ON SOME NON-PRINCIPAL LOCALLY ANALYTIC REPRESENTATIONS INDUCED BY WHITTAKER MODULES

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Abstract. Let G be a connected split adjoint semi simple p-adic Lie group. This paper can be seen as a continuation of [12] and is about the construction of locally analytic G-representations which do not lie in the principal series. Here we consider locally analytic representations which are induced by Whittaker modules of the attached Lie algebra. We prove that they are ind-admissible and topologically irreducible in case the Whittaker module is simple. On the other hand, we show that the naive Jacquet functor of these representations vanishes for all parabolic subgroups. However, they do not satisfy the definition of supercuspidality in the sense of Kohlhaase.

Contents

1. Introduction

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers O_L. Let $G = \mathbf{G}(L)$ be the group of L-valued points of a connected split adjoint semi simple algebraic group $\mathbf{G}$ over O_L. In [12] we have considered some locally analytic G-representations which do not lie in the principal series as treated in [15]. In fact those objects are globalized from cuspidal Lie algebra modules and satisfy the supercuspidality criterion of Kohlhaase apart from the possible lack of irreducibility. In this paper we globalize amongst other things Whittaker modules and seek to prove analogous results as in loc.cit. Indeed we prove that they are ind-admissible and topologically irreducible when the Whittaker module is simple. In contrast to loc.cit. they do not satisfy in general the the homological vanishing criterion in the definition of supercuspidality of Kohlhaase [7]. Here we exemplarily discuss the case of GL_2 and show that the homology groups for the opposite unipotent subgroup do not vanish in degree 1. However, the naive (Hausdorff) Jacquet vanishes with respect to all parabolic subgroups.

We shall give now more details on the construction and statements. Let $G_0 := \mathbf{G}(O_L)$ be the maximal compact open subgroup of O_L-valued points and fix a maximal unipotent subgroup $U \subset G$. Set $U_0 = U \cap G_0$ and denote by $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{u} = \mathrm{Lie}(U)$ the Lie algebras of G and U, respectively. Let K be a finite extension of L which serves as our coefficient field. Let $\eta : \mathfrak{u} \to K$ be a (non-degenerate) character which is induced by some locally L-analytic character $\hat{\eta} : U_0 \to K^*$. We consider the category $\mathcal{M}_\eta$ of all finitely generated $U(\mathfrak{g})$-modules M such that for each element $m \in M$ the expression $(x - \eta(x))^k m$ vanishes for some integer $k \geq 1$. Then the action of $\mathfrak{u}$ on M integrates to a locally analytic action of U_0.

Let $D(G_0)$ be the L-analytic distribution algebra of G_0 and denote by $D(\mathfrak{g}, U_0)$ the subalgebra of $D(G_0)$ generated by $\mathfrak{g}$ and U_0. Further we consider smooth admissible representations V of U_0. Then $M \otimes V'$ is a $D(\mathfrak{g}, U_0)$-module and we set

$$\mathcal{F}^G_{\hat{\eta}}(Z, V) := c - \mathrm{Ind}^G_{G_0}(D(G_0) \otimes_{D(\mathfrak{g},U_0)} M \otimes V')'$$

where $c - \mathrm{Ind}^G_{G_0}$ is the compact induction of locally analytic representations. Thus we have defined a functor

$$\mathcal{F}^G_{\hat{\eta}} : \mathcal{M}_\eta \times \mathrm{Rep}^{\infty,adm}_K(U_0) \to \mathrm{Rep}^{la}_K(G)$$

into the category of locally analytic G-representations.

Our first result is the following theorem:

Theorem A. *(i) The functor $\mathcal{F}^G_{\hat{\eta}}$ is functorial and exact in all arguments: contravariant in M and covariant in V.*

(ii) For all $M \in \mathcal{M}_\eta$ and for all smooth admissible representations V of U_0, the locally analytic representation $\mathcal{F}^G_{\hat{\eta}}(M, V)$ is ind-admissible and in particular of compact type.

(Here we recall from [12] *that a locally analytic representation is ind-admissible if it is a strict inductive limits of admissible representations when restricted to a compact open subgroup.)*

Concerning irreducibility statements for some of these representations we also have to consider the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Let $\Theta : Z(\mathfrak{g}) \to K$ be an algebra homomorphism. We then set $N_{\Theta,\eta} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})U(\mathfrak{u})} K_{\Theta,\eta}$ which is an object of $\mathcal{M}_\eta$. If η is non-degenerate then Kostant [8] proved that this module is simple. It is up to isomorphism the only simple object in the subcategory of $\mathcal{M}_\eta$ where the centre $Z(\mathfrak{g})$ acts via Θ.

