Micro-packets for real groups of type G_2

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Abstract

In their study of Arthur's conjectures for real groups, Adams, Barbasch, and Vogan introduced the notion of *micro-packets*. Micro-packets are finite sets of irreducible representations defined using microlocal geometric methods and characteristic cycles. We explore an action of the Weyl group on characteristic cycles to compute all micro-packets of real groups of type G_2.

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1 Introduction

Langlands introduced *L-packets* for connected real algebraic groups in [Lan89]. L-packets are finite sets (of equivalence classes) of irreducible admissible representations which were motivated by number-theoretic correspondences.

Arthur used the trace formula to explore such correspondences and was led to conjecture the existence of complementary packets for some unitary representations [Art84], [Art89]. These are commonly called *Arthur packets* or *A-packets*. Every A-packet contains a specific L-packet, but not every L-packet is obtained in this manner. A further conjecture of Arthur's is the existence of a stable distribution, in the sense of Langlands and Shelstad [She82], which is associated to each Arthur packet.

In their seminal work [ABV92], Adams, Barbasch and Vogan, gave a definition of A-packets for connected real reductive groups. In fact, they generalized the notion of an A-packet to that of a *micro-packet*, and gave precise definitions for these too. They also defined a stable distribution associated to each micro-packet. Every micro-packet contains a specific L-packet and all L-packets are obtained in this manner. In particular, there is a natural bijection between L-packets and micro-packets.

Our principal goal is to explicitly compute the micro-packets and the associated stable distributions for the real forms of the complex algebraic group of type G_2. Some analogues of micro-packets for p-adic groups of type G_2 are computed in [CFZ21, CFZ22].

To give an outline of how micro-packets are defined, we need to fix a few objects. First, we denote the complex group of type G_2 simply by G_2, and a real form of it by $\mathrm{G}_2(\mathbb{R})$. The real form is either split or compact [Tit66, Table II]. The Langlands dual group ${}^\vee\mathrm{G}_2$ of G_2 is isomorphic to G_2 itself. Nevertheless, it will be helpful in keeping track of the roots of one group and the coroots of the other to preserve the distinction in notation. Fix Borel subgroups $B \subset \mathrm{G}_2$, ${}^\vee B \subset {}^\vee\mathrm{G}_2$ and maximal tori $T \subset B$, ${}^\vee T \subset {}^\vee B$. Denote the root system of G_2 relative to T by $R(\mathrm{G}_2, T)$, and the positive roots determined by B as $R(B, T)$. We may identify $R(\mathrm{G}_2, T)$ with the roots $R(\mathfrak{g}_2, \mathfrak{t})$ of the respective complex Lie algebras, and this notation carries over to the dual objects in the obvious fashion.

Fix $\lambda \in {}^\vee\mathfrak{t}$. The element λ should be regarded as the infinitesimal character of a representation and we will refer to λ as an infinitesimal character. An infinitesimal character is actually the conjugacy class of a semisimple element. For this reason the Weyl group action makes it harmless to assume that λ is *integrally dominant* with respect to ${}^\vee B$, *i.e.*

$$ {}^\vee\alpha(\lambda) \in \mathbb{Z} \Longrightarrow {}^\vee\alpha(\lambda) \geq 0, \quad {}^\vee\alpha \in R({}^\vee B, {}^\vee T). \tag{1}$$

This is all we need to proceed with our outline, but the presentation is greatly simplified with the two additional assumptions that λ is *regular* and *integral*, *i.e.*

$$ {}^\vee\alpha(\lambda) \in \{1, 2, 3, \ldots\}, \quad {}^\vee\alpha \in R({}^\vee\mathrm{G}_2, {}^\vee T). \tag{2}$$

We continue with our outline under assumption (2) and will say a few words about how the removal of the regularity and integrality assumptions affects the presentation afterwards.

The framework of [ABV92], specialized to the group G_2, is summarized in the following diagram.

$$\begin{tikzcd}
\boxed{\text{Irreducible representations of } \mathrm{G}_2(\mathbb{R})} \arrow[d, leftrightarrow, "\text{Local Langlands correspondence}"] \\
\boxed{\text{Complete geometric parameters}} \arrow[d, leftrightarrow, "\text{Intermediate extension}"] \\
\boxed{\text{Irreducible perverse sheaves on } {}^\vee\mathrm{G}_2/{}^\vee B} \arrow[d, leftrightarrow, "\text{Riemann-Hilbert correspondence}"] \\
\boxed{\text{Irreducible D-modules on } {}^\vee\mathrm{G}_2/{}^\vee B} \arrow[d, leftrightarrow, "\text{Beilinson-Bernstein correspondence}"] \\
\boxed{\text{Irreducible representations of } {}^\vee\mathrm{G}_2(\mathbb{R})}
\end{tikzcd} \tag{3}$$

It is impossible to do justice to the immense amount of mathematics that is hinted at in this diagram. We must be content with a meagre description, emphasizing those points which appear in our computations later on. All arrows in the diagram indicate bijections.

The top bijection is [ABV92, Theorem 10.4]. It is an extension of Langlands' original correspondence [Lan89], which we assume the reader understands. On the one side of the bijection we have (infinitesimal equivalence classes of) irreducible admissible representations of both the split and compact forms of G_2. On the other side of the bijection we have *complete geometric parameters* [ABV92, Definition 7.6], whose description requires more effort. The underlying geometric object is the flag variety ${}^\vee\mathrm{G}_2/{}^\vee B$ [ABV92, Proposition 6.16]. The flag variety is acted upon by a symmetric subgroup ${}^\vee K \subset {}^\vee\mathrm{G}_2$, and each ${}^\vee K$-orbit $S \subset {}^\vee\mathrm{G}_2/{}^\vee B$ corresponds to a unique (equivalence class of an) L-parameter φ [ABV92, Proposition 6.17]. Conversely, every L-parameter (for the split form) corresponds to a unique ${}^\vee K$-orbit, for an appropriate choice of ${}^\vee K$. This yields a correspondence $\varphi \leftrightarrow S$ between L-parameters and symmetric orbits on the flag variety. The correspondence is enhanced by including an irreducible ${}^\vee K$-equivariant local system $\mathcal{V}$ of complex vector spaces on S. The monodromy representation of $\mathcal{V}$ is an irreducible representation of the Langlands component group for φ [ABV92, Lemma 7.5]. A complete geometric parameter is a pair $\xi = (S, \mathcal{V})$, and the enhanced local Langlands correspondence carries ξ to a unique (infinitesimal equivalence class of an) irreducible admissible representation $\pi(\xi)$ of a real form of G_2. Every such representation is obtained from a unique complete geometric parameter in this way. This completes our description of the bijection of the local Langlands correspondence.

We move to the second bijection in the diagram. Let $(S, \mathcal{V})$ be a complete geometric parameter and $j : S \hookrightarrow \overline{S}$ be the inclusion of S in its closure. To pass from $(S, \mathcal{V})$ to an irreducible perverse sheaf on ${}^\vee\mathrm{G}_2/{}^\vee B$ one applies the intermediate extension functor $j_{!*}$ to the shifted complex $\mathcal{V}[\dim S]$ followed by the direct image functor of the closed embedding $\overline{S} \hookrightarrow {}^\vee \mathrm{G}_2/{}^\vee B$ [ABV92, (7.10)]. The resulting irreducible perverse sheaf is an intersection cohomology complex of $(S, \mathcal{V})$ [Ach21, Definition 3.3.9]. Every irreducible perverse sheaf on ${}^\vee \mathrm{G}_2/{}^\vee B$ is obtained in this manner [Ach21, Theorem 3.4.5]. The intermediate extension functor is actually being applied to ${}^\vee K$-equivariant perverse sheaves. We refer to [BL94, I.5.2] and [Ach21, Section 6.5] for the equivariant versions of this functor. This completes our description of the second bijection.

We have little to say about the remaining two bijections. The D-modules in the diagram are ${}^\vee K$-equivariant regular holonomic sheaves of $\mathcal{D}$-modules on ${}^\vee \mathrm{G}_2/{}^\vee B$, where $\mathcal{D}$ is the sheaf of algebraic differential operators on ${}^\vee \mathrm{G}_2/{}^\vee B$ [ABV92, Definition 7.7], [HTT08, Chapter 6]. The Riemann-Hilbert correspondence is proven without reference to equivariance in [BGK$^+$87, Theorem VIII.14.4] and is asserted in the equivariant setting in [BL94, I.4.2].

The final bijection is the Beilinson-Bernstein correspondence [ABV92, Theorem 8.3], [HTT08, Sections 11.5-11.6]. Its image is the set of irreducible $({}^\vee \mathfrak{g}_2, {}^\vee K)$-modules whose infinitesimal character is ρ, the half-sum of the roots in $R({}^\vee B, {}^\vee T)$. These are the underlying modules of the irreducible admissible representations of ${}^\vee \mathrm{G}_2(\mathbb{R})$ with infinitesimal character ρ.

Having seen the general landscape in which we will be working, it is convenient to streamline the notation. Set

$$X = {}^\vee \mathrm{G}_2/{}^\vee B$$

and denote the set of ${}^\vee K$-orbits by ${}^\vee K \backslash X$. We use $\xi = (S, \mathcal{V})$ to denote a complete geometric parameter and the irreducible objects of (3) are denoted by

$$\begin{array}{cc}
\pi(\xi) & \text{representation of } \mathrm{G}_2(\mathbb{R}) \\
\updownarrow & \\
\xi & \text{complete geometric parameter} \\
\updownarrow & \\
P(\xi) & \text{perverse sheaf} \\
\updownarrow & \\
D(\xi) & \text{D-module} \\
\updownarrow & \\
\pi({}^\vee \xi) & \text{representation of } {}^\vee \mathrm{G}_2(\mathbb{R})
\end{array} \tag{4}$$

Continuing on our path to the definition of micro-packets, we focus on the D-modules. The definition of a micro-packet hinges on two invariants attached to an irreducible D-module $D = D(\xi)$. The first invariant is the *characteristic variety* $\mathrm{Ch}(D)$ [ABV92, Section 19], [HTT08, Sections 2.1-2.2]. It is a closed algebraic subvariety of the complex cotangent bundle T^*X. In fact, it is contained in the conormal bundle $T^*_{{}^\vee K}X$ to the ${}^\vee K$-action on T^*X [ABV92, (19.1), Proposition 19.12 (b)].

To every irreducible component C of $\mathrm{Ch}(D)$ there is an associated local module whose length we denote by $m_C(M)$ [ABV92, Definition 19.7]. The ${}^\vee K$-orbit of C is of the form $\overline{T^*_S X}$ for some $S \in {}^\vee K \backslash X$ [ABV92, Lemma 19.2 (b)], and every irreducible component in this orbit has length $m_C(D)$. This leads to the definition of the *characteristic cycle* of D as a formal sum of irreducible components

$$CC(D) = \sum_C m_C(D)\, C = \sum_{S \in {}^\vee K \backslash X} \chi_S^{\mathrm{mic}}(D)\, \overline{T^*_S X}, \tag{5}$$

where $\chi_S^{\mathrm{mic}}(D) = m_C(D)$ for any C in the ${}^\vee K$-orbit corresponding to $\overline{T_S^* X}$. The non-negative integer $\chi_S^{\mathrm{mic}}(D)$ is called the *microlocal multiplicity along* S.

We are now able to define the micro-packet attached to an L-parameter φ with infinitesimal character λ [ABV92, Definition 5.2, Proposition 5.6]. Recall that an L-parameter φ corresponds to a unique $S \in {}^\vee K\backslash X$. The micro-packet of φ is defined as

$$\Pi_\varphi^{\mathrm{mic}} = \{\pi(\xi') : \chi_S^{\mathrm{mic}}(D(\xi')) \neq 0\} \tag{6}$$

[ABV92, Definition 19.13]. It is a finite set of representations of possibly both the split and compact real forms of G_2. The associated stable distribution is

$$\eta_\varphi^{\mathrm{mic}} = \sum_{\pi(\xi')\in\Pi_\varphi^{\mathrm{mic}}} e(\xi')\,(-1)^{\dim S' - \dim S}\,\chi_S^{\mathrm{mic}}(D(\xi'))\,\pi(\xi'), \tag{7}$$

in which $\pi(\xi')$, for $\xi' = (S', \mathcal{V}')$, is identified with its distribution character, and $e(\xi')$ is the Kottwitz sign attached to the real form of which $\pi(\xi')$ is a representation [ABV92, (17.24)(g), Corollary 19.16]. It is obvious from these definitions that the computation of the micro-packets is equivalent to the computation of the microlocal multiplicities, or equivalently, the characteristic cycles. For this reason, we return to (5).

Taking a characteristic cycle defines a $\mathbb{Z}$-linear map

$$CC : \mathscr{K}(X, {}^\vee K) \to \mathscr{L}(X, {}^\vee K) \tag{8}$$

[HTT08, Theorem 2.2.3]. The domain of the map is the Grothendieck group $\mathscr{K}(X, {}^\vee K)$ of the category of ${}^\vee K$-equivariant regular holonomic sheaves of $\mathcal{D}$-modules on X. The codomain of the map is

$$\mathscr{L}(X, {}^\vee K) = \left\{ \sum_{S\in{}^\vee K\backslash X} m_S\, \overline{T_S^* X} : m_S \in \mathbb{Z} \right\} \tag{9}$$

the free $\mathbb{Z}$-module generated by $\overline{T_S^* X}$, $S \in {}^\vee K\backslash X$. Through the bijections of (4) we have isomorphisms between $\mathscr{K}(X, {}^\vee K)$ and the Grothendieck groups of the category of ${}^\vee K$-equivariant perverse sheaves on X, and the category of admissible representations of ${}^\vee\mathrm{G}_2(\mathbb{R})$ with infinitesimal character ρ. We identify $\mathscr{K}(X, {}^\vee K)$ with the other two isomorphic Grothendieck groups, so that

$$\begin{aligned} CC(P(\xi)) &= CC(\pi({}^\vee\xi)) = CC(D(\xi)) \\ \chi_S^{\mathrm{mic}}(P(\xi)) &= \chi_S^{\mathrm{mic}}(\pi({}^\vee\xi)) = \chi_S^{\mathrm{mic}}(D(\xi)) \\ \mathrm{Ch}(P(\xi)) &= \mathrm{Ch}(\pi({}^\vee\xi)) = \mathrm{Ch}(D(\xi)) \end{aligned} \tag{10}$$

One of the main tools in our computation of the micro-packets is the interaction of the Weyl group $W = W({}^\vee\mathrm{G}_2, {}^\vee T)$ with CC. In the setting of representations, there is a W-action on $\mathscr{K}(X, {}^\vee K)$ commonly called the *coherent continuation representation* [Vog81, Definition 7.2.28]. This W-action may be explicitly computed using the Atlas of Lie Groups and Representation software [APM$^+$]. On the other hand, there is a W-action on $\mathscr{L}(X, {}^\vee K)$ given in [Hot85], [Ros95]. By [Tan85, Theorem 1] the map CC is W-equivariant. In Section 5 we use the W-equivariance of CC to generate a number of linear equations whose coefficients involve microlocal multiplicities. We can then solve for some of the microlocal multiplicities by making elementary substitutions.

Another means of simplifying the equations is through the use of the *associated variety.* We define the associated variety of an irreducible object in $\mathscr{K}(X, {}^{\vee}K)$ by

$$\mathrm{AV}(P(\xi)) = \mu\left(\mathrm{Ch}(P(\xi))\right), \tag{11}$$

where $\mu : T^*X \to \mathfrak{g}_2^*$ is the *moment map* [ABV92, (20.3)]. Since $\mathrm{Ch}(P(\xi))$ is the support of $CC(P(\xi))$, information about $\mathrm{AV}(P(\xi))$ gives us information about which microlocal multiplicities can be non-zero. The image of the moment map lies in the cone of nilpotent elements in $\mathfrak{g}_2^*$ and is a union of finitely many known ${}^{\vee}K$-orbits [Đok00]. It follows from the ${}^{\vee}K$-equivariance of $P(\xi)$ that $\mathrm{AV}(P(\xi))$ is a finite union of ${}^{\vee}K$-orbits [ABV92, Lemma 20.16]. These are known in some cases [Sam14]. A compatibility property between the coherent continuation representation and the associated varieties, provides additional information about $\mathrm{AV}(P(\xi))$ in Propositions 4.2 and 4.3. This additional information allows us to compute all microlocal multiplicities. The micro-packets and associated stable distributions are given in Theorem 7.1.

This concludes our overview of micro-packets $\Pi_{\varphi}^{\mathrm{mic}}$ when the infinitesimal character λ of φ is regular and integral (2). The framework of (3) does not fundamentally change if we drop the assumption of integrality on λ. Consider the setting in which the infinitesimal character λ is merely integrally dominant (1) and regular, *i.e.*

$${}^{\vee}\alpha(\lambda) \neq 0, \quad {}^{\vee}\alpha \in R({}^{\vee}\mathrm{G}_2, {}^{\vee}T). \tag{12}$$

In this broader setting we replace ${}^{\vee}\mathrm{G}_2$ with the subgroup ${}^{\vee}\mathrm{G}_2(\lambda)$, which is the centralizer of $\exp(2\pi i\lambda)$ in ${}^{\vee}\mathrm{G}_2$ [ABV92, (6.2)(b), Proposition 6.16]. We determine the possibilities for ${}^{\vee}\mathrm{G}_2(\lambda)$, along with the possibilities for its symmetric subgroups ${}^{\vee}K(\lambda)$, in Section 2.4. The regular integrally dominant element λ fixes a Borel subgroup ${}^{\vee}B(\lambda) \subset {}^{\vee}\mathrm{G}_2(\lambda)$. To compute micro-packets with non-integral regular infinitesimal character λ, all we need to do is replace the groups ${}^{\vee}\mathrm{G}_2$, ${}^{\vee}K$ and ${}^{\vee}B$ with ${}^{\vee}\mathrm{G}_2(\lambda)$, ${}^{\vee}K(\lambda)$ and ${}^{\vee}B(\lambda)$ respectively, and apply the same methods. As ${}^{\vee}\mathrm{G}_2(\lambda)$ is a smaller group this is a much easier task. We compute these micro-packets and stable distributions in Section 6.

Removing the further assumption of regularity (12) has the effect of replacing the Borel subgroup ${}^{\vee}B(\lambda)$ with a parabolic subgroup ${}^{\vee}P(\lambda) \subset {}^{\vee}\mathrm{G}_2(\lambda)$ [ABV92, (6.2)(e), Proposition 6.16]. In this way the full flag variety ${}^{\vee}\mathrm{G}_2(\lambda)/{}^{\vee}B(\lambda)$ is replaced by a partial flag variety ${}^{\vee}\mathrm{G}_2(\lambda)/{}^{\vee}P(\lambda)$. One is able to obtain information in the singular setting from the regular setting using the *translation functors* of Jantzen, Vogan and Zuckerman [ABV92, Propositions 8.8 and 16.6]. In particular, the microlocal multiplicities, and so the micro-packets and stable distributions, are obtained through the translation functors [ABV92, Proposition 20.1 (e)].

Having ascertained the microlocal multiplicities, we finish by computing *Kashiwara's local index formula* for G_2. In essence, this formula provides a change of basis between the microlocal multiplicities and another set of maps called *local multiplicities.* This is a geometric relationship that may be motivated in the following representation-theoretic language. The stable distributions $\eta_{\varphi}^{\mathrm{mic}}$ in (7) form a basis for the lattice of stable virtual representations of the pure real forms of G_2 with infinitesimal character λ [ABV92, Corollary 19.16]. There is another basis for this space of stable virtual representations, namely the basis consisting of standard representations $\eta_{\varphi}^{\mathrm{loc}}$ [ABV92, Definition 18.9, Lemma 18.11]. In [ABV92, Lemma 18.15], the stable distribution $\eta_{\varphi}^{\mathrm{loc}}$ is expressed as a linear combination of representations whose coefficients are given by $\mathbb{Z}$-linear functionals

$$\chi_S^{\mathrm{loc}} : \mathscr{K}(X(\lambda), {}^{\vee}K(\lambda)) \longrightarrow \mathbb{Z},$$

called *local multiplicities* [ABV92, Definition 1.28]. The change of basis matrix between the two bases $\{\eta_\varphi^{\mathrm{mic}}\}$ and $\{\eta_\varphi^{\mathrm{loc}}\}$ entails a “change of basis”

$$\chi_S^{\mathrm{loc}} = \sum_{S'} (-1)^{\dim(S')} a(S,S')\, \chi_{S'}^{\mathrm{mic}} \tag{13}$$

between the sets of $\mathbb{Z}$-linear functionals $\{\chi_S^{\mathrm{mic}}\}$ and $\{\chi_S^{\mathrm{loc}}\}$. This equation is Kashiwara's local index formula [Kas73, Section 2]. In fact, the coefficients $a(S,S')$ are integers. The integer coefficients $a(S,S')$ are constructed by different methods in [Mac74, Section 3] and are individually called the *local Euler obstruction* of S at S'. The local Euler obstructions have been extensively studied from various perspectives [Gin86, §11.7], [GS81]. Nevertheless, to our knowledge, no method is known for computing them in full generality. By our work in the setting of ${}^\vee\mathrm{G}_2(\lambda)$, we know the values of $\chi_{S'}^{\mathrm{mic}}$ on the right of (13). In Section 9 we explain how the Atlas of Lie Groups and Representations software allows us to compute the values of χ_S^{loc} on the left of (13). Having these values on both sides of (13) allows us to compute the local Euler obstructions $a(S,S')$.

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2 ${}^\vee K$-orbits of the flag variety of ${}^\vee\mathrm{G}_2$

We begin this section by determining all possibilities for the symmetric groups ${}^\vee K \subset {}^\vee\mathrm{G}_2$ of the introduction, and concluding that there is only one such group of any interest. In Section 2.2 we parameterize the ${}^\vee K$-orbits of the flag variety, and describe the partial order on them given by a closure relation. This amounts to an application of [AdC09] and [RS90] to ${}^\vee\mathrm{G}_2$. In Section 2.3, we examine the correspondence between ${}^\vee K$-orbits and L-parameters, and compute the Langlands component group for each L-parameter. These computations give us a concrete picture of the geometric objects for regular and integral infinitesimal character λ. We close by repeating the computations when λ is regular and non-integral, that is finding the objects ${}^\vee\mathrm{G}_2(\lambda)$, ${}^\vee K(\lambda)$, *etc.* alluded to in the introduction.

There are some additional objects to fix beyond the Borel pairs $B \supset T$ and ${}^\vee B \supset {}^\vee T$ of the introduction. We identify the set of positive roots $R(B,T)$ with the set of positive coroots ${}^\vee R({}^\vee B, {}^\vee T)$. Similarly, the characters $X^*(T)$ are identified with the cocharacters $X_*({}^\vee T)$ of ${}^\vee T$. We identify the complex Lie algebra ${}^\vee\mathfrak{t}$ of ${}^\vee T$ with $X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}$. Define the surjection

$${}^\vee\mathfrak{t} \xrightarrow{e} {}^\vee T \tag{14}$$

by $e(X) = \exp(2\pi i X)$. Clearly $\ker e = X^*(T)$, which is also the root lattice spanned by $R(\mathrm{G}_2, T)$. Let α_1 and α_2 be the simple roots in $R(B,T)$, with α_1 being the long root. With this, $\rho = 5\alpha_2 + 3\alpha_1$ is the half-sum of the positive roots.

2.1 The group ${}^\vee K$

Let us review the definition of ${}^\vee K$ and specialize to the case at hand. Let ${}^\vee\mathrm{G}_2^\Gamma = {}^\vee\mathrm{G}_2 \times \Gamma$ be the Galois form of the L-group of G_2, and set

$$\mathcal{I} = \{y \in {}^\vee\mathrm{G}_2^\Gamma - {}^\vee\mathrm{G}_2 : y^2 = 1\}.$$

Since the L-group is a direct product, we may identify $\mathcal{I}$ with the set

$$\{y \in {}^\vee\mathrm{G}_2 : \ y^2 = 1\}.$$

The group ${}^\vee\mathrm{G}_2$ acts by conjugation on $\mathcal{I}$ with finitely many orbits [ABV92, Lemma 6.12]. By definition, a symmetric group ${}^\vee K$ is taken to be the centralizer in ${}^\vee\mathrm{G}_2$ of a representative y of a ${}^\vee\mathrm{G}_2$-orbit of $\mathcal{I}$ [ABV92, Proposition 6.16].

