Several classes of p-ary linear codes with few-weights derived from Weil sums

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Abstract

Linear codes with few weights have been a significant area of research in coding theory for many years, due to their applications in secret sharing schemes, authentication codes, association schemes, and strongly regular graphs. Inspired by the works of Cheng and Gao [11] and Wu, Li and Zeng [21], in this paper, we propose several new classes of few-weight linear codes over the finite field $\mathbb{F}_p$ through the selection of two specific defining sets. Consequently, we obtain five classes of 4-weight linear codes and one class of 2-weight linear codes from our first defining set. Furthermore, by employing weakly regular bent functions in our second defining set, we derive two classes of 6-weight codes, two classes of 8-weight codes, and one class of 9-weight codes. The parameters and weight distributions of all these constructed codes are wholly determined by detailed calculations on certain Weil sums over finite fields. In addition, we identify an optimal class of 2-weight codes that meet the Griesmer bound.

Keywords: Linear codes; Weight distributions; Weakly regular bent function; Weil sums.
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1 Introduction

Let p be a prime, and $q = p^m$, where m is a positive integer. Let $\mathbb{F}_q$ be a finite field of order q and $\mathbb{F}_q^* = \mathbb{F}_q \backslash \{0\}$ be the multiplicative group of order $q-1$. A linear code $\mathcal{C}$ with parameters $[n, k, d]$ over $\mathbb{F}_p$ is a k-dimensional subspace of $\mathbb{F}_p^n$, where n denotes

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the length of the code $\mathcal{C}$ whereas d represents the minimum (Hamming) distance of the code $\mathcal{C}$. Let $A_i = \{c \in \mathcal{C} : \mathrm{wt}_H(c) = i\}$, where $\mathrm{wt}_H(c)$ denotes the Hamming weight of the codeword c in $\mathcal{C}$. Obviously, $1 \leq i \leq n$ and $A_0 = 1$. The weight enumerator of the code $\mathcal{C}$ is defined by the polynomial $1 + A_1 z + A_2 z^2 + \cdots + A_n z^n$, and the sequence $(1, A_1, A_2, \cdots, A_n)$ is called the weight distribution of $\mathcal{C}$. We refer $\mathcal{C}$, to be a t-weight code if $|\{1 \leq i \leq n : A_i \neq 0\}| = t$. The weight distribution is an important research topic in coding theory, not only because the weight enumerator of a code contains crucial information about its error correction capabilities, but also allows the computation of the error probability of error detection and correction with respect to some algorithms [38]. Furthermore, linear codes with few weights have important applications in secret sharing schemes [1, 3, 26, 27], association schemes [2], strongly regular graphs [10], and authentication codes [32]. An $[n, k, d]$ linear code $\mathcal{C}$ is said to be optimal if n, k and d meet a bound on linear codes [16], and is called almost optimal if there exists an optimal $[n, k, d+1]$ linear code. The well-known Griesmer bound [12] for an $[n, k, d]$ code $\mathcal{C}$ over $\mathbb{F}_p$ with $k \geq 1$ states that $n \geq \sum_{i=0}^{k-1} \lceil \frac{d}{p^i} \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function. In 2007, Ding and Niederreiter [9] introduced a fundamental method for constructing linear codes based on the proper selection of a subset of finite fields. Let $D = \{d_1, d_2, \cdots, d_n\} \subseteq \mathbb{F}_{p^m}$ and define

$$\mathcal{C}_D = \{c(\gamma) = (\mathrm{Tr}_1^m(\gamma x))_{x \in D} : \gamma \in \mathbb{F}_{p^m}\}, \tag{1}$$

where $\mathrm{Tr}_1^m(x) = \sum_{i=0}^{m-1} x^{p^i}$ is the trace function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$. From the definition of (1), clearly $\mathcal{C}_D$ forms a linear code of length $|D|$ over $\mathbb{F}_p$ and the set D is called the defining set of $\mathcal{C}_D$. In fact, several classes of good linear codes with few weights have been constructed using the defining-set approach over the past few decades (see [7, 28, 6, 29, 4, 21, 27, 34, 26, 22, 8, 23, 24, 37]). Recently, Cheng and Gao [11] proposed a novel method for evaluating a binomial Weil sum given by

$$S_{\mathcal{N}}(a, b) = \sum_{x \in \mathbb{F}_q^*} \chi(a x^{\frac{q-1}{\mathcal{N}}} + bx), \tag{2}$$

where p is an odd prime, $\mathcal{N}$ is a positive integer not divisible by p, $q = p^{\phi(\mathcal{N})}$, $\mathrm{ord}_{\mathcal{N}}(p) = \phi(\mathcal{N})$ and $a, b \in \mathbb{F}_q$. Specifically, they provide the evaluation of $S_{\mathcal{N}}(a, b)$ for $\mathcal{N} = 2, 4, \ell^k$ or $2\ell^k$, where ℓ is an odd prime not divisible by p and k a positive integer. This novel evaluation of (2) facilitates them to construct a new class of two-weight ternary linear codes $\mathcal{C}_D$ of the form (1) by investigating the defining set $D = \{x \in \mathbb{F}_q^* : \mathrm{Tr}_1^{\phi(\ell^k)}(x^{(q-1)/\ell^k}) = 0\}$, where $q = 3^{\phi(\ell^k)}$, k is a positive integer and $\ell (> 3)$ is an odd prime such that 3 is a primitive root modulo ℓ^k. Thereafter, Cheng [24] further investigated the linear code $\mathcal{C}_D$ with the defining set $D = \{x \in \mathbb{F}_q^* : \mathrm{Tr}_1^e(x^{\frac{q-1}{2\ell^k}} + \beta x) = \alpha\}$, where $q = 3^e$, $e = \phi(2\ell^k)$, $\alpha \in \mathbb{F}_p$ and $\beta \in \mathbb{F}_q$. Considering ℓ be an odd prime such that $3 \nmid \ell$ and 3 be a primitive root modulo $2\ell^k$, Cheng [24] constructed some classes of two-weight and four-weight linear codes and explicitly determined their weight distributions. Very recently, Cheng and Sheng [36], generalising the defining set D as in [24] to an arbitrary odd prime p, investigated the linear code $\mathcal{C}_D$.

In 2017, Li et al. [7] considered the linear code of the form

$$\mathcal{C}_D = \{c(\gamma, \delta) = (\mathrm{Tr}_1^m(\gamma x + \delta y))_{(x,y)\in D} : \gamma, \delta \in \mathbb{F}_{p^m}\}, \tag{3}$$

where $D \subseteq \mathbb{F}_{p^m}^2$ is also called a defining set. They investigated the linear code $\mathcal{C}_D$ of the form (3) for $D = \{(x, y) \in \mathbb{F}_{p^m}^2 \backslash \{(0, 0)\} : \mathrm{Tr}_1^m(x^{N_1} + y^{N_2}) = 0\}$, where $N_1, N_2 \in \{1, 2, p^{\frac{m}{2}} + 1\}$. Motivated by the work of [7], Jian et al. [6] studied the linear code $\mathcal{C}_D$ of the form (3) for the defining sets $D = \{(x, y) \in \mathbb{F}_{p^m}^2 \backslash \{(0, 0)\} : \mathrm{Tr}_1^m(x^N + y^{p^u+1}) = 0\}$, where $N \in \{1, 2\}$ and u is a positive integer. In [21], Wu et al. presented two classes of two-weight and three-weight linear codes $\mathcal{C}_D$ of the form (3), selecting the defining set $D = \{(x, y) \in \mathbb{F}_{p^m}^2 \backslash \{(0, 0)\} : f(x) + g(y) = 0\}$ by considering the two cases (1) $f(x) = \mathrm{Tr}_1^m(x)$ and $g(y)$ is a weakly regular bent function; and (2) $f(x)$ and $g(y)$ are both weakly regular bent functions.

Throughout this paper, we assume $q = p^e$, where $e = \phi(\ell^k)$, p and ℓ are distinct odd primes, k is a positive integer and p is a primitive root modulo $2\ell^k$. That means $\mathrm{ord}_{2\ell^k}(p) = \phi(\ell^k)$. Inspired by the works of [21] and [11], in this paper, we investigate the linear code $\mathcal{C}_D$ of the form (3) by using the following two defining sets:

$$D_u = \{(x, y) \in \mathbb{F}_q^2 \backslash \{(0, 0)\} : \mathrm{Tr}_1^e(x + y^N) = u\}, \tag{4}$$

and

$$D^{'} = \{(x, y) \in \mathbb{F}_q^2 \backslash \{(0, 0)\} : f(x) + \mathrm{Tr}_1^e(y^N) = 0\}, \tag{5}$$

where p is any odd prime, $q = p^e$, $e = \phi(\ell^k) = (\ell - 1)\ell^{k-1}$, $u \in \mathbb{F}_p$, $N = \frac{q-1}{2\ell^k}$, and in (5), we consider $f : \mathbb{F}_{p^e} \to \mathbb{F}_p$ to be a weakly regular bent function such that f^* is a quadratic form, that is f satisfies the condition $f^*(cx) = c^2 f^*(x)$ for any $c \in \mathbb{F}_p^*$ and $x \in \mathbb{F}_{p^e}$, where f^* is the dual of f discussed in section 2.3. Through detailed calculations on certain exponential sums, we obtain five classes of 4-weight linear codes, one class of 2-weight linear codes, two classes of 6-weight linear codes, two classes of 8-weight linear codes and one class of 9-weight linear codes from our construction. It is noteworthy that the defining sets in (4) and (5) used to develop linear codes in this paper deal with a new binomial Weil sum $S_{\mathcal{N}}(a, b)$ defined in (2) compared to the previous works of [21, 6, 7] and others summarized in Table 1. The parameters and weight distributions of these codes are completely characterized by using Weil sums and quadratic Gauss sums. Moreover, we obtain a class of optimal codes that meet the Griesmer bound for a special case.

The rest of this paper is organized as follows. Section 2 introduces some basic knowledge on exponential sums, cyclotomic fields, and weakly regular bent functions which will be used for the subsequent sections. In section 3 and section 4, we first present some auxiliary results and then we respectively give the constructions of p-ary linear codes from the defining sets (4) and (5), and their parameters and weight distributions are explicitly determined. Section 5 concludes this paper.

2 Preliminaries

In this section, we present some preliminaries and important results related to exponential sums over finite fields, cyclotomic fields, and weakly regular bent functions, which will be used in the subsequent sections.

2.1 Exponential sums

Let $m \geq 1$ be an integer. Let χ_m be the canonical additive character of $\mathbb{F}_{p^m}$, and is defined by $\chi_m(x) = \zeta_p^{\mathrm{Tr}_1^m(x)}$, where $x \in \mathbb{F}_{p^m}$ and $\zeta_p = e^{\frac{2\pi\sqrt{-1}}{p}}$ is the complex p-th primitive root of unity. Let η_m be the quadratic multiplicative character of $\mathbb{F}_{p^m}$. It is known that for any $x \in \mathbb{F}_p^*$, $\eta_m(x) = 1$, if m is even, and $\eta_m(x) = \eta_1(x)$, if m is odd. The quadratic Gauss sum $G(\eta_m, \chi_m)$ over $\mathbb{F}_{p^m}$ is defined by

$$G(\eta_m, \chi_m) = \sum_{c \in \mathbb{F}_{p^m}^*} \eta_m(c)\chi_m(c).$$

Lemma 1. [30, Theorem 5.15] *Let $G(\eta_m, \chi_m)$ be defined as above, then*

$$G(\eta_m, \chi_m) = (-1)^{m-1}\sqrt{(p^*)^m} = \begin{cases} (-1)^{m-1}p^{\frac{m}{2}};\ \text{if } p \equiv 1 \pmod 4, \\ (-1)^{m-1}(\sqrt{-1})^m p^{\frac{m}{2}};\ \text{if } p \equiv 3 \pmod 4, \end{cases}$$

where $p^* = \eta_1(-1)p = (-1)^{\frac{p-1}{2}}p$.

Lemma 2. [30, Theorem 5.33] *Let $f(x) = r_2x^2 + r_1x + r_0 \in \mathbb{F}_{p^m}[x]$ with $r_2 \neq 0$. Then*

$$\sum_{x \in \mathbb{F}_{p^m}} \chi_m(f(x)) = \chi_m(r_0 - r_1^2(4r_2)^{-1})\eta_m(r_2)G(\eta_m, \chi_m).$$

Throughout we assume that α be a fixed primitive element of $\mathbb{F}_q$, $q = p^e$, $e = \phi(\ell^k)$, p is a primitive root modulo $2\ell^k$, $\mathrm{Ind}_\alpha(b)$ be the index of $b \in \mathbb{F}_q^*$ with respect to the base α, and $\xi = \alpha^{\frac{q-1}{2\ell^k}}$. One can check that $\mathbb{F}_q = \mathbb{F}_p(\xi)$. It follows that $\{\xi, \xi^2, \cdots, \xi^e\}$ forms a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$. In [11], Cheng and Gao defined the following two exponential sums over $\mathbb{F}_q$ given by

$$S_{2\ell^k}(a, b) := \sum_{x \in \mathbb{F}_q^*} \chi_e(ax^{\frac{q-1}{2\ell^k}} + bx), \tag{6}$$

and

$$S_{2\ell^k}(a) := \sum_{i=0}^{2\ell^k-1} \chi_e(a\xi^i), \text{ where } a, b \in \mathbb{F}_q. \tag{7}$$

