \leq 2x} \left| \sum_{n \leq y} b_n \left(\frac{n}{p} \right) \right|^q.$$

We extend the definition of Legendre symbol by

$$\left(\frac{m}{2} \right) = \begin{cases} 1 & \text{if } 2 \nmid m, \\ 0 & \text{if } 2 \mid m, \end{cases}$$

and

$$\left(\frac{m}{n} \right) = \prod_{p^\alpha \mid n} \left(\frac{m}{p} \right)^\alpha.$$

Note that this definition differs from the usual definition of Kronecker symbol.

Let $n = 2^\eta n_1$, $2 \nmid n_1$. We divide all integers n into four classes according to the parity of η , and whether $n_1 \equiv 1$ or $n_1 \equiv 3 \pmod{4}$. Let us denote by Σ_j ($j = 1, \dots, 4$) the summation over particular class.

The quadratic reciprocity law implies that for any odd integer m we have

$$\left(\frac{n}{m} \right) = \varepsilon \left(\frac{m}{n} \right),$$

where $\varepsilon = \pm 1$ and depends only on m and the class of n .

Hence

$$\left| \sum_{n \leq y} b_n \left(\frac{n}{m} \right) \right| = \left| \sum_{n \leq y} b_n \left(\frac{m}{n} \right) \right|.$$

Jensen's inequality applied to the function $\varphi(z) = z^q$ implies that

$$\sum_{x < p \leq 2x} \left| \sum_{n \leq y} b_n \left(\frac{n}{p} \right) \right|^q \leq 4^{q-1} \sum_{j=1}^4 \sum_{x < p \leq 2x} \left| \sum_{n \leq y} b_n \left(\frac{n}{p} \right) \right|^q.$$

It was shown in the proof of [9, Lemma 10] that

$$m \mapsto \left(\frac{m}{n} \right)$$

defines a non-principal character unless n or $\frac{1}{2}n$ is a perfect square.

Hence

$$(48) \quad \sum_{x < p \leq 2x} \left| \sum_{n \leq y} b_n \left(\frac{n}{p} \right) \right|^q = \sum_{x < p \leq 2x} \left| \sum_{n \leq y} b_n \left(\frac{p}{n} \right) \right|^q \leq \sum_{x < m \leq 2x} \left| \sum_{n \leq y} b_n \left(\frac{m}{n} \right) \right|^q \ll$$

$$x \sum_j b_{n_1} \dots b_{n_{q/2}} \overline{b_{n_{q/2+1}}} \dots b_{n_q} + \sum_j b_{n_1} \dots b_{n_{q/2}} \overline{b_{n_{q/2+1}}} \dots b_{n_q} \sum_{m \leq x} \left(\frac{m}{n_1 \dots n_q} \right).$$

$n_1, \dots, n_q \leq y$
 $n_1 \dots n_q = \square, 2\square$ $n_1, \dots, n_q \leq y$
 $n_1 \dots n_q \neq \square, 2\square$

Proposition 1 implies that

$$\sum_{\substack{n_1, \dots, n_q \leq y \\ n_1 \dots n_q = \square, 2\square}} 1 \leq (q+1) \sum_{\substack{n_1, \dots, n_q \leq y \\ n_1 \dots n_q = \square}} 1 = (q+1) \mathbb{E} \left[\left(\sum_{n \leq y} f(n) \right)^q \right] = o \left(\frac{y}{h(y) \log y} \right)^q,$$

for q in range (47).

If $n_1 \dots n_q \neq \square, 2\square$, then Pólya-Vinogradov inequality gives

$$\sum_{m \leq x} \left(\frac{m}{n_1 \dots n_q} \right) \ll y^{q/2} \log(y^q).$$

Combining all these estimates we obtain the desired inequality after redefining the function $h(y)$. \square

Let M be a subset of $[1, \infty)$. Denote by $\tilde{P}_x(M)$ the probability that for any $y \in M$ we have $\sum_{n \leq y} \frac{\chi_p(n)}{n} > 0$ and by $P(M)$ the probability that for any $y \in M$ we have $\sum_{n \leq y} \frac{f(n)}{n} > 0$, where f is a random completely multiplicative function.