Lemma 2.1. *There are exactly two ${}^\vee\mathrm{G}_2$-orbits of $\mathcal{I}$, and the two orbits are represented by the elements 1 and $e(\rho/2) = \exp(\pi i \rho)$.*

Proof. As the elements in $\mathcal{I}$ are semisimple they lie in maximal tori. Since ${}^\vee\mathrm{G}_2$ acts transitively on the set of maximal tori under conjugation, it follows that the ${}^\vee\mathrm{G}_2$-orbits of $\mathcal{I}$ are in bijection with the Weyl group orbits of

$$\{y \in {}^\vee T : y^2 = 1\}.$$

The condition $y^2 = 1$ is equivalent to y^2 being in the centre of ${}^\vee\mathrm{G}_2$, and the latter is equivalent to y^2 being in the kernel of ${}^\vee\alpha_1$ and ${}^\vee\alpha_2$. Using (14), we may write

$$y = e(c_1\alpha_1 + c_2\alpha_2) = \exp(2\pi i(c_1\alpha_1 + c_2\alpha_2))$$

for some $c_1, c_2 \in \mathbb{C}$. The kernel property for ${}^\vee\alpha_1$ and ${}^\vee\alpha_2$ translates into

$$c_1\langle {}^\vee\alpha_j, \alpha_1\rangle + c_2\langle {}^\vee\alpha_j, \alpha_2\rangle \in \frac{1}{2}\mathbb{Z}, \quad j = 1, 2.$$

As pairings between roots and coroots are integers, we see that $y^2 = 1$ if and only if $c_1, c_2 \in \frac{1}{2}\mathbb{Z}$. If c_1 or c_2 are integers then they may be removed from $e(c_1\alpha_1 + c_2\alpha_2)$. Thus, we are left with four possibilities for y

$$1, e((\alpha_1 + \alpha_2)/2), e(\alpha_1/2), e(\alpha_2/2).$$

However, $\alpha_2/2$ and $(\alpha_1 + \alpha_2)/2$ are Weyl group conjugates. In addition, $\alpha_1/2$ is a Weyl group conjugate of $(3\alpha_2 + 2\alpha_1)/2$, and

$$e((3\alpha_2 + 2\alpha_1)/2) = e(\alpha_2/2 + 2(\alpha_2 + \alpha_1)/2) = e(\alpha_2/2).$$

This proves that there are at most two ${}^\vee\mathrm{G}_2$-orbits of $\mathcal{I}$, with respective representatives 1 and $e(\alpha_2/2)$. It is easily verified that ${}^\vee\alpha_2(e(\alpha_2/2)) \neq 1$, so there are exactly two orbits. Finally,

$$e(\rho/2) = e((5\alpha_2 + 3\alpha_1)/2) = e((\alpha_1 + \alpha_2)/2)$$

is also a representative of the non-trivial orbit. $\square$

By Lemma 2.1, the group ${}^\vee K$ may be taken to equal either ${}^\vee\mathrm{G}_2$, when $y = 1$, or may be taken to equal the centralizer of $e(\rho/2)$. In the former case, there is only a single ${}^\vee K$-orbit of ${}^\vee\mathrm{G}_2/{}^\vee B$, namely ${}^\vee\mathrm{G}_2/{}^\vee B$ itself. For the rest of this section we will therefore assume that ${}^\vee K$ is the centralizer of $e(\rho/2)$.

The group ${}^\vee\mathrm{G}_2$ is simply connected and $e(\rho/2)$ is semisimple, so by Steinberg's Theorem, the centralizer ${}^\vee K$ is connected [Hum95, Theorem 2.11]. By [Hum95, Theorem 2.2], the group ${}^\vee K$ is generated by ${}^\vee T$ and the root subgroups $U_{{}^\vee\alpha} \subset {}^\vee\mathrm{G}_2$ such that ${}^\vee\alpha(e(\rho/2)) = 1$. These are given by the roots ${}^\vee\alpha \in R({}^\vee\mathrm{G}_2, {}^\vee T)$ such that $\langle {}^\vee\alpha, \rho\rangle \in 2\mathbb{Z}$. Using [Hum78, Lemma 13.3.A] one may prove that there are exactly two positive roots satisfying this property, ${}^\vee\alpha_1 + {}^\vee\alpha_2$ and $3{}^\vee\alpha_1 + {}^\vee\alpha_2$. (Note that ${}^\vee\alpha_1$ is a short root.) These two roots are orthogonal. Consequently,

$$ {}^\vee K = \langle {}^\vee T, U_{\pm({}^\vee\alpha_1+{}^\vee\alpha_2)}, U_{\pm(3{}^\vee\alpha_1+{}^\vee\alpha_2)} \rangle = \langle U_{\pm({}^\vee\alpha_1+{}^\vee\alpha_2)} \rangle \ \langle U_{\pm(3{}^\vee\alpha_1+{}^\vee\alpha_2)} \rangle \tag{15} $$

is the product of two commuting subgroups, each of which being isomorphic to SL_2. An element in the intersection of these two commuting subgroups must lie in the centre of each subgroup. The non-trivial central elements of $\langle U_{\pm({}^\vee\alpha_1+{}^\vee\alpha_2)} \rangle$ and $\langle U_{\pm(3{}^\vee\alpha_1+{}^\vee\alpha_2)} \rangle$ are $\exp(\pi i(3\alpha_2 + \alpha_1))$ and $\exp(\pi i(\alpha_1 + \alpha_2))$ respectively. Since

$$ \exp(\pi i(3\alpha_2 + \alpha_1)) = \exp(\pi i(2\alpha_2 + (\alpha_1 + \alpha_2))) = \exp(\pi i(\alpha_1 + \alpha_2)) $$

we see that the intersection of the two subgroups is a group of order two generated by $\exp(\pi i(\alpha_1 + \alpha_2))$. The intersection is isomorphic to $\mu_2 = \{\pm 1\}$ and so ${}^\vee K$ is isomorphic to the fibre product $\mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$.

2.2 ${}^\vee K$-orbits of ${}^\vee\mathrm{G}_2/{}^\vee B$

The algebraic group ${}^\vee K$ acts by left multiplication on ${}^\vee\mathrm{G}_2/{}^\vee B$. The closure of each ${}^\vee K$-orbit is a union of ${}^\vee K$-orbits. This closure relation gives rise to the *Bruhat order* on the ${}^\vee K$-orbits [RS90]. The goal of this section is to describe the ${}^\vee K$-orbits and the Bruhat order.

We begin by giving a complete set of representatives $y_0, \ldots, y_\ell \in {}^\vee\mathrm{G}_2$ for the ${}^\vee K$-orbits

$$ {}^\vee K \backslash {}^\vee\mathrm{G}_2/{}^\vee B = \{ {}^\vee K y_0 {}^\vee B, \ldots, {}^\vee K y_\ell {}^\vee B \} $$

by following [AdC09]. We shall also employ the Atlas of Lie Groups and Representations software [APM$^+$], which is an outgrowth of [AdC09]. There are a number of sets in [AdC09] which are pertinent to the description of ${}^\vee K \backslash {}^\vee\mathrm{G}_2/{}^\vee B$. We present them here, taking the simplifications of the special case of ${}^\vee\mathrm{G}_2$ into consideration. Set

$$ \widetilde{\mathcal{X}} = \{ \xi \in \mathrm{Norm}_{{}^\vee\mathrm{G}_2}({}^\vee T) : \xi^2 = 1 \}. $$

The torus ${}^\vee T$ acts on $\widetilde{\mathcal{X}}$ by conjugation. Let $\mathcal{X}$ be the set of ${}^\vee T$-conjugacy classes. The element $e(\rho/2) = \exp(\pi i \rho)$ defining ${}^\vee K$ belongs to $\widetilde{\mathcal{X}}$. Let $S_0 = \{e(\rho/2)\} \in \mathcal{X}$ be its ${}^\vee T$-conjugacy class. According to [AdC09, (9.8)], there is a bijection between ${}^\vee K \backslash {}^\vee\mathrm{G}_2/{}^\vee B$ and the subset $\mathcal{X}[S_0] \subset \mathcal{X}$ of elements whose pre-images in $\widetilde{\mathcal{X}}$ are ${}^\vee\mathrm{G}_2$-conjugate to $e(\rho/2)$. By [AdC09, (8.1), (9.1)], each element $S \in \mathcal{X}[S_0]$ is the ${}^\vee T$-conjugacy class of $y e(\rho/2) y^{-1}$ for some $y \in {}^\vee\mathrm{G}_2$, and the element $S \in \mathcal{X}[S_0]$ corresponds to the ${}^\vee K$-orbit ${}^\vee K y^{-1} {}^\vee B$

$$ S \longleftrightarrow {}^\vee K y^{-1} \, {}^\vee B. $$

Thus, the problem of determining ${}^\vee K \backslash {}^\vee\mathrm{G}_2/{}^\vee B$ is equivalent to determining the elements $S_0, \ldots, S_\ell \in \mathcal{X}[S_0]$ and $y_0, \ldots, y_\ell \in {}^\vee\mathrm{G}_2$ as indicated. Using the results [AdC09, Proposition 11.2, Lemma 14.2 and Lemma 14.11], the elements $S_0, \ldots, S_\ell \in \mathcal{X}[S_0]$ may be obtained by taking *cross-actions* [AdC09, (9.11 f)] and *Cayley transforms* [AdC09, Definition 14.1] from S_0. The command `KGB` in the Atlas of Lie Groups and Representations software implements this approach and produces ten elements, $y_0 = 1, y_1, \ldots, y_9 \in {}^\vee\mathrm{G}_2$ which parameterize the ${}^\vee K$-orbits.

Let us provide some more detail. From now on we fix a pair of simple root vectors $X_{{}^\vee\alpha_1}, X_{{}^\vee\alpha_2} \in {}^\vee\mathfrak{g}_2$ which then fixes the pinning

$$ ({}^\vee B, {}^\vee T, \{X_{{}^\vee\alpha_1}, X_{{}^\vee\alpha_2}\}) $$

of ${}^\vee\mathrm{G}_2$. Every element of $\widetilde{\mathcal{X}}$ belongs to a coset in the Weyl group

$$W = W({}^\vee\mathrm{G}_2, {}^\vee T) = \mathrm{Norm}_{{}^\vee\mathrm{G}_2}({}^\vee T)/{}^\vee T$$

and this correspondence determines a map

$$p : \mathcal{X}[S_0] \to W$$

[AdC09, (9.11i)]. It is clear that the image of p lies in the subset of elements in W which square to the identity. In consequence, the fibres of the map p partition $\mathcal{X}[S_0]$ into subsets $p^{-1}(w) = \mathcal{X}_w$, where $w \in W$ satisfies $w^2 = 1$. Every simple reflection $s_j = s_{{}^\vee\alpha_j} \in W$, $j = 1, 2$, has a Tits representative

$$\sigma_j = \exp\left(\frac{\pi}{2}(X_{{}^\vee\alpha_j} - X_{-{}^\vee\alpha_j})\right) \in \widetilde{\mathcal{X}} \tag{16}$$

[AV15, Section 12]. It follows that every element $S \in \mathcal{X}_w$ is a ${}^\vee T$-conjugacy class of an element of the form $t\sigma_w$, where $t \in {}^\vee T$, σ_w is a product of Tits representatives, and the element $t\sigma_w$ is ${}^\vee\mathrm{G}_2$-conjugate to $e(\rho/2)$.

We can describe $\mathcal{X}[S_0]$ by starting with $\mathcal{X}_1$, where the 1 in subscript is the identity element in W. The obvious choice of basepoint in $\mathcal{X}_1$ is S_0, the ${}^\vee T$-conjugacy class of $e(\rho/2)$. The remaining elements in $\mathcal{X}_1$ are obtained by cross-action of (imaginary) elements in W [AdC09, (12.17)]. This amounts to repeatedly conjugating $e(\rho/2)$ by the Tits representatives (16) of the simple reflections. The Atlas software tells us that these cross actions produce exactly three elements in $\mathcal{X}_1$: one corresponding to the base point S_0, and two further elements S_1 and S_2 equal to the ${}^\vee T$-conjugacy classes of $\sigma_2 e(\rho/2)\sigma_2^{-1}$ and $\sigma_1 e(\rho/2)\sigma_1^{-1}$ respectively.

We can pass from the basepoint S_0 for $\mathcal{X}_1$ to a basepoint in $\mathcal{X}_{s_j}$ by a Cayley transform with respect to ${}^\vee\alpha_j$, $j = 1, 2$. [AdC09, Equation (14.5)] expresses these Cayley transforms in terms of conjugation by an element

$$g_j = \exp\left(\frac{\pi}{4}(X_{-{}^\vee\alpha_j} - X_{{}^\vee\alpha_j})\right), \quad j = 1, 2.$$

(*cf.* [Kna02, (6.65a)]). In this manner, the basepoint of $\mathcal{X}_{s_j}$ is the ${}^\vee T$-conjugacy class of $g_j e(\rho/2) g_j^{-1}$. As with $\mathcal{X}_1$, the remaining elements of $\mathcal{X}_{s_j}$ are obtained from $g_j e(\rho/2) g_j^{-1}$ through cross-actions of Weyl group elements (which are imaginary with respect to s_j). There happen to be no such cross actions, and so $\mathcal{X}_{s_j}$ is a singleton. The elements of the remaining $\mathcal{X}_w$ are obtained through the similar iterations of Cayley transforms and cross-actions. A summary is given in the following table.

j	Representative of $S_j \in \mathcal{X}[S_0]$	y_j	$p(S_j)$
0	$e(\rho/2)$	1	1
1	$\sigma_2 e(\rho/2)\sigma_2^{-1}$	σ_2^{-1}	1
2	$\sigma_1 e(\rho/2)\sigma_1^{-1}$	σ_1^{-1}	1
3	$g_2 e(\rho/2) g_2^{-1}$	g_2^{-1}	s_2
4	$g_1 e(\rho/2) g_1^{-1}$	g_1^{-1}	s_1
5	$\sigma_2 g_1 e(\rho/2)(\sigma_2 g_1)^{-1}$	$(\sigma_2 g_1)^{-1}$	$s_2 s_1 s_2$
6	$\sigma_1 g_2 e(\rho/2)(\sigma_1 g_2)^{-1}$	$(\sigma_1 g_2)^{-1}$	$s_1 s_2 s_1$
7	$\sigma_2\sigma_1 g_2 e(\rho/2)(\sigma_2\sigma_1 g_2)^{-1}$	$(\sigma_2\sigma_1 g_2)^{-1}$	$s_2 s_1 s_2 s_1 s_2$
8	$\sigma_1\sigma_2 g_1 e(\rho/2)(\sigma_1\sigma_2 g_1)^{-1}$	$(\sigma_1\sigma_2 g_1)^{-1}$	$s_1 s_2 s_1 s_2 s_1$
9	$g_2\sigma_1\sigma_2 g_1 e(\rho/2)(g_2\sigma_1\sigma_2 g_1)^{-1}$	$(g_2\sigma_1\sigma_2 g_1)^{-1}$	$s_2 s_1 s_2 s_1 s_2 s_1$

Table 1: ${}^\vee K$-orbit data

We now turn to the description of the Bruhat order on these ten orbits. The minimal orbits in the Bruhat order are characterized as the closed orbits [Hum75, Proposition 8.3].

Lemma 2.2. *The closed* ${}^\vee K$*-orbits are* S_0*,* S_1 *and* S_2*.*

Proof. From (15) we see that ${}^\vee B \cap {}^\vee K = \langle {}^\vee T, U_{{}^\vee\alpha_1+{}^\vee\alpha_2}, U_{3{}^\vee\alpha_1+{}^\vee\alpha_2}\rangle$ is a Borel subgroup of ${}^\vee K$. Hence, ${}^\vee K/({}^\vee B \cap {}^\vee K)$ is a flag variety. As a flag variety, it is a projective variety. There is an obvious injection

$$ {}^\vee K/({}^\vee B \cap {}^\vee K) \to {}^\vee \mathrm{G}_2/{}^\vee B $$

and its image is ${}^\vee K\,{}^\vee B = {}^\vee K y_0 {}^\vee B$, *i.e.* the orbit S_0. Since it is the image of a projective variety, the orbit S_0 is closed. Similarly, the orbit S_2 is of the form

$$ {}^\vee K y_2 {}^\vee B = {}^\vee K \sigma_1^{-1}{}^\vee B = \sigma_1^{-1}\cdot(\sigma_1{}^\vee K\sigma_1^{-1})\,{}^\vee B. \tag{17} $$

From (15) we deduce that

$$ \sigma_1{}^\vee K\sigma_1^{-1} = \langle {}^\vee T, U_{\pm(2{}^\vee\alpha_1+{}^\vee\alpha_2)}, U_{\pm{}^\vee\alpha_2}\rangle. $$

Clearly, we may replace ${}^\vee K$ with $\sigma_1{}^\vee K\sigma_1^{-1}$ in the argument for S_0 to conclude that $(\sigma_1{}^\vee K\sigma_1^{-1})\,{}^\vee B$ is closed in ${}^\vee G_2/{}^\vee B$. The orbit of S_2 is then closed by (17). The same reasoning applies to prove that the orbit of S_1 is closed. $\square$

By Lemma 2.2, S_0, S_1 and S_2 minimal in the Bruhat order. With the aim of placing the remaining orbits $S_3, \ldots, S_9$ in the Bruhat order, we use an action defined by Richardson and Springer of simple root reflections on the orbits [RS90, 4.7]. Given a simple root ${}^\vee\alpha_j$ and an orbit ${}^\vee K y_\ell {}^\vee B$, they produce a union of orbits $P_{s_j}({}^\vee K y_\ell {}^\vee B)$ containing a unique open orbit $m(s_j)({}^\vee K y_\ell {}^\vee B)$. This defines an action m of the simple reflections on the set of ${}^\vee K$-orbits. By [RS90, Lemma 4.6], every orbit is obtained by iterating this action. More precisely, every orbit ${}^\vee K y_j {}^\vee B$ is of the form

$$ {}^\vee K y_j {}^\vee B = m(w_k)m(w_{k-1})\cdots m(w_1)({}^\vee K y_\ell {}^\vee B), \tag{18} $$

for some choice of simple root reflections $w_1, \ldots, w_k \in W$ and some closed orbit ${}^\vee K y_\ell {}^\vee B$. Furthermore, the closure of the orbit ${}^\vee K y_j {}^\vee B$ is

$$ \overline{{}^\vee K y_j {}^\vee B} = P_{w_k}\cdots P_{w_1}({}^\vee K y_\ell {}^\vee B). \tag{19} $$

An elementary and explicit description of the terms on the right is given in [RS90, 4.3]. It is therefore not difficult to compute that all of the remaining orbits are actually obtained as in (18) with $\ell = 0$, *i.e.* from the closed orbit ${}^\vee K y_0 {}^\vee B = {}^\vee K\,{}^\vee B$. The orbit closures are then easily computed from (19) with $\ell = 0$.

For example, the simple root ${}^\vee\alpha_1$ is imaginary for $p(S_0) = 1 \in W$ ([RS90, 1.6], where φ there is replaced by p here). [RS90, 4.3.4] tells us that for some $3 \le j \le 9$

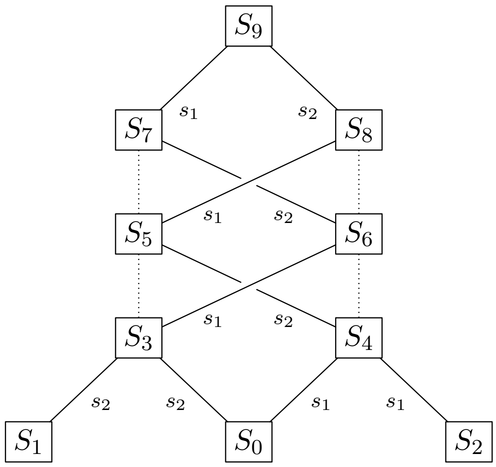
$$ \begin{aligned} \overline{{}^\vee K y_j {}^\vee B} &= \overline{m(s_1)({}^\vee K\,{}^\vee B)} \\ &= P_{s_1}({}^\vee K\,{}^\vee B) \\ &= m(s_1)({}^\vee K\,{}^\vee B) \cup ({}^\vee K\,{}^\vee B) \cup ({}^\vee K y_2 {}^\vee B) \end{aligned} $$

We then use [RS90, Lemma 7.4 (i)] to conclude that $p(S_j) = s_1$. Table 1 identifies the orbit number as $j = 4$, *i.e.* ${}^\vee K y_4 {}^\vee B = m(s_1)({}^\vee K\,{}^\vee B)$. Identical reasoning with the simple root α_2 shows that ${}^\vee K y_3 {}^\vee B = m(s_2)({}^\vee K\,{}^\vee B)$.

To go one step further, we argue from ${}^\vee K y_4 {}^\vee B$. The simple root α_2 is complex for $p(S_4) = s_1$. The orbit ${}^\vee K y_j {}^\vee B = m(s_2)({}^\vee K y_4 {}^\vee B)$ satisfies $p(S_j) = s_2 s_1 s_2$ and so $j = 5$ from Table 1. Thus, ${}^\vee K y_5 {}^\vee B = m(s_2)({}^\vee K y_4 {}^\vee B)$, and the closure of ${}^\vee K y_5 {}^\vee B$ is $P_{s_2} P_{s_1}({}^\vee K {}^\vee B)$. Our work on the earlier orbits and [RS90, 4.3.1] allow us to compute the closure as

$$\begin{aligned} &P_{s_2}\left(({}^\vee K y_4 {}^\vee B) \cup ({}^\vee K y_0 {}^\vee B) \cup ({}^\vee K y_1 {}^\vee B)\right) \\ &= ({}^\vee K y_5 {}^\vee B) \cup ({}^\vee K y_4 {}^\vee B) \cup ({}^\vee K y_3 {}^\vee B) \cup ({}^\vee K y_2 {}^\vee B) \cup ({}^\vee K y_1 {}^\vee B) \cup ({}^\vee K y_0 {}^\vee B). \end{aligned}$$

A summary of the closure relations between the orbits is given in the following Hasse diagram of the Bruhat order.



(20)

In this diagram a solid line indicates that the higher orbit is obtained from the lower one by applying m of the labelled simple reflection. For example $S_4 = m(s_1)S_0$. If one orbit is contained in the closure of the other, but is not related through the action of m then the line is dotted.

In the foregoing discussion we have not gone into any detail about what it means for a root to be (compact/non-compact) imaginary, real or complex relative to S_j. The reader may wish to review the equivalent definitions given in [RS90, 1.6] and [AdC09, 12]. Apart from the computations summarized above, in Section 4 it is important to know the nature of the simple roots relative to S_j. We have used the Atlas software to do the elementary computations for the following table.

S_j	${}^\vee\alpha_1$	${}^\vee\alpha_2$
S_0	imaginary non-compact	imaginary non-compact
S_1	imaginary compact	imaginary non-compact
S_2	imaginary non-compact	imaginary compact
S_3	complex	real
S_4	real	complex
S_5	complex	complex
S_6	complex	complex
S_7	imaginary non-compact	complex
S_8	complex	imaginary non-compact
S_9	real	real

Table 2: The nature of the simple roots relative to S_j

2.3 Langlands Component groups

To each L-parameter there is attached a Langlands component group. Recall from the introduction that each L-parameter corresponds to a ${}^\vee K$-orbit S. The Langlands component group is important as its irreducible characters parameterize the irreducible local systems $\mathcal{V}$ on S [ABV92, Lemma 7.3 (e)]. These local systems are the ones appearing in the complete geometric parameters $\xi = (S, \mathcal{V})$.

We review the definitions and compute the Langlands component groups in the context of G_2. In the present context, an L-parameter is a homomorphism $\varphi : W_{\mathbb{R}} \to {}^\vee\mathrm{G}_2$, where $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\,\mathbb{C}^\times$ is the Weil group of $\mathbb{C}/\mathbb{R}$. Without loss of generality, we may assume that

$$\varphi(z) = z^\lambda \bar{z}^{\mathrm{Ad}(y)\lambda}, \quad z \in \mathbb{C}^\times, \tag{21}$$

where $y = \exp(\pi i \lambda)\varphi(j)$ and $\lambda \in {}^\vee\mathfrak{t}$ is dominant relative to ${}^\vee B$ (*cf.* [ABV92, (5.7)]). Clearly, the L-parameter φ is uniquely determined by the pair (y, λ) and so we may write $\varphi = \varphi(y, \lambda)$.

In this section we assume that the infinitesimal character λ is dominant, regular and integral (2). Then $y^2 = e(\lambda) = 1$. According to [ABV92, Proposition 6.16 and Proposition 6.17], the equivalence classes of L-parameters with fixed regular integral infinitesimal character λ are in bijection with the union of the ${}^\vee K$-orbits of ${}^\vee\mathrm{G}_2/{}^\vee B$ and the ${}^\vee\mathrm{G}_2$-orbit of ${}^\vee\mathrm{G}_2/{}^\vee B$ (a singleton). We shall ignore the ${}^\vee\mathrm{G}_2$-orbit. The ${}^\vee K$-orbits are given explicitly in Table 1 and using the data listed there the bijection is induced by

$$\varphi(S_j, \lambda) \longmapsto {}^\vee K y_j {}^\vee B. \tag{22}$$

Set

$$\varphi_j = \varphi(S_j, \lambda), \quad 0 \le j \le 9$$

so that $\{\varphi_0, \ldots, \varphi_9\}$. Apart from the L-parameter corresponding to the ${}^\vee\mathrm{G}_2$-orbit, this is a complete set of representatives of the equivalence classes of L-parameters with infinitesimal character λ.

The Langlands component group of φ_j is defined as the component group of the centralizer of the image of φ_j in ${}^\vee\mathrm{G}_2$. By [ABV92, Corollary 5.9 (c) and Lemma 12.10], the Langlands component group of φ_j is isomorphic to the component group of the fixed-point subgroup ${}^\vee T^{S_j}$ of ${}^\vee T$ under conjugation by S_j (*cf.* [ABV92, (12.11)]). Evidently, the subgroup ${}^\vee T^{S_j}$ is equal to the fixed-point subgroup ${}^\vee T^{p(S_j)}$ under the Weyl group action of $p(S_j)$.

Let us find the Langlands component group of φ_j by identifying it with the component group of ${}^\vee T^{p(S_j)}$ and referring to Table 1. For $j = 0, 1, 2$ the Weyl group element $p(S_j)$ is trivial and so ${}^\vee T^{p(S_j)} = {}^\vee T$. Since the torus ${}^\vee T$ is connected, the Langlands component groups are trivial.