The defining set D	Conditions on D	Parameters of $\mathcal{C}_D$	Nonzero t-weights in $\mathcal{C}_D$	References
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x^{N_1} + y^{N_2}) = 0\}$	$N_1 = N_2 = 1$	$[p^{2m-1}-1, 2m-1, p^{2m-1}-p^{2m-2}]$	$t=1$	[7]
	$N_1 = 1$, $N_2 = 2$ and $m \geq 2$ is even	$[p^{2m-1}-1, 2m]$	$t=3$	[7, Table 1]
	$N_1 = 1$, $N_2 = 2$ and $m \geq 1$ is odd	$[p^{2m-1}-1, 2m]$	$t=3$	[7, Table 2]
	$N_1 = 1$ and $N_2 = p^{m/2}+1$, m is an even integer	$[p^{2m-1}-1, 2m, p^{2m-1}-p^{2m-2}-(p-1)p^{\frac{3m-4}{2}}]$	$t=3$	[7, Table 3]
	$N_1 = N_2 = 2$	$[p^{2m-1}+(-1)^{\frac{m(p-1)}{2}}(p-1)p^{m-1}-1, 2m]$	$t=2$	[7, Table 4]
	$N_1 = 2$ and $N_2 = p^{m/2}+1$, m is an even integer	$[p^{2m-1}+(-1)^{\frac{m(p-1)}{4}}(p-1)p^{m-1}-1, 2m]$	$t=2$	[7, Table 5]
	$N_1 = N_2 = p^{m/2}+1$, m is an even integer	$[p^{2m-1}+(p-1)p^{m-1}-1, 2m]$	$t=2$	[7, Table 6]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x + y^{p^u+1}) = 0\}$	m is odd	$[p^{2m-1}-1, 2m]$	$t=3$	[6, Table 1]
	$\frac{m}{\gcd(m,u)}$ is odd and $\gcd(m,u)$ is even	$[p^{2m-1}-1, 2m]$	$t=3$	[6, Table 2]
	$\frac{m}{\gcd(m,u)} \equiv 2 \pmod 4$	$[p^{2m-1}-1, 2m]$	$t=3$	[6, Table 3]
	$\frac{m}{\gcd(m,u)} \equiv 0 \pmod 4$	$[p^{2m-1}-1, 2m]$	$t=3$	[6, Table 4]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x^2 + y^{p^u+1}) = 0\}$	$\frac{m}{\gcd(m,u)}$ is odd or $\frac{m}{\gcd(m,u)} \equiv 2 \pmod 4$	$[n, 2m]$, where $n = (p^m+1)(p^{m-1}-1)$, if $p \equiv 3 \pmod 4$ and $\gcd(m,u)$ is odd; and $n = (p^m-1)(p^{m-1}+1)$, otherwise	$t=2$	[6, Table 5]
	$\frac{m}{\gcd(m,u)} \equiv 0 \pmod 4$	$[p^{2m-1}+p^{m+v}-p^{m+v-1}-1, 2m]$, where $v = \gcd(m,u)$	$t=3$	[6, Table 6]
$\{(x,y) \in \mathbb{F}_{p^m}^2 : x \in C_i, y \in C_j\}$, where C_i and C_j are any two cyclotomic classes of order e, $p^m \equiv 1 \pmod e$ and $0 \leq i,j \leq e-1$.	$m = 2d\gamma$ and d is the least positive integer satisfying $p^d \equiv -1 \pmod e$	$[\frac{(p^m-1)^2}{e^2}, 2m]$	$t=5$	[21, Table 1]
	$e = 2$ and m is even	$[\frac{(p^m-1)^2}{4}, 2m]$	$t=5$	[21, Table 2]
	$e = 2$ and m is odd	$[\frac{(p^m-1)^2}{4}, 2m]$	$t=2$	[21, Table 3]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : f(x) + g(y) = 0\}$, where f and g are weakly regular bent functions from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$.	$f(x) = \mathrm{Tr}_1^m(x)$ and m is even	$[p^{2m-1}-1, 2m]$	$t=3$	[21, Table 5]
	$f(x) = \mathrm{Tr}_1^m(x)$ and m is odd	$[p^{2m-1}-1, 2m]$	$t=3$	[21, Table 6]
	f and g are weakly regular bent functions in [21, Table 4], $l_f, l_g \in \{2, p-1\}$, $l_f \neq l_g$ and m is even	$[p^{2m-1}+\frac{p-1}{p}\epsilon_f\epsilon_g p^{*m}-1, 2m]$	$t=3$	[21, Table 7]
	f and g are weakly regular bent functions in [21, Table 4], $l_f, l_g \in \{2, p-1\}$, $l_f = l_g$ and m is even	$[p^{2m-1}+\frac{p-1}{p}\epsilon_f\epsilon_g p^{*m}-1, 2m]$	$t=2$	[21, Table 8]
$\{(x,y) \in \mathbb{F}_{p^{2m}}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^{2m}(x^{p^l+1}) = 1, \mathrm{Tr}_1^{2m}(y) \in C_i^{(2,p)}\}$, where l and m are positive integers, $C_i^{(2,p)}$ are the sets of all squares and non-squares in $\mathbb{F}_p^*$ resp. for $i = 0$ and 1.	$p \equiv 3 \pmod 4$, $\frac{2m}{\gcd(l,2m)}$ is even and $\frac{m}{\gcd(l,2m)} \equiv 1 \pmod 2$ is even	$[\frac{p-1}{2}(p^{4m-2}+p^{3m-2}), 4m]$	$t=6$	[14, Table 1]
	$p \equiv 3 \pmod 4$, $\frac{2m}{\gcd(l,2m)}$ is even, $m \geq \gcd(l,2m)+1$ and $\frac{m}{\gcd(l,2m)} \equiv 0 \pmod 2$ is even	$[\frac{p-1}{2}(p^{4m-2}+p^{3m+\gcd(l,2m)-2}), 4m]$	$t=6$	[14, Table 2]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x) + g(y) = 0\}$, where g is a weakly regular s-plateaued unbalanced function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$.	$m+s$ is even	$[p^{2m-1}-1, 2m]$	$t=3$	[15, Table 1]
	$m+s$ is odd	$[p^{2m-1}-1, 2m]$	$t=3$	[15, Table 2]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x)+g(y) \in SQ(NSQ)\}$, where g is a weakly regular s-plateaued unbalanced function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ and $SQ(NSQ)$ is the set of all squares (non-squares) in $\mathbb{F}_p^*$.	$m+s$ is even	$[\frac{p-1}{2}p^{2m-1}, 2m]$	$t=3$	[15, Table 3]
	$m+s$ is odd and $p \equiv 1 \pmod 4$	$[\frac{p-1}{2}p^{2m-1}, 2m]$	$t=3$	[15, Table 4 (Table 6)]
	$m+s$ is odd and $p \equiv 3 \pmod 4$	$[\frac{p-1}{2}p^{2m-1}, 2m]$	$t=3$	[15, Table 5 (Table 7)]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : f(x) + g(y) \in T\}$, where f and g are weakly regular s-plateaued unbalanced functions from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$, $T \subset \mathbb{F}_p$ and $SQ(NSQ)$ denote the set of all squares (non-squares) in $\mathbb{F}_p^*$.	$T = \{0\}$, $l_f = l_g$	$[p^{2m-1}+\frac{p-1}{p}\epsilon_f\epsilon_g p^{*(m+s)}-1, 2m]$	$t=3$	[15, Table 8]
	$T = \{0\}$, $l_f \neq l_g$ and $m-s$ is even	$[p^{2m-1}+\frac{p-1}{p}\epsilon_f\epsilon_g p^{*(m+s)}-1, 2m]$	$t=4$	[15, Table 9]
	$T = \{0\}$, $l_f \neq l_g$ and $m-s$ is odd and $p \equiv 1 \pmod 4$	$[p^{2m-1}+\frac{p-1}{p}\epsilon_f\epsilon_g p^{*(m+s)}-1, 2m]$	$t=4$	[15, Table 10]
	$T = \{0\}$, $l_f \neq l_g$ and $m-s$ is odd and $p \equiv 3 \pmod 4$	$[p^{2m-1}+\frac{p-1}{p}\epsilon_f\epsilon_g p^{*(m+s)}-1, 2m]$	$t=4$	[15, Table 11]
	$T = SQ(\text{or}, NSQ)$, $l_f = l_g$	$[p^{2m-1}\frac{p-1}{2}-\frac{p(p-1)}{2}\epsilon_f\epsilon_g p^{*(m+s-2)}, 2m]$	$t=3$	[15, Table 13]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x) + g(y) = 0\}$, where g is a weakly regular s_g-plateaued balanced function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ with $1 \leq s_g < m$.	$m+s_g$ is even	$[p^{2m-1}-1, 2m]$	$t=3$	[20, Table 1]
	$m+s_g$ is odd	$[p^{2m-1}-1, 2m]$	$t=3$	[20, Table 2]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : f(x) + g(y) = 0\}$, where f and g are weakly regular s_f and s_g-plateaued unbalanced ($\mathcal{WRP}$) function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ with $1 \leq s_f, s_g < m$.	$l_f = l_g$, $m+s_f$ is odd and $m+s_g$ is even	$[p^{2m-1}-1, 2m, (p-1)p^{2m-2}-\sqrt{p}^{2m+s_f+s_g-3}]$	$t=3$	[20, Table 3]
	$2m+s_f+s_g$ is even, $0 \leq s_f, s_g < m-1$ and $l_f = l_g$	$[p^{2m-1}-1+\epsilon_f\epsilon_g\frac{p-1}{p}\sqrt{p^*}^{2m+s_f+s_g}, 2m]$	$t=3$	[20, Table 4]
	$2m+s_f+s_g$ is even, $0 \leq s_f, s_g < m-1$, $p > 3$ and $l_f \neq l_g$	$[p^{2m-1}-1+\epsilon_f\epsilon_g\frac{p-1}{p}\sqrt{p^*}^{2m+s_f+s_g}, 2m]$	$t=4$	[20, Table 5]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : f(x) + g(y) = 0\}$, where f and g are weakly regular s_f and s_g-plateaued balanced ($\mathcal{WRPB}$) function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ with $1 \leq s_f, s_g < m-1$.	$l_f = l_g$ and $2m+s_f+s_g$ is even	$[p^{2m-1}-1, 2m]$	$t=3$	[20, Table 6]
	$l_f \neq l_g$ and $2m+s_f+s_g$ is even	$[p^{2m-1}-1, 2m]$	$t=4$	[20, Table 7]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x^p + xy) = u\}$, where $u \in \mathbb{F}_p$.	$u = 0$	$[p^{2m-1}+(p-1)p^{m-1}-1, 2m, (p-1)p^{2m-2}]$	$t=3$	[17, Table 1]
	$u \neq 0$	$[p^{2m-1}-p^{m-1}, 2m, (p-1)p^{2m-2}-2p^{m-1}]$	$t=3$	[17, Table 2]
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(x + y^N) = u\}$, where $m = \phi(\ell^k)$, $k \in \mathbb{N}$, $u \in \mathbb{F}_p$, $\ell(\neq p)$ is an odd integer such that $\mathrm{ord}_{2\ell^k}(p) = \phi(\ell^k)$ and $N = \frac{p^m-1}{2\ell^k}$.	$u = 0$ and $\ell \equiv 1 \pmod p$	$[p^{2m-1}-1, 2m]$	$t=4$	Table 2
	$u = 0$ and $\ell \not\equiv 1 \pmod p$	$[p^{2m-1}-1, 2m]$	$t=4$	Table 3
	$u \neq 0$ and $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$ and $u^2 \not\equiv \ell^{2(k-1)} \pmod p$	$[p^{2m-1}, 2m, p^{2m-2}(p-1)]$	$t=2$	Table 4
	$u \neq 0$ and $u^2 \equiv \phi(\ell^k)^2 \pmod p$ and $u^2 \not\equiv \ell^{2(k-1)} \pmod p$	$[p^{2m-1}, 2m]$	$t=4$	Table 5
	$u \neq 0$ with $u \equiv \phi(\ell^k) \equiv \ell^{k-1} \pmod p$ or $u \equiv -\phi(\ell^k) \equiv -\ell^{k-1} \pmod p$	$[p^{2m-1}, 2m]$	$t=4$	Table 6
	$u \neq 0$ and $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$ and $u^2 \equiv \ell^{2(k-1)} \pmod p$	$[p^{2m-1}, 2m]$	$t=4$	Table 7
$\{(x,y) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\} : \mathrm{Tr}_1^m(f(x) + y^N) = 0\}$, where $m = \phi(\ell^k)$, $k \in \mathbb{N}$, $u \in \mathbb{F}_p$, $\ell(\neq p)$ is an odd integer such that $\mathrm{ord}_{2\ell^k}(p) = \phi(\ell^k)$, $N = \frac{p^m-1}{2\ell^k}$ and f is a weakly regular bent function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ such that f^* is a quadratic form.	$p \equiv 1 \pmod 4$ and $\ell \equiv 1 \pmod p$	$[p^{2m-1} + \epsilon_f\sqrt{p^*}^m\left(p^{m-1}(p-1)-(\ell-1)\frac{p^m-1}{\ell^k}\right) - 1, 2m]$	$t=8$	Table 8
	$p \equiv 3 \pmod 4$ and $\ell \equiv 1 \pmod p$	$[p^{2m-1} + \epsilon_f\sqrt{p^*}^m\left(p^{m-1}(p-1)-(\ell-1)\frac{p^m-1}{\ell^k}\right) - 1, 2m]$	$t=6$	Table 9
	$p \equiv 1 \pmod 4$, $\ell \not\equiv 1 \pmod p$ and $\eta_1(t_1) = \eta_1(t_2)$, where $t_1 = \phi(\ell^k) \pmod p$ and $t_2 = \ell^{k-1} \pmod p$	$[p^{2m-1} + \epsilon_f\sqrt{p^*}^m\left(p^{m-1}(p-1)-\frac{p^m-1}{\ell^{k-1}}\right) - 1, 2m]$	$t=8$	Table 10
	$p \equiv 1 \pmod 4$, $\ell \not\equiv 1 \pmod p$ and $\eta_1(t_1) = -\eta_1(t_2)$, where $t_1 = \phi(\ell^k) \pmod p$ and $t_2 = \ell^{k-1} \pmod p$	$[p^{2m-1} + \epsilon_f\sqrt{p^*}^m\left(p^{m-1}(p-1)-\frac{p^m-1}{\ell^{k-1}}\right) - 1, 2m]$	$t=9$	Table 11
	$p \equiv 3 \pmod 4$ and $\ell \not\equiv 1 \pmod p$	$[p^{2m-1} + \epsilon_f\sqrt{p^*}^m\left(p^{m-1}(p-1)-\frac{p^m-1}{\ell^{k-1}}\right) - 1, 2m]$	$t=6$	Table 12

Table 1: Known p-ary linear code $\mathcal{C}_D$ with t-weights of the form (3) by using the defining set $D \subset \mathbb{F}_{p^m}^2$, where p is an odd prime.