Lemma 4.3. *For*

$$\exp \left(\frac{\log_3 x \log_4 x}{o(1) \log_5 x} \right) \leq N \leq \exp \left(\frac{\log_2 x \log_3 x}{3 \log_4 x} \right)$$

we have

$$1 - \tilde{P}_x([N, \infty)) \ll \exp \left(- \exp \left(\frac{\log N \log_3 N}{(1 + o(1)) \log_2 N} \right) \right).$$

Proof. For each $p \in (x, 2x]$ Pólya-Vinogradov inequality implies that

$$\sum_{n > y} \frac{\chi_p(n)}{n} \ll \frac{\sqrt{x} \log x}{y}.$$

By Siegel's theorem [7, Chapter 21] for any $\varepsilon > 0$ we have $L(1, \chi_p) > C(\varepsilon)x^{-\varepsilon}$.

Hence for any $\varepsilon > 0$, sufficiently large x , and $y > x^{1/2+\varepsilon}$

$$\sum_{n \leq y} \frac{\chi_p(n)}{n} = L(1, \chi_p) - \sum_{n > y} \frac{\chi_p(n)}{n} > 0.$$

Let us denote $g_p = \chi_p * 1$.

We have

$$\sum_{n \leq y} \frac{\chi_p(n)}{n} = \frac{1}{y} \sum_{n \leq y} g_p(n) + \frac{1}{y} \sum_{n \leq y} \chi_p(n) \left\{ \frac{y}{n} \right\}.$$

Let us denote

$$Pr_1(y) = \mathbb{P} \left(\frac{1}{y} \left| \sum_{n \leq y} \chi_p(n) \left\{ \frac{y}{n} \right\} \right| > \frac{0.1}{\log y} \right), \quad Pr_2(y) = \mathbb{P} \left(\frac{1}{y} \sum_{n \leq y} g_p(n) < \frac{0.2}{\log y} \right).$$

Note that if both events do not take place, then $\sum_{n \leq y} \frac{\chi_p(n)}{n} > \frac{0.1}{\log y}$. Thus the inequality $\sum_{n \leq y'} \frac{\chi_p(n)}{n} > 0$ holds for all $y' \in \left[y, y + \frac{y}{10^2 \log y} \right]$.

Let $y_0 = N$ and $y_{i+1} = y_i + \frac{y_i}{10^2 \log y_i}$. Suppose that k is the least number such that $y_k > x^{1/2+\varepsilon}$. Note that $k \ll (\log x)^2$. It is enough to prove that

$$\sum_{i=0}^k (Pr_1(y_i) + Pr_2(y_i)) \ll \exp \left(- \exp \left(\frac{\log N \log_3 N}{(1 + o(1)) \log_2 N} \right) \right).$$

By the assumption $N \leq \exp \left(\frac{\log_2 x \log_3 x}{3 \log_4 x} \right)$ and thus

$$\exp \left(- \exp \left(\frac{\log N \log_3 N}{(1 + o(1)) \log_2 N} \right) \right) \gg B(x) := \exp \left(-(\log x)^{\frac{1}{3}+o(1)} \right).$$

Since

$$\sum_{n \leq y} g_p(n) \geq \sum_{q \leq y} g_p(q) = \sum_{q \leq y} (1 + \chi_p(q)),$$

where q ranges over prime numbers, we get

$$Pr_2(y) \leq \mathbb{P} \left(\frac{1}{y} \left| \sum_{q \leq y} \chi_p(q) \right| > \frac{0.7}{\log y} \right).$$

For $y_i > \exp(\sqrt{\log x})$ we apply Lemma 4.1 and the moment inequality to obtain

$$(Pr_1(y_i) + Pr_2(y_i)) \ll (\log x)(\log y_i)^3 \left(\frac{1}{y_i} + \frac{y_i}{x} \right) \ll \exp \left(-(\log x)^{\frac{1}{2}+o(1)} \right).$$