For the other extreme, consider φ_9. In this case $p(S_9)$ is the long Weyl group element which negates all roots. As in the proof of Lemma 2.1, we write an element in ${}^\vee T$ as $e(c_1\alpha_1 + c_2\alpha_2)$ where $c_1, c_2 \in \mathbb{C}$. Then $e(c_1\alpha_1 + c_2\alpha_2)$ belongs to ${}^\vee T^{p(S_9)}$ if and only if

$$\begin{aligned}
&p(S_9) \cdot (c_1\alpha_1 + c_2\alpha_2) - (c_1\alpha_1 + c_2\alpha_2) \in \ker e = X^*(T) \\
&\iff -2c_1\alpha_1 - 2c_2\alpha_2 \in X^*(T) \\
&\iff c_1, c_2 \in \frac{1}{2}\mathbb{Z}.
\end{aligned}$$

It follows that ${}^\vee T^{p(S_9)}$ is generated by $e(\alpha_1/2)$ and $e(\alpha_2/2)$–a Klein 4-group. We conclude that the Langlands component group of φ_9 is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Now consider φ_4, for which $p(S_4) = s_2$. Arguing as we did for φ_9, we see that $e(c_1\alpha_1 + c_2\alpha_2)$ belongs to ${}^\vee T^{s_2}$ if and only if

$$\begin{aligned} & s_2 \cdot (c_1\alpha_1 + c_2\alpha_2) - (c_1\alpha_1 + c_2\alpha_2) \in \ X^*(T) \\ \iff & c_1(\alpha_1 + \alpha_2) - c_2\alpha_2 - (c_1\alpha_1 + c_2\alpha_2) \in X^*(T) \\ \iff & c_1 - 2c_2 \in \mathbb{Z}. \end{aligned}$$

This implies that

$$\begin{aligned} {}^\vee T^{s_2} &= \{e((2c_2 + \ell)\alpha_1 + c_2\alpha_2) : c_2 \in \mathbb{C},\, \ell \in \mathbb{Z}\} \\ &= \{e(c_2(2\alpha_1 + \alpha_2)) : c_2 \in \mathbb{C}\}. \end{aligned}$$

The group ${}^\vee T^{s_2}$ is the image of the connected space $\mathbb{C}$, and so is itself connected. We conclude that the Langlands component group of φ_4 is trivial.

The remaining cases are $\varphi_5, \varphi_6, \varphi_7$ and φ_8. In each of these cases $p(S_j) \in W$ is a reflection. One may argue as we did for ${}^\vee T^{s_2}$ to prove that ${}^\vee T^{p(S_j)}$ is connected. In summary, the only non-trivial Langlands component group is the Langlands component group of φ_9, which is a Klein 4-group.

2.4 Orbits on flag varieties for regular non-integral infinitesimal character

In this section we consider L-parameters $\varphi = \varphi(y, \lambda)$ as in (21), but remove the assumption of integrality on $\lambda \in {}^\vee\mathfrak{t}$. We merely assume that λ is regular 12. In this more general setting, [ABV92, Section 6] tells us that the equivalence class of φ corresponds to a certain orbit of the the flag variety ${}^\vee\mathrm{G}_2(\lambda)/{}^\vee B(\lambda)$. Here, ${}^\vee\mathrm{G}_2(\lambda)$ is the centralizer of $e(\lambda)$ in G_2, and ${}^\vee B(\lambda)$ is the unique Borel subgroup of ${}^\vee\mathrm{G}_2(\lambda)$ determined by the regular element λ. Our first task is to determine all of the possible groups ${}^\vee\mathrm{G}_2(\lambda)$. Once this is complete, we will find the analogues of ${}^\vee K$ for each ${}^\vee\mathrm{G}_2(\lambda)$ and describe their orbits on ${}^\vee\mathrm{G}_2(\lambda)/{}^\vee B(\lambda)$.

Proposition 2.3. *The group ${}^\vee\mathrm{G}_2(\lambda)$ is isomorphic to one of the following complex algebraic groups:*

$${}^\vee T,\ \mathrm{GL}_2,\ \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2,\ \mathrm{SL}_3,\ \mathrm{G}_2.$$

Proof. By [Hum95, Theorem 2.2 and Theorem 2.11] the centralizer ${}^\vee\mathrm{G}_2(\lambda)$ of $e(\lambda)$ is connected, reductive and equal to

$$\langle {}^\vee T, U_{\pm{}^\vee\alpha} : {}^\vee\alpha(e(\lambda)) = 1\rangle = \langle {}^\vee T, U_{\pm{}^\vee\alpha} : \langle {}^\vee\alpha, \lambda\rangle \in \mathbb{Z}\rangle.$$

The root system of ${}^\vee\mathrm{G}_2(\lambda)$ is

$$R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T) = \{{}^\vee\alpha \in R({}^\vee\mathrm{G}_2, {}^\vee T) : \langle {}^\vee\alpha, \lambda\rangle \in \mathbb{Z}\}.$$

As a subsystem of $R({}^\vee\mathrm{G}_2, {}^\vee T)$, there are only three nonempty possibilities for $R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T)$, namely those of type A_1, $\mathrm{A}_1 \times \mathrm{A}_1$, A_2 and G_2. Writing $\lambda = c_1\alpha_1 + c_2\alpha_2$ for $c_1, c_2 \in \mathbb{C}$, we see that a positive root ${}^\vee\alpha \in R({}^\vee\mathrm{G}_2, {}^\vee T)$ belongs to $R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T)$ if and only if

$$c_1\langle {}^\vee\alpha, \alpha_1\rangle + c_2\langle {}^\vee\alpha, \alpha_2\rangle = m \in \mathbb{Z}.$$

This complex equation is equivalent to the two families of real lines

$$\begin{aligned} & a_1\langle {}^\vee\alpha, \alpha_1\rangle + a_2\langle {}^\vee\alpha, \alpha_2\rangle = m \in \mathbb{Z}, \\ & b_1\langle {}^\vee\alpha, \alpha_1\rangle + b_2\langle {}^\vee\alpha, \alpha_2\rangle = 0, \end{aligned}$$

where $c_1 = a_1+ib_1$ and $c_2 = a_2+ib_2$ are in standard form. One may always take $b_1 = b_2 = 0$ to obtain a point on the second line. We therefore focus on the first family of lines. As m runs over all integers and ${}^\vee\alpha$ runs over all positive roots the first equation tessellates the real plane. By choosing $(a_1, a_2) \in \mathbb{R}^2$ so that it does not lie on any of the lines we obtain λ such that $R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T) = \emptyset$ and ${}^\vee\mathrm{G}_2(\lambda) = {}^\vee T \cong \mathrm{GL}_1 \times \mathrm{GL}_1$. A nonempty root system is given by choosing (a_1, a_2) to lie on a unique line for a fixed $m \in \mathbb{Z}$ and positive root ${}^\vee\alpha$. In this case $R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T) = \{\pm{}^\vee\alpha\}$ and ${}^\vee\mathrm{G}_2(\lambda) = \langle {}^\vee T, U_{\pm{}^\vee\alpha}\rangle$. Setting ${}^\vee\alpha_\perp$ to be the positive root orthogonal to ${}^\vee\alpha$, we see that $\alpha, \alpha_\perp \in X_*({}^\vee T)$ generate ${}^\vee T$. Consequently,

$$ {}^\vee\mathrm{G}_2(\lambda) = \langle {}^\vee T, U_{\pm{}^\vee\alpha}\rangle = \langle \alpha_\perp(\mathbb{C}^\times)\rangle\ \langle \alpha(\mathbb{C}^\times), U_{\pm{}^\vee\alpha}\rangle, $$

a product of commuting groups. Computations similar to those made for ${}^\vee K$ at the end of Section 2.1 reveal that $\langle \alpha_\perp(\mathbb{C}^\times)\rangle \cap \langle \alpha(\mathbb{C}^\times), U_{\pm{}^\vee\alpha}\rangle$ equals $\langle \alpha(-1)\rangle \cong \mu_2$. Hence,

$$ {}^\vee\mathrm{G}_2(\lambda) \cong \mathrm{GL}_1 \times_{\mu_2} \mathrm{SL}_2 \cong \mathrm{GL}_2. $$

Finally, distinct positive roots ${}^\vee\alpha, {}^\vee\beta \in R({}^\vee\mathrm{G}_2, {}^\vee T)$ produce non-parallel lines and so we may choose (a_1, a_2) so that

$$ \begin{aligned} a_1\langle {}^\vee\alpha, \alpha_1\rangle + a_2\langle {}^\vee\alpha, \alpha_2\rangle &= m \\ a_1\langle {}^\vee\beta, \alpha_1\rangle + a_2\langle {}^\vee\beta, \alpha_2\rangle &= \ell \end{aligned} $$

for some $\ell, m \in \mathbb{Z}$. This implies that ${}^\vee\alpha, {}^\vee\beta \in R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T)$. Clearly,

$$ a_1\langle {}^\vee\alpha \pm {}^\vee\beta, \alpha_1\rangle + a_2\langle {}^\vee\alpha \pm^\vee \beta, \alpha_2\rangle = m \pm \ell \in \mathbb{Z}, $$

which implies that the root subsystem of $R({}^\vee\mathrm{G}_2, {}^\vee T)$ generated by ${}^\vee\alpha$ and ${}^\vee\beta$ lies in $R({}^\vee\mathrm{G}_2(\lambda), {}^\vee T)$ [Hum78, 9.4]. The resulting root subsystem is of type $\mathrm{A}_1 \times \mathrm{A}_1, \mathrm{A}_2$ or G_2. As explained at the end of Section 2.1, if the root subsystem is of type $\mathrm{A}_1 \times \mathrm{A}_1$, the group ${}^\vee\mathrm{G}_2(\lambda)$ is isomorphic to ${}^\vee K \cong \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$. If the root subsystem is of type A_2 then

$$ {}^\vee\mathrm{G}_2(\lambda) = \langle {}^\vee T, {}^\vee U_{\pm{}^\vee\alpha_2}, U_{\pm(3{}^\vee\alpha_1+{}^\vee\alpha_2)}\rangle = \langle {}^\vee U_{\pm{}^\vee\alpha_2}, U_{\pm(3{}^\vee\alpha_1+{}^\vee\alpha_2)}\rangle \cong \mathrm{SL}_3. $$

□

Having identified the groups ${}^\vee\mathrm{G}_2(\lambda)$, we may determine the analogues of the group ${}^\vee K$ for ${}^\vee\mathrm{G}_2(\lambda)$. We denote these analogues by ${}^\vee K(\lambda)$. Each ${}^\vee K(\lambda)$ is defined as the centralizer ${}^\vee\mathrm{G}_2(\lambda)^y$ of an element $y \in {}^\vee\mathrm{G}_2(\lambda)$ satisfying $y^2 = e(\lambda)$ [ABV92, Propositions 6.13 and 6.16]. Furthermore, the elements y are only of interest up to conjugation by ${}^\vee\mathrm{G}_2(\lambda)$, and there are finitely many such conjugacy classes. To put it more clearly, let

$$ \mathcal{I}(\lambda) = \{y \in {}^\vee\mathrm{G}_2(\lambda) : y^2 = e(\lambda)\}. $$

The group ${}^\vee\mathrm{G}_2(\lambda)$ acts by conjugation on $\mathcal{I}(\lambda)$ with finitely many orbits, and the relevant groups ${}^\vee K(\lambda)$ are given by taking the centralizer of a representative for each orbit.

Proposition 2.4. *The possibilities for ${}^\vee K(\lambda)$, up to isomorphism, are given in the following table*

${}^\vee\mathrm{G}_2(\lambda)$	${}^\vee K(\lambda)$
${}^\vee T$	${}^\vee T$
GL_2	${}^\vee T$, GL_2
$\mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$	GL_2
SL_3	GL_2, SL_3
${}^\vee\mathrm{G}_2$	$\mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$, ${}^\vee\mathrm{G}_2$

In the row for $\mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$*, the group* GL_2 *is isomorphic to* $\mathrm{SL}_2 \times_{\mu_2} {}^\vee T$*. In the row for* SL_3*, the group* GL_2 *is identified with a block diagonal subgroup of* SL_3*. In all cases,* ${}^\vee T \subset {}^\vee K(\lambda)$*.*

Proof. The final row of the table was established in Lemma 2.1. We therefore assume ${}^\vee \mathrm{G}_2(\lambda) \ncong \mathrm{G}_2$ and offer a sketch of the elementary computations involved in the remaining cases. Let $W(\lambda)$ be the Weyl group of $({}^\vee \mathrm{G}_2(\lambda), {}^\vee T)$. As in the proof of Lemma 2.1, we have a natural bijection between the ${}^\vee \mathrm{G}_2(\lambda)$-orbits of $\mathcal{I}(\lambda)$ and the $W(\lambda)$-orbits of

$$\{y \in {}^\vee T : y^2 = e(\lambda)\}. \tag{23}$$

The groups ${}^\vee K(\lambda)$ we are seeking are the centralizers of representatives of the $W(\lambda)$-orbits. As we are determining ${}^\vee K(\lambda)$ only up to isomorphism, we may ignore the $W(\lambda)$-action. Setting $y' = y\, e(-\lambda/2)$, we may identify (23) with

$$\{y' \in {}^\vee T : (y')^2 = 1\}. \tag{24}$$

For each instance of ${}^\vee \mathrm{G}_2(\lambda)$ it is easy to show that the elements that square to the identity may be taken to be diagonal with ± 1 as diagonal entries. This determines the set (24), and therefore the set (23), on a case-by-case basis. The centralizers of the resulting elements are ascertained using [Hum95, Theorem 2.2 and Theorem 2.11]. We omit the computations. $\square$

Our last objective is to describe the ${}^\vee K(\lambda)$-orbits of the flag variety ${}^\vee \mathrm{G}_2(\lambda)/{}^\vee B(\lambda)$ for each row in the table of Proposition 2.4. We disregard all cases in which ${}^\vee K(\lambda) = {}^\vee \mathrm{G}_2(\lambda)$, as there is only a single orbit in these cases. In view of this and Section 2.1, we need only consider rows 2-4 of the table.

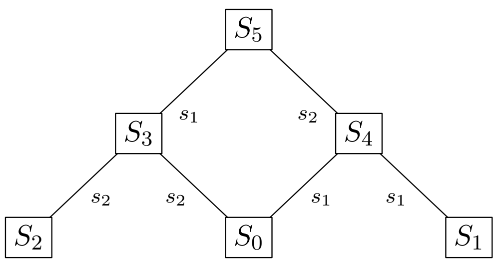
If ${}^\vee \mathrm{G}_2(\lambda) \cong \mathrm{GL}_2$ then ${}^\vee \mathrm{G}_2(\lambda)/{}^\vee B(\lambda)$ is isomorphic to GL_2/B where $B \subset \mathrm{GL}_2$ is the upper-triangular subgroup. The flag variety GL_2/B is isomorphic to the complex projective line $\mathbb{P}^1$ via the map that sends gB to the line spanned by $g \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. If ${}^\vee K(\lambda)$ is isomorphic to the diagonal subgroup ${}^\vee T \subset \mathrm{GL}_2$ then there are three orbits: two orbits correspond to the points $0, \infty \in \mathbb{P}^1$ and the remaining orbit corresponds to $\mathbb{P}^1 - \{0, \infty\}$.

If ${}^\vee \mathrm{G}_2(\lambda) \cong \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$, the flag variety is isomorphic to

$$(\mathrm{GL}_2 \times \mathrm{GL}_2)/(B \times B) \cong \mathrm{GL}_2/B \times \mathrm{GL}_2/B$$

as central subgroups of reductive groups are contained in Borel subgroups. The ${}^\vee K(\lambda)$-orbits may be recovered as in the previous paragraph. There are three orbits for ${}^\vee K(\lambda) \cong \mathrm{GL}_2 \cong \mathrm{SL}_2 \times_{\mu_2} {}^\vee T$.

If ${}^\vee \mathrm{G}_2(\lambda) \cong \mathrm{SL}_3$ then we may argue as in the previous paragraph to take the flag variety equal to GL_3/B, where $B \subset \mathrm{GL}_3$ is the upper-triangular subgroup. We may replace the block diagonal subgroup of SL_3 which is isomorphic to GL_2, with the block diagonal subgroup of GL_3 which is isomorphic to $\mathrm{GL}_2 \times \mathrm{GL}_1$. With these superficial changes, the ${}^\vee K(\lambda)$-orbits of the flag variety may be identified with the $\mathrm{GL}_2 \times \mathrm{GL}_1$-orbits of GL_3/B. The orbits in this example, together with their Bruhat order, are a special case of [Yam97, Section 2.4] (see [Yam97, Figure 1]). Alternatively, we may argue using [AdC09], as we did in Section 2.2. The Hasse diagram for the Bruhat order is



(25)

In this diagram for ${}^\vee\mathrm{G}_2(\lambda) \cong \mathrm{SL}_3$ we have again labelled the orbits with S_j and the simple reflections with s_j, as we did for the Hasse diagram for ${}^\vee\mathrm{G}_2$. This abuse of notation should not cause any confusion as the orbits for the different groups will be treated in different sections.

We conclude by making some general observations about the orbits introduced in this section. The reason the groups ${}^\vee\mathrm{G}_2(\lambda)$ are of interest is that the ${}^\vee K(\lambda)$-orbits on the flag variety of ${}^\vee\mathrm{G}_2(\lambda)$ are in bijection with the equivalence classes of L-parameters for ${}^\vee\mathrm{G}_2$ with infinitesimal character λ [ABV92, Proposition 6.16 and Proposition 6.17]. The Langlands component group of a particular L-parameter is the component group of a centralizer in ${}^\vee K(\lambda)$ of the semisimple element $\lambda \in {}^\vee\mathfrak{t}$ [ABV92, Corollary 5.9 (c)]. The case of ${}^\vee K(\lambda) \cong \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2$ was treated in Section 2.3. In all remaining cases, the derived subgroup of ${}^\vee K(\lambda)$ is simply connected. By a theorem of Steinberg, the centralizer of λ is connected and the Langlands component group is trivial.

3 Complete geometric parameters and the moment map

In this section we return to the general framework of (3) and (4), establishing some more notation and fleshing out some details concerning the moment map. We conclude by computing the characteristic cycle (5) of a particularly simple perverse sheaf on X.

Define the set of complete geometric parameters

$$\Xi\left(X, {}^\vee K\right) = \{(S, \mathcal{V}) : S \in {}^\vee K \backslash X\} \tag{26}$$

to be the set of pairs $\xi = (S, \mathcal{V})$ with S a ${}^\vee K$-orbit on X and $\mathcal{V}$ an irreducible ${}^\vee K$-equivariant local system on S. We are identifying the sets

$$\left\{P(\xi) : \xi \in \Xi\left(X, {}^\vee K\right)\right\}, \quad \left\{D(\xi) : \xi \in \Xi\left(X, {}^\vee K\right)\right\}$$

through the Riemann-Hilbert correspondence. These sets are bases of the Grothendieck group $\mathscr{K}(X, {}^\vee K)$.

As we saw in Section 2.2, there are ten ${}^\vee K$-orbits on X, which we continue to denote somewhat abusively by S_i, $0 \leq i \leq 9$. According to the description of the Langlands component groups given in Section 2.3 and [ABV92, Lemma 7.3 (e)], the trivial local system $\underline{\mathbb{C}}_{S_i}$ is the unique irreducible local system supported in orbits S_i, $0 \leq i \leq 8$. For orbit S_9 the Langlands component group is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The trivial character of the Langlands component group corresponds to the constant sheaf $\underline{\mathbb{C}}_{S_9}$. There are three remaining irreducible non-constant local systems that we denote by $\mathcal{L}_i$, $i = 10, 11, 12$. Therefore, $\Xi(X, {}^\vee K)$ consists of 13 complete geometric parameters

$$\Xi(X, {}^\vee K) \;=\; \{\xi_i = (S_i, \underline{\mathbb{C}}_{S_i}) : 0 \leq i \leq 9\} \cup \{\xi_i = (S_9, \mathcal{L}_i) : i = 10, 11, 12\} \tag{27}$$

and $\mathscr{K}(X, {}^\vee K)$ has a basis given by $\{P(\xi_i) : 0 \leq i \leq 12\}$.

We now specialize the general framework (3) to the setting of Section 2.4. Let $\lambda \in {}^\vee\mathfrak{t}$ be a non-integral regular element. Denote the flag variety by $X(\lambda) = {}^\vee\mathrm{G}_2(\lambda)/{}^\vee B(\lambda)$ and take the group ${}^\vee K(\lambda)$ to be one of those described in Proposition 2.4. Under these conditions the situation is slightly simpler. For any of the pairs $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda))$ described in Proposition 2.4, and any ${}^\vee K(\lambda)$-orbit S on $X(\lambda)$ the only irreducible local system is the trivial local system $\underline{\mathbb{C}}_S$. Consequently, the Grothendieck group $\mathscr{K}(X(\lambda), {}^\vee K(\lambda))$ has a basis given by

$$\{P(\xi_i) : 0 \leq i \leq n\}, \tag{28}$$

where n is the number of ${}^\vee K(\lambda)$-orbits on $X(\lambda)$, and $\xi_i = (S_i, \underline{\mathbb{C}}_{S_i})$.

Our goal is to compute the characteristic cycles (5) of the basis elements (27) and (28). Some well-known properties of these characteristic cycles which narrow the computations are given in the following lemma.

Lemma 3.1 (Lemma 19.14 [ABV92]). *Let S, S' be a pair of ${}^\vee K(\lambda)$-orbits in $X(\lambda)$. Fix an irreducible local system $\mathcal{V}$ on S and write $\xi = (S, \mathcal{V})$. Then $\chi_S^{\mathrm{mic}}(P(\xi))$ is equal to 1 (the rank of the local system $\mathcal{V}$). Moreover, if $\chi_{S'}^{\mathrm{mic}}(P(\xi)) \neq 0$, then*

$$S' \subset \overline{S}.$$

For the remainder of this section we survey the moment map and use it to help compute $CC(P(\xi_9))$ from (27). To simplify the notation we work in the context of an arbitrary connected complex reductive group G and Borel subgroup B. We will take $X = G/B$ and the symmetric subgroup $K \subset G$ to be the fixed-point subgroup of an involutive automorphism of G. Once we have completed our survey, we will specialize to $G = {}^\vee\mathrm{G}_2$, *etc.* as in Section 2.2.

At each point $x \in X$, the differential of the action of G on X, defines a map

$$a_x : \mathfrak{g} \longrightarrow T_x X.$$

Since X is a homogeneous space, the map a_x extends to a surjective morphism

$$a : \mathfrak{g} \times X \longrightarrow TX.$$

The dual of this map

$$a^* : T^*X \longrightarrow \mathfrak{g}^* \times X,$$

is a closed immersion. Composing a^* with the projection onto the first factor yields the *moment map*

$$\mu : T^*X \longrightarrow \mathfrak{g}^*.$$

We have an alternative description of the cotangent bundle T^*X, and therefore have an alternative definition of the moment map. Indeed, let $\mathcal{N}(\mathfrak{g}^*)$ be the nilpotent cone in $\mathfrak{g}^*$. We may identify $X = G/B$ with the variety of Borel subalgebras of $\mathfrak{b}' \subset \mathfrak{g}$ under the map $gB \mapsto \mathrm{Ad}(g)\mathfrak{b}$. As shown in [CG10, Proposition 1.4.11 and Lemma 3.2.2], we have isomorphisms

$$\begin{aligned} T^*X &\cong \{(\mathfrak{b}', \nu) : \mathfrak{b}' \in X,\ \nu \in (\mathfrak{g}/\mathfrak{b}')^*\} \\ &\cong \{(\mathfrak{b}', \nu) \in X \times \mathcal{N}(\mathfrak{g}^*) : \nu \in (\mathfrak{b}')^*\}. \end{aligned} \tag{29}$$

Under this identification the moment map μ is simply the projection

$$(\mathfrak{b}', \nu) \quad \longmapsto \quad \nu,$$

from T^*X to $\mathcal{N}(\mathfrak{g}^*)$ [CG10, Lemma 3.2.3].

Let $\mathfrak{k}$ be the Lie algebra of the symmetric subgroup K. The conormal bundle to the K-action on X is defined as

$$T_K^*X \;=\; \left\{(\mathfrak{b}', \lambda) : \lambda \in T_{\mathfrak{b}'}^*X,\ \lambda(a_{\mathfrak{b}'}(\mathfrak{k})) = 0\right\}.$$

For any K-orbit S in X, the fibre of T_K^*X at $y = \mathfrak{b}' \in S$ is $T_{S,y}^*X$, the conormal bundle to S at y, or equivalently, the annihilator of T_yS in T_y^*X. It follows that

$$T_K^*X \;=\; \bigcup_{S \in K\backslash X} T_S^*X,$$

where $T^*_S X = \cup_{y\in S} T^*_{S,y}X$. Under the identification (29)

$$T^*_S X \;=\; \{(\mathfrak{b}', \nu) : \mathfrak{b}' \in S,\, \nu \in (\mathfrak{g}/(\mathfrak{k}+\mathfrak{b}'))^*\}. \tag{30}$$

The image of $T^*_K X$ under the moment map μ is $\mathcal{N}_{X,K}(\mathfrak{g}^*)$, the cone of nilpotent elements in the intersection

$$(G\cdot(\mathfrak{g}/\mathfrak{b})^*)\cap(\mathfrak{g}/\mathfrak{k})^*.$$

By [KR71] (see also [ABV92, Lemma 20.14]), the group K has finitely many orbits on $\mathcal{N}_{X,K}(\mathfrak{g}^*)$, and for any $\nu \in \mathcal{N}_{X,K}(\mathfrak{g}^*)$

$$\dim(K\cdot\nu) \;=\; \frac{1}{2}\dim(G\cdot\nu).$$

Additionally, by [ABV92, Lemma 20.16], the image $\mu\left(\overline{T^*_S X}\right)$ is the closure of a single orbit of K on $\mathcal{N}_{X,K}(\mathfrak{g}^*)$. When X is the flag variety of ${}^\vee\mathrm{G}_2$, we will provide a detailed description of the conormal bundles in $\mu^{-1}(\mathcal{O})$, for each nilpotent ${}^\vee K$-orbit $\mathcal{O}$ on $\mathcal{N}_{X,{}^\vee K}(\mathfrak{g}_2^*)$ (Proposition 4.6). Understanding the image of μ will play a central role in our computations of characteristic cycles.