Define a set $P = \{i \in \mathbb{Z} : 0 \leq i \leq 2\ell^k - 1\}$. Let P_1, P_2, P_3, and P_4 be the subsets of P defined as follows:

$$\begin{aligned}
P_1 &= \{i \in P : 1 \leq i \leq (\ell-1)\ell^{k-1}\},\\
P_2 &= \{i \in P : (\ell-1)\ell^{k-1} < i \leq \ell^k\},\\
P_3 &= \{i \in P : \ell^k < i \leq 2\ell^k - \ell^{k-1}\},\\
P_4 &= \{i \in P : 2\ell^k - \ell^{k-1} < i \leq 2\ell^k - 1\}.
\end{aligned}$$

It is clear that $(\{0\}, P_1, P_2, P_3, P_4)$ is a partition of P. Now we further partition the two sets P_1 and P_3 into the union of three disjoint subsets as $P_1 = P_1^{(1)} \cup P_1^{(2)} \cup P_1^{(3)}$ and $P_3 = \{\ell^k + u\ell^{k-1} - v : 1 \leq u \leq \ell - 1 \text{ and } 0 \leq v \leq \ell^{k-1} - 1\} = P_3^{(1)} \cup P_3^{(2)} \cup P_3^{(3)}$ respectively, where

$$P_1^{(1)} = \{i \in P_1 : \ell^{k-1} \nmid i\},\ P_1^{(2)} = \{i \in P_1 : \ell^{k-1} \mid i \text{ and } 2 \nmid i\},\ P_1^{(3)} = \{i \in P_1 : 2\ell^{k-1} \mid i\};$$

and

$$\begin{aligned}
P_3^{(1)} &= \{i \in P_3 : \ell^k + u\ell^{k-1} - v : 1 \leq u \leq \ell - 1 \text{ and } 0 < v \leq \ell^{k-1} - 1\},\\
P_3^{(2)} &= \{i \in P_3 : \ell^k + u\ell^{k-1} : 1 \leq u \leq \ell - 1 \text{ and } 2 \mid u\},\\
P_3^{(3)} &= \{i \in P_3 : \ell^k + u\ell^{k-1} : 1 \leq u \leq \ell - 1 \text{ and } 2 \nmid u\}.
\end{aligned}$$

The following fundamental results from [11, 36] will be useful in the subsequent sections.

Lemma 3. *If $S_{2\ell^k}(a,b)$ and $S_{2\ell^k}(a)$ be defined as in* (6) *and* (7) *respectively. Then, we have*

(a) *For any $a \in \mathbb{F}_q$, we have*

$$S_{2\ell^k}(a, 0) = \frac{q-1}{2\ell^k} S_{2\ell^k}(a).$$

(b) *For any $a \in \mathbb{F}_p$, we have*

$$S_{2\ell^k}(a) = \zeta_p^{\ell^{k-1}(\ell-1)a} + \zeta_p^{-\ell^{k-1}(\ell-1)a} + (\ell-1)(\zeta_p^{\ell^{k-1}a} + \zeta_p^{-\ell^{k-1}a}) + 2\ell^k - 2\ell.$$

(c) *For every $i \in P = \{i \in \mathbb{Z} : 0 \leq i \leq 2\ell^k - 1\}$ and $a \in \mathbb{F}_p$, we have*

$$S_{2\ell^k}(a\xi^i) = S_{2\ell^k}(a).$$

(d) *For any $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_q^*$, we have*

$$S_{2\ell^k}(a, b) = \chi_e(c)\sqrt{q} - \frac{\sqrt{q}+1}{2\ell^k} S_{2\ell^k}(c), \ \textit{where, } c = ab^{-\frac{q-1}{2\ell^k}}$$

(e) *For any* $a \in \mathbb{F}_q$ *and* $y \in \mathbb{F}_p^*$*, we have* $S_{2\ell^k}(a,b) = S_{2\ell^k}(a,yb)$.

Lemma 4. [11, Lemma 2.8] *Let* $i \in P = \{i \in \mathbb{Z} : 0 \leq i \leq 2\ell^k - 1\}$*. Then*

$$\mathrm{Tr}_1^e(\xi^i) = \begin{cases} (\ell-1)\ell^{k-1}, & \text{if } i = 0, \\ -(\ell-1)\ell^{k-1}, & \text{if } i = \ell^k, \\ -\ell^{k-1}, & \text{if } i \in P_1^{(3)} \cup P_3^{(3)}, \\ \ell^{k-1}, & \text{if } i \in P_1^{(2)} \cup P_3^{(2)}, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1. *If* $b \in \mathbb{F}_q^*$ *and let* $i_b = -\mathrm{Ind}_\alpha(b) \pmod{2\ell^k}$*, then* $b^{-\frac{q-1}{2\ell^k}} = \alpha^{-\frac{q-1}{2\ell^k}\mathrm{Ind}_\alpha(b)} = \xi^{i_b}$*. Clearly,* $i_b \in P = \{i \in \mathbb{Z} : 0 \leq i \leq 2\ell^k - 1\}$ *and* $\chi_e(zb^{-\frac{q-1}{2\ell^k}}) = \zeta_p^{z\,\mathrm{Tr}_1^e(\xi^{i_b})}$ *for all* $z \in \mathbb{F}_p^*$*. The value of* $\mathrm{Tr}_1^e(\xi^{i_b})$ *can be calculated from Lemma 4. Hence, one can evaluate* $S_{2\ell^k}(z,b)$ *for* $z \in \mathbb{F}_p^*$ *and* $b \in \mathbb{F}_q^*$.

Define the following useful notation $w(u,b)$ as follows:

$$w(u,b) = \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-uz} S_{2\ell^k}(z,b), \text{ where } u \in \mathbb{F}_p \text{ and } b \in \mathbb{F}_q. \tag{8}$$

Lemma 5. [36, Proposition 3.2] *For* $b \in \mathbb{F}_q^*$*, let* $i_b = -\mathrm{Ind}_\alpha(b) \pmod{2\ell^k}$*. Suppose* $w(u,b)$ *is defined as in* (8)*. Then the explicit values of* $w(u,b)$ *are as follows:*

(a) *When* $u = b = 0$*. Then we have*

$$w(0,0) = \begin{cases} \frac{q-1}{\ell^k}\left((p-1)\ell^k - p\ell + p\right); & \text{if } \ell \equiv 1 \pmod{p}, \\ \frac{q-1}{\ell^{k-1}}\left((p-1)\ell^{k-1} - p\right); & \text{if } \ell \not\equiv 1 \pmod{p}. \end{cases}$$

(b) *When* $u = 0$ *and* $b \neq 0$*. Then we have*

$$w(0,b) = \begin{cases} \frac{\sqrt{q}+1}{\ell^k}(-p\ell^k + p\ell - p) + 1; & \text{if } i_b \in P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)}, \\ \frac{\sqrt{q}+1}{\ell^k}(p\ell - p) - p + 1; & \text{otherwise,} \end{cases}$$

if $\ell \equiv 1 \pmod{p}$*; and*

$$w(0,b) = \begin{cases} \frac{\sqrt{q}+1}{\ell^{k-1}}(-p\ell^{k-1} + p) + 1; & \text{if } i_b \in \{0, \ell^k\} \cup P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)}, \\ \frac{p(\sqrt{q}+1)}{\ell^{k-1}} - p + 1; & \text{otherwise,} \end{cases}$$

if $\ell \not\equiv 1 \pmod{p}$.

(c) *When $u \neq 0$ and $b = 0$. Then we have*

$$w(u,0) = \begin{cases} \frac{q-1}{2\ell^k}\left(-2\ell^k + p\right); & \text{if } u^2 \equiv \phi(\ell^k)^2 \pmod p \text{ but } u^2 \not\equiv \ell^{2(k-1)} \pmod p, \\ \frac{q-1}{2\ell^{k-1}}\left(-2\ell^{k-1} + p\right); & \text{if } u \equiv \phi(\ell^k) \equiv \ell^{k-1} \pmod p \text{ or } \\ & \quad u \equiv -\phi(\ell^k) \equiv -\ell^{k-1} \pmod p, \\ \frac{q-1}{2\ell^k}(-2\ell^k + \ell p - p); & \text{if } u^2 \not\equiv \phi(\ell^k)^2 \pmod p \text{ but } u^2 \equiv \ell^{2(k-1)} \pmod p, \\ 1 - q; & \text{if } u^2 \not\equiv \phi(\ell^k)^2 \text{ and } u^2 \not\equiv \ell^{2(k-1)} \pmod p. \end{cases}$$

(d) *When $u \neq 0$ and $b \neq 0$. Then $w(u,b) = 1$ if $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$ and $u^2 \not\equiv \ell^{2(k-1)} \pmod p$. Moreover, we have the following results.*

(i)

$$w(u,b) = \begin{cases} 1 + p\sqrt{q} - p\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{if } i_b = 0, \\ 1 - p\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{otherwise,} \end{cases}$$

if $u \equiv \phi(\ell^k) \pmod p$ with $u^2 \not\equiv \ell^{2(k-1)} \pmod p$;

(ii)

$$w(u,b) = \begin{cases} 1 + p\sqrt{q} - p\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{if } i_b = \ell^k, \\ 1 - p\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{otherwise,} \end{cases}$$

if $u \equiv -\phi(\ell^k) \pmod p$ with $u^2 \not\equiv \ell^{2(k-1)} \pmod p$;

(iii)

$$w(u,b) = \begin{cases} 1 + p\sqrt{q} - p(\ell-1)\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{if } i_b \in P_1^{(2)} \cup P_3^{(2)}, \\ 1 - p(\ell-1)\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{otherwise,} \end{cases}$$

if $u \equiv \ell^{k-1} \pmod p$ with $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$;

(iv)

$$w(u,b) = \begin{cases} 1 + p\sqrt{q} - p(\ell-1)\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{if } i_b \in P_1^{(3)} \cup P_3^{(3)}, \\ 1 - p(\ell-1)\left(\frac{\sqrt{q}+1}{2\ell^k}\right); & \text{otherwise,} \end{cases}$$

if $u \equiv -\ell^{k-1} \pmod p$ with $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$;

(v)

$$w(u,b) = \begin{cases} 1 + p\sqrt{q} - p\left(\frac{\sqrt{q}+1}{2\ell^{k-1}}\right); & \text{if } i_b \in \{0\} \cup P_1^{(2)} \cup P_3^{(2)}, \\ 1 - p\left(\frac{\sqrt{q}+1}{2\ell^{k-1}}\right); & \text{otherwise,} \end{cases}$$

if $u \equiv \phi(\ell^k) \equiv \ell^{k-1} \pmod p$; and

(vi)

$$w(u,b)=\begin{cases}1+p\sqrt{q}-p\left(\frac{\sqrt{q}+1}{2\ell^{k-1}}\right);\ \textit{if } i_b\in\{\ell^k\}\cup P_1^{(3)}\cup P_3^{(3)},\\ 1-p\left(\frac{\sqrt{q}+1}{2\ell^{k-1}}\right);\ \textit{otherwise},\end{cases}$$

if $u\equiv-\phi(\ell^k)\equiv-\ell^{k-1}\pmod p$.

It is to be noted that when $u\neq 0$, then $u\equiv\phi(\ell^k)\equiv\ell^{k-1}\pmod p$ would imply $u\not\equiv-\phi(\ell^k)\pmod p$ and $u\not\equiv-\ell^{k-1}\pmod p$; and $u\equiv-\phi(\ell^k)\equiv-\ell^{k-1}\pmod p$ would imply $u\not\equiv\phi(\ell^k)\pmod p$ and $u\not\equiv\ell^{k-1}\pmod p$.

2.2 Cyclotomic fields $\mathbb{Q}(\zeta_p)$

Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{Z}$ be the set of integers. The cyclotomic field $\mathbb{Q}(\zeta_p)$ is obtained from the field $\mathbb{Q}$ by adjoining the p-th primitive root of unity ζ_p.

Lemma 6. [18] *Let $F=\mathbb{Q}(\zeta_p)$ be the p-th cyclotomic field over $\mathbb{Q}$. Then*

(1) *The ring of integers in $F=\mathbb{Q}(\zeta_p)$ is defined as $\mathbb{Z}(\zeta_p)$, and an integral basis of $\mathbb{Z}(\zeta_p)$ is the set $\{\zeta_p^i:1\leq i\leq p-1\}$.*

(2) *The field extension $F/\mathbb{Q}$ is Galois of degree $p-1$ and Galois group $Gal(F/\mathbb{Q})=\{\sigma_z:z\in\mathbb{F}_p^*\}$, where the automorphism σ_z of F is defined by $\sigma_z(\zeta_p)=\zeta_p^z$.*

(3) *The field F has a unique quadratic subfield $L=\mathbb{Q}(\sqrt{p^*})$, where $p^*=\eta_1(-1)p$, η_1 is the quadratic character over $\mathbb{F}_p^*$. For $z\in\mathbb{F}_p^*$, $\sigma_z(\sqrt{p^*})=\eta_1(z)\sqrt{p^*}$. Therefore, the Galois group $Gal(L/\mathbb{Q})$ is $\{1,\sigma_w\}$, where w is any quadratic non-residue in $\mathbb{F}_p$.*

By Lemma 6, we get, $\sqrt{p^*}^m=\eta_1^{\frac{m}{2}}(-1)p^{\frac{m}{2}}$, $\sigma_z(\zeta_p^y)=\zeta_p^{zy}$ and $\sigma_z(\sqrt{p^*}^m)=\eta_1^m(z)\sqrt{p^*}^m=\sqrt{p^*}^m$, if m is an even positive integer.