Theorem B. *Let Θ and η as above. Let V be an irreducible smooth U_0-representation, then $\mathcal{F}^G_{\hat{\eta}}(N_{\Theta,\eta}, V)$ is topologically irreducible.*

The proof of the above theorems follows the same strategy as in the principal series case [15] and [12], respectively. More precisely, as for Theorem A part ii) we use a recent result of Agrawal and Strauch [1] which applies to more general closed subgroups of G_0. Concerning Theorem B we first treat the situation where V is the trivial representation. We prove as in [12] that the module $D(G_0) \otimes_{D(\mathfrak{g},U_0)} N_{\Theta,\eta}$ is simple. Then we apply the levelwise Mackey Criterion of Bode [12, Appendix] to prove the topologically irreducibility. If V is arbitrary irreducible we mimic the proof of [15] to deduce the statement.

As for the list of content we recall in section 2 some basic facts in locally analytic representation theory and some material developed in [12], e.g. the notion of ind-admissibility. Section 3 is devoted to review the necessary background on Whittaker modules over the field of complex numbers. Here we follow the paper of Miličić and Soergel [10]. In section 4 we define our functor $\mathcal{F}^G_{\hat{\eta}}$ and prove assertions i)-ii) from Theorem A above. The irreducibility result is shown in section 5. Finally the last section addresses the question of supercuspidality in the sense of Kohlhaase.

Notation and conventions: We denote by p a prime number and consider fields $L \subset K$ which are both finite extensions of $\mathbb{Q}_p$. Let O_L and O_K be the rings of integers of L, resp. K, and let $|\cdot|_K$ be the absolute value on K such that $|p|_K = p^{-1}$. The field L is our "base field", whereas we consider K as our "coefficient field". For a locally convex K-vector space V we denote by V'_b its strong dual, i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence. Sometimes, in particular when V is finite-dimensional, we simplify notation and write V' instead of V'_b. All finite-dimensional K-vector spaces are equipped with the unique Hausdorff locally convex topology.

We let $\mathbf{G}_0$ be a split reductive group scheme over O_L and $\mathbf{T}_0 \subset \mathbf{B}_0 \subset \mathbf{G}_0$ a maximal split torus and a Borel subgroup scheme, respectively. We denote by $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$ the base change of $\mathbf{G}_0$, $\mathbf{B}_0$ and $\mathbf{T}_0$ to L. By $G_0 = \mathbf{G}_0(O_L)$, $B_0 = \mathbf{B}_0(O_L)$, etc., and $G = \mathbf{G}(L)$, $B = \mathbf{B}(L)$, etc., we denote the corresponding groups of O_L-valued points and L-valued points, respectively. Standard parabolic subgroups of $\mathbf{G}$ (resp. G) are those which contain $\mathbf{B}$ (resp. B). For each standard parabolic subgroup $\mathbf{P}$ (or P) we let $\mathbf{L_P}$ (or L_P) be the unique Levi subgroup which contains $\mathbf{T}$ (resp. T) and $\mathbf{U_P}$ its unipotent radical. The opposite unipotent radical is denoted by $\mathbf{U}^-_{\mathbf{P}}$. Finally, Gothic letters $\mathfrak{g}$, $\mathfrak{p}$, etc., will denote the Lie algebras of $\mathbf{G}$, $\mathbf{P}$, etc.: $\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$, $\mathfrak{t} = \mathrm{Lie}(\mathbf{T})$, $\mathfrak{b} = \mathrm{Lie}(\mathbf{B})$, $\mathfrak{p} = \mathrm{Lie}(\mathbf{P})$, $\mathfrak{l}_P = \mathrm{Lie}(\mathbf{L_P})$, etc.. Base change to K is usually denoted by the subscript $_K$, for instance, $\mathfrak{g}_K = \mathfrak{g} \otimes_L K$.

We make the general convention that we denote by $U(\mathfrak{g})$, $U(\mathfrak{p})$, etc., the corresponding enveloping algebras, *after base change to* K, i.e., what would be usually denoted by $U(\mathfrak{g}) \otimes_L K$, $U(\mathfrak{p}) \otimes_L K$ etc. Similarly, we use the abbreviations $D(G) = D(G,K), D(P) = D(P,K)$ etc. for the locally L-analytic distributions with values in K.