Another object that will play a key role in the computation of these characteristic cycles is the associated variety $\mathrm{AV}(P) = \mu(\mathrm{Ch}(P)))$ of an irreducible perverse sheaf P (11). (We could have defined $\mathrm{AV}(P)$ independently of $\mathrm{Ch}(P)$, as in [Vog91]. That the two definitions agree follows from [BB85, Theorem 1.9(c)]). The set $\mathrm{AV}(P)$ is a closed K-invariant subvariety of $\mathcal{N}_{X,K}(\mathfrak{g}^*)$. Since there are finitely many K-orbits on $\mathcal{N}_{X,K}(\mathfrak{g}^*)$, we may write $\mathrm{AV}(P)$ as a finite union of closures of K-orbits of equal dimension

$$\mathrm{AV}(P) \;=\; \overline{\mathcal{O}}_1 \cup \cdots \cup \overline{\mathcal{O}}_n. \tag{31}$$

By [Vog91, Theorem 8.4], there exists a nilpotent G-orbit $\mathcal{O}_G \subset \mathcal{N}(\mathfrak{g}^*)$, such that for each nilpotent K-orbit $\mathcal{O}_i$ appearing in (31), we have

$$G\cdot\mathcal{O}_i = \mathcal{O}_G.$$

Let us say a few words about the G-orbit $\mathcal{O}_G$. Write π for the irreducible $(\mathfrak{g}, K)$-module corresponding to P under the Riemann-Hilbert and Beilinson-Bernstein correspondences (3). Denote by $\mathrm{Ann}(\pi)$ the annihilator of π in the universal enveloping algebra $U(\mathfrak{g})$. By definition, such an ideal is called *primitive*. Associated to $\mathrm{Ann}(\pi)$ is a subvariety $\mathrm{AV}(\mathrm{Ann}(\pi))$ of $\mathcal{N}(\mathfrak{g}^*)$, commonly referred to as the associated variety of $\mathrm{Ann}(\pi)$ [Vog91, (1.4)]. By results of Joseph and Borho-Brylinski (see [CM93, Theorem 10.2.2]), the variety $\mathrm{AV}(\mathrm{Ann}(\pi))$ is the closure $\overline{\mathcal{O}}_G$ of $\mathcal{O}_G$ in $\mathcal{N}(\mathfrak{g}^*)$. Moreover, by the work of Joseph and Barbasch-Vogan (see, for example, [CM93, Theorem 10.3.2]), the orbit $\mathcal{O}_G$ is *special* in the sense of Lusztig–that is, $\mathcal{O}_G$ lies in the image of the Lusztig-Spaltenstein map [Spa82]. Simply put, the associated variety $\mathrm{AV}(P)$ decomposes as a union of closures of nilpotent K-orbits $\mathcal{O}_i$ whose G-saturations $G\cdot\mathcal{O}_i$ are special nilpotent orbits.

We come back to the setting of $G = {}^\vee\mathrm{G}_2$ and give a partial description of the associated varieties of irreducible perverse sheaves on X. The full description will be given in Proposition 4.3 in the next section. To this end, we review the work of Đoković [Đok00], which classifies the nilpotent orbits for groups of type G_2. Let ${}^\vee K$ be as in Section 2.1. There are six nilpotent ${}^\vee K$-orbits: one of dimension six, denoted $\mathcal{O}_6$; two of dimension five, $\mathcal{O}_{5,1}$ and $\mathcal{O}_{5,2}$; one of dimension four, $\mathcal{O}_4$; one of dimension three, $\mathcal{O}_3$; and finally, the zero-dimensional orbit $\mathcal{O}_0$. Among the two five-dimensional nilpotent orbits, the component group of the centralizer of any $N \in \mathcal{O}_{5,1}$ in ${}^\vee K$ is S_3, while the component group of the centralizer of any $N \in \mathcal{O}_{5,2}$ in ${}^\vee K$ is $\mathbb{Z}_2$. According to [Đok00, Theorem 2.1], the closure ordering on these nilpotent ${}^\vee K$-orbits is represented by the following Hasse diagrams

$$
\begin{tikzcd}
6 & & \boxed{\mathcal{O}_6} \arrow[dl, dash] \arrow[dr, dash] & & & \boxed{\mathcal{O}_{6,{}^\vee\mathrm{G}_2}} \arrow[d, dash] & 12 \\
5 & \boxed{\mathcal{O}_{5,1}} \arrow[rr, dashed, dash] \arrow[dr, dash] & & \boxed{\mathcal{O}_{5,2}} \arrow[dl, dash] & & \boxed{\mathcal{O}_{5,{}^\vee\mathrm{G}_2}} \arrow[d, dash] & 10 \\
4 & & \boxed{\mathcal{O}_4} \arrow[d, dash] & & & \boxed{\mathcal{O}_{4,{}^\vee\mathrm{G}_2}} \arrow[d, dash] & 8 \\
3 & & \boxed{\mathcal{O}_3} \arrow[d, dash] & & & \boxed{\mathcal{O}_{3,{}^\vee\mathrm{G}_2}} \arrow[d, dash] & 6 \\
0 & & \boxed{\mathcal{O}_0} & & & \boxed{\mathcal{O}_{0,{}^\vee\mathrm{G}_2}} & 0
\end{tikzcd} \tag{32}
$$

In the diagram on the right, we display the closure ordering among the ${}^\vee\mathrm{G}_2$-saturations of the nilpotent ${}^\vee K$-orbits. The ${}^\vee K$-orbits $\mathcal{O}_{5,1}$ and $\mathcal{O}_{5,2}$ lie in the same ${}^\vee\mathrm{G}_2$-orbit, as indicated by the horizontal dotted line in the diagram on the left. The dimension of each orbit is shown alongside it.

According to the table for type G_2 on [Car85, page 427], only the orbits $\mathcal{O}_{6,{}^\vee\mathrm{G}_2}$, $\mathcal{O}_{5,{}^\vee\mathrm{G}_2}$ and $\mathcal{O}_{0,{}^\vee\mathrm{G}_2}$ are special. These are precisely the orbits whose closures correspond to the associated varieties of certain primitive ideals. Consequently, only the orbits $\mathcal{O}_6$, $\mathcal{O}_{5,1}$, $\mathcal{O}_{5,2}$, and $\mathcal{O}_0$ can appear in the decomposition (31) of the associated variety $\mathrm{AV}(P)$ of a irreducible perverse sheaf P.

We now have sufficient information to determine the associated varieties of the perverse sheaves $P(\xi_i)$, where $\xi_i = (S_i, \underline{\mathbb{C}}_{S_i})$ for $i = 0, 1, 2$ and 9 in (27). By Proposition 2.2, the orbits S_0, S_1, and S_2 are closed and minimal in the Bruhat order. Therefore Lemma 3.1 implies

$$
CC(P(\xi_i)) = \overline{T^*_{S_i} X}, \tag{33}
$$

and

$$
\mathrm{AV}(P(\xi_i)) = \mu\left(\overline{T^*_{S_i} X}\right) \quad i = 0, 1, 2.
$$

Fortunately, these images are computed by Samples. According to [Sam14, Proposition 4.1, tables 1 and 4][1], we have

$$
\mu\left(\overline{T^*_{S_0} X}\right) = \overline{\mathcal{O}}_6, \quad \mu\left(\overline{T^*_{S_1} X}\right) = \overline{\mathcal{O}}_{5,1}, \quad \mu\left(\overline{T^*_{S_2} X}\right) = \overline{\mathcal{O}}_{5,2}. \tag{34}
$$

To treat the perverse sheaf $P(\xi_9)$, we proceed differently. The perverse sheaf $P(\xi_9)$ corresponds to the trivial representation $\pi({}^\vee\xi_9)$ of the split real form ${}^\vee\mathrm{G}_2(\mathbb{R})$ (4) following the correspondences of [Vog83, Proposition 2.7]. The trivial representation belongs to a family of representations commonly denoted by $A_\mathfrak{q}$ [VZ84, Section 2]. In fact, $\pi({}^\vee\xi_9) = A_{{}^\vee\mathfrak{g}_2}$. (see the paragraph following [VZ84, Theorem 2.5]). The associated variety of $A_\mathfrak{q}$ is known [BV83, Proposition 3.4], and yields

$$
\mathrm{AV}(P(\xi_9)) = \mathcal{O}_0. \tag{35}
$$

[1]In Samples' notation orbit our orbit S_1 corresponds a closed orbit labelled with the number one. However, our orbits S_0 and S_2 correspond to closed orbits number two and three, respectively.

We can go further with $P(S_9)$. Since

$$\mu\left(\mathrm{Ch}(P(\xi_9))\right) = \mathrm{AV}(P(\xi_9)) = \mathcal{O}_0,$$

it follows that for every ${}^\vee K$-orbit S on X with $\chi_S^{\mathrm{mic}}(P(\xi_9)) \neq 0$, we have

$$\mu(\overline{T_S^* X}) = \mathcal{O}_0. \tag{36}$$

However, if $S \neq S_9$ and we take the Borel subalgebra $\mathfrak{b}'$ in S then the vector space $T^*_{S,\mathfrak{b}'}X$ is non-zero, and so the projection given by μ is non-zero on T_S^*X (30). In consequence (36) holds only when $S = S_9$, and

$$CC(P(\xi_9)) = \overline{T^*_{S_9}X}.$$

In equation (51) below, we will present an alternative method to obtain this equality.

4 Weyl group actions and the characteristic cycles map

Recall from the introduction that the characteristic cycle map

$$CC : \mathscr{K}(X, {}^\vee K) \to \mathscr{L}(X, {}^\vee K)$$

is W-equivariant. The W-action on the domain stems from the coherent continuation representation, and the W-action on the codomain is due to Hotta and Rossmann. We examine each of these two W-actions in turn and then use them to draw conclusions about characteristic cycles of the perverse sheaves on orbits S_3, S_4 and S_9 (20).

We conclude by returning to the W-action on $\mathscr{L}(X, {}^\vee K)$. The definition of this W-action involves several unknown integers. Using the properties of the characteristic cycles map and the moment map, we are able to give a table of values for the W-action with fewer unknown integers.

4.1 The coherent continuation representation

We begin by describing the coherent continuation representation in the context of $\mathscr{K}(X, {}^\vee K)$ (27). We then introduce two concepts related to the coherent continuation representation that are particularly valuable to us: the tau-invariant, and Harish-Chandra cells. The Harish-Chandra cells are valuable as they provide information about the associated varieties of irreducible perverse sheaves.

Although the definition of the coherent continuation representation is fascinating, we need not present it here. For our computations on ${}^\vee\mathrm{G}_2$ the only information about coherent continuation that we need, is an explicit depiction of the action of the simple reflections in W on the irreducible perverse sheaves on X. Luckily, the Atlas of Lie Groups and Representations software has command called `coherent_irr`, which, upon entering the parameter for an irreducible $({}^\vee\mathfrak{g}_2, {}^\vee K)$-module ${}^\vee\pi$ and a simple root α as input, displays the value of the coherent continuation representation of α applied to ${}^\vee\pi$. The value is a virtual module, displayed as a $\mathbb{Z}$-linear combination of irreducible $({}^\vee\mathfrak{g}_2, {}^\vee K)$-modules. We take the image of this virtual module under the Beilinson-Bernstein and Riemann-Hilbert correspondences ($P(\xi) \leftrightarrow \pi({}^\vee\xi)$ in (4)) to obtain a $\mathbb{Z}$-linear combination of irreducible perverse sheaves on X, *i.e.* an element in $\mathscr{K}(X, {}^\vee K)$. The same procedure applies to ${}^\vee\mathrm{G}_2(\lambda)$ and $X(\lambda)$ as in Section 2.4.

The most elaborate case arises for ${}^\vee\mathrm{G}_2$ and X. The action of the simple reflections on the set of irreducible perverse sheaves $P(\xi_i)$ as in (27) is summarized in the following table. For simplicity, we write P_i instead of $P(\xi_i)$ and s_i instead of $s_{{}^\vee\alpha_i}$

Table 3: $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda)) \;=\; (\mathrm{G}_2, \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2)$

i	$s_1 \cdot P_i$	$s_2 \cdot P_i$
0	$P_0 + P_4$	$P_0 + P_3$
1	$-P_1$	$P_1 + P_3$
2	$P_2 + P_4$	$-P_2$
3	$P_1 + P_3 + P_6$	$-P_3$
4	$-P_4$	$P_2 + P_4 + P_5$
5	$P_4 + P_5 + P_8$	$-P_5$
6	$-P_6$	$P_3 + P_6 + P_7$
7	$P_6 + P_7 + P_9 + P_{10}$	$-P_7$
8	$-P_8$	$P_5 + P_8 + P_9 + P_{11}$
9	$-P_9$	$-P_9$
10	$-P_{10}$	$P_7 + P_{10}$
11	$P_8 + P_{11}$	$-P_{11}$

For any of the other pairs $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda))$ described in Proposition 2.4, the situation is considerably simpler. A description of the coherent continuation action in those cases will be given in Section 6.

We will make use of the notion of *tau-invariant*, following [Vog81, Definition 7.3.8]. For an irreducible $P \in \mathscr{K}(X, {}^\vee K)$, we say that the simple root ${}^\vee\alpha_j$ is in the (Borho-Jantzen-Duflo) tau-invariant $\tau(P)$ of P if

$$s_j \cdot P = -P, \quad j = 1, 2. \tag{37}$$

By Table 3, these tau-invariants are given as follows

$$\begin{aligned} {}^\vee\alpha_1 \in \tau(P(\xi_i)) \quad &\text{if and only if} \quad i \in \{1, 4, 6, 8, 9, 10\}, \\ {}^\vee\alpha_2 \in \tau(P(\xi_i)) \quad &\text{if and only if} \quad i \in \{2, 3, 5, 7, 9, 11\}. \end{aligned} \tag{38}$$

Another notion related to the coherent continuation representation is that of *Harish-Chandra cells.* The notion of Harish-Chandra cells was originally formulated in the setting of $({}^\vee\mathfrak{g}, {}^\vee K)$-modules. However, we identify these modules with perverse sheaves as we did earlier in the discussion on coherent continuation. Harish-Chandra cells partition the set of irreducible perverse sheaves as follows. Suppose $P, P' \in \mathscr{K}(X, {}^\vee K)$ are irreducible. Following the description in [BV83, Definition 2.5], as enabled by [BV83, Proposition 2.6], we write

$$P \underset{LR}{\leq} P'$$

if there exists $w \in W$ such that P appears as a summand in coherent continuation representation of w applied to P' (a $\mathbb{Z}$-linear combination of irreducible perverse sheaves). We write $P \sim P'$ if both $P \underset{LR}{\leq} P'$ and $P' \underset{LR}{\leq} P$ hold. It is straightforward to verify that $\sim$ is an equivalence relation. The equivalence classes in this partition are called Harish-Chandra cells. Of particular significance to us is their connection to the associated varieties (11).

Using Table 3, it is straightforward to verify the following description of the Harish-Chandra cells for the irreducible perverse sheaves on $X = {}^\vee\mathrm{G}_2/{}^\vee B$.

Proposition 4.1. *The Harish-Chandra cells of $\mathscr{K}(X, {}^\vee K)$ are*

$$\begin{aligned}
\mathrm{cell}_0 &= [P_0] \\
\mathrm{cell}_1 &= [P_1, P_3, P_6, P_7, P_{10}] \\
\mathrm{cell}_2 &= [P_2, P_4, P_5, P_8, P_{11}] \\
\mathrm{cell}_3 &= [P_9]
\end{aligned}$$

The relationship between Harish-Chandra cells and associated varieties is described in the following theorem due to Vogan.

Proposition 4.2. *Let P and P' be two irreducible perverse sheaves that belong to the same Harish-Chandra cell. Then their corresponding associated varieties* (11) *coincide*

$$\mathrm{AV}(P) = \mathrm{AV}(P').$$

The combination of Propositions 4.1 and 4.2 leads us to

Proposition 4.3. *The associated varieties of the irreducible perverse sheaves in $\mathscr{K}(X, {}^\vee K)$ are given as follows:*

$$\begin{aligned}
\mathrm{AV}(P(\xi_0)) &= \overline{\mathcal{O}}_6, \\
\mathrm{AV}(P) &= \overline{\mathcal{O}}_{5,1}, \quad P \in \mathrm{cell}_1, \\
\mathrm{AV}(P) &= \overline{\mathcal{O}}_{5,2}, \quad P \in \mathrm{cell}_2, \\
\mathrm{AV}(P(\xi_9)) &= \overline{\mathcal{O}}_0.
\end{aligned}$$

Proof. As established in (34) and (35), we have

$$\mathrm{AV}(P(\xi_0)) = \overline{\mathcal{O}}_6, \quad \mathrm{AV}(P(\xi_1)) = \overline{\mathcal{O}}_{5,1}, \quad \mathrm{AV}(P(\xi_2)) = \overline{\mathcal{O}}_{5,2}, \quad \mathrm{AV}(P(\xi_9)) = \overline{\mathcal{O}}_0.$$

The desired result now follows from Proposition 4.2, □

4.2 The W-action on $\mathscr{L}(X, {}^\vee K)$

We have reviewed the W-action on $\mathscr{K}(X, {}^\vee K)$, the domain of the characteristic cycle map (8). We now turn to the study of the W-action on its codomain, namely the $\mathbb{Z}$-module $\mathscr{L}(X, {}^\vee K)$ (9). As we shall see, there is a compatibility between the W-action and the moment map μ which allows one to construct subquotients of $\mathscr{L}(X, {}^\vee K)$. These subquotients can be characterized using the Springer correspondence and the work of Rossmann. This characterization will be exploited in the proof of Lemma 5.2 below.

There is a natural isomorphism between the formal $\mathbb{Z}$-module $\mathscr{L}(X, {}^\vee K)$ and the top Borel-Moore homology $H^{\mathrm{BM}}_{\mathrm{top}}(T^*_{{}^\vee K}X, \mathbb{Z})$ [Max19, Section 2.2]. By [Ful98, Lemma 19.1.1], the set $\left\{[\overline{T^*_S X}] : S \in {}^\vee K \setminus X\right\}$ of fundamental classes of conormal bundle closures form a basis for the top Borel-Moore homology $H^{\mathrm{BM}}_{\mathrm{top}}(T^*_{{}^\vee K}X, \mathbb{Z})$,

$$H^{\mathrm{BM}}_{\mathrm{top}}(T^*_{{}^\vee K}X, \mathbb{Z}) = \bigoplus_{S \in {}^\vee K \setminus X} \mathbb{Z}\,[\overline{T^*_S X}].$$

We identify $\overline{T^*_S X}$ with $[\overline{T^*_S X}]$ and write $\mathscr{L}(X, {}^\vee K) = H^{\mathrm{BM}}_{\mathrm{top}}(T^*_{{}^\vee K}X, \mathbb{Z})$.

In [Hot85], [Hot84] and [Ros95], the topological construction of the Springer representation is expanded in order to define an action of the Weyl group on the Borel-Moore homology space. According to [Hot85, Lemma 1 §3 and Theorem §4]), this action is graded in the following sense. If $w \in W$ and $\mu\left(\overline{T^*_S X}\right) = \overline{\mathcal{O}}$, then $w \cdot \overline{T^*_S X}$ is a linear combination of conormal bundles $\overline{T^*_{S'} X}$, such that

$$\mu\left(\overline{T^*_{S'} X}\right) \subset \overline{\mathcal{O}}.$$

Let us present the action of the simple reflections on $H^{\mathrm{BM}}_{\mathrm{top}}(T^*_K X, \mathbb{Z})$. As in the previous section it simplifies the presentation to work in the context of an arbitrary connected complex reductive group G and the associated objects $X = G/B$, K *etc.* Let α be a simple root relative to B, $s_\alpha \in W$ be its simple reflection, and consider the natural projection

$$\pi_\alpha : G/B \longrightarrow G/(B \sqcup B s_\alpha B). \tag{39}$$

Following [Hot85, Section 2] and [Tan85, §3, Lemma 2], we say that a K-orbit S is s_α-*vertical* if for every $x \in S$ the intersection

$$\pi_\alpha^{-1}(\pi_\alpha(x)) \cap S,$$

is open and dense in $\pi_\alpha^{-1}(\pi_\alpha(x))$. If this condition is not satisfied, we say that S is s_α-*horizontal.* The following theorem gives a partial description of the action.

Theorem 4.4 ([Hot85, Sections 3 and 4], [Tan85, Proposition 8])**.** *Let S be a K-orbit in X. Fix a simple root $\alpha \in R(B,T)$.*

1. *If S is s_α-vertical, then*

$$s_\alpha \cdot \overline{T^*_S X} \;=\; -\overline{T^*_S X}. \tag{40}$$

2. *If S is s_α-horizontal, then for any s_α-vertical orbit $S' \subset \overline{\pi_\alpha^{-1}(\pi_\alpha(S))}$ with*

$$\mu\left(\overline{T^*_{S'} X}\right) \;\subset\; \mu\left(\overline{T^*_S X}\right).$$

there is a non-negative integer $n_{S,S'}$, such that

$$s_\alpha \cdot \overline{T^*_S X} \;=\; \overline{T^*_S X} \;+\; \sum_{S'} n_{S,S'}\, \overline{T^*_{S'} X}. \tag{41}$$

Although the W-action on Borel-Moore homology has been the subject of extensive study, we do not know of any general method for explicitly determining the action of a simple reflection on the conormal orbits. In Table 5, at the end of the next section, we present an explicit depiction of this action in the case of $X = {}^\vee\mathrm{G}_2/{}^\vee B$.

Before turning to the discussion of the W-equivariance of the characteristic cycle map (8), we briefly elaborate on the relationship between this W-action and the Springer correspondence, as described in [Ros91, Section 4]. Let $N \in \mathfrak{g}$ be a nilpotent element. Denote by $\mathcal{O}_K$ the K-orbit of N, and let $A_K(N)$ and $A_G(N)$ denote the component groups of the centralizers of N in K and G, respectively.

According to (41), the $\mathbb{Z}$-module

$$\bigoplus_{\{S \in K\backslash X : \mu\left(\overline{T^*_S X}\right) \subset \overline{\mathcal{O}}_K\}} \mathbb{Z}\, \overline{T^*_S X}, \tag{42}$$

is W-invariant. Hence, the quotient

$$M(\mathcal{O}_K) := \bigoplus_{S \in K\backslash X : \mu\left(\overline{T^*_S X}\right) \subset \overline{\mathcal{O}}_K} \mathbb{Z}\, \overline{T^*_S X} \Big/ \bigoplus_{S \in K\backslash X : \mu\left(\overline{T^*_S X}\right) \subsetneq \overline{\mathcal{O}}_K} \mathbb{Z}\, \overline{T^*_S X} \tag{43}$$

is naturally a W-module, with basis indexed by the K-orbits S such that $\mu(T_S^*X) = \overline{\mathcal{O}}_K$. By Springer's correspondence, in the formulation of [Ros91, Theorem 4.1], there is a representation of $W \times A_G(N)$ on $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(N), \mathbb{Z})$, which decomposes as a direct sum

$$\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(\eta), \mathbb{Z}) = \bigoplus_{\mu \in \widehat{A_G(N)}} \phi_\mu \otimes \mu, \tag{44}$$

where $\widehat{A_G(N)}$ denotes the set of irreducible $\mathbb{Z}[A_G(N)]$-modules, and ϕ_μ is either zero or an irreducible $\mathbb{Z}[W]$-module.

Since $A_K(N)$ maps to $A_G(N)$, we may consider the space of $A_K(N)$-invariants in $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(N), \mathbb{Z})$, denoted

$$\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(N), \mathbb{Z})^{A_K(N)}.$$

By [Tra05, Theorem 2.9.1], there is a $\mathbb{Z}[W]$-module isomorphism

$$M(\mathcal{O}_K) \cong \mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(N), \mathbb{Z})^{A_K(N)}. \tag{45}$$

It follows that

$$\#\left\{S \in K \setminus X \,:\, \mu(\overline{T_S^*X}) = \overline{\mathcal{O}}_K\right\} = \mathrm{rank}\left(\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(N), \mathbb{Z})^{A_K(N)}\right). \tag{46}$$

(*cf.* [CNT12, Proposition 2.15 and Remark 2.16]).

For groups of type G_2, Carter attributes the computation of the Springer correspondence to Springer himself. Carter displays the description of the W-action on $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(T^*_{{}^\vee K}({}^\vee\mathrm{G}_2/{}^\vee B), \mathbb{Z})$ on [Car85, page 427]. In Section 5 this description will allow us to explicitly describe the W-action on a basis of $\mathrm{H}_{\mathrm{top}}^{\mathrm{BM}}(\mu^{-1}(N), \mathbb{Z})^{A_{{}^\vee K}(N)}$ for N in any nilpotent ${}^\vee K$-orbit $\mathcal{O}$.