2.3 Weakly regular bent functions

Let $f:\mathbb{F}_{p^m}\to\mathbb{F}_p$ be a p-ary function. The Walsh transform of f is defined by

$$\mathcal{W}_f(\lambda)=\sum_{x\in\mathbb{F}_q}\zeta_p^{f(x)-\mathrm{Tr}_1^m(\lambda x)},$$

where $\lambda\in\mathbb{F}_{p^m}$ and ζ_p is a complex primitive p-th root of unity. The p-ary function f is said to be bent if $|\mathcal{W}_f(\lambda)|^2=p^m$ for all $\lambda\in\mathbb{F}_{p^m}$. A bent function f is said to be regular bent if there exists some p-ary function $f^*:\mathbb{F}_{p^m}\to\mathbb{F}_p$ such that $\mathcal{W}_f(\lambda)=p^{m/2}\zeta_p^{f^*(\lambda)}$ for every $\lambda\in\mathbb{F}_{p^m}$. The bent function f is called weakly regular bent if there exists a p-ary function f^* and a complex number u with unit magnitude satisfying $\mathcal{W}_f(\lambda)=up^{\frac{m}{2}}\zeta_p^{f^*(\lambda)}$ for every $\lambda\in\mathbb{F}_{p^m}$. Such a function f^* is called the dual of f. The dual of a weakly regular bent function is also weakly regular bent. From [31], a weakly regular bent function $f(x)$ satisfies $\mathcal{W}_f(\lambda) = \epsilon_f \sqrt{p^*}^m \zeta_p^{f^*(\lambda)}$, where $\epsilon_f \in \{\pm 1\}$ is called the sign of the Walsh transform of $f(x)$ and $p^* = (-1)^{\frac{p-1}{2}} p$, and the sign of the Walsh transform of $f^*(x)$ is $(-1)^{(\frac{p-1}{2})m}\epsilon_f$. We denote $\mathcal{RF}$ to be the set of p-ary weakly regular bent functions $f(x)$ with $f(0) = 0$ and $f(cx) = c^{k_f} f(x)$ for any $c \in \mathbb{F}_p^*$, where k_f is a positive even integer with $\gcd(k_f - 1, p - 1) = 1$. It is well known that $f^*(0) = 0$ when $f \in \mathcal{RF}$, and there is an even positive integer l_f satisfying $\gcd(l_f - 1, p - 1) = 1$ such that $f^*(cx) = c^{l_f} f^*(x)$ for any $c \in \mathbb{F}_p^*$ and $x \in \mathbb{F}_{p^m}$ (see [34, Proposition 2.4 and 2.5]). Weakly regular bent functions can be constructed from known planar functions. A function g from $\mathbb{F}_{p^m}$ to $\mathbb{F}_{p^m}$ is said to be planar if $|\{x \in \mathbb{F}_{p^m} : g(x + a) - g(x) = b\}| = 1$ for any $a \in \mathbb{F}_{p^m}^*$ and $b \in \mathbb{F}_{p^m}$. It is known that the p-ary functions $f(x) = \mathrm{Tr}_1^m(\alpha g(x))$, $\alpha \in \mathbb{F}_{p^m}$ for which $g(x) = \sum_{0 \le i,j \le m-1} a_{ij} x^{p^i + p^j}$ are (DO-type) planar, are all weakly regular quadratic bent functions, when m is even. For more information, we refer to [5, 19, 31].

In this paper, we mainly focus on those functions $f \in \mathcal{RF}$ such that f^* is a quadratic form (i.e., $l_f = 2$). The known weakly regular bent functions satisfying $l_f = 2$ are provided in [21, Table 4].

Lemma 7. [34, Lemma 8] *Let $f(x) \in \mathcal{RF}$ such that $\mathcal{W}_f(0) = \epsilon_f \sqrt{p^*}^m$, where $\epsilon_f \in \{1, -1\}$. Let f^* be the dual of f. Define $N_{f^*}(\lambda) = |\{a \in \mathbb{F}_{p^m} : f^*(a) = \lambda\}|$, where $\lambda \in \mathbb{F}_p$. Then we have the following.*

(1) *When m is even, we have*

$$N_{f^*}(\lambda) = \begin{cases} p^{m-1} + \epsilon_f (p-1) \frac{\sqrt{p^*}^m}{p}; & \text{if } \lambda = 0, \\ p^{m-1} - \epsilon_f \frac{\sqrt{p^*}^m}{p}; & \text{if } \lambda \in \mathbb{F}_p^*. \end{cases}$$

(2) *When m is odd, we have*

$$N_{f^*}(\lambda) = \begin{cases} p^{m-1}; & \text{if } \lambda = 0, \\ p^{m-1} + \epsilon_f \eta_1(-1) \sqrt{p^*}^{m-1}; & \text{if } \eta_1(\lambda) = 1, \\ p^{m-1} - \epsilon_f \eta_1(-1) \sqrt{p^*}^{m-1}; & \text{if } \eta_1(\lambda) = -1. \end{cases}$$

3 Linear codes from the defining set D_u

Lemma 8. *If $D_u = \{(x, y) \in \mathbb{F}_q^2 \backslash \{(0, 0)\} : \mathrm{Tr}_1^e(x + y^{\frac{q-1}{2\ell^k}}) = u\}$, where $u \in \mathbb{F}_p$. Let $n_1 = |D_u|$. Then*

$$n_1 = \begin{cases} p^{2e-1} - 1; & \text{if } u = 0, \\ p^{2e-1}; & \text{if } u \in \mathbb{F}_p^*. \end{cases}$$

Proof. From the properties of the canonical additive characters and using the fact $\sum_{x \in \mathbb{F}_q} \zeta_p^{z \, \mathrm{Tr}_1^e(x)} =$

0 for all $z \in \mathbb{F}_p^*$, we obtain

$$
\begin{aligned}
n_1 &= \frac{1}{p}\sum_{x,y\in\mathbb{F}_q}\sum_{z\in\mathbb{F}_p}\zeta_p^{z\left(\mathrm{Tr}_1^e(x+y^{\frac{q-1}{2\ell^k}})-u\right)} - \frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{-uz}\\
&= \frac{1}{p}\left(q^2 + \sum_{z\in\mathbb{F}_p^*}\zeta_p^{-uz}\sum_{x\in\mathbb{F}_q}\zeta_p^{z\,\mathrm{Tr}_1^e(x)}\sum_{y\in\mathbb{F}_q}\zeta_p^{z\,\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}})}\right) - \frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{-uz}\\
&= p^{2e-1} - \frac{1}{p}\sum_{z\in\mathbb{F}_p}\zeta_p^{-uz}\\
&= \begin{cases} p^{2e-1}-1; \text{ if } u=0,\\ p^{2e-1}; \text{ if } u\in\mathbb{F}_p^*. \end{cases}
\end{aligned}
$$

□

Define a notation

$$N_{\gamma,\delta}^{(1)} = |\{(x,y)\in D_u : \mathrm{Tr}_1^e(\gamma x+\delta y)=0\}|, \tag{9}$$

and, the Hamming weight of the codeword $c(\gamma,\delta)\in\mathcal{C}_{D_u}$ is as follows:

$$\mathrm{wt}(c(\gamma,\delta)) = n_1 - N_{\gamma,\delta}^{(1)}. \tag{10}$$

Lemma 9. *Let $\gamma,\delta\in\mathbb{F}_q$ with $(\gamma,\delta)\neq(0,0)$ and $N_{\gamma,\delta}^{(1)}$ be defined in (9). For $b\in\mathbb{F}_q^*$, let $\mathrm{Ind}_\alpha(b)$ be the index of b with respect to the base α, and $i_b = -\mathrm{Ind}_\alpha(b) \pmod{2\ell^k}$. Assume that the set P and its partitions be defined as in section 2.1. Then the explicit value of $N_{\gamma,\delta}^{(1)}$ is given as follows:*

(a) *If $u=0$, then*

(i) *When $\ell\equiv 1 \pmod p$, then*

$$
N_{\gamma,\delta}^{(1)} = \begin{cases}
p^{2e-1}-1-p^{e-1}(\ell-1)\frac{q-1}{\ell^k}; \textit{ if } \gamma\in\mathbb{F}_p^* \textit{ and } \delta=0,\\
p^{2e-2}-1+p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{\ell^k}-p^{e-1}\sqrt{q}; \textit{ if } \gamma\in\mathbb{F}_p^* \textit{ and } \delta\neq 0\\
\qquad\qquad \textit{with } i_\delta\in P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\
p^{2e-2}-1+p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{\ell^k}; \textit{ if } \gamma\in\mathbb{F}_p^* \textit{ and } \delta\neq 0\\
\qquad\qquad \textit{with } i_\delta\in P\backslash(P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)}),\\
p^{2e-2}-1; \textit{ otherwise.}
\end{cases}
$$

(ii) *When $\ell\not\equiv 1 \pmod p$, then*

$$
N_{\gamma,\delta}^{(1)} = \begin{cases}
p^{2e-1}-1-p^{e-1}\frac{q-1}{\ell^{k-1}}; \textit{ if } \gamma\in\mathbb{F}_p^* \textit{ and } \delta=0,\\
p^{2e-2}-1+p^{e-1}\frac{\sqrt{q}+1}{\ell^{k-1}}-p^{e-1}\sqrt{q}; \textit{ if } \gamma\in\mathbb{F}_p^* \textit{ and } \delta\neq 0 \textit{ with}\\
\qquad\qquad i_\delta\in\{0,\ell^k\}\cup P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\
p^{2e-2}-1+p^{e-1}\frac{\sqrt{q}+1}{\ell^{k-1}}; \textit{ if } \gamma\in\mathbb{F}_p^* \textit{ and } \delta\neq 0 \textit{ with}\\
\qquad\qquad i_\delta\in P\backslash(\{0,\ell^k\}\cup P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)}),\\
p^{2e-2}-1; \textit{ otherwise.}
\end{cases}
$$

(b) *If $u \neq 0$, then*

(i) *When $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$ and $u^2 \not\equiv \ell^{2(k-1)} \pmod p$, then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} 0; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2}; & \text{otherwise.} \end{cases}$$

(ii) *When $u \equiv \phi(\ell^k) \pmod p$ and $u^2 \not\equiv \ell^{2(k-1)} \pmod p$, then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} p^{e-1}\frac{q-1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2} + p^{e-1}\sqrt{q} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta = 0, \\ p^{2e-2} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P\backslash\{0\}, \\ p^{2e-2}; & \text{otherwise.} \end{cases}$$

(iii) *When $u \equiv -\phi(\ell^k) \pmod p$ and $u^2 \not\equiv \ell^{2(k-1)} \pmod p$, then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} p^{e-1}\frac{q-1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2} + p^{e-1}\sqrt{q} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta = \ell^k, \\ p^{2e-2} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P\backslash\{\ell^k\}, \\ p^{2e-2}; & \text{otherwise.} \end{cases}$$

(iv) *When $u \equiv \phi(\ell^k) \equiv \ell^{k-1} \pmod p$, then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} p^{e-1}\frac{q-1}{2\ell^{k-1}}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2} + p^{e-1}\sqrt{q} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^{k-1}}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in \{0\} \cup P_1^{(2)} \cup P_3^{(2)}, \\ p^{2e-2} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^{k-1}}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P\backslash(\{0\} \cup P_1^{(2)} \cup P_3^{(2)}), \\ p^{2e-2}; & \text{otherwise.} \end{cases}$$

(v) *When $u \equiv -\phi(\ell^k) \equiv -\ell^{k-1} \pmod p$, then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} p^{e-1}\frac{q-1}{2\ell^{k-1}}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2} + p^{e-1}\sqrt{q} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^{k-1}}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in \{\ell^k\} \cup P_1^{(3)} \cup P_3^{(3)}, \\ p^{2e-2} - p^{e-1}\frac{\sqrt{q}+1}{2\ell^{k-1}}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P\backslash(\{\ell^k\} \cup P_1^{(3)} \cup P_3^{(3)}), \\ p^{2e-2}; & \text{otherwise.} \end{cases}$$

(vi) *When $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$ and $u \equiv \ell^{(k-1)} \pmod p$, then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} p^{e-1}(\ell-1)\frac{q-1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2} + p^{e-1}\sqrt{q} - p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P_1^{(2)} \cup P_3^{(2)}, \\ p^{2e-2} - p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{2\ell^k}; & \text{if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P\backslash(P_1^{(2)} \cup P_3^{(2)}), \\ p^{2e-2}; & \text{otherwise.} \end{cases}$$

(vii) *When* $u^2 \not\equiv \phi(\ell^k)^2 \pmod{p}$ *and* $u \equiv -\ell^{(k-1)} \pmod{p}$, *then*

$$N_{\gamma,\delta}^{(1)} = \begin{cases} p^{e-1}(\ell-1)\frac{q-1}{2\ell^k};\ \textit{if } \gamma \in \mathbb{F}_p^* \textit{ and } \delta = 0, \\ p^{2e-2} + p^{e-1}\sqrt{q} - p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{2\ell^k};\ \textit{if } \gamma \in \mathbb{F}_p^* \textit{ and } \delta \neq 0 \textit{ with } i_\delta \in P_1^{(3)} \cup P_3^{(3)}, \\ p^{2e-2} - p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{2\ell^k};\ \textit{if } \gamma \in \mathbb{F}_p^* \textit{ and } \delta \neq 0 \textit{ with } i_\delta \in P \backslash (P_1^{(3)} \cup P_3^{(3)}), \\ p^{2e-2};\ \textit{otherwise.} \end{cases}$$

Proof. According to the definition, we have

$$\begin{aligned} N_{\gamma,\delta}^{(1)} &= |\{(x,y) \in \mathbb{F}_q^2 \backslash \{(0,0)\} : \operatorname{Tr}_1^e(x + y^{\frac{q-1}{2\ell^k}}) = u \text{ and } \operatorname{Tr}_1^e(\gamma x + \delta y) = 0\}| \\ &= \sum_{x,y \in \mathbb{F}_q} \left(\frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{z\{\operatorname{Tr}_1^e(x+y^{\frac{q-1}{2\ell^k}})-u\}} \right) \times \left(\frac{1}{p} \sum_{\delta \in \mathbb{F}_p} \zeta_p^{\delta \operatorname{Tr}_1^e(\gamma x + \delta y)} \right) - \frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{-uz} \\ &= \frac{1}{p^2} \sum_{x,y \in \mathbb{F}_q} \left(\sum_{z \in \mathbb{F}_p^*} \zeta_p^{z\{\operatorname{Tr}_1^e(x+y^{\frac{q-1}{2\ell^k}})-u\}} + 1 \right) \left(\sum_{\delta \in \mathbb{F}_p^*} \zeta_p^{\delta \operatorname{Tr}_1^e(\gamma x + \delta y)} + 1 \right) - \frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{-uz} \\ &= \frac{1}{p^2} \left(q^2 + \sum_{x,y \in \mathbb{F}_q} \sum_{z,\delta \in \mathbb{F}_p^*} \zeta_p^{z(\operatorname{Tr}_1^e(x+y^{\frac{q-1}{2\ell^k}})-u)+\delta \operatorname{Tr}_1^e(\gamma x + \delta y)} \right) - \frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{-uz} \\ &= p^{2e-2} + \frac{1}{p^2}\Omega_1 - \frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{-uz}, \end{aligned} \tag{11}$$

where $\Omega_1 = \sum_{x,y \in \mathbb{F}_q} \sum_{z,w \in \mathbb{F}_p^*} \zeta_p^{z(\operatorname{Tr}_1^e(x+y^{\frac{q-1}{2\ell^k}})-u)+w \operatorname{Tr}_1^e(\gamma x + \delta y)}$.