As $k \ll (\log x)^2$, we have

$$\sum_{\substack{y_i > \exp(\sqrt{\log x}) \\ i \leq k}} (Pr_1(y_i) + Pr_2(y_i)) \ll \exp \left(-(\log x)^{\frac{1}{2}+o(1)} \right) \ll B(x).$$

Now for $y_i \leq \exp(\sqrt{\log x})$ we apply Lemma 4.2 with $h(y) = 10$. We take q as large as possible with the restrictions

$$q \leq D(y_i) := \exp \left(\frac{\log y_i \log_3 y_i}{(1 + o_h(1)) \log_2 y_i} \right), \quad q \leq E(y_i) := \frac{\log x}{10 \log(4y_i^{1/2} \log y_i)}.$$

The last restriction in view of Lemma 4.2 and the moment inequality implies that

$$(Pr_1(y_i) + Pr_2(y_i)) \ll (\log x) \exp(-q) + O(x^{-1/2}).$$

Therefore

$$(49) \quad \sum_{y_i \leq \exp(\sqrt{\log x})} (Pr_1(y_i) + Pr_2(y_i)) \ll \sum_{y_i \leq \exp(\sqrt{\log x})} ((\log x) (\exp(-D(y_i)) + \exp(-E(y_i))) + O(x^{-1/2})) \ll (\log x) \sum_{y_i \leq \exp(\sqrt{\log x})} (\exp(-D(y_i)) + \exp(-E(y_i))) + O(x^{-1/2}(\log x)^2).$$

We have

$$(\log x) \sum_{y_i \leq \exp(\sqrt{\log x})} \exp(-E(y_i)) \ll (\log x)^3 \exp(-\sqrt{\log x}) \ll B(x).$$

Also

$$(50) \quad (\log x) \sum_{y_i \leq \exp(\sqrt{\log x})} \exp(-D(y_i)) \ll (\log x)^3 \exp\left(-\exp\left(\frac{\log N \log_3 N}{(1+o(1)) \log_2 N}\right)\right) \ll \exp\left(-\exp\left(\frac{\log N \log_3 N}{(1+o(1)) \log_2 N}\right)\right).$$

Here we used the assumed lower bound on N .

The result follows. \square

Let

$$\pi(x; k, l) := \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} 1.$$

Lemma 4.4. *Let $k \leq \exp(C\sqrt{\log x})$.*

$$\pi(x; k, l) = \frac{\text{Li}(x)}{\varphi(k)} - E_1 \frac{\chi_1(l)}{\varphi(k)} \int_2^x \frac{u^{\beta_1-1}}{\log u} du + O\left(x \exp^{-c'\sqrt{\log x}}\right),$$

where $E_1 = 1$ if there exists a quadratic Dirichlet character $\chi_1 \pmod{k}$ with real zero β_1 such that $\beta_1 > 1 - \frac{c}{\log k}$ and $E_1 = 0$ otherwise.

Moreover if such character exists, then it is unique and $\chi_1(l) = \left(\frac{d}{l}\right)$, where the symbol on the right is Kronecker symbol, and d is the product of relatively prime factors of the form

$$-4, 8, -8, (-1)^{(p-1)/2} p \quad (p > 2).$$

Also $|d|$ is the conductor of χ_1 and $d \log^4 d \gg \log x$.

Proof. The first part of the Lemma follows from [7, Chapter 20, equation 9] after integration by parts. For the second part see [7, Chapter 5, equation 9] and [7, Chapter 20, equation 12]. \square

Proof of Theorem 4. Let us take $N = c_1 \sqrt{\log x}$ and $k = 8 \prod_{2 < q \leq N} q$, where c_1 is sufficiently small. Clearly $k \ll \exp(c_2 \sqrt{\log x})$.

Now we want to compare $\tilde{P}_x([1, N])$ and $P([1, N])$. Let $f(n)$ be a sample of Rademacher random completely multiplicative function. Also let us set $\mathbb{P}(f(-1) = 1) = \mathbb{P}(f(-1) = -1) = 1/2$.