4.3 The W-equivariance of the characteristic cycle map

It is time to put the W-actions to good use. We maintain the general setting of G being a connected complex reductive algebraic group as in the previous section. [Tan85, Theorem 1], asserts that taking characteristic cycle map is W-equivariant, *i.e.*

$$CC(w \cdot P) = w \cdot CC(P), \quad P \in \mathscr{K}(X, K),\ w \in W. \tag{47}$$

The W-equivariance of $CC(\cdot)$ is the central tool upon which our method for computing the characteristic cycles of irreducible perverse sheaves on ${}^\vee\mathrm{G}_2/{}^\vee B$ relies.

To simplify the notation, from now on, for each K-orbit S in X and each irreducible perverse sheaf P on X, we write

$$[T_S] = \overline{T_S^*X} \quad \text{and} \quad \chi_S(P) = \chi_S^{\mathrm{mic}}(P).$$

The following result plays an important role in simplifying the computations in the next section.

Proposition 4.5. *Let $P \in \mathscr{K}(X, K)$ be irreducible. Suppose $\alpha \in \tau(P)$ (see (37)). If $\chi_S(P) \neq 0$ then S is s_α-vertical. That is, all conormal bundles appearing in $CC(P)$ have s_α-vertical orbits.*

Proof. We have

$$CC(P) = \sum_{S,\, s_\alpha\text{-vert.}} \chi_S(P)\,[T_S] + \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\,[T_S].$$

Now, since $\alpha \in \tau(P)$, equalities (41) and (47) imply

$$s_\alpha \cdot CC(P) = CC(s_\alpha \cdot P) = \mathrm{Ch}(-P).$$

Therefore

$$\sum_{S,\, s_\alpha\text{-vert.}} \chi_S(P)\, s_\alpha \cdot [T_S] + \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, s_\alpha \cdot [T_S] = -\sum_{S,\, s_\alpha\text{-vert.}} \chi_S(P)\, [T_S] \; - \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, [T_S],$$

and by (40) and (41), we write

$$\begin{aligned}
&\sum_{S,\, s_\alpha\text{-vert.}} -\chi_S(P)\, [T_S] + \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, s_\alpha \cdot [T_S] = -\sum_{S,\, s_\alpha\text{-vert.}} \chi_S(P)\, [T_S] \; - \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, [T_S] \\
&\Longrightarrow \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, s_\alpha \cdot [T_S] = -\sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, [T_S] \\
&\Longrightarrow \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P) \left([T_S] + \sum_{S'} n_{S,S'}\, [T_{S'}] \right) = -\sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P)\, [T_S] \\
&\Longrightarrow \sum_{S,\, s_\alpha\text{-hor.}} \chi_S(P) \left(2[T_S] + \sum_{S'} n_{S,S'}\, [T_{S'}] \right) = 0.
\end{aligned}$$

Therefore, for all s_α-horizontal orbit S, we have $\chi_S(P) = 0$. The result follows. $\square$

Now we return to the setting of $G = {}^\vee\mathrm{G}_2$. Using Proposition 4.5 we can deduce which of the ${}^\vee K$-orbits S are horizontal and which ones are vertical. To see this, recall that by Lemma 3.1 the conormal bundle of orbit S_i appears with multiplicity one in the characteristic cycle of $P(\xi_i)$ for $1 \leq i \leq 9$ (27). Combining this observation with the tau-invariants in (38) and Proposition 4.5, we deduce that

$$\begin{aligned}
&S_i \text{ is } s_1\text{-vertical} \quad \text{if and only if} \quad i \in \{1,4,6,8,9\}, \\
&S_i \text{ is } s_2\text{-vertical} \quad \text{if and only if} \quad i \in \{2,3,5,7,9\}.
\end{aligned} \tag{48}$$

As a warm-up for the computations in the next section, we compute the characteristic cycles of $P(\xi_3)$, $P(\xi_4)$, and $P(\xi_9)$. By the Bruhat ordering relation described in Diagram (20), we have

$$\overline{S_3} = S_0 \cup S_1 \cup S_3 \quad \text{and} \quad \overline{S_4} = S_0 \cup S_2 \cup S_4.$$

Consequently, Lemma 3.1 tells us that

$$\begin{aligned}
CC(P(\xi_3)) &= \chi_{S_0}(P(\xi_3))\, [T_{S_0}] + \chi_{S_1}(P(\xi_3))\, [T_{S_1}] + [T_{S_3}], \\
CC(P(\xi_4)) &= \chi_{S_0}(P(\xi_4))\, [T_{S_0}] + \chi_{S_2}(P(\xi_4))\, [T_{S_2}] + [T_{S_4}].
\end{aligned}$$

Since $s_2 \in \tau(P(\xi_3))$, and S_0 and S_1 are s_2-horizontal, the contrapositive of Proposition 4.5 implies that

$$CC(P(\xi_3)) = [T_{S_3}]. \tag{49}$$

Similarly, since $s_1 \in \tau(P(\xi_4))$, and S_0 and S_2 are s_1-horizontal, we deduce that

$$CC(P(\xi_4)) = [T_{S_4}]. \tag{50}$$

The case of $P(\xi_9)$ is simpler, since $\tau(P(\xi_9)) = \{{}^\vee\alpha_1, {}^\vee\alpha_2\}$, and S_9 is the only orbit that is s_α-vertical for both simple roots. By the contrapositive of Proposition 4.5, we deduce that $\chi_{S_i}(P(\xi_9)) = 0$ for $0 \leq i \leq 8$, that is

$$CC(P(\xi_9)) \;=\; [T_{S_9}]. \tag{51}$$

4.4 Interim progress on the W-action on $\mathscr{L}(X, {}^\vee K)$

Using the W-equivariance of the characteristic cycles map, we are able to provide a better description of the W-action on $\mathscr{L}(X, {}^\vee K)$. The complete description, provided in Table (5), will be obtained alongside the computation of the characteristic cycles of the remaining perverse sheaves $P(\xi_i)$, for $5 \leq i \leq 11$ (27).

Let α_ℓ, $\ell = 1, 2$, be a simple root and recall the projection $\pi_\ell = \pi_{\alpha_\ell}$ (39). For each s_ℓ-horizontal orbit $S \in {}^\vee K\backslash X$, let $S' \in {}^\vee K\backslash X$ denote the unique open orbit in the closure $\overline{\pi_\ell^{-1}(\pi_\ell(S))}$. Then for any other ${}^\vee K$-orbit $S'' \subset \overline{\pi_\ell^{-1}(\pi_\ell(S))}$, we have

$$\dim(S'') \leq \dim(S') - 1. \tag{52}$$

Notice from Table 3 and (38), that if $\alpha_\ell \notin \tau(P(\xi_i))$ for $0 \leq i \leq 6$, then there exists a unique perverse sheaf $P(\xi_j)$ such that $\dim(S_j) = \dim(S_i) + 1$ and $P(\xi_j)$ appears with multiplicity one in $s_\ell \cdot P(\xi_i)$. It follows from the linearity of the characteristic cycle map and Lemma 3.1 that $[T_{S_j}]$ appears with multiplicity one in

$$CC(s_\ell \cdot P(\xi_i)).$$

On the other hand, using equation (41) and (52) we deduce that $[T_{S_j}]$ does not appear in $s_\ell \cdot [T_S]$ for any orbit $S \neq S_i$ with $\chi_S(P(\xi_i)) \neq 0$. By equation (47) we have

$$CC(s_\ell \cdot P(\xi_i)) = s_\ell \cdot CC(P(\xi_i)), \tag{53}$$

we conclude that

$$n_{S_i,S_j} = 1, \quad 0 \leq i \leq 6. \tag{54}$$

In the case of $P(\xi_i)$ for $i = 7, 8$, we see from Table 3 that $[T_{S_9}]$ appears with multiplicity two in $CC(s_\alpha \cdot P(\xi_i))$. As before, equation (52) implies that $[T_{S_9}]$ does not appear in $s_\alpha \cdot [T_S]$ for any orbit S with $\dim(S) < 5$. Once again, using equality (53), we deduce that

$$n_{S_i,S_9} = 2, \quad i = 7, 8. \tag{55}$$

In order to gain more information about further coefficients n_{S_i,S_j} we make use of the moment map μ as it appears in Theorem 4.4. We first determine the image under μ of the conormal bundles to the ${}^\vee K$-orbits on X. The reader should review (32) before reading the following proposition.

Proposition 4.6. *In the setting of Sections 2.2 and 3, the moment map takes the following values*

$$\begin{aligned}
\mu([T_{S_9}]) &= \mathcal{O}_0 \\
\mu([T_{S_8}]) &= \overline{\mathcal{O}}_3 \\
\mu([T_{S_6}]) &= \mu([T_{S_7}]) = \overline{\mathcal{O}}_4 \\
\mu([T_{S_1}]) &= \mu([T_{S_3}]) = \overline{\mathcal{O}}_{5,1} \\
\mu([T_{S_2}]) &= \mu([T_{S_4}]) = \mu([T_{S_5}]) = \overline{\mathcal{O}}_{5,2} \\
\mu([T_{S_0}]) &= \overline{\mathcal{O}}_6.
\end{aligned}$$

Proof. Since S_9 is open in X, it follows from (30) and the definition of moment map, that $\mu([T_{S_9}]) = \mathcal{O}_0$. The statement concerning the image of the conormal bundle corresponding to any of the three closed orbits S_0, S_1 and S_2 has been verified in (34). Moving to the image of $[T_{S_4}]$, we saw in equality (50) that $CC(P(\xi_4)) = [T_{S_4}]$. Consequently, $\mu([T_{S_4}]) = \mu(\mathrm{Ch}(P(\xi_4)))$ and by Propositions 4.1 and 4.3, we obtain

$$\mu([T_{S_4}]) = \mu(\mathrm{Ch}(P(\xi_4))) = \mathrm{AV}(P(\xi_4)) = \overline{\mathcal{O}}_{5,2}.$$

By (49), the same argument proves that $\mu([T_{S_3}]) = \overline{\mathcal{O}}_{5,1}$.

We see that μ sends both $[T_{S_1}]$ and $[T_{S_3}]$ to $\overline{\mathcal{O}}_{5,1}$. We next prove that these are the only two conormal bundles sent to $\overline{\mathcal{O}}_{5,1}$. Let $N_{5,1} \in \mathcal{O}_{5,1}$. According to column three of [Sam14, Table 1], we have

$$A_K(N_{5,1}) = S_3.$$

By the table for type G_2 on [Car85, page 427], we obtain $A_K(N_{5,1}) = A_G(N_{5,1})$. Consequently, by (46) and (44), we can write

$$\#\left\{S \in {}^\vee K \setminus X : \mu([T_S]) = \mathcal{O}_{5,1}\right\} = \operatorname{rank}\left(\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N_{5,1}), \mathbb{Z})^{S_3}\right) = \operatorname{rank} \phi_{\mathrm{triv}},$$

where triv denotes the trivial $\mathbb{Z}[S_3]$-module. As noted in the table for type G_2 on [Car85, page 427], the rank of ϕ_{triv} is two. Hence, $[T_{S_1}]$ and $[T_{S_3}]$ are the only two conormal bundles whose moment map image is equal to $\overline{\mathcal{O}}_{5,1}$.

To determine $\mu([T_{S_8}])$ we appeal directly to the definition of the moment map and identify X with the set of Borel subalgebras of ${}^\vee\mathfrak{g}$. The element S_8 in Table 1 corresponds to the ${}^\vee K$-orbit ${}^\vee K y_8^{-1}\,{}^\vee B$, where $y_8^{-1} = \sigma_1\sigma_2 g_1$. This ${}^\vee K$-orbit is identified with the ${}^\vee K$-orbit of the Borel subalgebra ${}^\vee\mathfrak{b}_8 = \mathrm{Ad}(y_8^{-1})\,{}^\vee\mathfrak{b}$. Set ${}^\vee\mathfrak{t}_8 = \mathrm{Ad}(y_8^{-1})\,{}^\vee\mathfrak{t}$ and ${}^\vee\mathfrak{n}_8 = \mathrm{Ad}(y_8^{-1})\,{}^\vee\mathfrak{n}$ so that ${}^\vee\mathfrak{b}_8 = {}^\vee\mathfrak{t}_8 + {}^\vee\mathfrak{n}_8$. Through equation (30) (see the remark before [BZ08, Definition 2.2]), we deduce

$$\begin{aligned}\mu([T_{S_8}]) &= \overline{{}^\vee K \cdot ({}^\vee\mathfrak{g}_2/({}^\vee\mathfrak{k} + {}^\vee\mathfrak{b}_8))^*} \\ &\cong \overline{{}^\vee K \cdot \left({}^\vee\mathfrak{n}_8^- \cap ({}^\vee\mathfrak{g}_2/{}^\vee\mathfrak{k})\right)}.\end{aligned}$$

Let θ_8 be the involution of ${}^\vee\mathfrak{g}_2$ defined by conjugation by the element S_8 given in Table 1. Then

$$\mathrm{Ad}(y_8)({}^\vee\mathfrak{n}_8^- \cap ({}^\vee\mathfrak{g}_2/{}^\vee\mathfrak{k})) \cong \mathrm{Ad}(y_8)({}^\vee\mathfrak{n}_8^- \cap ({}^\vee\mathfrak{g}_2)^{-\mathrm{Ad}(e(\rho/2))}) = {}^\vee\mathfrak{n}^- \cap ({}^\vee\mathfrak{g}_2)^{-\theta_8}.$$

By Table 2, the simple roots ${}^\vee\alpha_1$ and ${}^\vee\alpha_2$ are, respectively, complex and imaginary noncompact for S_8. The action of θ_8 on the roots is given by the Weyl group element $p(S_8) = s_1s_2s_1s_2s_1$ from Table 1. It is then a straightforward computation to verify that $\theta_8 \cdot {}^\vee\alpha$ is negative for every positive root ${}^\vee\alpha \neq {}^\vee\alpha_2$, and that every positive non-simple root is also complex. It is immediate that for all negative roots ${}^\vee\alpha \neq -{}^\vee\alpha_2$, the root $\theta_8 \cdot {}^\vee\alpha$ is positive, and that for these negative roots ${}^\vee\alpha$

$$X_{{}^\vee\alpha} \notin ({}^\vee\mathfrak{g}_2)^{-\theta_8}$$

for any non-zero root vector $X_{{}^\vee\alpha}$. Consequently, ${}^\vee\mathfrak{n}^- \cap ({}^\vee\mathfrak{g}_2)^{-\theta_8} = \mathbb{C}X_{-{}^\vee\alpha_2}$, and

$$\mathfrak{n}_8^- \cap ({}^\vee\mathfrak{g}_2/{}^\vee\mathfrak{k}) = \mathbb{C}\,(\mathrm{Ad}(y_8^{-1})X_{-{}^\vee\alpha_2}).$$

Moreover, since ${}^\vee\alpha_2$ is a long root, it follows from [Đok00, Table 1] that the orbit ${}^\vee K \cdot X_{-{}^\vee\alpha_2}$ is three-dimensional. This implies that $X_{-{}^\vee\alpha_2}$ belongs to the six-dimensional nilpotent ${}^\vee\mathrm{G}_2$-orbit $\mathcal{O}_{3,{}^\vee\mathrm{G}_2}$. As $y_8 \in {}^\vee\mathrm{G}_2$ the nilpotent element $\mathrm{Ad}(y_8^{-1})X_{-{}^\vee\alpha_2}$ also belongs to $\mathcal{O}_{3,{}^\vee\mathrm{G}_2}$. Since $\mathcal{O}_3$ is the unique nilpotent ${}^\vee K$-orbit such that ${}^\vee\mathrm{G}_2 \cdot \mathcal{O}_3 = \mathcal{O}_{3,{}^\vee\mathrm{G}_2}$, we deduce that ${}^\vee K \cdot (gX_{-{}^\vee\alpha_2}g^{-1}) = \mathcal{O}_3$. This implies $\mu([T_{S_8}]) = \overline{\mathcal{O}}_3$.

On the other hand, according to the table for type G_2 on [Car85, page 427], for any $N_3 \in \mathcal{O}_3$, equality (46) reduces to

$$\#\left\{S \in {}^\vee K \setminus {}^\vee G/{}^\vee B : \mu([T_S]) = \mathcal{O}_3\right\} \;=\; \mathrm{rank}\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N_3), \mathbb{Z})^{\{1\}} \;=\; \mathrm{rank}\phi_{\mathrm{triv}} = 1.$$

Hence, $[T_{S_8}]$ is the unique conormal bundle mapping to $\mathcal{O}_3$ under the moment map.

To conclude, since $S_3 \leq S_6 \leq S_7$ in the weak order closure—that is, the partial order on ${}^\vee K$-orbits obtained by considering only the solid arrows in Diagram 20—it follows from [CNT12, Proposition 2.6], together with the fact that $\overline{\mathcal{O}}_3 \subset \mu([T_S])$ for all $S \neq S_9$, that

$$\overline{\mathcal{O}}_3 \subset \mu([T_{S_7}]) \subset \mu([T_{S_6}]) \subset \mu([T_{S_3}]) = \overline{\mathcal{O}}_{5,1}.$$

The only possibilities for $\mu([T_{S_7}])$ and $\mu([T_{S_6}])$ are $\overline{\mathcal{O}}_3, \overline{\mathcal{O}}_4$ and $\overline{\mathcal{O}}_{5,1}$. However, $[T_{S_1}]$ and $[T_{S_3}]$ are the only conormal bundles whose moment map image equals $\overline{\mathcal{O}}_{5,1}$, and we have just verified that $[T_{S_8}]$ is the unique conormal bundle whose image under the moment map is $\overline{\mathcal{O}}_3$. Therefore

$$\mu([T_{S_6}]) = \mu([T_{S_7}]) = \overline{\mathcal{O}}_4.$$

We may prove that these are the only two conormal bundles whose moment map image is equal to $\overline{\mathcal{O}}_4$ following the argument we gave for $\overline{\mathcal{O}}_3$. Indeed, according to the table for type G_2 on [Car85, page 427], for any $N_4 \in \mathcal{O}_4$, equality (46) gives

$$\#\left\{S \in {}^\vee K \setminus {}^\vee G/{}^\vee B : \mu([T_S]) = \mathcal{O}_4\right\} = \dim \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N_4), \mathbb{Z})^{\{1\}} = \dim \phi_{\mathrm{triv}} = 2.$$

By the process of elimination the conormal bundle of S_5 is taken to $\overline{\mathcal{O}}_{5,2}$ by μ. $\square$

Suppose S is an s_ℓ-horizontal ${}^\vee K$-orbit so that (41) describes the s_ℓ-action on $[T_S]$. Using Table 3, Theorem 4.4.2, equations (54) and (55), and Proposition 4.6, we can rule out some of the conormal bundles from appearing on the right-hand side of (41). For example

$$s_2 \cdot [T_{S_8}] = [T_{S_8}] + \sum_{S'} n_{S_8,S'}\,[T_{S'}]$$

where S' is s_2-vertical and $\mu([T_{S'}]) \subset \mu([T_{S_8}]) = \overline{\mathcal{O}}_3$. It follows from (32) that $\mu([T_{S'}])$ equals $\overline{\mathcal{O}}_0$ or $\overline{\mathcal{O}}_3$. By Proposition 4.6 the only s_2-vertical ${}^\vee K$-orbit satisfying this property is $S' = S_9$. By equation (55) $n_{S_8,S_9} = 2$, so that

$$s_2 \cdot [T_{S_8}] = [T_{S_8}] + 2[T_{S_9}].$$

Arguing in this manner we obtain the following table.

Table 4: $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda)) \;=\; (\mathrm{G}_2, \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2)$

i	$s_1 \cdot [T_{S_i}]$	$s_2 \cdot [T_{S_i}]$
0	$[T_{S_0}] + [T_{S_4}]$	$[T_{S_0}] + [T_{S_3}]$
1	$-[T_{S_1}]$	$[T_{S_1}] + [T_{S_3}]$
2	$[T_{S_2}] + [T_{S_4}]$	$-[T_{S_2}]$
3	$n_{S_3,S_1}[T_{S_1}] + [T_{S_3}] + [T_{S_6}]$	$-[T_{S_3}]$
4	$-[T_{S_4}]$	$n_{S_4,S_2}[T_{S_2}] + [T_{S_4}] + [T_{S_5}]$
5	$n_{S_5,S_4}[T_{S_4}] + T_{S_5} + n_{S_5,S_6}[T_{S_6}] + [T_{S_8}]$	$-[T_{S_5}]$
6	$-[T_{S_6}]$	$[T_{S_6}] + [T_{S_7}]$
7	$n_{S_7,S_6}[T_{S_6}] + [T_{S_7}] + n_{S_7,S_8}[T_{S_8}] + 2[T_{S_9}]$	$-[T_{S_7}]$
8	$-[T_{S_8}]$	$[T_{S_8}] + 2[T_{S_9}]$
9	$-[T_{S_9}]$	$-[T_{S_9}]$

5 Computing characteristic cycles for ${}^{\vee}\mathrm{G}_2$

In this section, we work within the framework of Section 2.2 and complete the description of the characteristic cycles of the irreducible perverse sheaves (27). In view of equations (33), (49)-(51), it remains to treat the cases of the irreducible perverse sheaves $P(\xi_i)$, associated to the parameters

$$\xi_i = \left(S_i, \underline{\mathbb{C}}_{S_i}\right), \; i = 5,6,7,8 \quad \text{and} \quad \xi_i = \left(S_9, \mathcal{L}_i\right), \; i = 10, 11.$$

To simplify notation throughout this section, we denote

$$P_i := P(\xi_i),$$

and continue to write

$$[T_S] := \overline{T^*_S X} \quad \text{and} \quad \chi_S(P) := \chi_S^{\mathrm{mic}}(P).$$

The remaining characteristic cycles are given by the next theorem.

Theorem 5.1.

$$\begin{aligned}
CC(P_5) &= [T_{S_5}] \\
CC(P_6) &= 2[T_{S_1}] \,+\, [T_{S_6}] \\
CC(P_7) &= [T_{S_3}] \,+\, [T_{S_7}] \\
CC(P_8) &= [T_{S_4}] \,+\, [T_{S_6}] \,+\, [T_{S_8}] \\
CC(P_{10}) &= [T_{S_1}] \,+\, [T_{S_6}] \,+\, [T_{S_8}] \,+\, [T_{S_9}] \\
CC(P_{11}) &= [T_{S_2}] \,+\, [T_{S_7}] \,+\, [T_{S_9}].
\end{aligned}$$

The proof of this theorem will occupy the entire section and relies heavily on the W-equivariance of the characteristic cycle map (47). Before we roll up our sleeves and delve into the intricacies of the argument, let us appeal to some results from the previous section to provide a finer description of the cycles in Theorem 5.1 beyond that of Lemma 3.1. These results impose additional restrictions on the conormal bundles that may appear in the characteristic cycle of an irreducible perverse sheaf. The first restriction arises from the relationship between the τ-invariant and the horizontal/vertical nature of the orbits, as seen in Proposition 4.5. The second restriction depends on the structure of the Harish-Chandra cells and our understanding of the associated varieties.

By Lemma 3.1, the microlocal multiplicity $\chi_S(P_i)$ can be nonzero only if S is contained in the support of P_i. Using an argument similar to the one leading to equations (49)-(51), another family of conormal bundles can be removed from $CC(P_i)$. Indeed, Proposition 4.5 implies that if $\alpha_\ell \in \tau(P_i)$ (37) and S is s_ℓ-horizontal, then

$$\chi_S(P_i) = 0.$$

Therefore, from the description of the τ-invariants in (38) and of the vertical orbits in (48), we deduce

$$\begin{aligned}
CC(P_5) &= \chi_{S_2}(P_5)[T_{S_2}] \,+\, \chi_{S_3}(P_5)[T_{S_3}] \,+\, [T_{S_5}] \\
CC(P_6) &= \chi_{S_1}(P_6)[T_{S_1}] \,+\, \chi_{S_4}(P_6)[T_{S_4}] \,+\, [T_{S_6}] \\
CC(P_7) &= \chi_{S_2}(P_7)[T_{S_2}] \,+\, \chi_{S_3}(P_7)[T_{S_3}] \,+\, \chi_{S_5}(P_7)[T_{S_5}] \,+\, [T_{S_7}] \\
CC(P_8) &= \chi_{S_1}(P_8)[T_{S_1}] \,+\, \chi_{S_4}(P_8)[T_{S_4}] \,+\, \chi_{S_6}(P_8)[T_{S_6}] \,+\, [T_{S_8}] \\
CC(P_{10}) &= \chi_{S_1}(P_{10})[T_{S_1}] \,+\, \chi_{S_4}(P_{10})[T_{S_4}] \,+\, \chi_{S_6}(P_{10})[T_{S_6}] \,+\, \chi_{S_8}(P_{10})[T_{S_8}] \,+\, [T_{S_9}] \\
CC(P_{11}) &= \chi_{S_2}(P_{11})[T_{S_2}] \,+\, \chi_{S_3}(P_{11})[T_{S_3}] \,+\, \chi_{S_5}(P_{11})[T_{S_5}] \,+\, \chi_{S_7}(P_{11})[T_{S_7}] + [T_{S_9}].
\end{aligned} \tag{56}$$

These formulas can be further simplified by appealing to the structure of the Harish-Chandra cells described in Proposition 4.1, together with the image of the moment map given in Proposition 4.6. More precisely, since $\mu(\mathrm{Ch}(P_i)) = \mathrm{AV}(P_i)$ the conormal bundles contributing to $CC(P_i)$ must map under μ to subsets of $\mathrm{AV}(P_i)$. There are two simplifying conclusions to be made from this observation.