Observe that

$$\begin{aligned} \Omega_1 &= \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-uz} \sum_{w \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{z \operatorname{Tr}_1^e\left((1+\gamma\frac{w}{z})x\right)} \times \sum_{y \in \mathbb{F}_q} \zeta_p^{z \operatorname{Tr}_1^e(y^{\frac{q-1}{2\ell^k}} + \delta\frac{w}{z}y)} \\ &= \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-uz} \sum_{w \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{z \operatorname{Tr}_1^e((1+\gamma w)x)} \times \sum_{y \in \mathbb{F}_q} \zeta_p^{z \operatorname{Tr}_1^e(y^{\frac{q-1}{2\ell^k}} + \delta w y)} \end{aligned} \tag{12}$$

Eq. (12) holds because for a fixed z, as w varies over $\mathbb{F}_p^*$, so does $\frac{w}{z}$. If $\gamma = 0$ and $\delta \neq 0$, it is easy to see that $\Omega_1 = 0$. It then implies that $N_{0,\delta}^{(1)} = \begin{cases} p^{2e-2} - 1;\ \text{if } u = 0, \\ p^{2e-2};\ \text{if } u \in \mathbb{F}_p^*. \end{cases}$

If $\gamma \neq 0$, then from (12), we have

$$\Omega_1 = \begin{cases} 0;\ \text{if } \gamma \notin \mathbb{F}_p^*, \\ q \sum_{z \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_q} \zeta_p^{z\{\operatorname{Tr}_1^e(y^{\frac{q-1}{2\ell^k}} - \frac{\delta}{\gamma}y) - u\}};\ \text{if } \gamma \in \mathbb{F}_p^*. \end{cases}$$

When $\gamma \in \mathbb{F}_p^*$, from the definition in Eq. (6) and Lemma 3 (e), we have

$$\begin{aligned}
\Omega_1 &= q \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-uz} \left(S_{2\ell^k}(z, -\frac{\delta}{\gamma} z) + 1 \right) \\
&= q \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-uz} \left(S_{2\ell^k}(z, \delta) + 1 \right) \\
&= q \left(w(u, \delta) + \sum_{z \in \mathbb{F}_p^*} \zeta_p^{-uz} \right)
\end{aligned} \tag{13}$$

For the case of $u = 0$ and $\ell \equiv 1 \pmod p$, by using Lemma 5, we obtain

$$\Omega_1 = \begin{cases} q\left(q(p-1) - p(\ell-1)\frac{q-1}{\ell^k}\right); \text{ if } \delta = 0, \\ pq\left((\ell-1)\frac{\sqrt{q}+1}{\ell^k} - \sqrt{q}\right); \text{ if } \delta \neq 0 \text{ with } i_\delta \in P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)}, \\ pq(\ell-1)\frac{\sqrt{q}+1}{\ell^k}; \text{ otherwise.} \end{cases} \tag{14}$$

This implies that when $u = 0$ and $\ell \equiv 1 \pmod p$, by combining Eq. (11) and (14), we obtain

$$\begin{aligned}
N_{\gamma,\delta}^{(1)} &= \begin{cases} p^{2e-2} + \frac{1}{p^2}\Omega_1 - 1; \text{ if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \in \mathbb{F}_q, \\ p^{2e-2} - 1; \text{ otherwise.} \end{cases} \\
&= \begin{cases} p^{2e-1} - 1 - p^{e-1}(\ell-1)\frac{q-1}{\ell^k}; \text{ if } \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0, \\ p^{2e-2} - 1 + p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{\ell^k} - p^{e-1}\sqrt{q}; \text{ if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \\ \qquad \text{with } i_\delta \in P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)}, \\ p^{2e-2} - 1 + p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{\ell^k}; \text{ if } \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \\ \qquad \text{with } i_\delta \in P \backslash (P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)}), \\ p^{2e-2} - 1; \text{ otherwise.} \end{cases}
\end{aligned}$$

Consequently, $N_{\gamma,\delta}^{(1)}$ can be determined accordingly for the remaining cases. The proof is completed. □

Theorem 1. *Assume that $q = p^e$ and $e = \phi(\ell^k)$. Let $\mathcal{C}_{D_u} = \{(\mathrm{Tr}_1^e(\gamma x + \delta y))_{(x,y) \in D_u} : \gamma, \delta \in \mathbb{F}_q\}$, where D_u is defined in (4) with $u \in \mathbb{F}_p$. Then,*

(1) *If $u = 0$ and $\ell \equiv 1 \pmod p$, then $\mathcal{C}_{D_0}$ is a four-weight $[p^{2e-1} - 1, 2e]$ linear code over $\mathbb{F}_p$ and its weight distribution given in Table 2.*

(2) *If $u = 0$ and $\ell \not\equiv 1 \pmod p$, then $\mathcal{C}_{D_0}$ is a four-weight $[p^{2e-1} - 1, 2e]$ linear code over $\mathbb{F}_p$ and its weight distribution given in Table 3.*

Table 2:

Weight	Frequency
0	1
$p^{e-1}(\ell-1)\frac{q-1}{\ell^k}$	$p-1$
$p^{2e-2}(p-1)+p^{e-1}\left(\sqrt{q}-(\ell-1)\frac{\sqrt{q}+1}{\ell^k}\right)$	$(p-1)(\ell-1)\frac{q-1}{\ell^k}$
$p^{2e-2}(p-1)-p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{\ell^k}$	$(p-1)(\ell^k-\ell+1)\frac{q-1}{\ell^k}$
$p^{2e-2}(p-1)$	$(q-1)+(q-p)q$

Table 3:

Weight	Frequency
0	1
$p^{e-1}\frac{q-1}{\ell^{k-1}}$	$p-1$
$p^{2e-2}(p-1)+p^{e-1}\left(\sqrt{q}-\frac{\sqrt{q}+1}{\ell^{k-1}}\right)$	$(p-1)\frac{q-1}{\ell^{k-1}}$
$p^{2e-2}(p-1)-p^{e-1}\frac{\sqrt{q}+1}{\ell^{k-1}}$	$(p-1)(\ell^{k-1}-1)\frac{q-1}{\ell^{k-1}}$
$p^{2e-2}(p-1)$	$(q-1)+(q-p)q$

Table 4:

Weight	Frequency
0	1
p^{2e-1}	$p-1$
$p^{2e-2}(p-1)$	q^2-p

(3) *If* $u \neq 0$, $u^2 \not\equiv \phi(\ell^k)^2 \pmod{p}$ *and* $u^2 \not\equiv \ell^{2(k-1)} \pmod{p}$, *then* $\mathcal{C}_{D_u}$ *is a two-weight* $[p^{2e-1}, 2e]$ *linear code over* $\mathbb{F}_p$ *and its weight distribution given in Table* 4.

(4) *If* $u \neq 0$, $u^2 \equiv \phi(\ell^k)^2 \pmod{p}$ *and* $u^2 \not\equiv \ell^{2(k-1)} \pmod{p}$, *then* $\mathcal{C}_{D_u}$ *is a four-weight* $[p^{2e-1}, 2e]$ *linear code over* $\mathbb{F}_p$ *and its weight distribution given in Table* 5.

Table 5:

Weight	Frequency
0	1
$p^{2e-1} - p^{e-1}\frac{q-1}{2\ell^k}$	$p-1$
$p^{2e-2}(p-1) - p^{e-1}\sqrt{q} + p^{e-1}\frac{\sqrt{q}+1}{2\ell^k}$	$(p-1)\frac{q-1}{2\ell^k}$
$p^{2e-2}(p-1) + p^{e-1}\frac{\sqrt{q}+1}{2\ell^k}$	$(p-1)(2\ell^k-1)\frac{q-1}{2\ell^k}$
$p^{2e-2}(p-1)$	$(q-1)+q(q-p)$

(5) *If* $u \neq 0$ *with either* $u \equiv \phi(\ell^k) \equiv \ell^{k-1} \pmod{p}$ *or* $u \equiv -\phi(\ell^k) \equiv -\ell^{k-1} \pmod{p}$, *then* $\mathcal{C}_{D_u}$ *is a four-weight* $[p^{2e-1}, 2e]$ *linear code over* $\mathbb{F}_p$ *and its weight distribution given in Table* 6.

Table 6:

Weight	Frequency
0	1
$p^{2e-1} - p^{e-1}\frac{q-1}{2\ell^{k-1}}$	$p-1$
$p^{2e-2}(p-1) - p^{e-1}\sqrt{q} + p^{e-1}\frac{\sqrt{q}+1}{2\ell^{k-1}}$	$(p-1)\frac{q-1}{2\ell^{k-1}}$
$p^{2e-2}(p-1) + p^{e-1}\frac{\sqrt{q}+1}{2\ell^{k-1}}$	$(p-1)(2\ell^{k-1}-1)\frac{q-1}{2\ell^{k-1}}$
$p^{2e-2}(p-1)$	$(q-1)+q(q-p)$

(6) *If* $u \neq 0$, $u^2 \not\equiv \phi(\ell^k)^2 \pmod{p}$ *and* $u^2 \equiv \ell^{2(k-1)} \pmod{p}$, *then* $\mathcal{C}_{D_u}$ *is a four-weight* $[p^{2e-1}, 2e]$ *linear code over* $\mathbb{F}_p$ *and its weight distribution given in Table* 7.

Proof. We will prove only for the case $u = 0$ and $\ell \equiv 1 \pmod{p}$ as the remaining cases can be proved in a similar manner. From Lemma 9 and Eq. (10), we conclude that

Table 7:

Weight	Frequency
0	1
$p^{2e-1} - p^{e-1}(\ell-1)\frac{q-1}{2\ell^k}$	$p-1$
$p^{2e-2}(p-1) - p^{e-1}\sqrt{q} + p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{2\ell^k}$	$(p-1)(\ell-1)\frac{q-1}{2\ell^k}$
$p^{2e-2}(p-1) + p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{2\ell^k}$	$(p-1)(2\ell^k-\ell+1)\frac{q-1}{2\ell^k}$
$p^{2e-2}(p-1)$	$(q-1)+q(q-p)$

$\mathrm{wt}(c(\gamma,\delta)) \in \{w_1, w_2, w_3, w_4\}$ for every $(\gamma,\delta) \in \mathbb{F}_q^2 \backslash \{(0,0)\}$, where

$$\begin{aligned}
w_1 &= p^{e-1}(\ell-1)\frac{q-1}{\ell^k},\\
w_2 &= p^{2e-2}(p-1) + p^{e-1}\left(\sqrt{q} - (\ell-1)\frac{\sqrt{q}+1}{\ell^k}\right),\\
w_3 &= p^{2e-2}(p-1) - p^{e-1}(\ell-1)\frac{\sqrt{q}+1}{\ell^k},\\
w_4 &= p^{2e-2}(p-1).
\end{aligned}$$

In the following, we will show their frequencies A_{w_1}, A_{w_2}, A_{w_3} and A_{w_4}, respectively.

Recall that, for $\delta \in \mathbb{F}_q^*$, $\mathrm{Ind}_\alpha(\delta)$ be the index of δ with respect to the base α, where α generates $\mathbb{F}_q^*$ and $i_\delta = -\mathrm{Ind}_\alpha(\delta) \pmod{2\ell^k}$. Since $i_\delta \in P = \{0, 1, \cdots, 2\ell^k - 1\}$, then for any subset Q of P, it is easy to see that $|\{\delta \in \mathbb{F}_q^* : i_\delta \in Q\}| = \frac{q-1}{2\ell^k}|Q|$. Note that $|P_1^{(2)}| = |P_1^{(3)}| = |P_3^{(2)}| = |P_3^{(3)}| = \frac{\ell-1}{2}$. Therefore, we obtain

$$\begin{aligned}
A_{w_1} &= |\{(\gamma,\delta) \in \mathbb{F}_q^2 : \gamma \in \mathbb{F}_p^* \text{ and } \delta = 0\}| = (p-1),\\
A_{w_2} &= |\{(\gamma,\delta) \in \mathbb{F}_q^2 : \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)}\}|\\
&= (p-1)(\ell-1)\frac{q-1}{\ell^k},\\
A_{w_3} &= |\{(\gamma,\delta) \in \mathbb{F}_q^2 : \gamma \in \mathbb{F}_p^* \text{ and } \delta \neq 0 \text{ with } i_\delta \in P\backslash(P_1^{(2)} \cup P_1^{(3)} \cup P_3^{(2)} \cup P_3^{(3)})\}|\\
&= (p-1)(\ell^k-\ell+1)\frac{q-1}{\ell^k},\\
A_{w_4} &= |\{(\gamma,\delta) \in \mathbb{F}_q^2 : \gamma = 0 \text{ and } \delta \in \mathbb{F}_q^*\}| + |\{(\gamma,\delta) \in \mathbb{F}_q^2 : \gamma \in \mathbb{F}_q\backslash\mathbb{F}_p \text{ and } \delta \in \mathbb{F}_q\}|\\
&= (q-1) + (q-p)q.
\end{aligned}$$

From the above arguments, we see that $\mathrm{wt}(c(\gamma,\delta)) \neq 0$ for all $(\gamma,\delta) \in \mathbb{F}_q^2\backslash\{(0,0)\}$. Therefore, $|\mathcal{C}_{D_0}| = q^2$ and then $\mathcal{C}_{D_0}$ is of dimension $2e$. The length of the code $\mathcal{C}_{D_u}$ is obvious from Lemma 8. This completes the proof. □

Example 1. *Consider $p=5$, $\ell=3$, $k=1$ and $u=0$, then $\ell \not\equiv 1 \pmod p$ and $e = \phi(6) = 2$. The code $\mathcal{C}_{D_0}$ has parameters $[124, 4, 95]$ and weight enumerator $1 + 96z^{95} + 524z^{100} + 4z^{120}$.*

Example 2. *Consider* $p = 7$, $\ell = 5$, $k = 1$, *and* $u(\neq 0) \in \mathbb{F}_7$ *such that* $u^2 \equiv 2 \pmod 7$ *and* $u^2 \not\equiv 1 \pmod 7$, *then* $u \in \{3, 4\}$ *and* $e = \phi(10) = 4$. *The code* $\mathcal{C}_{D_u}$ *has parameters* $[823543, 8, 690802]$ *and weight enumerator* $1 + 1440z^{690802} + 5750394z^{705894} + 12960z^{707609} + 6z^{741223}$.

Example 3. *Consider* $p = 5$, $\ell = 3$, $k = 1$ *and* $u(\neq 0) \in \mathbb{F}_5$ *such that* $u^2 \equiv 1 \pmod 5$ *and* $u^2 \not\equiv 4 \pmod 5$, *then* $u \in \{1, 4\}$ *and* $e = \phi(6) = 2$. *The code* $\mathcal{C}_{D_u}$ *has parameters* $[125, 4, 85]$ *and weight enumerator* $1 + 36z^{85} + 524z^{100} + 64z^{110}$.