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2. Preliminaries on locally analytic representations

In the first half of this chapter we recall some basic facts on locally analytic representations as introduced by Schneider and Teitelbaum [18]. In the second half we continue with some additional background on these representations established in [12].

For a locally L-analytic group H, let $C^{an}(H,K)$ be the locally convex vector space of locally L-analytic K-valued functions. The dual space $D(H) = D(H,K) = C^{an}(H,K)'$ is a topological K-algebra which has the structure of a Fréchet-Stein algebra when H is compact [19]. More generally, if V is a Hausdorff locally convex K-vector space, let $C^{an}(H,V)$ be the K-vector space consisting of locally analytic functions with values in V. It has the structure of a Hausdorff locally convex vector space, as well.

A locally analytic H-representation is a Hausdorff barrelled locally convex K-vector space together with a homomorphism $\rho : H \to \mathrm{GL}_K(V)$ such that the action of H on V is continuous and the orbit maps $\rho_v : H \to V,\ h \mapsto \rho(h)(v)$, are elements in $C^{an}(H,V)$ for all $v \in V$. We denote by $\mathrm{Rep}_K^{la}(H)$ the category of locally analytic H-representations on K-vector spaces where the morphisms are the continuous H-linear maps.

We recall that a Hausdorff locally convex K-vector space V is called of *compact type* if it is an inductive limit of countably many Banach spaces with injective and compact transition maps, cf. [18]. In this case, the strong dual V_b' is a nuclear Fréchet space, cf. [17]. We denote by $\mathrm{Rep}_K^{la,c}(H)$ the full subcategory of $\mathrm{Rep}_K^{la}(H)$ consisting of objects of compact type. By [18, 3.3], the duality functor gives an equivalence of categories

$$\mathrm{Rep}_K^{la,c}(H) \quad \xrightarrow{\sim} \quad \left\{ \begin{array}{c} \text{separately continuous } D(H)\text{-} \\ \text{modules on nuclear Fréchet} \\ \text{spaces with continuous} \\ D(H)\text{-module maps} \end{array} \right\}^{op}.$$

In particular, V is topologically irreducible if V_b' is a topologically simple $D(H,K)$-module.

For any closed subgroup H' of H and any locally analytic representation V of H', we denote by

$$\mathrm{Ind}^H_{H'}(V) := \Big\{ f \in C^{an}(H,V) \mid \forall h' \in H', \forall h \in H : f(h \cdot h') = (h')^{-1} \cdot f(h) \Big\}$$

the induced locally analytic representation. The group H acts on this vector space by $(h \cdot f)(x) = f(h^{-1}x)$. If V is of compact type and H/H' is compact then $\mathrm{Ind}^H_{H'}(V)$ is of compact type again and $\mathrm{Ind}^H_{H'}(V)' = D(H) \otimes_{D(H')} V_b'$ is a nuclear Fréchet space.

If H is second countable and $H' \subset H$ is a compact open subgroup, then we set

$$c - \mathrm{Ind}^H_{H'}(V) := \{ f \in \mathrm{Ind}^H_{H'}(V) \mid f \text{ has compact support} \}.$$

This is an H-stable subspace, and for any $f \in c-\mathrm{Ind}^H_{H'}(V)$ there are only finitely many elements $h_1, \dots h_r$ such that $\mathrm{supp}(f) = \bigcup_{i=1}^r h_i H'$. Since H is second countable we may write $c-\mathrm{Ind}^H_{H'}(V)$ as a countable direct sum $\bigoplus_{g \in H/H'} h \cdot V$ and supply this space with the locally convex direct sum topology. Then $c-\mathrm{Ind}_{H'}^{H}(V)$ is barrelled, Hausdorff and the action is locally analytic so that we get a locally analytic H-representation, cf. [12]. The construction is functorial and we get a functor

$$c-\mathrm{Ind}_{H'}^{H}:\mathrm{Rep}_K^{la}(H')\to\mathrm{Rep}_K^{la}(H).$$

The functor $c-\mathrm{Ind}_{H'}^{H}$ is left adjoint to the restriction functor

$$\mathrm{Rep}(H)_K^{la}\to\mathrm{Rep}_K^{la}(H').$$

The representations $c-\mathrm{Ind}_{H'}^{H}(V)$ are of compact type by [18, Prop 1.2. ii] but in general not admissible. However, they are ind-admissible [12, Lemma A.20]:

Definition 2.1. A locally analytic H-representation V is called ind-(strongly)admissible if there is some compact open subgroup $C\subset H$ such that $V_{|C}=\varinjlim_n V_n$ is a strict inductive limit of (strongly) admissible locally analytic C-representations V_n.