The first conclusion is a result of the discussion surrounding (32) and reads as follows. If $\mu([T_S]) = \overline{\mathcal{O}}$ and ${}^\vee\mathrm{G}_2 \cdot \mathcal{O}$ is special with ${}^\vee\mathrm{G}_2 \cdot \mathcal{O} \neq {}^\vee\mathrm{G}_2 \cdot \mathrm{AV}(P_i)$ then $\chi_S(P_i) = 0$. For example, by Proposition 4.6, $\mu([T_{S_5}]) = \overline{\mathcal{O}}_{5,2}$ and ${}^\vee\mathrm{G}_2 \cdot \mathcal{O}_{5,2} = \mathcal{O}_{5,{}^\vee\mathrm{G}_2}$ is special. Therefore, by Proposition 4.3, the conormal bundle $\chi_{S_5}(P_i) \neq 0$ only if $P_i \in \mathrm{cell}_2$. The same holds for $[T_{S_2}]$ and $[T_{S_4}]$, the other two conormal bundles whose image under the moment map is also $\overline{\mathcal{O}}_{5,2}$. In other words, for any $P_i \notin \mathrm{cell}_2$, we have

$$0 = \chi_{S_2}(P_i) = \chi_{S_4}(P_i) = \chi_{S_5}(P_i).$$

The same reasoning applies to any conormal bundle whose image under the moment map is $\overline{\mathcal{O}}_{5,1}$. Therefore, the formulas in (56) simplify to

$$\begin{aligned}
CC(P_5) &= \chi_{S_2}(P_5)[T_{S_2}] + [T_{S_5}]\\
CC(P_6) &= \chi_{S_1}(P_6)[T_{S_1}] + [T_{S_6}]\\
CC(P_7) &= \chi_{S_3}(P_7)[T_{S_3}] + [T_{S_7}]\\
CC(P_8) &= \chi_{S_4}(P_8)[T_{S_4}] + \chi_{S_6}(P_8)[T_{S_6}] + [T_{S_8}]\\
CC(P_{10}) &= \chi_{S_1}(P_{10})[T_{S_1}] + \chi_{S_6}(P_{10})[T_{S_6}] + \chi_{S_8}(P_{10})[T_{S_8}] + [T_{S_9}]\\
CC(P_{11}) &= \chi_{S_2}(P_{11})[T_{S_2}] + \chi_{S_5}(P_{11})[T_{S_5}] + \chi_{S_7}(P_{11})[T_{S_7}] + [T_{S_9}].
\end{aligned} \tag{57}$$

The second conclusion to be drawn from the identity $\mu(\mathrm{Ch}(P_i)) = \mathrm{AV}(P_i)$ is that if $\mathrm{AV}(P_i) = \overline{\mathcal{O}}$ then at least one $[T_S]$ appearing in $CC(P_i)$ must satisfy $\mu([T_S]) = \mathcal{O}$. Applying this property to (57) with the aid of Propositions 4.3 and 4.6, we conclude that the microlocal multiplicities

$$\chi_{S_1}(P_6), \quad \chi_{S_3}(P_7), \quad \chi_{S_4}(P_8), \quad \chi_{S_1}(P_{10}), \tag{58}$$

are all nonzero, and that at least one of two values

$$\chi_{S_2}(P_{11}) \quad \text{and} \quad \chi_{S_2}(P_5)$$

is nonzero.

To compute the remaining coefficients, we use the W-equivariance of the characteristic cycle map $CC(\cdot)$, as described in (47), to establish relationships among the microlocal multiplicities. We first consider $CC(P_i)$ for $i = 7, 8, 10$, and 11, and postpone the computation of $CC(P_5)$ and $CC(P_6)$ to the end of this section.

Let us start by looking at row 11 of Table 3. Equality (47) implies

$$s_1 \cdot CC(P_{11}) = CC(s_1 \cdot P_{11}) = CC(P_{11}) + CC(P_8). \tag{59}$$

By Equality (57) and Table 4, the left-hand side of the previous equation is equal to the sum of the right-hand sides of the following three equations

$$\begin{aligned}
\chi_{S_2}(P_{11})(s_1 \cdot [T_{S_2}]) &= \chi_{\mathbf{S_2}}(\mathbf{P_{11}})[\mathbf{T_{S_2}}] + \chi_{S_2}(P_{11})[T_{S_4}]\\
\chi_{S_5}(P_{11})(s_1 \cdot [T_{S_5}]) &= \chi_{\mathbf{S_5}}(\mathbf{P_{11}})[\mathbf{T_{S_5}}] + \chi_{S_5}(P_{11})\left(n_{S_5,S_4}[T_{S_4}] + n_{S_5,S_6}[T_{S_6}] + [T_{S_8}]\right)\\
\chi_{S_7}(P_{11})(s_1 \cdot [T_{S_7}]) &= \chi_{\mathbf{S_7}}(\mathbf{P_{11}})[\mathbf{T_{S_7}}] + \chi_{S_7}(P_{11})\left(n_{S_7,S_6}[T_{S_6}] + n_{S_7,S_8}[T_{S_8}]\right)\\
&\qquad + \mathbf{2} \cdot \chi_{\mathbf{S_7}}(\mathbf{P_{11}})[\mathbf{T_{S_9}}]\\
s_1 \cdot [T_{S_9}] &= -[\mathbf{T_{S_9}}].
\end{aligned}$$

In bold, we have written the constituents of $CC(P_{11})$, which cancel with the right-hand side of (59). Comparing with the right-hand side of (59), we conclude

$$\begin{aligned}\chi_{S_4}(P_8) &= \chi_{S_2}(P_{11}) + \chi_{S_5}(P_{11})n_{S_5,S_4} && (60)\\ \chi_{S_6}(P_8) &= \chi_{S_5}(P_{11})n_{S_5,S_6} + \chi_{S_7}(P_{11})n_{S_7,S_6} && (61)\\ 1 &= \chi_{S_5}(P_{11}) + \chi_{S_7}(P_{11})n_{S_7,S_8} && (62)\\ 1 &= 2\cdot\chi_{S_7}(P_{11}) - 1. && (63)\end{aligned}$$

Since by Table 3 row 11, the action of s_2 on P_{10} is given by $P_{10}+P_7$, the exact same argument to the one leading to equalities (60)-(63), yields the following list of equations

$$\begin{aligned}\chi_{S_3}(P_7) &= \chi_{S_1}(P_{10}) && (64)\\ 1 &= \chi_{S_6}(P_{10}) && (65)\\ 1 &= 2\cdot\chi_{S_8}(P_{10}) - 1. && (66)\end{aligned}$$

Note that four equations appear in (60)- (63), and three equations appear in (64)-(66). The fewer and simpler equations in the latter case are due to the image of the moment map and its relationship with the W-action on the conormal bundles, as described in Table 4. For example, since

$$\mu([T_{S_8}]) = \overline{\mathcal{O}}_3 \subset \overline{\mathcal{O}}_4 = \mu([T_{S_7}]),$$

it follows from (41) that the conormal bundle $[T_{S_7}]$ does not appear in $s_1\cdot[T_{S_8}]$, while $[T_{S_8}]$ may appear in $s_2\cdot[T_{S_7}]$. This results in the asymmetry between Equations (62) and (65).

Looking at equalities (63) and (66), we deduce that

$$\chi_{S_7}(P_{11}) = \chi_{S_8}(P_{10}) = 1. \tag{67}$$

Above, we applied s_2 to P_{10} and s_1 to P_{11}. We now apply s_1 to P_7, and s_2 to P_8. Once again, using the W-equivariance (47) and Table (3), we write

$$\begin{aligned}s_1\cdot CC(P_7) &= CC(P_6) + CC(P_7) + CC(P_9) + CC(P_{10}), && (68)\\ s_2\cdot CC(P_8) &= CC(P_5) + CC(P_8) + CC(P_9) + CC(P_{11}). && (69)\end{aligned}$$

According to (57), the left-hand side of (68) contains a multiple of $s_1\cdot[T_{S_3}]$. We can say more about this term. By (49), we have $CC(P_3) = [T_{S_3}]$. The W-equivariance (47) combined with row three of Table 3 therefore implies

$$\begin{aligned}s_1\cdot[T_{S_3}] = s_1\cdot CC(P_3) &= CC(s_1\cdot P_3)\\ &= CC(P_3) + CC(P_1) + CC(P_6)\\ &= [T_{S_3}] + [T_{S_1}] + (\chi_{S_1}(P_6)[T_{S_1}] + [T_{S_6}]).\end{aligned}$$

The same argument with P_4 and s_2 in place of P_3 and s_1, can be used to obtain

$$s_2\cdot[T_{S_4}] = [T_{S_4}] + [T_{S_2}] + \left(\chi_{S_2}(P_5)[T_{S_2}] + [T_{S_5}]\right). \tag{70}$$

Consequently, by (57) and Table 4, the left-hand side of (68) is the sum of the right-hand sides of the following two equations

$$\begin{aligned}\chi_{S_3}(P_7)(s_1\cdot[T_{S_3}]) &= \chi_{\mathbf{S_3}}(\mathbf{P_7})[\mathbf{T_{S_3}}] + \chi_{S_3}(P_7)[T_{S_1}] + \\ &\qquad \chi_{S_3}(P_7)\left(\chi_{S_1}(P_6)[T_{S_1}] + [T_{S_6}]\right) && (71)\\ s_1\cdot[T_{S_7}] &= [\mathbf{T_{S_7}}] + n_{S_7,S_6}[T_{S_6}] + n_{S_7,S_8}[T_{S_8}] + 2[T_{S_9}]. && (72)\end{aligned}$$

The summands of $CC(P_7)$ are indicated in bold.

Similarly, the left-hand side of (69) is equal to the sum of the right-hand sides of the following list of equations

$$\chi_{S_4}(P_8)(s_2\cdot[T_{S_4}]) = \chi_{\mathbf{S_4}}(\mathbf{P_8})[\mathbf{T_{S_4}}]+\chi_{S_4}(P_8)[T_{S_2}]+ \chi_{S_4}(P_8)\left(\chi_{S_2}(P_5)[T_{S_2}]+[T_{S_5}]\right) \tag{73}$$

$$\chi_{S_6}(P_8)(s_2\cdot[T_{S_6}]) = \chi_{\mathbf{S_6}}(\mathbf{P_8})[\mathbf{T_{S_6}}]+\chi_{S_6}(P_8)[T_{S_7}] \tag{74}$$

$$s_2\cdot[T_{S_8}] = [\mathbf{T_{S_8}}]+2[T_{S_9}].$$

The summands of $CC(P_8)$ are indicated in bold.

Now, by (51), (57) and (67), $[T_{S_8}]$ appears on the right-hand-side of (68) with multiplicity given by $\chi_{S_8}(P_{10})=1$. Meanwhile, from (71) and (72), the conormal bundle $[T_{S_8}]$ appears on the left-hand side of (68) with multiplicity n_{S_7,S_8}. Hence,

$$n_{S_7,S_8}=1,$$

and then from (62) and (67), that

$$\chi_{S_5}(P_{11})=0. \tag{75}$$

To finish with equation (68), we notice that:

1. $[T_{S_1}]$ appears on the right-hand-side of (68), multiplied by $\chi_{S_1}(P_{10})+\chi_{S_1}(P_6)$. Hence, by (71) and (64), we have

   $$\chi_{S_3}(P_7)(1+\chi_{S_1}(P_6))=\chi_{S_1}(P_{10})+\chi_{S_1}(P_6) \quad\Longrightarrow\quad \chi_{S_3}(P_7)\chi_{S_1}(P_6)=\chi_{S_1}(P_6).$$

   By (58) the microlocal multiplicity $\chi_{S_1}(P_6)$ is nonzero and so $1=\chi_{S_3}(P_7)=\chi_{S_1}(P_{10})$.

2. $[T_{S_6}]$ appears on the right-hand-side of (68), multiplied by $1+\chi_{S_6}(P_{10})$. Hence, by (65), (71), (72) and the previous point, we have

   $$\chi_{S_3}(P_7)+n_{S_7,S_6}=1+\chi_{S_6}(P_{10})=2 \quad\Longrightarrow\quad n_{S_7,S_6}=1,$$

Applying the same reasoning to equation (69), we deduce the following:

1. By (75), the conormal bundle $[T_{S_5}]$ appears only once on the right-hand side of (69). Consequently, equation (73) tells us that

   $$\chi_{S_4}(P_8)=1,$$

   and the from (60) and (75), we obtain $\chi_{S_2}(P_{11})=1$.

2. By (67), the conormal bundle $[T_{S_7}]$ appears only once on the right-hand-side of (69). By (74), we therefore obtain

   $$\chi_{S_6}(P_8)=1.$$

Putting all the above equations together, we conclude that:

$$\begin{aligned}CC(P_7)&=[T_{S_3}]+[T_{S_7}]\\ CC(P_8)&=[T_{S_4}]+[T_{S_6}]+[T_{S_8}]\\ CC(P_{10})&=[T_{S_1}]+[T_{S_6}]+[T_{S_8}]+[T_{S_9}]\\ CC(P_{11})&=[T_{S_2}]+[T_{S_7}]+[T_{S_9}].\end{aligned}$$

To complete our computations for Theorem 5.1, it remains to determine $CC(P_5)$ and $CC(P_6)$. Recall from (57) that

$$\begin{aligned} CC(P_5) &= \chi_{S_2}(P_5)[T_{S_2}] \,+\, [T_{S_5}], \\ CC(P_6) &= \chi_{S_1}(P_6)[T_{S_1}] \,+\, [T_{S_6}]. \end{aligned}$$

Using the W-equivariance (47) and tables (3) and (4), we can write

$$\begin{aligned} CC(P_6) \,+\, CC(P_3) \,+\, CC(P_7) &= s_2 \cdot CC(P_6) \\ &= \chi_{S_1}(P_6)([T_{S_1}] \,+\, [T_{S_3}]) \,+\, [T_{S_6}] \,+\, [T_{S_7}] \\ &= CC(P_6) \,+\, CC(P_3) \,+\, (\chi_{S_1}(P_6) - 1)\,[T_{S_3}] \,+\, [T_{S_7}] \\ CC(P_5) \,+\, CC(P_4) \,+\, CC(P_8) &= s_1 \cdot CC(P_5) \\ &= \chi_{S_2}(P_5)([T_{S_2}] \,+\, [T_{S_4}]) \,+\, n_{S_5,S_4}[T_{S_4}] \\ &\qquad + [T_{S_5}] \,+\, n_{S_5,S_6}[T_{S_6}] \,+\, [T_{S_8}] \\ &= CC(P_5) \,+\, CC(P_4) \,+\, (\chi_{S_2}(P_5) + n_{S_5,S_4} - 1)\,[T_{S_4}] \\ &\qquad + n_{S_5,S_6}[T_{S_6}] \,+\, [T_{S_8}]. \end{aligned}$$

Since $\chi_{S_3}(P_7) = \chi_{S_4}(P_8) = \chi_{S_6}(P_8) = 1$, we conclude

$$\begin{aligned} \chi_{S_1}(P_6) - 1 = \chi_{S_3}(P_7) = 1 \quad &\Longrightarrow \quad \chi_{S_1}(P_6) = 2 \\ \chi_{S_2}(P_5) + n_{S_5,S_4} - 1 = \chi_{S_4}(P_8) = 1 \quad &\Longrightarrow \quad \chi_{S_2}(P_5) + n_{S_5,S_4} = 2 \\ n_{S_5,S_6} = \chi_{S_6}(P_8) = 1. & \end{aligned}$$

Therefore

$$\begin{aligned} s_2 \cdot [T_{S_4}] &= (1 + \chi_{S_2}(P_5))[T_{S_2}] + [T_{S_4}] + [T_{S_5}] \\ s_1 \cdot [T_{S_5}] &= (2 - \chi_{S_2}(P_5))[T_{S_4}] + [T_{S_5}] + [T_{S_6}] + [T_{S_8}], \end{aligned} \tag{76}$$

where the first equation is simply a rearrangement of (70), and

$$\begin{aligned} CC(P_5) &= \chi_{S_2}(P_5)[T_{S_2}] \,+\, [T_{S_5}], \quad \text{where} \;\; \chi_{S_2}(P_5) \leq 2, \\ CC(P_6) &= 2[T_{S_1}] \,+\, [T_{S_6}]. \end{aligned} \tag{77}$$

In the following Lemma, we prove that $\chi_{S_2}(P_5) = 0$, which in turn implies that $n_{S_5,S_4} = 2$. This lemma concludes the proof of Theorem 5.1.

Lemma 5.2. *We have*

$$\chi_{S_2}(P_5) = 0.$$

Proof. We use the theory presented in Section 4.2. Let $N_{5,2} \in \mathcal{O}_{5,2}$. From column three of [Sam14, Table 1], we have

$$A_K(N_{5,2}) \cong \mathbb{Z}_2$$

and equation (45) tells us that

$$M(\mathcal{O}_{5,2}) \;\cong\; \mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N_{5,2}), \mathbb{Z})^{\mathbb{Z}_2}. \tag{78}$$

By the table for type G_2 on [Car85, page 427], we obtain

$$A_K(N_{5,2}) \subset A_G(N_{5,2}) \cong \mathcal{S}_3,$$

the symmetric group of order six. The irreducible $\mathbb{Z}[\mathcal{S}_3]$-modules are indexed by the partitions of 3:

$$\psi_{(3)} = \mathrm{triv}, \quad \psi_{(1,1,1)} = \mathrm{sgn}, \quad \psi_{(1,2)},$$

where triv and sgn denote the trivial and sign modules, respectively, and $\psi_{(1,2)}$ is the unique two-dimensional irreducible module. According to the table for type G_2 on [Car85, page 427], the Springer correspondence (44) assigns to the pair $\left(\mu^{-1}(N_{5,2}), \psi_{(3)}\right)$ a two-dimensional irreducible character $\phi_{\psi_{(3)}}$ of W, and to $\left(\mu^{-1}(N_{5,2}), \psi_{(1,2)}\right)$ a one-dimensional irreducible character $\phi_{\psi_{(1,2)}}$ of W. The pair $\left(\mu^{-1}(N_{5,2}), \psi_{(1,1,1)}\right)$, however, is not associated to any irreducible character of W via the Springer correspondence. In summary, the top Borel-Moore homology decomposes as the direct sum

$$\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N_{5,2}), \mathbb{Z}) = \left(\phi_{\psi_{(3)}} \otimes \psi_{(3)}\right) \oplus \left(\phi_{\psi_{(1,2)}} \otimes \psi_{(1,2)}\right).$$

Moreover, since the restriction $\psi_{(1,2)}\big|_{\mathbb{Z}_2} = \mathrm{triv} \oplus \mathrm{sgn}$, by (78), we have

$$\begin{aligned} M(\mathcal{O}_{5,2}) &\cong \left(\left(\phi_{\psi_{(3)}} \otimes \psi_{(3)}\right) \oplus \left(\phi_{\psi_{(1,2)}} \otimes \psi_{(1,2)}\right)\right)^{\mathbb{Z}_2} \\ &\cong \phi_{\psi_{(3)}} \oplus \phi_{\psi_{(1,2)}} \end{aligned} \tag{79}$$

as $\mathbb{Z}(W)$-modules. Consider the projection q from the $\mathbb{Z}$-module (42) to the quotient (43), and let σ denote the $\mathbb{Z}[W]$-module on (43) induced by the W-action on (42). By Proposition 4.6, we have

$$\mu([T_{S_2}]) = \mu([T_{S_4}]) = \mu([T_{S_5}]) = \overline{\mathcal{O}}_{5,2},$$

and so from (79) we deduce that

$$(\langle q([T_{S_2}]), q([T_{S_4}]), q([T_{S_5}])\rangle, \sigma) \cong \phi_{\psi_{(3)}} \oplus \phi_{\psi_{(1,2)}}. \tag{80}$$

Additionally, Table 4 and (76) imply

$$\begin{aligned} &\sigma(s_1) \cdot q([T_{S_2}]) = q([T_{S_2}]) + q([T_{S_4}]) \\ &\sigma(s_2) \cdot q([T_{S_2}]) = -q([T_{S_2}]) \\ &\sigma(s_1) \cdot q([T_{S_4}]) = -q([T_{S_4}]) \\ &\sigma(s_2) \cdot q([T_{S_4}]) = (1 + \chi_{S_2}(P_5))q([T_{S_2}]) + q([T_{S_4}]) + q([T_{S_5}]) \\ &\sigma(s_1) \cdot q([T_{S_5}]) = (2 - \chi_{S_2}(P_5))q([T_{S_4}]) + q([T_{S_5}]) \\ &\sigma(s_2) \cdot q([T_{S_5}]) = -q([T_{S_5}]). \end{aligned} \tag{81}$$

Now, by (80), there exists a $\mathbb{Z}$-linear combination with of $q([T_{S_2}])$, $q([T_{S_4}])$, and $q([T_{S_5}])$ that spans a one-dimensional W-invariant subspace isomorphic to $\phi_{\psi_{(1,2)}}$. Examining the action defined in (81), it is straightforward to verify that

$$\begin{aligned} \sigma(s_1) \cdot ((\chi_{S_2}(P_5) - 2)\, q([T_{S_2}]) + q([T_{S_5}])) &= ((2 - \chi_{S_2}(P_5))\, q([T_{S_2}]) + q([T_{S_5}])), \\ \sigma(s_2) \cdot ((\chi_{S_2}(P_5) - 2)\, q([T_{S_2}]) + q([T_{S_5}])) &= -((2 - \chi_{S_2}(P_5))\, q([T_{S_2}]) + q([T_{S_5}])). \end{aligned}$$

Therefore, the vector

$$v = (\chi_{S_2}(P_5) - 2)\, q([T_{S_2}]) + q([T_{S_5}])$$

spans a rank one $\mathbb{Z}[W]$-module on which s_1 acts trivially and s_2 acts by multiplication by -1. This module corresponds to $\phi_{\psi_{(1,2)}}$. Returning to (80), we obtain the decomposition

$$(\langle q([T_{S_2}]), q([T_{S_4}]), q([T_{S_5}])\rangle, \sigma) \cong (\langle v\rangle, \sigma) \oplus \phi_{\psi_{(3)}}.$$

Following (77), the only possible values for $\chi_{S_2}(P_5)$ are 0, 1, or 2. We will show that any value other than 0 leads to a contradiction. Suppose $\chi_{S_2}(P_5) = 2$ so that

$$\phi_{\psi_{(1,2)}} \cong (\langle v \rangle, \sigma) = (\langle q([T_{S_5}]) \rangle, \sigma)\,.$$

Any complement of $\langle q([T_{S_5}]) \rangle$ is of the form

$$\langle q([T_{S_2}]) + n\, q([T_{S_5}]),\ q([T_{S_4}]) + m\, q([T_{S_5}]) \rangle \quad \text{with } n, m \in \mathbb{Z}. \tag{82}$$

Let us show that the complement cannot be W-invariant. Indeed, by (81), the element

$$\sigma(s_2) \cdot (q([T_{S_4}]) + m\, q([T_{S_5}]))$$

only lies in the submodule (82) when $3n + 2m = 1$, and if this condition is satisfied, then

$$\sigma(s_1) \cdot (q([T_{S_4}]) + m\, q([T_{S_5}]))$$

does not lie in (82).

Similarly, if $\chi_{S_2}(P_5) = 1$, then

$$\phi_{\psi_{(1,2)}} \cong (\langle -q([T_{S_2}]) + q([T_{S_5}]) \rangle, \sigma)\,.$$

Any complement of this submodule is of the form

$$\langle q([T_{S_2}]) + q([T_{S_5}]) + n\, z,\ q([T_{S_4}]) + m\, z \rangle\,, \quad \text{with } n, m \in \mathbb{Z}, \tag{83}$$

where $z = -q([T_{S_2}]) + q([T_{S_5}])$. By (81), we have

$$\begin{aligned}\sigma(s_2) \cdot (q([T_{S_4}]) + mv) &= 2q([T_{S_2}]) + q([T_{S_4}]) + q([T_{S_5}]) - mz \\ &= \frac{3}{2}\,(q([T_{S_2}]) + q([T_{S_5}])) + q([T_{S_4}]) - \frac{1+2m}{2} z.\end{aligned}$$

and this element does not belong to the $\mathbb{Z}$-module (83). We are thus left with the remaining possibility that $\chi_{S_2}(P_5) = 0$. This proves Lemma 5.2. Additionally, we observe that equations (76) reduce to

$$\begin{aligned} s_2 \cdot [T_{S_4}] &= [T_{S_2}] + [T_{S_4}] + [T_{S_5}] \\ s_1 \cdot [T_{S_5}] &= 2[T_{S_4}] + [T_{S_5}] + [T_{S_6}] + [T_{S_8}] \end{aligned}$$

yielding

$$\phi_{\psi_{(1,2)}} \oplus \phi_{\psi_{(3)}} \cong (\langle -2q([T_{S_2}]) + q([T_{S_5}]) \rangle, \sigma) \oplus (\langle q([T_{S_2}]) + q([T_{S_5}]), q([T_{S_4}]) \rangle, \sigma)$$

□

In computing these characteristic cycles, we have determined all the coefficients appearing in equation (41) of Theorem 4.4. Table 4, which describes the W-action of the simple reflections on the conormal bundles to the ${}^\vee K$-orbits in the flag variety $X = {}^\vee \mathrm{G}_2/{}^\vee B$, can now be completed as follows.