The following corollary provides an optimal class of linear codes under a special condition and can be proved directly from Theorem 1.

Corollary 1. *Let* $u \in \mathbb{F}_p^*$ *be such that* $u^2 \not\equiv \phi(\ell^k)^2 \pmod p$ *and* $u^2 \not\equiv \ell^{2(k-1)} \pmod p$, *then the code* $\mathcal{C}_{D_u}$ *defined in Theorem 1 is an optimal two-weight* $[p^{2e-1}, 2e, p^{2e-2}(p-1)]$ *linear code over* $\mathbb{F}_p$ *that meets the Griesmer bound, and its weight enumerator is* $1 + (p^{2e} - p)z^{p^{2e-2}(p-1)} + (p-1)z^{p^{2e-1}}$.

Example 4. *Consider* $p = 7$, $\ell = 5$, $k = 1$, *and* $u(\neq 0) \in \mathbb{F}_7$ *such that* $u^2 \not\equiv 1, 2 \pmod 7$, *then* $u \in \{2, 5\}$ *and* $e = \phi(10) = 4$. *The code* $\mathcal{C}_{D_u}$ *has parameters* $[823543, 8, 705894]$ *and weight enumerator* $1 + 5764794z^{705894} + 6z^{823543}$. *The code* $\mathcal{C}_{D_u}$ *is optimal according to the Griesmer bound.*

4 Linear codes from the defining set D'

In this section, we always assume $f \in \mathcal{RF}$ such that f^* is a quadratic form (as discussed in section 2.3) and ϵ_f be the sign of the Walsh transform of $f(x)$.

Lemma 10. *If* $D' = \{(x, y) \in \mathbb{F}_q^2 \backslash (0, 0) : f(x) + \mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}}) = 0\}$, *where* $u \in \mathbb{F}_p$. *Let* $n_2 = |D'|$. *Then*

$$n_2 = \begin{cases} \frac{q^2}{p} - 1 + \frac{\epsilon_f \sqrt{p^*}^e}{p}(q(p-1) - p(\ell-1)\frac{q-1}{\ell^k}); & \text{if } \ell \equiv 1 \pmod p, \\ \frac{q^2}{p} - 1 + \frac{\epsilon_f \sqrt{p^*}^e}{p}(q(p-1) - p\frac{q-1}{\ell^{k-1}}); & \text{if } \ell \not\equiv 1 \pmod p. \end{cases}$$

Proof. Since, $e = \phi(\ell^k)$ is even, we can say $\sigma_z(\sqrt{p^*}^e) = \sqrt{p^*}^e$. By using the definition of

(8), we have

$$
\begin{aligned}
n_2 &= \frac{1}{p}\sum_{x,y\in\mathbb{F}_q}\sum_{z\in\mathbb{F}_p}\zeta_p^{z(f(x)+\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}}))}-1\\
&= \frac{q^2}{p}-1+\frac{1}{p}\sum_{x,y\in\mathbb{F}_q}\sum_{z\in\mathbb{F}_p^*}\zeta_p^{z(f(x)+\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}}))}\\
&= \frac{q^2}{p}-1+\frac{1}{p}\sum_{z\in\mathbb{F}_p^*}\sigma_z\left(\epsilon_f\sqrt{p^*}^e\right)\left(S_{2\ell^k}(z,0)+1\right)\\
&= \frac{q^2}{p}-1+\frac{\epsilon_f\sqrt{p^*}^e}{p}(w(0,0)+p-1) \qquad (15)
\end{aligned}
$$

Hence, the result follows from Lemma 5 (a) and Eq. (15). □

Define a notation

$$N_{\gamma,\delta}^{(2)} = |\{(x,y)\in D^{'} : \mathrm{Tr}_1^e(\gamma x+\delta y)=0\}|, \qquad (16)$$

and, the Hamming weight of the codeword $c(\gamma,\delta)\in\mathcal{C}_{D'}$ is as follows:

$$\mathrm{wt}(c(\gamma,\delta)) = n_2 - N_{\gamma,\delta}^{(2)}. \qquad (17)$$

Lemma 11. *Let $\gamma,\delta\in\mathbb{F}_q$ with $(\gamma,\delta)\neq(0,0)$ and $N_{\gamma,\delta}^{(1)}$ be defined in (16). For $b\in\mathbb{F}_q^*$, let $\mathrm{Ind}_\alpha(b)$ be the index of b with respect to the base α, and $i_b=-\mathrm{Ind}_\alpha(b) \pmod{2\ell^k}$. Assume that the set P and its partitions be defined as in section 2.1. Then the explicit value of $N_{\gamma,\delta}^{(2)}$ is given as follows:*

1) *When $f^*(\gamma)=0$ $(\gamma\neq0)$ and $\delta=0$, we have*

$$N_{\gamma,\delta}^{(2)} = \begin{cases} \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p}(p-1)-\frac{(q-1)(\ell-1)}{\ell^k}\right)-1; & \text{if } \ell\equiv1 \pmod{p},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p}(p-1)-\frac{(q-1)}{\ell^{k-1}}\right)-1; & \text{if } \ell\not\equiv1 \pmod{p}.\end{cases}$$

2) *When $f^*(\gamma)=0$ and $\delta\neq0$, we have*

$$N_{\gamma,\delta}^{(2)} = \begin{cases} \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{\sqrt{q}(\sqrt{q}-p)(p-1)}{p^2}-\frac{(\ell-1)(\sqrt{q}+1)(\sqrt{q}-p)}{p\ell^k}\right)-1; & \text{if } i_\delta\in P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}-\frac{(\ell-1)(\sqrt{q}+1)(\sqrt{q}-p)}{p\ell^k}\right)-1; & \text{otherwise},\end{cases}$$

if $\ell\equiv1 \pmod{p}$; and

$$N_{\gamma,\delta}^{(2)} = \begin{cases} \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{\sqrt{q}(p-1)(\sqrt{q}-p)}{p^2}-\frac{(\sqrt{q}+1)(\sqrt{q}-p)}{p\ell^{k-1}}\right)-1; & \text{if } i_\delta\in\{0,\ell^k\}\cup P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}-\frac{(\sqrt{q}+1)(\sqrt{q}-p)}{p\ell^{k-1}}\right)-1; & \text{otherwise},\end{cases}$$

if $\ell\not\equiv1 \pmod{p}$.

3) *When $f^*(\gamma)\neq 0$ and $\delta=0$, we have*

$$N^{(2)}_{\gamma,\delta}=\begin{cases}\frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\eta_1(f^*(a))\frac{(\ell-1)(q-1)}{p\ell^k}-1; \text{ if } \ell\equiv 1 \pmod p \text{ and } p\equiv 1 \pmod 4,\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\eta_1(f^*(a))(\eta_1(t_1)+(\ell-1)\eta_1(t_2))\frac{q-1}{p\ell^k}-1; \text{ if } \ell\not\equiv 1 \pmod p\\ \qquad\qquad\qquad\qquad \text{and } p\equiv 1 \pmod 4,\\ \frac{q^2}{p^2}-1; \text{ if } p\equiv 3 \pmod 4,\end{cases}$$

where $t_1=\phi(\ell^k) \pmod p$ and $t_2=\ell^{k-1} \pmod p$.

4) *When $f^*(\gamma)\neq 0$ and $\delta\neq 0$, we have*

$$N^{(2)}_{\gamma,\delta}=\begin{cases}\frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1+\eta_1(f^*(a)))-\frac{(\ell-1)(\sqrt{q}+1)}{p\ell^k}(\sqrt{q}+\eta_1(f^*(a)))\right)-1;\\ \qquad\qquad \text{if } i_\delta\in P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}-\frac{(\ell-1)(\sqrt{q}+1)}{p\ell^k}(\sqrt{q}+\eta_1(f^*(a)))\right)-1; \text{ otherwise},\end{cases}$$

if $\ell\equiv 1 \pmod p$ and $p\equiv 1 \pmod 4$,

$$N^{(2)}_{\gamma,\delta}=\begin{cases}\frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\times\\ \left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1+\eta_1(f^*(\gamma))\eta_1(t_1))-\frac{\sqrt{q}+1}{p\ell^k}(\ell\sqrt{q}+\eta_1(f^*(\gamma))(\eta_1(t_1)+(\ell-1)\eta_1(t_2)))\right)-1;\\ \qquad\qquad \text{if } i_\delta\in\{0,\ell^k\},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\times\\ \left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1+\eta_1(f^*(\gamma))\eta_1(t_2))-\frac{\sqrt{q}+1}{p\ell^k}(\ell\sqrt{q}+\eta_1(f^*(\gamma))(\eta_1(t_1)+(\ell-1)\eta_1(t_2)))\right)-1;\\ \qquad\qquad \text{if } i_\delta\in P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}-\frac{\sqrt{q}+1}{p\ell^k}(\ell\sqrt{q}+\eta_1(f^*(\gamma))(\eta_1(t_1)+(\ell-1)\eta_1(t_2)))\right)-1; \text{ otherwise},\end{cases}$$

if $\ell\not\equiv 1 \pmod p$ and $p\equiv 1 \pmod 4$,

$$N^{(2)}_{\gamma,\delta}=\begin{cases}\frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1+\eta_1(f^*(\gamma)))-\frac{(\ell-1)(q+\sqrt{q})}{p\ell^k}\right)-1; \text{ if } i_\delta\in P_1^{(3)}\cup P_3^{(3)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1-\eta_1(f^*(\gamma)))-\frac{(\ell-1)(q+\sqrt{q})}{p\ell^k}\right)-1; \text{ if } i_\delta\in P_1^{(2)}\cup P_3^{(2)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}-\frac{(\ell-1)(q+\sqrt{q})}{p\ell^k}\right)-1; \text{ otherwise},\end{cases}$$

if $\ell\equiv 1 \pmod p$ and $p\equiv 3 \pmod 4$,

$$N^{(2)}_{\gamma,\delta}=\begin{cases}\frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1-\eta_1(f^*(\gamma))\eta_1(t_1))-\frac{(q+\sqrt{q})}{p\ell^{k-1}}\right)-1; \text{ if } i_\delta=0,\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1+\eta_1(f^*(\gamma))\eta_1(t_1))-\frac{(q+\sqrt{q})}{p\ell^{k-1}}\right)-1; \text{ if } i_\delta=\ell^k,\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1+\eta_1(f^*(\gamma))\eta_1(t_2))-\frac{(q+\sqrt{q})}{p\ell^{k-1}}\right)-1; \text{ if } i_\delta\in P_1^{(3)}\cup P_3^{(3)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}+\frac{\sqrt{q}}{p}(1-\eta_1(f^*(\gamma))\eta_1(t_2))-\frac{(q+\sqrt{q})}{p\ell^{k-1}}\right)-1; \text{ if } i_\delta\in P_1^{(2)}\cup P_3^{(2)},\\ \frac{q^2}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p^2}-\frac{(q+\sqrt{q})}{p\ell^{k-1}}\right)-1; \text{ otherwise},\end{cases}$$

if $\ell \not\equiv 1 \pmod{p}$ *and* $p \equiv 3 \pmod{4}$,

where $t_1 = \phi(\ell^k) \pmod{p}$ *and* $t_2 = \ell^{k-1} \pmod{p}$.

Proof. By the definition in (6) and using the fact $S_{2\ell^k}(z, w\delta) = S_{2\ell^k}(z, \delta)$ for any $\delta \in \mathbb{F}_q$ and $z, w \in \mathbb{F}_p^*$, we obtain

$$\begin{aligned}
N_{\gamma,\delta}^{(2)} &= |\{(x,y) \in \mathbb{F}_q^2 \backslash \{(0,0)\} : f(x) + \mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}}) = 0 \text{ and } \mathrm{Tr}_1^e(\gamma x + \delta y) = 0\}| \\
&= \sum_{x,y \in \mathbb{F}_q} \left(\frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{z\{f(x)+\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}})\}} \right) \times \left(\frac{1}{p} \sum_{w \in \mathbb{F}_p} \zeta_p^{w\,\mathrm{Tr}_1^e(\gamma x + \delta y)} \right) - 1 \\
&= \frac{1}{p^2} \sum_{x,y \in \mathbb{F}_q} \left(\sum_{z \in \mathbb{F}_p^*} \zeta_p^{z\{f(x)+\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}})\}} + 1 \right) \left(\sum_{w \in \mathbb{F}_p^*} \zeta_p^{w\,\mathrm{Tr}_1^e(\gamma x + \delta y)} + 1 \right) - 1 \\
&= \frac{1}{p^2} \left(q^2 + \sum_{x,y \in \mathbb{F}_q} \sum_{z \in \mathbb{F}_p^*} \zeta_p^{z\{f(x)+\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}})\}} + \sum_{x,y \in \mathbb{F}_q} \sum_{z,w \in \mathbb{F}_p^*} \zeta_p^{z\{f(x)+\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}})\} + w\,\mathrm{Tr}_1^e(\gamma x + \delta y)} \right) - 1 \\
&= \frac{1}{p^2} \left(q^2 + \epsilon_f \sqrt{p^*}^e (w(0,0) + p - 1) + \sum_{z,w \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{z\{f(x)+\mathrm{Tr}_1^e(\gamma \frac{w}{z} x)\}} \sum_{y \in \mathbb{F}_q} \zeta_p^{z\,\mathrm{Tr}_1^e(y^{\frac{q-1}{2\ell^k}} + \delta \frac{w}{z} y)} \right) - 1 \\
&= \frac{1}{p^2} \left(q^2 + \epsilon_f \sqrt{p^*}^e (w(0,0) + p - 1) + \sum_{z,w \in \mathbb{F}_p^*} \sigma_z \left(\epsilon_f \sqrt{p^*}^e \zeta_p^{f^*(-\gamma w)} \right) (S_{2\ell^k}(z, \delta w z) + 1) \right) - 1 \\
&= \frac{1}{p^2} \left(q^2 + \epsilon_f \sqrt{p^*}^e (w(0,0) + p - 1) + \epsilon_f \sqrt{p^*}^e \sum_{z \in \mathbb{F}_p^*} (S_{2\ell^k}(z, \delta) + 1) \sum_{w \in \mathbb{F}_p^*} \zeta_p^{w^2 z f^*(\gamma)} \right) - 1 \\
&= \frac{1}{p^2} \left(q^2 + \epsilon_f \sqrt{p^*}^e (w(0,0) + p - 1) + \epsilon_f \sqrt{p^*}^e \Omega_2 \right) - 1, \qquad (18)
\end{aligned}$$

where $\Omega_2 = \sum_{z \in \mathbb{F}_p^*} (S_{2\ell^k}(z, \delta) + 1) \sum_{w \in \mathbb{F}_p^*} \zeta_p^{w^2 z f^*(\gamma)}$. The sixth equality above holds because for a fixed z, as w varies over $\mathbb{F}_p^*$, also does $\frac{w}{z}$.