The above definition does not depend on the chosen compact open subgroup cf. [12, Corollary A.14] for a proof. It turns out that the subspaces V_n are closed subrepresentation of $V_{|C}$ for all $n\in\mathbb{N}$. Moreover, V is of compact type, cf. [12, Lemma 2.4] and $V_b'=\varprojlim_n (V_n)_b'$ is the locally convex projective limit of the Fréchet spaces $(V_n)_b'$. In particular V_b' is a Fréchet space, cf. [12, Proposition 2.5].

3. Whittaker Lie algebra representations

In this section we recall the theory of Whittaker modules of a Lie algebra $\mathfrak{g}$ over the field of complex numbers $\mathbb{C}$ in the sense of Miličić and Soergel [10, 11]. In particular we assume that $\mathfrak{g}$ is semi-simple. We thus replace here our p-adic field L by $\mathbb{C}$. Finally we denote by $Z(\mathfrak{g})$ the centre of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

Let $\mathfrak{b}\subset\mathfrak{g}$ be a Borel subalgebra and $\mathfrak{u}$ its nilradical. We let $\mathfrak{t}$ be a Cartan algebra inside $\mathfrak{b}$. We denote by $\Phi\subset\mathfrak{t}^*$ the attached root system and by Φ^+ resp. Φ^- its subset of positive resp. negative roots. Further we denote by Δ its set of simple roots. Let $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha\in\mathfrak{t}^*$. For every $\alpha\in\Phi^+$ let $x_\alpha\in\mathfrak{u}_\alpha$ and $y_\alpha\in\mathfrak{u}_{-\alpha}$ be the standard generators of the root spaces, respectively.

Following Miličić and Soergel [11] we consider the following category.

Definition 3.1. Let $\mathcal{N}$ be the full subcategory of all $U(\mathfrak{g})$-modules M such that

i) M is a finitely generated $U(\mathfrak{g})$-module.

ii) M is $U(\mathfrak{u})$-finite.

iii) M is $Z(\mathfrak{g})$-finite.

The category $\mathcal{N}$ is abelian and every object is of finite length [11, Thm 2.5].

For a maximal ideal $\Theta \subset Z(\mathfrak{g})$ we let $\mathcal{N}_\Theta$ be the full subcategory of objects M such that $\Theta M = 0$. Further $\mathcal{N}_{\overline{\Theta}}$ be the full subcategory consisting of objects M for which there is an integer $k \geq 1$ such that $\Theta^k M = 0$.

For a character η of $\mathfrak{u}$ we let $\mathcal{N}_\eta$ be the subcategory of $\mathcal{N}$ consisting of objects M such that $M = \{m \in M \mid \text{there exists an integer } k \geq 1 \text{ with } (x - \eta(x))^k m = 0 \text{ for all } x \in \mathfrak{u}\}$. We set

$$\mathcal{N}_{\overline{\Theta},\eta} = \mathcal{N}_{\overline{\Theta}} \cap \mathcal{N}_\eta \text{ and } \mathcal{N}_{\Theta,\eta} = \mathcal{N}_\Theta \cap \mathcal{N}_\eta.$$

The category $\mathcal{N}$ behaves semi-simple:

Lemma 3.2. *We have* $\mathcal{N} = \bigoplus_{\Theta \in \mathrm{Max} Z(\mathfrak{g})} \mathcal{N}_{\overline{\Theta}}$, $\mathcal{N} = \bigoplus_{\eta \in \mathfrak{u}^*} \mathcal{N}_\eta$ *and*

$$\mathcal{N} = \bigoplus_{\Theta,\eta} \mathcal{N}_{\overline{\Theta},\eta}.$$

Proof. This follow from [11, Lemma 2.1, Lemma 2.2]. See also [10]. □

Example 3.3. Let Θ be given by an algebra homomorphism $\psi_\Theta : Z(\mathfrak{g}) \to \mathbb{C}$. Then $N_{\Theta,\eta} := U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})U(\mathfrak{u})} \mathbb{C}_{\psi_\Theta,\eta}$ is an object of $\mathcal{N}_{\Theta,\eta}$ where $\mathbb{C}_{\psi_\Theta,\eta}$ is $\mathbb{C}$ with the module structure of $Z(\mathfrak{g})U(\mathfrak{u})$ given by $\psi_\Theta \cdot \eta$.

The following theorem is due to Kostant.