Table 5: $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda)) = (\mathrm{G}_2, \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2)$

i	$s_1 \cdot [T_{S_i}]$	$s_2 \cdot [T_{S_i}]$
0	$[T_{S_0}] + [T_{S_4}]$	$[T_{S_0}] + [T_{S_3}]$
1	$-[T_{S_1}]$	$[T_{S_1}] + [T_{S_3}]$
2	$[T_{S_2}] + [T_{S_4}]$	$-[T_{S_2}]$
3	$3[T_{S_1}] + [T_{S_3}] + [T_{S_6}]$	$-[T_{S_3}]$
4	$-[T_{S_4}]$	$[T_{S_2}] + [T_{S_4}] + [T_{S_5}]$
5	$2[T_{S_4}] + [T_{S_5}] + [T_{S_6}] + [T_{S_8}]$	$-[T_{S_5}]$
6	$-[T_{S_6}]$	$[T_{S_6}] + [T_{S_7}]$
7	$[T_{S_6}] + [T_{S_7}] + [T_{S_8}] + 2[T_{S_9}]$	$-[T_{S_7}]$
8	$-[T_{S_8}]$	$[T_{S_8}] + 2[T_{S_9}]$
9	$-[T_{S_9}]$	$-[T_{S_9}]$

In (81), using the identification in (45), we described the action of W on a basis of $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N), \mathbb{Z})^{A_K(N)}$ for $N \in \mathcal{O}_{5,2}$. Similarly, for any nilpotent ${}^\vee K$-orbit $\mathcal{O}$ and $N \in \mathcal{O}$, the action of W on a basis of $\mathrm{H}^{\mathrm{BM}}_{\mathrm{top}}(\mu^{-1}(N), \mathbb{Z})^{A_K(N)}$ can be readily deduced from Table 5. We leave the details to the interested reader.

6 Characteristic cycles in the case of a regular non-integral infinitesimal character

Let $\lambda \in {}^\vee\mathfrak{t}$ be a regular integrally dominant element (1). In the previous section we assumed that λ was integral (1). According to Proposition 2.3, the integrality of λ is equivalent to ${}^\vee\mathrm{G}_2(\lambda) = {}^\vee\mathrm{G}_2$. In this section we assume that λ is *not* integral. By Proposition 2.3, there are four possibilities for ${}^\vee\mathrm{G}_2(\lambda)$:

$$ {}^\vee T,\ \mathrm{GL}_2,\ \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2,\ \mathrm{SL}_3. $$

Write $X(\lambda)$ for the flag variety ${}^\vee\mathrm{G}_2(\lambda)/{}^\vee B(\lambda)$ of ${}^\vee\mathrm{G}_2(\lambda)$. As in Proposition 2.4, we continue to denote ${}^\vee K(\lambda)$ for the analogues of the subgroup group ${}^\vee K \subset {}^\vee\mathrm{G}_2$.

The set $\{P(\xi) : \xi \in \Xi(X(\lambda), {}^\vee K(\lambda))\}$ of ${}^\vee K(\lambda)$-equivariant irreducible perverse sheaves on $X(\lambda)$, was described in (28). The goal of this section is to compute the characteristic cycles of all these irreducible perverse sheaves. This turns out to be an easy exercise, which is a direct consequence of Lemma 3.1 and Proposition 4.5.

Theorem 6.1. *Let P be an irreducible perverse sheaf in $\mathcal{P}(X(\lambda), {}^\vee K(\lambda))$ with support in the ${}^\vee K(\lambda)$-orbit closure $\overline{S}$. Then*

$$ CC(P) = [T_S]. \tag{84} $$

Proof. We provide a proof based on the cases listed in Proposition 2.4. In the case that ${}^\vee K(\lambda) = {}^\vee\mathrm{G}_2(\lambda)$ there is only one ${}^\vee K(\lambda)$-orbit on $X(\lambda)$, and the result follows from Lemma 3.1.

For all other pairs $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda))$, equality (84) ends up being a direct consequence of Proposition 4.5. Among those pairs, the most interesting case is the pair $(\mathrm{SL}_3, \mathrm{GL}_2)$. We provide the details for this case. As orbits S_0, S_1, and S_2 are closed (25), equality (84 follows for these orbits from Lemma 3.1.

As we did in Table 3, in the case of the pair $(\mathrm{G}_2, \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2)$, we summarize in the following table, the coherent continuation representation of the simple reflections on the set

Table 6: $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda)) = (\mathrm{SL}_3, \mathrm{GL}_2)$

i	$s_1 \cdot P_i$	$s_2 \cdot P_i$
0	$P_0 + P_4$	$P_0 + P_3$
1	$-P_1$	$P_1 + P_4$
2	$P_2 + P_3$	$-P_2$
3	$-P_3$	$P_2 + P_3 + P_5$
4	$P_1 + P_4 + P_5$	$-P_4$
5	$-P_5$	$-P_5$

irreducible perverse sheaves $\{P(\xi_i) : 0 \leq i \leq 5\}$ (28). For simplicity, we write P_i instead of $P(\xi_i)$.

Looking at Table 6 and (37), we see that

$$\alpha_1 \in \tau(P(\xi_i)) \quad \text{if and only if} \quad i \in \{1, 3, 5\},$$
$$\alpha_2 \in \tau(P(\xi_i)) \quad \text{if and only if} \quad i \in \{2, 4, 5\}.$$

As a consequence, equality (84) follows for $P(\xi_3)$ and $P(\xi_4)$ by the exact same argument used to prove (49) and (50). The case of $P(\xi_5)$ is similar and follows by the same argument used to obtain (51).

For the remaining pairs described in Proposition 2.4, the situation is less exciting. In these cases, the open orbit S carries only the trivial local system $\underline{\mathbb{C}}_S$, and the action of any simple reflection on $P(\xi)$, where $\xi = (S, \underline{\mathbb{C}}_S)$, satisfies

$$s_\alpha \cdot P(\xi) = -P(\xi).$$

Equality $CC(P(\xi)) = [T_S]$ can then be proved as equality (51). This completes the proof of Theorem 6.1, since all the other orbits are closed, and Lemma 3.1 takes care of Equality (84) for closed orbits. □

As an application of Theorem 6.1, we determine the W-action of the simple reflections on the conormal bundles for the pair $(\mathrm{SL}_3, \mathrm{GL}_2)$. Since the characteristic cycle of every irreducible perverse sheaf reduces to a single conormal bundle, the W-equivariance (47) of the characteristic cycle map implies that the W-action on the conormal bundles follows the same pattern as Table 6. More precisely,

Table 7: $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda)) = (\mathrm{SL}_3, \mathrm{GL}_2)$

i	$s_1 \cdot [T_{S_i}]$	$s_2 \cdot [T_{S_i}]$
0	$[T_{S_0}] + [T_{S_4}]$	$[T_{S_0}] + [T_{S_3}]$
1	$-[T_{S_1}]$	$[T_{S_1}] + [T_{S_4}]$
2	$[T_{S_1}] + [T_{S_3}]$	$-[T_{S_2}]$
3	$-[T_{S_3}]$	$[T_{S_2}] + [T_{S_3}] + [T_{S_5}]$
4	$[T_{S_1}] + [T_{S_4}] + [T_{S_5}]$	$-[T_{S_4}]$
5	$-[T_{S_5}]$	$-[T_{S_5}]$

7 Computing Micro-Packets

Let $\lambda \in {}^\vee\mathfrak{t}$ be a regular integrally dominant element (1). In this section we compute the micro-packets of each of the ${}^\vee K(\lambda)$-orbits S on the flag variety $X(\lambda)$ of ${}^\vee\mathrm{G}_2(\lambda)$. We begin by briefly recalling the definition of the micro-packet corresponding to S as given in the introduction. The definition of characteristic cycles (5) involves a map

$$\chi_S^{\text{mic}} : \mathscr{K}(X, H) \to \mathbb{Z}$$

called the microlocal multiplicity along S (10). The micro-packet Π_S^{mic} is defined in the generalized setting of (3) and (4) as follows. By the Local Langlands Correspondence, to each complete geometric parameter $\xi \in \Xi(X(\lambda), {}^\vee K(\lambda))$ there corresponds an irreducible representation $\pi(\xi)$ of either the split or compact form of G_2. The micro-packet is defined as

$$\Pi_S^{\text{mic}} = \{\pi(\xi) : \chi_S^{\text{mic}}(P(\xi)) \neq 0\}. \tag{85}$$

(*cf.* (6)). For each S there is also a strongly stable distribution given by

$$\eta_S^{\text{mic}} = \sum_{\xi \in \Xi(X(\lambda), {}^\vee K(\lambda))} e(\xi)\, (-1)^{\dim(S_\xi) - \dim(S)}\, \chi_S^{\text{mic}}(P(\xi))\; \pi(\xi)$$

(*cf.* (7)). We shall give an explicit description of these micro-packets and stable distributions.

The most interesting case arises when λ is regular and integral. In this case

$$\left({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda)\right) = ({}^\vee\mathrm{G}_2, {}^\vee K) \cong (\mathrm{G}_2, \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2),$$

and $X(\lambda)$ is the flag variety ${}^\vee\mathrm{G}_2/{}^\vee B$ as in Sections 2.2 and 5 . The microlocal multiplicities along ${}^\vee K$-orbits of $X(\lambda)$ were computed in Theorem 5.1. The following theorem is therefore an immediate consequence of Theorem 5.1 and the definitions of the micro-packets and stable distributions.

Theorem 7.1. *The micro-packet and stable distribution, associated to each* ${}^\vee K$*-orbit of the flag variety of* ${}^\vee\mathrm{G}_2$ *are described in the following table.*

i	$\Pi_{S_i}^{\text{mic}}$	$\eta_{S_i}^{\text{mic}}$
0	$\{\pi(\xi_0)\}$	$\pi(\xi_0)$
1	$\{\pi(\xi_{10}), \pi(\xi_6), \pi(\xi_1)\}$	$\pi(\xi_{10}) + 2\pi(\xi_6) + \pi(\xi_1)$
2	$\{\pi(\xi_{11}), \pi(\xi_2)\}$	$\pi(\xi_{11}) + \pi(\xi_2)$
3	$\{\pi(\xi_7), \pi(\xi_3)\}$	$\pi(\xi_7) + \pi(\xi_3)$
4	$\{\pi(\xi_8), \pi(\xi_4)\}$	$\pi(\xi_8) + \pi(\xi_4)$
5	$\{\pi(\xi_5)\}$	$\pi(\xi_5)$
6	$\{\pi(\xi_{10}), \pi(\xi_8), \pi(\xi_6)\}$	$\pi(\xi_{10}) - \pi(\xi_8) + \pi(\xi_6)$
7	$\{\pi(\xi_{11}), \pi(\xi_7)\}$	$-\pi(\xi_{11}) + \pi(\xi_7)$
8	$\{\pi(\xi_{10}), \pi(\xi_8)\}$	$-\pi(\xi_{10}) + \pi(\xi_8)$
9	$\{\pi(\xi_9), \pi(\xi_{10}), \pi(\xi_{11}), \boldsymbol{\pi(\xi_{12})}\}$	$\pi(\xi_9) + \pi(\xi_{10}) + \pi(\xi_{11}) - \boldsymbol{\pi(\xi_{12})}$

We have written $\pi(\xi_{12})$ *in bold to indicate that it is a representation of the compact pure real form of* G_2*. All other representations are representations of the split real form.*

It is of interest to know which of these micro-packets can be Arthur packets in the sense of [ABV92, Section 22]. We refer the reader to [ABV92, Section 22] for more details in the following investigation. In the setting of G_2, an A-parameter reduces to a homomorphism

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}_2 \to {}^\vee\mathrm{G}_2$$

whose restriction $\psi_{|W_{\mathbb{R}}}$ is a tempered L-parameter, and whose restriction $\psi_{|\mathrm{SL}_2}$ is algebraic. The Arthur packet Π_ψ only depends on ψ up to ${}^\vee\mathrm{G}_2$-conjugacy. The restriction $\psi_{|\mathrm{SL}_2}$

is determined by the value of its differential at the nilpotent element $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathfrak{sl}_2$, and the conjugacy class of this value falls into one of five possible nilpotent orbits [Hum95, Section 6.20, Section 7.18]. Since $\psi(W_{\mathbb{R}})$ commutes with $\psi(\mathrm{SL}_2)$, the image of $\psi_{|W_{\mathbb{R}}}$ lies in the (Levi subgroup of the) centralizer of $\psi(\mathrm{SL}_2)$. These centralizers are well-understood [Car85, Section 13.1] and place restrictions on the nature of ψ.

By definition, each Arthur packet Π_ψ is equal to some micro-packet $\Pi^{\mathrm{mic}}_{S_\ell}$. In order to determine which S_ℓ this is in the present setting, we need to first ensure that the infinitesimal character λ of ψ [ABV92, (22.8)(c)], is dominant, regular and integral (*cf.* Section 2.3). Let S be the ${}^\vee K$-orbit which corresponds to the tempered L-parameter $\psi_{|W_{\mathbb{R}}}$ as in (22). Then it turns out that $\Pi_\psi = \Pi^{\mathrm{mic}}_{S_\ell}$ only if $p(S_\ell) = p(S)$ (see Table 1). We provide a summary of how to determine S_ℓ for each of the five nilpotent orbits mentioned above. The terminology of the headings follows [Hum95, Table 7.18].

Trivial (1) In this case the image of $\psi_{|\mathrm{SL}_2}$ is trivial and we may identify $\psi = \psi_{|W_{\mathbb{R}}}$, a tempered L-parameter of G_2. In this case the Arthur packet is equal to the L-packet of ψ. As a tempered L-parameter with dominant, regular and integral infinitesimal character, the L-packet Π_ψ consists of discrete series [Kna86, Theorem 14.91, Proposition 8.22]. The ${}^\vee K$-orbit S_ℓ corresponding to the L-parameter ψ, as outlined (21)-(22), must therefore have $p(S_\ell)$ equal to the long Weyl group element ([Lan89, proof of Lemma 3.4]). Consulting Table 1 we see that $\ell = 9$ and so $\Pi_\psi = \Pi^{\mathrm{mic}}_{S_9}$.

Long/short root $(A_1/\tilde{A}_1)$ As an example, suppose $\psi(\mathrm{SL}_2)$ is given by the SL_2-triple of the long root $3{}^\vee\alpha_1 + 2{}^\vee\alpha_2$. As the group $\psi(W_{\mathbb{R}})$ centralizes $\psi(\mathrm{SL}_2)$, $\psi(W_{\mathbb{R}})$ lies in the group generated by the SL_2-triple of the orthogonal short root ${}^\vee\alpha_1$. The infinitesimal character of ψ is integral only if the infinitesimal character of $\psi_{|W_{\mathbb{R}}}$ is integral. Since $\psi_{|W_{\mathbb{R}}}$ is also tempered, it must be a discrete parameter. As a discrete parameter with values in a group isomorphic to SL_2, it has infinitesimal character equal to $m\alpha_1$ where m is a positive odd integer. If we write $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ as in Section 2.3, the image of $\psi_{|W_{\mathbb{R}}}(j)$ in the Weyl group is the simple reflection s_1 generated by ${}^\vee\alpha_1$. According to [ABV92, (22.8)(c)], the infinitesimal character of ψ is

$$\lambda = m\alpha_1 + {}^\vee(3{}^\vee\alpha_1 + 2{}^\vee\alpha_2) = m\alpha_1 + (2\alpha_2 + \alpha_1) \in {}^\vee\mathfrak{t}$$

which is not dominant. Conjugation of ψ by the Weyl group element $s_1 s_2$ produces an equivalent A-parameter with dominant infinitesimal character $s_1 s_2 \cdot \lambda$. The infinitesimal character is regular for $m > 1$. Following the outline of (21)-(22), the Weyl group element $p(S_\ell)$ must be equal to the conjugate of s_1 by $s_1 s_2$. This corresponds to orbit S_8, as seen in Table 1. Consequently, $\Pi_\psi = \Pi^{\mathrm{mic}}_{S_8}$.

Similar arguments show that when $\psi(\mathrm{SL}_2)$ is given by the SL_2-triple of a short root, then $\Pi_\psi = \Pi^{\mathrm{mic}}_{S_7}$.

Subregular $({}^\vee\mathrm{G}_2(a_1))$ In this case the identity component of the centralizer of $\psi(\mathrm{SL}_2)$ consists of unipotent elements [Hum95, Table 7.12]. This forces $\psi_{|W_{\mathbb{R}}}(\mathbb{C}^\times) = 1$ and so $\psi_{|W_{\mathbb{R}}}$ has infinitesimal character zero. The infinitesimal character of ψ is then

$$\lambda = d\psi\left(0, \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}\right) \in {}^\vee\mathfrak{t} \tag{86}$$

[ABV92, (22.8)(c)]. The first column of [Hum95, Table 7.12] indicates that ${}^\vee\alpha_1(\lambda) = 0$. Therefore λ is singular and Π_ψ does not equal any of the micro-packets above.

We will need a description of subregular A-parameters without restriction on the infinitesimal character in Section 8, so we may as well do that now. The first column of [Hum95, Table 7.12] also indicates that ${}^\vee\alpha_2(2\lambda) = 2$. This implies

$$\lambda = \alpha_1 + 2\alpha_2 = (2\alpha_1 + 3\alpha_2)/2 + \alpha_2/2, \tag{87}$$

a dominant element in ${}^\vee\mathfrak{t}$. The roots α_2 and $2\alpha_1 + 3\alpha_2$ are orthogonal, and their respective coroots are ${}^\vee\alpha_2$ and $2\,{}^\vee\alpha_1 + {}^\vee\alpha_2$. There is a natural choice for $\psi(1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix})$, namely a product of nontrivial elements in the root subgroups $U_{{}^\vee\alpha_2}$ and $U_{2\,{}^\vee\alpha_1 + {}^\vee\alpha_2}$. With this choice

$$\psi(1, \mathrm{SL}_2) = \langle U_{\pm {}^\vee\alpha_2} \rangle \langle U_{\pm(2\,{}^\vee\alpha_1 + {}^\vee\alpha_2)} \rangle \cong \mathrm{SL}_2 \times_{\mu_2} \mathrm{SL}_2 \tag{88}$$

(*cf.* Section 2.1). This being fixed, the only remaining freedom in choosing ψ is in the value of the semisimple element $\psi_{|W_\mathbb{R}}(j)$. As indicated above, the identity component of the image $\psi(W_\mathbb{R})$ consists of unipotent elements. Therefore the square of $\psi_{|W_\mathbb{R}}(j)$, being both unipotent and semisimple, is trivial. This shows that $\psi_{|W_\mathbb{R}}(j)$ can have order at most two. Let ψ_a be a subregular A-parameter for which $(\psi_a)_{|W_\mathbb{R}}(j) = 1$. The only remaining possibility is a subregular A-parameter for which $\psi_{|W_\mathbb{R}}(j)$ has order two. This semisimple element must then lie outside of the identity component of the centralizer of $\psi(\mathrm{SL}_2)$. [Hum95, Table 7.18] indicates that the component group of the centralizer $\psi(1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix})$ is $\mathcal{S}_3$–the symmetric group of order six. The elements of order two in $\mathcal{S}_3$ form a single conjugacy class, so are all conjugate in the centralizer. One of the elements of order two is the unique nontrivial central element in (88), which is $\exp(\pi i \alpha_2)$. We define ψ_b to be the unique A-parameter with $(\psi_b)_{|W_\mathbb{R}}(j) = \exp(\pi i \alpha_2)$. This exhausts all subregular A-parameters up to conjugacy.

Regular (${}^\vee\mathrm{G}_2$) In this case the entire centralizer of $\psi(\mathrm{SL}_2)$ consists of unipotent elements [Hum95, Tables 7.12 and 7.18]. This forces $\psi_{|W_\mathbb{R}}$ to be trivial. As in the subregular case, the infinitesimal character λ of ψ is given by (86). The first column of [Hum95, Table 7.12] then indicates that ${}^\vee\alpha_1(2\lambda) = {}^\vee\alpha_2(2\lambda) = 2$. [Hum78, Lemma A 13.3] then tells us that $\lambda = \rho$, which is regular, dominant and integral. Following (21)-(22) and [ABV92, (22.8)], the Weyl group element $p(S_\ell)$ is equal to the image of $\psi_{|W_\mathbb{R}}(j) = 1$ in the Weyl group. As the image is trivial, Table 1 indicates that $\Pi_\psi = \Pi^{\mathrm{mic}}_{S_\ell}$ for some $0 \leq \ell \leq 2$. A careful examination of the terms in [ABV92, (22.8)(c)], together with [ABV92, (5.7)(c) and Proposition 6.17], reveal that $\Pi_\psi = \Pi^{\mathrm{mic}}_{S_0}$, where S_0 corresponds to the Langlands parameter of the trivial representation of the split form of G_2. The details are left to the reader.

We note that, according to [AR22, Corollary 4.18] the Arthur packets corresponding to the orbits S_0, S_7, S_8 and S_9 above all fall within the class of packets described by Adams and Johnson in [AJ87, Definition 2.11]. Each such packet is associated with a parabolic subgroup $Q \subset \mathrm{G}_2$, and to each strong real form of the Levi subgroup of Q, there corresponds a unitary character. The packets are then constructed by cohomologically inducing these unitary characters to G_2. That these packets are of Adams-Johnson type may be computed using the `Aq_packet` command in the Atlas of Lie Groups and Representations software.

Using the Atlas software command `is_unitary`, we may verify that each of the remaining micro-packets contains at least one representation that is not unitary. In the next section, we will see that some of these micro-packets are nonetheless useful in describing

Arthur packets with singular infinitesimal character. This occurs for the singular Arthur packet associated to the subregular nilpotent orbit discussed above.

We conclude with the case that λ is regular, but not integral. By (28) and Theorem 6.1, the microlocal multiplicity $\chi_S^{\mathrm{mic}}(P(\xi))$, for any ${}^\vee K(\lambda)$-orbit S in $X(\lambda)$ and $\xi \in \Xi(X(\lambda), {}^\vee K(\lambda))$, is nonzero only when $\xi = (S, \underline{\mathbb{C}}_S)$. Therefore, the micro-packet corresponding to S reduces to its associated L-packet Π_S. This proves the following theorem.

Theorem 7.2. *Suppose $\lambda \in {}^\vee\mathfrak{t}$ is integrally dominant, regular, but non-integral. Then*

$$\Pi_S^{\mathrm{mic}} = \Pi_S.$$

In more familiar notation, Theorem 7.2 asserts $\Pi_S^{\mathrm{mic}} = \Pi_\varphi$, where φ is the L-parameter corresponding to the ${}^\vee K(\lambda)$-orbit S.

8 Micro-packets with singular infinitesimal character

To conclude our description of the micro-packets of G_2, we remove the regularity hypothesis on the infinitesimal character $\lambda \in {}^\vee\mathfrak{t}$. In other words, the integrally dominant (1) element $\lambda \in {}^\vee\mathfrak{t}$ may now satisfy

$${}^\vee\alpha(\lambda) = 0$$

for some roots ${}^\vee\alpha \in R({}^\vee\mathrm{G}_2, {}^\vee T)$. To compute the micro-packets with singular infinitesimal character, we extend the description of the microlocal multiplicities χ_S^{mic} in Sections 5 and 6 to incorporate ${}^\vee K$-orbits S of a *generalized* flag variety of ${}^\vee\mathrm{G}_2$. The main tool for this extension is the *translation principle*, which allows one to transfer results from regular infinitesimal characters to those for singular infinitesimal characters ([Jan79], [Vog81, Chapter 7]). The reader is assumed to be somewhat familiar with the translation principle. We will begin by presenting the geometric perspective of the translation principle, as described in [ABV92, Chapter 8]. Applying the translation principle to Theorems 7.1 and 7.2 will allow us to describe all micro-packets of G_2 with singular infinitesimal character.

Let $\mathcal{O} \subset {}^\vee\mathfrak{g}_2$ be the ${}^\vee\mathrm{G}_2$-orbit of $\lambda \in {}^\vee\mathfrak{t}$ under the adjoint action. The translation principle begins with the existence of a regular element $\lambda' \in {}^\vee\mathfrak{t}$, with ${}^\vee\mathrm{G}_2$-orbit $\mathcal{O}' \subset {}^\vee\mathfrak{g}_2$, and a *translation datum* $\mathcal{T}$ from $\mathcal{O}$ to $\mathcal{O}'$ ([ABV92, Definition 8.6, Lemma 8.7]). Two requirements of a translation datum are that

$$\lambda - \lambda' \in X_*({}^\vee T), \tag{89}$$

and that for every root $\alpha \in R({}^\vee\mathrm{G}_2, {}^\vee T)$

$$\alpha(\lambda) \in \{1, 2, 3, \dots\} \quad \Longrightarrow \quad \alpha(\lambda') \in \{1, 2, 3, \dots\}. \tag{90}$$

Let ${}^\vee P(\lambda)$ be the parabolic subgroup of ${}^\vee\mathrm{G}_2(\lambda)$ given by the root spaces of the roots $\alpha \in R({}^\vee\mathrm{G}_2{}^\vee T)$, satisfying

$$\alpha(\lambda) \in \{0, 1, 2, 3, \dots\}.$$

We define ${}^\vee P(\lambda')$ similarly [ABV92, (6.1), (6.2)]. Due to equations (89) and (90), when λ' is integral, ${}^\vee P(\lambda')$ is the Borel subgroup ${}^\vee B \subset {}^\vee\mathrm{G}_2$ fixed at the beginning of Section 2. Otherwise, ${}^\vee P(\lambda')$ is one of the Borel subgroups ${}^\vee B(\lambda')$ introduced in Section 2.4. In any case, we have

$${}^\vee P(\lambda') \subset {}^\vee P(\lambda).$$

Let $({}^\vee\mathrm{G}_2(\lambda'), {}^\vee K(\lambda'))$ be one of the pairs described in Proposition 2.4. Recall from Section 2.4 that ${}^\vee\mathrm{G}_2(\lambda')$ is the centralizer of $e(\lambda')$ in G_2, and that ${}^\vee K(\lambda')$ is the centralizer of an element $y \in {}^\vee\mathrm{G}_2(\lambda')$ with $y^2 = e(\lambda')$. It follows from (89) and (90) that $e(\lambda) = e(\lambda')$, which in turn implies that the corresponding centralizers coincide, that is

$$ {}^\vee\mathrm{G}_2(\lambda) = {}^\vee\mathrm{G}_2(\lambda') \quad \text{and} \quad {}^\vee K(\lambda) = {}^\vee\mathrm{G}_2(\lambda)^y = {}^\vee K(\lambda'). $$

Despite these equalities, we will persist in writing ${}^\vee\mathrm{G}_2(\lambda)$ and ${}^\vee\mathrm{G}_2(\lambda')$, and likewise ${}^\vee K(\lambda)$ and ${}^\vee K(\lambda')$, so that the reader can easily distinguish when the theory pertains to the singular or to the regular setting.