Next, we determine the value of Ω_2 by considering the following three cases while (γ, δ) runs through $\mathbb{F}_q^2 \backslash \{(0,0)\}$.

Case 1 : When $f^*(\gamma) = 0$, we have

$$\begin{aligned}
\Omega_2 &= (p-1) \sum_{z \in \mathbb{F}_p^*} (S_{2\ell^k}(z, \delta) + 1) \\
&= \begin{cases} (p-1)w(0,0) + (p-1)^2; \text{ if } \delta = 0 \text{ with } \gamma \neq 0, \\ (p-1)w(0,\delta) + (p-1)^2; \text{ if } \delta \neq 0. \end{cases} \qquad (19)
\end{aligned}$$

Case 2 : When $f^*(\gamma)\neq 0$ and $\delta=0$, by Lemma 2, we have

$$\begin{aligned}\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}(S_{2\ell^k}(z,0)+1)(\eta_1(zf^*(\gamma))\sqrt{p^*}-1)\\ &= \eta_1(f^*(\gamma))\sqrt{p^*}\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,0)-w(0,0)-(p-1) \end{aligned}\tag{20}$$

From Lemma 3$(a),(b)$ and the fact $\sum_{z\in\mathbb{F}_p^*}\eta_1(z)=0$, it is to be noted that

$$\begin{aligned}\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,0) &= \frac{q-1}{2\ell^k}\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z)\\ &= \begin{cases}\frac{q-1}{2\ell^k}(\ell-1)\sqrt{p^*}(1+(-1)^{\frac{p-1}{2}});\ \text{if } \ell\equiv 1 \pmod p,\\ \frac{q-1}{2\ell^k}\sqrt{p^*}\left(\eta_1(t_1)+(-1)^{\frac{p-1}{2}}\eta_1(t_1)+(\ell-1)(\eta_1(t_2)+(-1)^{\frac{p-1}{2}}\eta_1(t_2))\right);\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{if } \ell\not\equiv 1 \pmod p,\end{cases}\\ &= \begin{cases}\frac{q-1}{\ell^k}(\ell-1)\sqrt{p^*};\ \text{if } \ell\equiv 1 \pmod p \text{ and } p\equiv 1 \pmod 4,\\ \frac{q-1}{\ell^k}\sqrt{p^*}(\eta_1(t_1)+(\ell-1)\eta_1(t_2));\ \text{if } \ell\not\equiv 1 \pmod p\\ \qquad\qquad\qquad\qquad\qquad \text{and } p\equiv 1 \pmod 4,\\ 0;\ \text{if } p\equiv 3 \pmod 4,\end{cases}\end{aligned}\tag{21}$$

where $t_1=\phi(\ell^k) \pmod p$ and $t_2=\ell^{k-1} \pmod p$.

Case 3 : When $f^*(\gamma)\neq 0$ and $\delta\neq 0$, by Lemma 2, we have

$$\begin{aligned}\Omega_2 &= \sum_{z\in\mathbb{F}_p^*}(S_{2\ell^k}(z,\delta)+1)(\eta_1(zf^*(\gamma))\sqrt{p^*}-1)\\ &= \eta_1(f^*(\gamma))\sqrt{p^*}\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,\delta)-w(0,\delta)-(p-1) \end{aligned}\tag{22}$$

From Lemma 3$(c),(d)$ and (21) it is to be noted that

$$\begin{aligned}\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,\delta) &= \sum_{z\in\mathbb{F}_p^*}\eta_1(z)(\chi_e(z\xi^{i_\delta})\sqrt{q}-\frac{\sqrt{q}+1}{2\ell^k}S_{2\ell^k}(z))\\ &= \sum_{z\in\mathbb{F}_p^*}\eta_1(z)\left(\zeta_p^{z\,\mathrm{Tr}_1^e(\xi^{i_\delta})}\sqrt{q}-\frac{\sqrt{q}+1}{2\ell^k}S_{2\ell^k}(z)\right)\\ &= \begin{cases}\sqrt{p^*}\left(\eta_1(\mathrm{Tr}_1^e(\xi^{i_\delta}))\sqrt{q}-\frac{\sqrt{q}+1}{\ell^k}(\ell-1)\right);\ \text{if } \ell\equiv 1 \pmod p\\ \qquad\qquad\qquad\qquad\qquad \text{and } p\equiv 1 \pmod 4,\\ \sqrt{p^*}\left(\eta_1(\mathrm{Tr}_1^e(\xi^{i_\delta}))\sqrt{q}-\frac{\sqrt{q}+1}{\ell^k}(\eta_1(t_1)+(\ell-1)\eta_1(t_2))\right);\ \text{if } \ell\not\equiv 1 \pmod p\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } p\equiv 1 \pmod 4,\\ \sqrt{p^*}\eta_1(\mathrm{Tr}_1^e(\xi^{i_\delta}))\sqrt{q};\ \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}\tag{23}$$

By Lemma 4, one can easily check

When $\ell\equiv 1 \pmod p$ and $p\equiv 1 \pmod 4$, we have

$$\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,\delta)=\begin{cases}\sqrt{p^*}\left(\sqrt{q}-\frac{\sqrt{q}+1}{\ell^k}(\ell-1)\right);\ \text{if } i_\delta\in P_1^{(3)}\cup P_3^{(3)}\cup P_1^{(2)}\cup P_3^{(2)},\\ -\sqrt{p^*}\frac{\sqrt{q}+1}{\ell^k}(\ell-1);\ \text{otherwise},\end{cases}$$

when $\ell\not\equiv 1 \pmod p$ and $p\equiv 1 \pmod 4$, we have

$$\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,\delta)=\begin{cases}\sqrt{p^*}\left(\eta_1(t_1)\sqrt{q}-\frac{\sqrt{q}+1}{\ell^k}(\eta_1(t_1)+(\ell-1)\eta_1(t_2))\right);\ \text{if } i_\delta\in\{0,\ell^k\},\\ \sqrt{p^*}\left(\eta_1(t_2)\sqrt{q}-\frac{\sqrt{q}+1}{\ell^k}(\eta_1(t_1)+(\ell-1)\eta_1(t_2))\right);\ \text{if } i_\delta\in P_1^{(3)}\cup P_3^{(3)}\cup P_1^{(2)}\cup P_3^{(2)},\\ -\sqrt{p^*}\frac{\sqrt{q}+1}{\ell^k}(\eta_1(t_1)+(\ell-1)\eta_1(t_2));\ \text{otherwise},\end{cases}$$

when $\ell\equiv 1 \pmod p$ and $p\equiv 3 \pmod 4$, we have

$$\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,\delta)=\begin{cases}-\sqrt{p^*q};\ \text{if } i_\delta\in P_1^{(3)}\cup P_3^{(3)},\\ \sqrt{p^*q};\ \text{if } i_\delta\in P_1^{(2)}\cup P_3^{(2)},\\ 0;\ \text{otherwise},\end{cases}$$

and when $\ell\not\equiv 1 \pmod p$ and $p\equiv 3 \pmod 4$, we have

$$\sum_{z\in\mathbb{F}_p^*}\eta_1(z)S_{2\ell^k}(z,\delta)=\begin{cases}\sqrt{p^*}\eta_1(t_1)\sqrt{q};\ \text{if } i_\delta=0,\\ -\sqrt{p^*}\eta_1(t_1)\sqrt{q};\ \text{if } i_\delta=\ell^k,\\ -\sqrt{p^*}\eta_1(t_2)\sqrt{q};\ \text{if } i_\delta\in P_1^{(3)}\cup P_3^{(3)},\\ \sqrt{p^*}\eta_1(t_2)\sqrt{q};\ \text{if } i_\delta\in P_1^{(2)}\cup P_3^{(2)},\\ 0;\ \text{otherwise},\end{cases}$$

where $t_1=\phi(\ell^k) \pmod p$ and $t_2=\ell^{k-1} \pmod p$.

Then, one can explicitly calculate the values of Ω_2 in (19), (20) and (22) by the help of Lemma 5; Lemma 5 and (21); and Lemma 5 and (23), respectively.

Therefore, $N_{\gamma,\delta}^{(2)}$ is easily determined for each of the above cases by substituting the respective values of Ω_2 in (18). □

Theorem 2. *Assume that $q=p^e$ and $e=\phi(\ell^k)$. Let $\mathcal{C}_{D'}=\{(\mathrm{Tr}_1^e(\gamma x+\delta y))_{(x,y)\in D'}:\gamma,\delta\in\mathbb{F}_q\}$, where D' is defined in (5). Define $T:=\eta_1(t_1)+(\ell-1)\eta_1(t_2)$, where $t_1=\phi(\ell^k) \pmod p$ and $t_2=\ell^{k-1} \pmod p$. Then,*

(1) *If $\ell\equiv 1 \pmod p$ and $p\equiv 1 \pmod 4$, then $\mathcal{C}_{D'}$ is a eight-weight $[\frac{q^2}{p}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(\ell-1)(q-1)}{\ell^k}\right)-1,2e]$ linear code over $\mathbb{F}_p$ and its weight distribution given in Table 8.*

(2) *If $\ell\equiv 1 \pmod p$ and $p\equiv 3 \pmod 4$, then $\mathcal{C}_{D'}$ is a six-weight $[\frac{q^2}{p}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(\ell-1)(q-1)}{\ell^k}\right)-1,2e]$ linear code over $\mathbb{F}_p$ and its weight distribution given in Table 9.*

Table 8:

Weight	Frequency
0	1
$\frac{q^2}{p^2}(p-1)$	$p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}-1$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2+\frac{\sqrt{q}(p-1)}{p}-\frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right)$	$\frac{(q-1)(\ell-1)}{\ell^k}\left(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right)$	$\frac{(q-1)(\ell^k-\ell+1)}{\ell^k}\left(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p}(p-1)-\frac{(\ell-1)(q-1)}{p\ell^k}(p+1)\right)$	$\frac{(p-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p}(p-1)-\frac{(\ell-1)(q-1)}{p\ell^k}(p-1)\right)$	$\frac{(p-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{2\sqrt{q}}{p}-\frac{(\ell-1)(\sqrt{q}+1)}{p\ell^k}(\sqrt{q}(p-1)-(p+1))\right)$	$\frac{(q-1)(\ell-1)(p-1)}{2\ell^k}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(\ell-1)(\sqrt{q}+1)}{p\ell^k}(\sqrt{q}(p-1)-(p+1))\right)$	$\frac{(q-1)(\ell^k-\ell+1)(p-1)}{2\ell^k}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(\ell-1)(q-1)}{p\ell^k}(p-1)\right)$	$\frac{(p-1)(q-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$

Table 9:

Weight	Frequency
0	1
$\frac{q^2}{p^2}(p-1)$	$p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}-1$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2+\frac{\sqrt{q}(p-1)}{p}-\frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right)$	$\frac{(q-1)(\ell-1)}{\ell^k}\left(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right)$	$\frac{(q-1)(\ell^k-\ell+1)}{\ell^k}\left(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p}(p-1)-\frac{(\ell-1)(q-1)}{\ell^k}\right)$	$(p-1)\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{2\sqrt{q}}{p}-\frac{(\ell-1)(\sqrt{q}+1)}{p\ell^k}(\sqrt{q}(p-1)-p)\right)$	$\frac{(q-1)(\ell-1)(p-1)}{2\ell^k}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(\ell-1)(\sqrt{q}+1)}{p\ell^k}(\sqrt{q}(p-1)-p)\right)$	$\frac{(q-1)(2\ell^k-\ell+1)(p-1)}{2\ell^k}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right)$

(3) *If* $\ell \not\equiv 1 \pmod p$ *and* $p \equiv 1 \pmod 4$, *then* $\mathcal{C}_{D'}$ *is a eight-weight* (when $\eta_1(t_1)=\eta_1(t_2)$) *and a nine-weight* (when $\eta_1(t_1)=-\eta_1(t_2)$) $[\frac{q^2}{p}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(q-1)}{\ell^{k-1}}\right)-1, 2e]$ *linear code over* $\mathbb{F}_p$ *and their respective weight distributions are given in Table* 10 *and* 11.

(4) *If* $\ell \not\equiv 1 \pmod p$ *and* $p \equiv 3 \pmod 4$, *then* $\mathcal{C}_{D'}$ *is a six-weight* $[\frac{q^2}{p}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(q-1)}{\ell^{k-1}}\right)-1, 2e]$ *linear code over* $\mathbb{F}_p$ *and its weight distribution given in Table* 12.

Proof. We will prove only for the case $\ell \equiv 1 \pmod p$ and $p \equiv 1 \pmod 4$ as the remaining cases can be proved in a similar manner.

It is obvious from Lemma 10 that when $\ell \equiv 1 \pmod p$, $\mathcal{C}_{D'}$ has length $n_2 = |D'| = \frac{q^2}{p}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(\ell-1)(q-1)}{\ell^k}\right)-1$.

From (17), we have the Hamming weight of $c(\gamma,\delta)$ in $\mathcal{C}_{D'}$ is $\mathrm{wt}(c(\gamma,\delta)) = n_2 - N^{(2)}_{\gamma,\delta}$, where $N^{(2)}_{\gamma,\delta}$ is defined in (16). Next, we determine the all possible values of $\mathrm{wt}(c(\gamma,\delta))$ with the help of Lemma 11 while (γ,δ) runs through $\mathbb{F}_q^2\backslash\{(0,0)\}$.