Let us denote

$$S(f) = \left\{ l \pmod{k} : (l, k) = 1, p \equiv l \pmod{k} \Rightarrow \forall q \leq N \left(\frac{q}{p}\right) = f(q), \left(\frac{-1}{p}\right) = f(-1) \right\}.$$

Quadratic reciprocity law implies that

$$|S(f)| = \frac{\varphi(k)}{2^{\pi(N)+1}}.$$

Also we note that for each $l \in S(f)$ we have

$$\chi_1(l) = \prod_{q^{\alpha_q} \parallel d} \left(\frac{q}{l}\right)^{\alpha_q} = f(d),$$

where $q = -1, \alpha_{-1} = 1$ is included in the product if $d < 0$.

Putting all this together we obtain for a fixed f

(51)

$$\mathbb{P}(\forall n \in [1, N] \chi_p(n) = f(n)) = \frac{1}{2^{\pi(N)}} - E_0 \frac{f(d)}{2^{\pi(N)}} \frac{\int_x^{2x} \frac{u^{\beta_1-1}}{\log u} du}{\text{Li}(2x) - \text{Li}(x)} + O\left(\exp^{-c'\sqrt{\log x}}\right).$$

where $E_0 = 0$ if $E_1 = 0$ or $d < 0$, and $E_0 = 1$ otherwise.

Let A_N the set of completely multiplicative functions f defined on $[1, N]$, that take values ± 1 and such that $\sum_{n \leq y} \frac{f(n)}{n}$ is positive for $1 \leq y \leq N$.

From (51) we deduce that

$$(52) \quad \tilde{P}_x([1, N]) = P([1, N]) - E_0 \frac{\sum_{f \in A_N} f(d)}{2^{\pi(N)}} \frac{\int_x^{2x} \frac{u^{\beta_1-1}}{\log u} du}{\text{Li}(2x) - \text{Li}(x)} + O\left(\exp^{-c'\sqrt{\log x}}\right).$$

We have

$$(53) \quad \frac{\sum_{f \in A_N} f(d)}{2^{\pi(N)}} = \text{Cov}(\mathbf{1}_{A_N}, f(d)).$$

The right-hand side should be interpreted as the covariance in \mathcal{F} .

Also

$$(54) \quad |\text{Cov}(\mathbf{1}_{A_N}, f(d)) - \text{Cov}(\mathbf{1}_A, f(d))| \leq 1 - P([N, \infty)).$$

Finally by Lemma 4.3 and Theorem 2 (or [22, Theorem 1.2]) we obtain

$$(55) \quad (1 - P((N, \infty))) + (1 - \tilde{P}_x((N, \infty))) \ll \exp\left(-\exp\left(\frac{\log_2 x \log_4 x}{(2 + o(1)) \log_3 x}\right)\right).$$

Since

$$\tilde{P}_x = \tilde{P}_x((1, N]) + O(1 - \tilde{P}_x((N, \infty))), \quad P = P((1, N]) + O(1 - P((N, \infty))),$$

the theorem follows from (52), (53), (54), (55). \square

4.1. Proof of Corollary 2. If $E_0 = 0$, then the result is obvious. Assume that $E_0 = 1$ and d is the conductor of character with Siegel zero.

Denote by p_0 the greatest prime divisor of d . Lemma 4.4 implies that $p_0 \geq (1 + o(1)) \log d \geq (1 + o(1)) \log_2(x)$. Take $N' = p_0 - 1$. Note that

$$\text{Cov}(\mathbf{1}_{A_{N'}}, f(d)) = 0,$$

and

$$\text{Cov}(\mathbf{1}_A, f(d)) - \text{Cov}(\mathbf{1}_{A_{N'}}, f(d)) \ll 1 - P((N', \infty)) \ll \exp\left(-\exp\left(\frac{\log_3 x \log_6 x}{(1 + o(1)) \log_5 x}\right)\right),$$