In keeping with the notation of the previous sections, we denote by $X(\lambda)$ the generalized flag variety ${}^\vee\mathrm{G}_2(\lambda)/{}^\vee P(\lambda)$ of ${}^\vee\mathrm{G}_2(\lambda)$, and by $X(\lambda')$ the flag variety ${}^\vee\mathrm{G}_2(\lambda')/{}^\vee P(\lambda')$. By [ABV92, Proposition 8.8], the surjection

$$ f_{\mathcal{T}} : X(\lambda') \longrightarrow X(\lambda) \tag{91} $$

defines a smooth and proper morphism. In addition, the morphism $f_{\mathcal{T}}$ has connected fibres of fixed dimension d. According to [ABV92, Proposition 7.15], the morphism $f_{\mathcal{T}}$ induces an inclusion

$$ f_{\mathcal{T}}^* : \Xi(X(\lambda), {}^\vee K(\lambda)) \hookrightarrow \Xi(X(\lambda'), {}^\vee K(\lambda)) $$

of complete geometric parameters (26). We refer the reader to [ABV92, page 95] for a detailed description of the image

$$ f_{\mathcal{T}}^*(\xi) = (f_{\mathcal{T}}^* S, f_{\mathcal{T}}^* \tau) $$

of a complete geometric parameter $\xi = (S, \tau) \in \Xi(X(\lambda), {}^\vee K(\lambda))$. We only mention here that, since there are only a finite number of ${}^\vee K(\lambda')$-orbits on $X(\lambda')$, and $f_{\mathcal{T}}^*$ has connected fibres, there is a unique open ${}^\vee K(\lambda')$-orbit in $f_{\mathcal{T}}^{-1}(S)$. The orbit $f_{\mathcal{T}}^* S$ is defined to be this unique open orbit.

We can now introduce the geometric version of the translation functor. As defined in [ABV92, Proposition 8.8(b)], this functor maps the category of ${}^\vee K(\lambda)$–equivariant perverse sheaves on $X(\lambda)$ to the category of ${}^\vee K(\lambda')$–equivariant perverse sheaves on $X(\lambda')$. It is given by the inverse image of the morphism $f_{\mathcal{T}}^*$ in (91), shifted by the relative dimension d

$$ f_{\mathcal{T}}^*[d] : \mathcal{P}\left(X(\lambda), {}^\vee K(\lambda)\right) \longrightarrow \mathcal{P}\left(X(\lambda'), {}^\vee K(\lambda')\right). \tag{92} $$

This is a fully faithful exact functor, sending irreducible perverse sheaves to irreducible perverse sheaves

$$ f_{\mathcal{T}}^*[d]\,(P(\xi)) = P(f_{\mathcal{T}}^*(\xi)), \quad \xi \in \Xi(X(\lambda), {}^\vee K(\lambda)) $$

[ABV92, Proposition 7.15(b)]. Another important property of the translation functor is that it preserves the microlocal multiplicities. Indeed, it is shown in [ABV92, Proposition 20.1 (e)] that for all $\xi \in \Xi(X(\lambda), {}^\vee K(\lambda))$ and for all ${}^\vee K(\lambda)$-orbits S in $X(\lambda)$, we have

$$ \begin{aligned} \chi_S^{\mathrm{mic}}(P(\xi)) &= \chi_{f_{\mathcal{T}}^* S}^{\mathrm{mic}}(f_{\mathcal{T}}^* P(\xi)) \\ &= \chi_{f_{\mathcal{T}}^* S}^{\mathrm{mic}}(P(f_{\mathcal{T}}^* \xi)). \end{aligned} \tag{93} $$

As a consequence of (93), we can write

$$ CC(P(\xi)) = \sum_S \chi_{f_{\mathcal{T}}^* S}^{\mathrm{mic}}(P(f_{\mathcal{T}}^* \xi)) \cdot \overline{T_S^* X(\lambda)} $$

(*cf.* (5)). In addition, by the definition of a micro-packet (85), we have

$$\pi(\xi) \in \Pi_S^{\mathrm{mic}} \quad \Longleftrightarrow \quad \pi(f_{\mathcal{T}}^{*}\xi) \in \Pi_{f_{\mathcal{T}}^{*}S}^{\mathrm{mic}}. \tag{94}$$

To complete the description of the micro-packets of G_2, we need to relate the geometric translation functor (92) with its representation-theoretic counterpart. The (Jantzen–Zuckerman) translation functor in representation theory may be regarded as a homomorphism

$$\Psi_{\mathcal{T}} : \ K\Pi\left(\lambda', \mathrm{G}_2/\mathbb{R}\right) \longrightarrow K\Pi\left(\lambda, \mathrm{G}_2/\mathbb{R}\right),$$

from the Grothendieck group of representations of pure real forms of G_2 with infinitesimal character λ', to that of pure real forms of G_2 with infinitesimal character λ. The functor $\Psi_{\mathcal{T}}$ is surjective (see, for example, [AvLTV20, Corollary 17.9.8]). [ABV92, Proposition 16.4(b)] tells us that for any complete geometric parameter $\gamma \in \Xi(X(\lambda'), {}^{\vee}K(\lambda'))$ the image $\Psi_{\mathcal{T}}(\pi(\gamma))$ is either irreducible or zero—the former occurring if and only if $\gamma = f_{\mathcal{T}}^{*}(\xi)$ for some $\xi \in \Xi(X(\lambda), {}^{\vee}K(\lambda))$. By [ABV92, Proposition 16.6], we have the identity

$$\Psi_{\mathcal{T}}\left(\pi(f_{\mathcal{T}}^{*}(\xi))\right) = \pi(\xi). \tag{95}$$

An immediate consequence of this identity and the definition of the L-packet Π_S is

$$\Pi_S = \left\{ \Psi_{\mathcal{T}}(\pi) : \pi \in \Pi_{f_{\mathcal{T}}^{*}S}, \ \Psi_{\mathcal{T}}(\pi) \neq 0 \right\}. \tag{96}$$

Identity (95) applies to micro-packets as well. Indeed, the equivalence of (94) implies that the image of the micro-packet of $f_{\mathcal{T}}^{*}S$ under the translation functor is equal to the micro-packet of S

$$\Pi_S^{\mathrm{mic}} \ = \ \left\{ \Psi_{\mathcal{T}}(\pi) : \ \pi \in \Pi_{f_{\mathcal{T}}^{*}S}^{\mathrm{mic}}, \ \Psi_{\mathcal{T}}(\pi) \neq 0 \right\}. \tag{97}$$

The parallel statement for the strongly stable distributions is

$$\begin{aligned} \eta_S^{\mathrm{mic}} &= \sum_{\xi \in \Xi(X(\lambda), {}^{\vee}K(\lambda))} e(\xi)(-1)^{\dim(S_\xi) - \dim(S)} \chi_S^{\mathrm{mic}}(P(\xi))\, \pi(\xi) \\ &= \sum_{\xi \in \Xi(X(\lambda), {}^{\vee}K(\lambda))} e(\xi)(-1)^{\dim(f_{\mathcal{T}}^{*}S_\xi) - \dim(f_{\mathcal{T}}^{*}S)} \chi_{f_{\mathcal{T}}^{*}S}^{\mathrm{mic}}(P(f_{\mathcal{T}}^{*}\xi)) \ \Psi_{\mathcal{T}}(\pi(f_{\mathcal{T}}^{*}(\xi))) \\ &= \Psi_{\mathcal{T}}(\eta_{f_{\mathcal{T}}^{*}S}^{\mathrm{mic}}). \end{aligned} \tag{98}$$

By Theorems 7.1 and 7.2, we know how to compute $\Pi_{S'}^{\mathrm{mic}}$ for all ${}^{\vee}K(\lambda')$-orbits S' in $X(\lambda')$. Applying the translation functors as above, the only difficulty in computing Π_S^{mic} and η_S^{mic} lies in recognizing $f_{\mathcal{T}}^{*}S$ among the ${}^{\vee}K(\lambda')$-orbits in $X(\lambda')$. Identity (95) converts the problem of recognizing $f_{\mathcal{T}}^{*}S$ among the ${}^{\vee}K(\lambda')$-orbits into the equivalent problem of computing the representation-theoretic translation functor $\Psi_{\mathcal{T}}$. The Atlas of Lie Groups and Representations software allows one to compute the image of an irreducible or standard representation under $\Psi_{\mathcal{T}}$ using the commands `T_irr` and `T_std`. One may use these commands to identify the orbits $f_{\mathcal{T}}^{*}S$.

One does not need software for these computations. As an example, we sketch the computation of $\Pi_\psi = \Pi_{S_\psi}^{\mathrm{mic}}$ for ψ an A-parameter of G_2 whose restriction to SL_2 is determined by the subregular nilpotent orbit as presented on page 40. Following this presentation, there are two unipotent A-parameters, ψ_a and ψ_b, of G_2 that satisfy

$$(\psi_a)_{|\mathrm{SL}_2} = (\psi_b)_{|\mathrm{SL}_2}.$$

Here, ψ_a denotes the A-parameter whose restriction to $W_{\mathbb{R}}$ is trivial, and ψ_b denotes the unique A-parameter such that $(\psi_b)_{|W_{\mathbb{R}}}(j) = \exp(\pi i \alpha_2)$. Let λ be as in (87). We may then choose $\lambda' = \rho$, the half-sum of the positive roots in $R(\mathrm{G}_2, T)$, and use the translation datum $\mathcal{T}$ from $\mathcal{O}$ to $\mathcal{O}'$. The parabolic subgroup ${}^\vee P(\lambda')$ is the Borel subgroup ${}^\vee B$. Since ${}^\vee\alpha_1(\lambda) = 0$, the parabolic subgroup ${}^\vee P(\lambda)$ is given by

$$^\vee P(\lambda) = {}^\vee B \sqcup {}^\vee B \sigma_1 {}^\vee B = \langle U_{\pm^\vee\alpha_1}\rangle\, {}^\vee B,$$

where σ_1 is the Tits representative (16) of the reflection of ${}^\vee\alpha_1$, and $U_{^\vee\alpha_1}$ is the root subgroup of ${}^\vee\alpha_1$. The prescriptions of [ABV92, (22.8)(c), (5.7)(c) and Proposition 6.17] imply that

$$S_{\psi_a} = {}^\vee K(\lambda)\, {}^\vee P(\lambda) = {}^\vee K(\lambda)\, \langle U_{\pm^\vee\alpha_1}\rangle\, {}^\vee B$$

The column labelled with y_j in Table 1 indicates that $f_{\mathcal{T}}(S_0) = f_{\mathcal{T}}(S_2) = S_{\psi_a}$. Moreover, since $g_1 \in \langle U_{\pm^\vee\alpha_1}\rangle$ the table shows that $f_{\mathcal{T}}(S_4) = S_{\psi_a}$. In summary, Table 1 shows that

$$f_{\mathcal{T}}^{-1}(S_{\psi_a}) = S_0 \cup S_2 \cup S_4 \quad \text{and} \quad f_{\mathcal{T}}^{*}(S_{\psi_a}) = S_4.$$

We may now substitute the information from Theorem 7.1 into equations (97) and (98) to conclude that

$$\Pi^{\mathrm{mic}}_{S_{\psi_a}} = \{\pi(f_{\mathcal{T}}^*(\xi_8)), \pi(f_{\mathcal{T}}^*(\xi_4))\} \quad \text{and} \quad \eta^{\mathrm{mic}}_{S_{\psi_a}} = \pi(f_{\mathcal{T}}^*(\xi_8)) + \pi(f_{\mathcal{T}}^*(\xi_4)).$$

Similar considerations with ψ_b bring to light that

$$S_{\psi_b} = {}^\vee K(\lambda)\sigma_2^{-1}\, {}^\vee P(\lambda) = {}^\vee K(\lambda)\sigma_2^{-1}\, \langle U_{\pm^\vee\alpha_1}\rangle\, {}^\vee B,$$
$$f_{\mathcal{T}}^{-1}(S_{\psi_b}) = S_1, \quad f_{\mathcal{T}}^{*}(S_{\psi_b}) = S_1.$$

Substituting Theorem 7.1 into (97) and (98), we obtain

$$\Pi^{\mathrm{mic}}_{S_{\psi_b}} = \{\pi(f_{\mathcal{T}}^*(\xi_{10})), \pi(f_{\mathcal{T}}^*(\xi_6)), \pi(f_{\mathcal{T}}^*(\xi_1))\} \quad \text{and} \quad \eta^{\mathrm{mic}}_{S_{\psi_b}} = \pi(f_{\mathcal{T}}^*(\xi_{10})) + 2\pi(f_{\mathcal{T}}^*(\xi_6)) + \pi(f_{\mathcal{T}}^*(\xi_1)).$$

The stable sums $\eta^{\mathrm{mic}}_{S_{\psi_a}}$ and $\eta^{\mathrm{mic}}_{S_{\psi_b}}$ are the ones appearing in [Vog94, Theorem 18.10]. However, Vogan uses entirely different methods.

We finish this section by supposing that λ is non-integral. In this case it follows from (89) that λ' is also non-integral. By Theorem 7.2, the micro-packets with infinitesimal character λ' are equal to their corresponding L-packets. According to equations (96) and (97), the same can be said for any micro-packet Π^{mic}_S for S a ${}^\vee K(\lambda)$-orbit in $X(\lambda)$.

9 Kashiwara's local index theorem

We continue working in the setting of Section 8. As an application of Theorem 5.1, we provide an explicit description of Kashiwara's local index formula (13) stated in the introduction.

We give more details to the notion of *local multiplicity* along a ${}^\vee K(\lambda)$-orbit in $X(\lambda)$ as in [ABV92, Definition 23.6]. To this end, fix a ${}^\vee K(\lambda)$-orbit $S \subset X(\lambda)$ and a point $x \in S$. For any constructible sheaf $\mathcal{C}$ of finite-dimensional complex vector spaces on $X(\lambda)$, we denote by $\mathcal{C}_x$ the stalk of $\mathcal{C}$ at x. Let $\mathcal{C}(X(\lambda), {}^\vee K(\lambda))$ be the category of ${}^\vee K(\lambda)$-equivariant constructible sheaves on $X(\lambda)$. The map

$$\chi^{\mathrm{loc}}_S : \mathcal{C}(X(\lambda), {}^\vee K(\lambda)) \longrightarrow \mathbb{N},$$

defined by $\chi_S^{\mathrm{loc}}(\mathcal{C}) = \dim(\mathcal{C}_x)$, is independent of the choice of $x \in S$, and is additive with respect to short exact sequences. It extends to a $\mathbb{Z}$-linear map

$$\chi_S^{\mathrm{loc}} : \mathscr{K}(X(\lambda), {}^\vee K(\lambda)) \longrightarrow \mathbb{Z},$$

which we refer to as the *local multiplicity* along S [ABV92, Definition 1.28, Lemma 7.8].

There is a close relationship between the values of χ_S^{loc} on irreducible perverse sheaves and the values on irreducible constructible sheaves. To see this, let $\xi = (S_\xi, \mathcal{V}_\xi) \in \Xi(X(\lambda), {}^\vee K(\lambda))$ be a complete geometric parameter. Consider the irreducible constructible sheaf

$$\mu(\xi) = i_* j_! \mathcal{V}_\xi,$$

where $j \colon S \hookrightarrow \overline{S}$ is the inclusion of S into its closure, and $i \colon \overline{S} \hookrightarrow X(\lambda)$ is the embedding of $\overline{S}$ into X. By [ABV92, Corollary 23.3(a)-(b)], we have

$$\chi_S^{\mathrm{loc}}(\mu(\xi)) = \dim((\mathcal{V}_\xi)_x) = 1,$$

where $x \in S_\xi$. By [ABV92, (7.11)(a)], any irreducible perverse sheaf $P(\gamma)$, $\gamma \in \Xi(X(\lambda), {}^\vee K(\lambda))$ decomposes as a $\mathbb{Z}$-linear combination

$$P(\gamma) = \sum_{\xi \in \Xi(X(\lambda), {}^\vee K(\lambda))} (-1)^{\dim(S_\xi)}\ c_g(\xi, \gamma)\, \mu(\xi)$$

in $\mathscr{K}(X(\lambda), {}^\vee K(\lambda))$. Applying χ_S^{loc} to this equation, we see that

$$\chi_S^{\mathrm{loc}}(P(\gamma)) = \sum_{\xi \in \Xi(X(\lambda), {}^\vee K(\lambda))} (-1)^{\dim(S_\xi)}\ c_g(\xi, \gamma)\, \chi_S^{\mathrm{loc}}(\mu(\xi)).$$

[ABV92, Theorem 16.19] tells us that the coefficients $c_g(\xi, \gamma)$ are given by Kazhdan–Lusztig–Vogan polynomials, and the Atlas of Lie Groups and Representations software can compute them explicitly. This provides an effective method for evaluating $\chi_S^{\mathrm{loc}}(P(\gamma))$.

The local multiplicity map χ_S^{loc} is related to the microlocal multiplicity map χ_S^{mic} through Kashiwara's local index formula (13) which we state again here. For every ${}^\vee K(\lambda)$-orbit $S' \subset \overline{S}$, there exists an integer $a(S, S')$ such that

$$\chi_S^{\mathrm{loc}}(P(\gamma)) = \sum_{S' \subset \overline{S}} (-1)^{\dim(S')}\, a(S, S')\ \chi_{S'}^{\mathrm{mic}}(P(\gamma)). \tag{99}$$

In [Mac74, Section 3] MacPherson points out that the local Euler obstructions $a(S, S')$ may be used to determine the singularity of $\overline{S}$ at a point $x \in S'$. We also mention that the inverse of the relationship in (99) appears in [Gin86, Section 8] as the index formula of Dubson-Kashiwara. This index formula expresses χ_S^{mic} in terms of the various $\chi_{S'}^{\mathrm{loc}}$. There, the matrix entries are denoted by $c_{S,S'}$, and these entries form the inverse of the matrix with entries $(-1)^{\dim(S)} a(S', S)$.

In what follows, we explain how to compute them in the specific setting of our generalized flag variety $X(\lambda)$. Suppose first that $\lambda \in {}^\vee\mathfrak{t}$ is regular and integral (2). Then $X(\lambda) = X$ is the full flag variety of ${}^\vee\mathrm{G}_2$. We fix ${}^\vee K$ as in Section 2.2. Since the left-hand side of (99) can be computed using the Atlas software, and Theorem 5.1 gives the values $\chi_{S'}^{\mathrm{mic}}(P(\gamma))$ on the right, we can compute all the local Euler obstructions $a(S, S')$ by a simple recursive argument on the dimension of the ${}^\vee K$-orbit S. We record these values in the form of a matrix, where the entry in position (i,j) is the local Euler obstruction $a(S_i, S_j)$.

$$\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1\\
0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 2 & 1\\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Examining columns 6, 7, and 8, we deduce from [Kas73, Example 1] that the ${}^\vee K$-orbit closures $\overline{S_6}$ and $\overline{S_8}$ have a singularity at S_1, and that the closure $\overline{S_7}$ has a singularity at S_2. All other ${}^\vee K$-orbits have smooth closures. Indeed, orbits S_0, S_1 and S_2 are closed homogeneous spaces, and thus smooth. The open orbit S_9 has the smooth variety X as its closure. We cannot determine whether orbit closures $\overline{S}_3$, $\overline{S}_4$ and $\overline{S}_5$ are smooth or not using [Kas73, Example 1]. However, we have verified that these orbit closures are smooth by using Magma software; we omit the details.

Suppose now that $\lambda \in {}^\vee\mathfrak{t}$ is regular, but not integral. The possibilities for the pairs $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda))$ are listed in Proposition 2.4. When $({}^\vee\mathrm{G}_2(\lambda), {}^\vee K(\lambda))$ is not isomorphic to $(\mathrm{SL}_3, \mathrm{GL}_2)$, the orbits are either open or closed in $X(\lambda)$. Hence, they are smooth. For the pair $(\mathrm{SL}_3, \mathrm{GL}_2)$, the smoothness of the ${}^\vee K(\lambda)$-orbits follows from [McG23, Theorem 7.3.3]. It follows from [Kas73, Example 1] that all local Euler obstructions are equal to one in these cases.

Suppose finally that $\lambda \in {}^\vee\mathfrak{t}$ is singular. Once again the Atlas software can be used to compute the local multiplicities on the left-hand side of equation (99). We know the microlocal multiplicities on the right-hand side of equation (99) by Theorem 5.1, Theorem 6.1 and equation (93). In this manner we may compute the local Euler obstructions $a(S, S')$.

As an example, we consider λ as in equation (87). This is the infinitesimal character corresponding to the subregular nilpotent orbit which we studied at the end of Section 8. As we saw there, taking $\lambda' = \rho$ furnishes a translation datum $\mathcal{T}$ from the ${}^\vee\mathrm{G}_2$-orbit of λ to the ${}^\vee\mathrm{G}_2$-orbit of λ'. Recall the set $\{\xi_i : 0 \leq i \leq 11\}$ of complete geometric parameters for the flag variety X of ${}^\vee\mathrm{G}_2$ listed in (27). The ${}^\vee K(\lambda)$-orbit structure on $X(\lambda)$ is given by the image of the ${}^\vee K(\lambda)$-orbit structure on X under (91). There are five ${}^\vee K(\lambda)$-orbits on $X(\lambda)$ of ${}^\vee\mathrm{G}_2$, which we denote by Q_i for $0 \leq i \leq 4$. Among these orbits, Q_0 and Q_1 are closed and are equal to S_{ψ_a} and S_{ψ_b} respectively. The orbit Q_4 is open, and the closure relations are given as follows

$$Q_0, Q_1 \subset \overline{Q_2} \subset \overline{Q_3} \subset \overline{Q_4} = X(\lambda).$$

For orbits Q_i with $0 \leq i \leq 3$, the constant sheaf $\underline{\mathbb{C}}_{Q_i}$ is the only irreducible ${}^\vee K(\lambda)$-equivariant local system supported on Q_i. In orbit Q_4, in addition to $\underline{\mathbb{C}}_{Q_4}$, there exists a non-trivial local system which we denote by $\mathcal{L}_5$. In short, there are exactly six complete geometric parameters for $X(\lambda)$

$$\gamma_i = (Q_i, \underline{\mathbb{C}}_{Q_i}) \quad \text{for } 0 \leq i \leq 3, \quad \gamma_4 = (Q_4, \underline{\mathbb{C}}_{Q_4}), \quad \text{and} \quad \gamma_5 = (Q_4, \mathcal{L}_5).$$

Using the Atlas software command `T_irr` in combination with (95), we obtain

$$f^*_{\mathcal{T}}(\gamma_0) = \xi_1,\ f^*_{\mathcal{T}}(\gamma_1) = \xi_4,\ f^*_{\mathcal{T}}(\gamma_2) = \xi_6,\ f^*_{\mathcal{T}}(\gamma_3) = \xi_8,\ f^*_{\mathcal{T}}(\gamma_4) = \xi_9,\ f^*_{\mathcal{T}}(\gamma_5) = \xi_{10}.$$

By Theorem 5.1 and equation (93), we deduce that $CC(P(\gamma_i)) = \overline{T^*_{Q_i}X}$ for $i = 0, 1, 4$, and that

$$
\begin{aligned}
CC(P(\gamma_2)) &= 2 \cdot \overline{T^*_{Q_0}X} + \overline{T^*_{Q_2}X}, \\
CC(P(\gamma_3)) &= \overline{T^*_{Q_1}X} + \overline{T^*_{Q_2}X} + \overline{T^*_{Q_3}X}, \\
CC(P(\gamma_5)) &= \overline{T^*_{Q_0}X} + \overline{T^*_{Q_2}X} + \overline{T^*_{Q_3}X} + \overline{T^*_{Q_4}X}.
\end{aligned}
$$

From these equations we can read off the values of $\chi^{\mathrm{mic}}_{Q_j}$. Finally, using the Atlas software to compute the local multiplicities in equation (99) we arrive to the matrix

$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & -1 & 1 & 1 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

where the entry in position (i, j) is the local Euler obstruction $a(Q_i, Q_j)$.

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