Table 10: When $\eta_1(t_1) = \eta_1(t_2)$

Weight	Frequency
0	1
$\frac{q^2}{p^2}(p-1)$	$p^{e-1} + \epsilon_f (p-1)\frac{\sqrt{p^*}^e}{p} - 1$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 + \frac{\sqrt{q}(p-1)}{p} - \frac{(q+\sqrt{q})(p-1)}{p\ell^{k-1}} \right)$	$\frac{q-1}{\ell^{k-1}}(p^{e-1} + \epsilon_f (p-1)\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{(q+\sqrt{q})(p-1)}{p\ell^{k-1}} \right)$	$\frac{(q-1)(\ell^{k-1}-1)}{\ell^{k-1}}(p^{e-1} + \epsilon_f (p-1)\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p}(p-1) - \frac{(q-1)(p\ell+T)}{p\ell^k} \right)$	$\frac{(p-1)}{2}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p}(p-1) - \frac{(q-1)(p\ell-T)}{p\ell^k} \right)$	$\frac{(p-1)}{2}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{2\sqrt{q}}{p} - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}+\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(q-1)(p-1)}{2\ell^{k-1}}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}-\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(p-1)(q-1)}{2}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}+\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(p-1)(q-1)(\ell^{k-1}-1)}{2\ell^{k-1}}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$

Table 11: When $\eta_1(t_1) = -\eta_1(t_2)$

Weight	Frequency
0	1
$\frac{q^2}{p^2}(p-1)$	$p^{e-1} + \epsilon_f (p-1)\frac{\sqrt{p^*}^e}{p} - 1$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 + \frac{\sqrt{q}(p-1)}{p} - \frac{(q+\sqrt{q})(p-1)}{p\ell^{k-1}} \right)$	$\frac{q-1}{\ell^{k-1}}(p^{e-1} + \epsilon_f (p-1)\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{(q+\sqrt{q})(p-1)}{p\ell^{k-1}} \right)$	$\frac{(q-1)(\ell^{k-1}-1)}{\ell^{k-1}}(p^{e-1} + \epsilon_f (p-1)\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p}(p-1) - \frac{(q-1)(p\ell+T)}{p\ell^k} \right)$	$\frac{(p-1)}{2}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p}(p-1) - \frac{(q-1)(p\ell-T)}{p\ell^k} \right)$	$\frac{(p-1)}{2}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{2\sqrt{q}}{p} - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}+\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(q-1)(p-1)}{2\ell^k}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}+\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(q-1)(p-1)(\ell^k-1)}{2\ell^k}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{2\sqrt{q}}{p} - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}-\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(q-1)(p-1)(\ell-1)}{2\ell^k}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1) + \epsilon_f \sqrt{p^*}^e \left(\frac{q}{p^2}(p-1)^2 - \frac{(q-1)}{\ell^{k-1}} + \frac{(\sqrt{q}+1)(\ell\sqrt{q}-\eta_1(t_1)T)}{p\ell^k} \right)$	$\frac{(q-1)(p-1)(\ell^k-\ell+1)}{2\ell^k}(p^{e-1} - \epsilon_f \frac{\sqrt{p^*}^e}{p})$

Table 12:

Weight	Frequency
0	1
$\frac{q^2}{p^2}(p-1)$	$p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}-1$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2+\frac{\sqrt{q}(p-1)}{p}-\frac{(q+\sqrt{q})(p-1)}{p\ell^{k-1}}\right)$	$\frac{q-1}{\ell^{k-1}}(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(q+\sqrt{q})(p-1)}{p\ell^{k-1}}\right)$	$\frac{(q-1)(\ell^{k-1}-1)}{\ell^{k-1}}(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p}(p-1)-\frac{(q-1)}{\ell^{k-1}}\right)$	$(p-1)(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{2\sqrt{q}}{p}-\frac{(\sqrt{q}+1)}{p\ell^{k-1}}(\sqrt{q}(p-1)-p)\right)$	$\frac{(q-1)(p-1)}{2\ell^{k-1}}(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p})$
$\frac{q^2}{p^2}(p-1)+\epsilon_f\sqrt{p^*}^e\left(\frac{q}{p^2}(p-1)^2-\frac{(\sqrt{q}+1)}{p\ell^{k-1}}(\sqrt{q}(p-1)-p)\right)$	$\frac{(q-1)(p-1)(2\ell^{k-1}-1)}{2\ell^{k-1}}(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p})$

For $f^*(\gamma)=0$ ($\gamma\neq 0$) and $\delta=0$, we have

$$\mathrm{wt}(c(\gamma,\delta))=\frac{q^2(p-1)}{p^2},$$

For $f^*(\gamma)=0$ and $\delta\neq 0$, we have

$$\mathrm{wt}(c(\gamma,\delta))=\begin{cases}\frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2}+\frac{\sqrt{q}(p-1)}{p}-\frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right);\ \text{if } i_\delta\in P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\ \frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2}-\frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right);\ \text{otherwise},\end{cases}$$

For $f^*(\gamma)\neq 0$ and $\delta=0$, we have

$$\mathrm{wt}(c(\gamma,\delta))=\begin{cases}\frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(\ell-1)(q-1)(p+1)}{p\ell^k}\right);\ \text{if } \eta_1(f^*(\gamma))=1,\\ \frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p}-\frac{(\ell-1)(q-1)(p-1)}{p\ell^k}\right);\ \text{if } \eta_1(f^*(\gamma))=-1,\end{cases}$$

For $f^*(\gamma)\neq 0$ and $\delta\neq 0$, we have

$$\mathrm{wt}(c(\gamma,\delta))=\begin{cases}\frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2}-\frac{(\ell-1)(q-1)(p-1)}{p\ell^k}\right);\ \text{if } \eta_1(f^*(\gamma))=-1,\\ \frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2}-\frac{2\sqrt{q}}{p}-\frac{(\ell-1)(\sqrt{q}+1)(\sqrt{q}(p-1)-(p+1))}{p\ell^k}\right);\\ \qquad\qquad \text{if } \eta_1(f^*(\gamma))=1 \text{ and } i_\delta\in P_1^{(2)}\cup P_1^{(3)}\cup P_3^{(2)}\cup P_3^{(3)},\\ \frac{q^2(p-1)}{p^2}+\epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2}-\frac{(\ell-1)(\sqrt{q}+1)(\sqrt{q}(p-1)-(p+1))}{p\ell^k}\right);\ \text{otherwise}.\end{cases}$$

Therefore, the eight nonzero weights of $\mathcal{C}_{D'}$ are as follows:

$$\begin{aligned}
w_1 &= \frac{q^2(p-1)}{p^2},\\
w_2 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2} + \frac{\sqrt{q}(p-1)}{p} - \frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right),\\
w_3 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2} - \frac{(\ell-1)(q+\sqrt{q})(p-1)}{p\ell^k}\right),\\
w_4 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p} - \frac{(\ell-1)(q-1)(p+1)}{p\ell^k}\right),\\
w_5 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)}{p} - \frac{(\ell-1)(q-1)(p-1)}{p\ell^k}\right),\\
w_6 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2} - \frac{2\sqrt{q}}{p} - \frac{(\ell-1)(\sqrt{q}+1)(\sqrt{q}(p-1)-(p+1))}{p\ell^k}\right),\\
w_7 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2} - \frac{(\ell-1)(\sqrt{q}+1)(\sqrt{q}(p-1)-(p+1))}{p\ell^k}\right),\\
w_8 &= \frac{q^2(p-1)}{p^2} + \epsilon_f\sqrt{p^*}^e\left(\frac{q(p-1)^2}{p^2} - \frac{(\ell-1)(q-1)(p-1)}{p\ell^k}\right).
\end{aligned}$$

In the following, we will show their frequencies $A_{w_1}, A_{w_2}, A_{w_3}, A_{w_4}, A_{w_5}, A_{w_6}, A_{w_7}$ and A_{w_8} respectively.

Note that e is even, then from Lemma 7, we obtain

$$\begin{aligned}
A_{w_1} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)=0 \text{ with } \gamma\neq 0 \text{ and } \delta=0\}| = p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}-1,\\
A_{w_2} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)=0 \text{ and } \delta\neq 0 \text{ with } i_\delta\in P_1^{(2)}\cup P_1^{(3))}\cup P_3^{(2)}\cup P_3^{(3)}\}|\\
&= \frac{(q-1)(\ell-1)}{\ell^k}\left(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}\right),\\
A_{w_3} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)=0 \text{ and } \delta\neq 0 \text{ with } i_\delta\notin P_1^{(2)}\cup P_1^{(3))}\cup P_3^{(2)}\cup P_3^{(3)}\}|\\
&= \frac{(q-1)(\ell^k-\ell+1)}{\ell^k}\left(p^{e-1}+\epsilon_f(p-1)\frac{\sqrt{p^*}^e}{p}\right),\\
A_{w_4} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)\neq 0 \text{ with } \eta_1(f^*(\gamma))=1 \text{ and } \delta=0\}|\\
&= \frac{(p-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right),\\
A_{w_5} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)\neq 0 \text{ with } \eta_1(f^*(\gamma))=-1 \text{ and } \delta=0\}|\\
&= \frac{(p-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right),
\end{aligned}$$

$$
\begin{aligned}
A_{w_6} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)\neq 0 \text{ such that } \eta_1(f^*(\gamma))=1 \text{ and } \delta\neq 0 \text{ with } i_\delta\in P_1^{(2)}\cup P_1^{(3))}\cup P_3^{(2)}\cup P_3^{(3)}\}|\\
&= \frac{(q-1)(\ell-1)}{\ell^k}\times\frac{(p-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right),\\
A_{w_7} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : f^*(\gamma)\neq 0 \text{ such that } \eta_1(f^*(\gamma))=1 \text{ and } \delta\neq 0 \text{ with } i_\delta\notin P_1^{(2)}\cup P_1^{(3))}\cup P_3^{(2)}\cup P_3^{(3)}\}|\\
&= \frac{(q-1)(\ell^k-\ell+1)}{\ell^k}\times\frac{(p-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right),\\
A_{w_8} &= |\{(\gamma,\delta)\in\mathbb{F}_q^2 : \delta\neq 0 \text{ and } f^*(\gamma)\neq 0 \text{ such that } \eta_1(f^*(\gamma))=-1\}|\\
&= \frac{(p-1)(q-1)}{2}\left(p^{e-1}-\epsilon_f\frac{\sqrt{p^*}^e}{p}\right).
\end{aligned}
$$

Obviously, for $\gamma=\delta=0$ the zero codeword occurs only once in $\mathcal{C}_{D'}$, which has exactly p^{2e} codewords, so the dimension of $\mathcal{C}_{D'}$ is $2e$. This completes the proof. □

Example 5. *Let $f(x)=\mathrm{Tr}_1^e(x^2)$ over $\mathbb{F}_{p^e}$, then $f(x)$ is a weakly regular bent function with $\epsilon_f=-1$ and $l_f=2$ from [21, Table 4]. Consider $p=3$, $\ell=7$ and $k=1$. Clearly, $\ell\equiv 1 \pmod 3$, $p\equiv 3 \pmod 4$ and $e=\phi(14)=6$. Therefore, $\mathcal{C}_{D'}$ is a ternary $[173420,12,114372]$ linear code and its weight enumerator is $1+468z^{114372}+27144z^{115182}+146016z^{115344}+162864z^{115668}+194688z^{115830}+260z^{118098}$.*

Example 6. *Let $f(x)=\mathrm{Tr}_1^e(\alpha x^{p^i+1})$, where α is a primitive element over $\mathbb{F}_{p^e}$. Then $f(x)$ is a weakly regular bent function with $\epsilon_f=1$ and $l_f=2$ from [21, Table 4]. Consider $p=5$, $\ell=3$, $k=1$ and $i=2$. Clearly, $\ell\not\equiv 1 \pmod 5$, $p\equiv 1 \pmod 4$, $e=\phi(6)=2$, $\eta_1(2)=-\eta_1(1)=-1$ and $T=\eta_1(2)+2\eta_1(1)=1$. Therefore, $\mathcal{C}_{D'}$ is a quinary $[104,4,72]$ linear code and its weight enumerator is $1+8z^{72}+64z^{78}+216z^{80}+128z^{82}+136z^{88}+64z^{92}+8z^{100}$.*

Example 7. *Let $f(x)=\mathrm{Tr}_1^e(x^{p^i+1})$ over $\mathbb{F}_{p^e}$. Then $f(x)$ is a weakly regular bent function with $\epsilon_f=-1$ and $l_f=2$ from [21, Table 4]. Consider $p=5$, $\ell=3$, $k=1$ and $i=2$. Clearly, $\ell\not\equiv 1 \pmod 5$, $p\equiv 1 \pmod 4$, $e=\phi(6)=2$, $\eta_1(2)=-\eta_1(1)=-1$ and $T=\eta_1(2)+2\eta_1(1)=1$. Therefore, $\mathcal{C}_{D'}$ is a quinary $[144,4,108]$ linear code and its weight enumerator is $1+96z^{108}+204z^{112}+192z^{118}+24z^{120}+96z^{122}+12z^{128}$.*

Example 8. *Let $f(x)=\mathrm{Tr}_1^e(x^{\frac{3^i+1}{2}})$ over $\mathbb{F}_{3^e}$. Then $f(x)$ is a weakly regular bent function with $\epsilon_f=-1$ and $l_f=2$ from [21, Table 4]. Consider $p=3$, $\ell=5$, $k=1$ and $i=5$. Clearly, $\ell\not\equiv 1 \pmod 3$, $p\equiv 3 \pmod 4$ and $e=\phi(10)=4$. Therefore, $\mathcal{C}_{D'}$ is a ternary $[2420,8,1458]$ linear code and its weight enumerator is $1+20z^{1458}+2400z^{1584}+1680z^{1620}+2400z^{1638}+60z^{1692}$.*

5 Concluding remarks

Inspired by the pioneering works in [21, 6] and [11], which explored the construction of p-ary linear codes of the form (3) from specific defining sets and the evaluation of the Weil sum $S_{\mathcal{N}}(a, b)$ defined in (2), respectively, this paper focuses on constructing new classes of p-ary linear codes of the form (3). By carefully selecting two different defining sets, defined by (4) and (5), we systematically analyze their algebraic structures and employ character sum techniques to determine their corresponding code parameters and weight distributions.

Through detailed computation and application of number-theoretic tools such as quadratic Gauss sums and Weil sums over finite fields, we successfully construct several new classes of t-weight linear codes over $\mathbb{F}_p$, where $t = 2, 4, 6, 8, 9$. For each class of codes, we explicitly determined their length, dimension, and weight distributions. These computations not only verify the correctness of our theoretical framework but also demonstrate the diversity and richness of the proposed code families.

To the best of our knowledge, the obtained codes are non-equivalent to any few-weight linear codes previously known in the literature. Moreover, their parameters are flexible, allowing for various code lengths and weight structures by adjusting the defining sets and field characteristics. This flexibility makes them suitable for a wide range of practical applications, such as secret sharing schemes, authentication codes, and combinatorial design constructions.

In addition, we identify a special class of optimal two-weight linear codes within our construction that achieves the Griesmer bound, thereby confirming their optimality with respect to length and minimum distance. These results not only enrich the current library of known optimal and near-optimal linear codes but also provide new insights into the relationship between character sums and code weight distributions in finite fields.

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