where we used Theorem 2. The result follows. \square

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REFERENCES

- [1] Krishnaswami Alladi. Asymptotic estimates of sums involving the moebius function. II. *Transactions of the American Mathematical Society*, 272(1):87–105, 1982.
- [2] Rodrigo Angelo and Max Wenqiang Xu. On a Turán conjecture and random multiplicative functions. *The Quarterly Journal of Mathematics*, 74(2):767–777, 2023.
- [3] Marco Aymone. Sign changes of the partial sums of a random multiplicative function II. *Comptes Rendus. Mathématique*, 362(G8):895–901, 2024.
- [4] Marco Aymone. On the positivity of some weighted partial sums of a random multiplicative function. *Mathematische Zeitschrift*, 311(3):1–13, 2025.
- [5] Aline Bonami. Étude des coefficients de fourier des fonctions de $L_p(G)$. In *Annales de l'institut Fourier*, volume 20, pages 335–402, 1970.
- [6] Peter Borwein, Ron Ferguson, and Michael Mossinghoff. Sign changes in sums of the Liouville function. *Mathematics of computation*, 77(263):1681–1694, 2008.
- [7] Harold Davenport. *Multiplicative number theory*, volume 74. Springer Science & Business Media, 2013.
- [8] Régis de La Bretèche and Gérald Tenenbaum. Propriétés statistiques des entiers friables. *The Ramanujan Journal*, 9(1):139–202, 2005.
- [9] P.D.T.A. Elliott. On the mean value of $f(p)$. *Proceedings of the London Mathematical Society*, 3(1):28–96, 1970.
- [10] Sidney West Graham and C.J. Ringrose. Lower bounds for least quadratic non-residues. In *Analytic number theory: Proceedings of a conference in honor of Paul T. Bateman*, pages 269–309. Springer, 1990.
- [11] Andrew Granville, Adam Harper, and Kannan Soundararajan. A more intuitive proof of a sharp version of Halász’s theorem. *Proceedings of the American Mathematical Society*, 146(10):4099–4104, 2018.
- [12] Andrew Granville and Kannan Soundararajan. Negative values of truncations to $L(1, \chi)$. *Clay Mathematics Proceedings*, 7, 2007.
- [13] Gábor Halász. On random multiplicative functions. *Publ. Math. Orsay*, 83(4):74–96, 1983.
- [14] H. Halberstam and H.-E. Richert. On a result of R. R. Hall. *Journal of Number Theory*, 11(1):76–89, 1979.
- [15] Adolf Hildebrand. Integers free of large prime divisors in short intervals. *The Quarterly Journal of Mathematics*, 36(1):57–69, 1985.
- [16] Adolf Hildebrand. On the number of positive integers $\leq x$ and free of prime factors $> y$. *Journal of Number Theory*, 22(3):289–307, 1986.
- [17] Adolf Hildebrand and Gérald Tenenbaum. On integers free of large prime factors. *Transactions of the American mathematical society*, 296(1):265–290, 1986.
- [18] Adolf Hildebrand and Gérald Tenenbaum. Integers without large prime factors. *Journal de théorie des nombres de Bordeaux*, 5(2):411–484, 1993.
- [19] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American statistical association*, 58(301):13–30, 1963.
- [20] Ayesha Hussain and Youness Lamzouri. The limiting distribution of Legendre paths. *arXiv preprint arXiv:2304.13025*, 2023.
- [21] A. B. Kalmynin. Quadratic characters with positive partial sums. *Mathematika*, 69(1):90–99, 2023.
- [22] Bryce Kerr and Oleksiy Klurman. How negative can $\sum_{n \leq x} \frac{f(n)}{n}$ be? *arXiv preprint arXiv:2211.05540*, 2022.
- [23] Oleksiy Klurman, Youness Lamzouri, and Marc Munsch. Sign changes of short character sums and real zeros of Fekete polynomials. *arXiv preprint arXiv:2403.02195*, 2024.
- [24] Sergei Konyagin and Carl Pomerance. On primes recognizable in deterministic polynomial time. In *The Mathematics of Paul Erdős I*, pages 176–198. Springer, 1997.
- [25] Kaisa Matomäki and Xuancheng Shao. When the sieve works II. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2020(763):1–24, 2020.
- [26] Gérald Tenenbaum. Moyennes effectives de fonctions multiplicatives complexes. *The Ramanujan Journal*, 44(3):641–701, 2017.
- [27] Edward Charles Titchmarsh and David Rodney Heath-Brown. *The theory of the Riemann zeta-function*. Oxford university press, 1986.

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