Theorem 3.4. *Let η be non-degenerate, i.e. $\eta(x_\alpha) \neq 0$ for all simple roots $\alpha \in \Delta$. Then $N_{\Theta,\eta}$ is a simple $U(\mathfrak{g})$-module.*

Proof. This is [8, Thm 3.6.1]. □

The generator of $N_{\Theta,\eta}$ is up to scalar multiplication the only Whittaker vector in $N_{\Theta,\eta}$, i.e., a vector $v \in N_{\Theta,\eta}$ for which $x \cdot v = \eta(x)v$ for all $x \in \mathfrak{u}$, cf. [8, Thm 3.4].

The following theorem of Miličić and Soergel is also useful for intrinsic reasons and will be applied in the next section.

Theorem 3.5. *The universal enveloping algebra $U(\mathfrak{g})$ is free over $Z(\mathfrak{g}) \otimes U(\mathfrak{u}) \cong Z(\mathfrak{g})U(\mathfrak{u})$.*

Proof. This is [11, Lemma 5.7]. □

For a finite-dimensional $Z(\mathfrak{g})$-representation Z, we consider as in loc.cit. the module $I_\eta(Z) = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})U(\mathfrak{u})} (Z \otimes \mathbb{C}_\eta)$. Denote by $\mathrm{Rep}^{fd}(Z(\mathfrak{g}))$ the category of finite-dimensional $Z(\mathfrak{g})$-modules.

Theorem 3.6. *Let η be non-degenerate.*

i) There is an equivalence of categories $I_\eta : \mathrm{Rep}^{fd}(Z(\mathfrak{g})) \to \mathcal{N}_\eta$.

ii) Any $M \in \mathcal{N}_{\Theta,\eta}$ is isomorphic to a finite sum of objects isomorphic to $N_{\Theta,\eta}$.

Proof. Part i) is [11, Thm. 5.9] whereas part ii) is [11, Thm. 5.6]. □

For later applications it is also useful to consider the following lift of the categories $\mathcal{N}_\eta$.

We let $\mathcal{M}$ be the category of all finitely generated $U(\mathfrak{g})$-modules M which are $U(\mathfrak{u})$-finite. Hence in contrast to $\mathcal{N}$ we do not assume the $Z(\mathfrak{g})$-finite condition. It follows by the same reasoning [6] for the BGG-category $\mathcal{O}$ that $\mathcal{M}$ is an abelian, noetherian category closed under submodules, quotients and finite direct sums.

We let $\mathcal{M}_\eta$ be the subcategory of $\mathcal{M}$ consisting of objects M such that $M = \{m \in M \mid$ there exists an integer $k \geq 1$ with $(x - \eta(x))^k m = 0$ for all $x \in \mathfrak{u}\}$. Then $\mathcal{N}_{\overline{\Theta},\eta}$ is a full subcategory of $\mathcal{M}_\eta$ for all maximal ideals Θ of $Z(\mathfrak{g})$. By the same argument as for $\mathcal{N}$ there is a decomposition

$$\mathcal{M} = \bigoplus_{\eta \in \mathfrak{u}^*} \mathcal{M}_\eta.$$

Indeed, this follows from [2, Ch. VII, §1, no. 3].

Example 3.7. For a character η of $\mathfrak{u}$, let

$$M_\eta = U(\mathfrak{g}) \otimes_{U(\mathfrak{u})} K_\eta.$$

Then $M_\eta \in \mathcal{M}_\eta$ and there is a canonical epimorphism $M_\eta \to N_{\Theta,\eta}$.

4. The functor $\mathcal{F}^G_{\hat{\eta}}$

Now we come back to our p-adic situation. We replace $\mathbb{C}$ by our coefficient field K and consider all objects defined in the previous section over K. This is harmless as all definitions are algebraic.

Let $\eta : \mathfrak{u} \to K$ be a character. This extends to a homomorphism $\eta : U(\mathfrak{u}) \to K$. We suppose that η is non-degenerate (i.e, $\eta(x_\alpha) \neq 0$ for $\alpha \in \Delta$) and that it is induced (via derivation) by a locally L-analytic character $\hat{\eta} : U_0 \to K^*$ of the unipotent subgroup $U_0 = U \cap \mathbb{G}(O_L)$.

Proposition 4.1. *For any $M \in \mathcal{M}_\eta$ there is a locally L-analytic action of U_0 on M.*

Proof. We consider the twisted module $M \otimes K_{-\eta}$ which is a $U(\mathfrak{n})$-finite module where $\mathfrak{n}$ acts trivially. Here the action of the Lie algebra $\mathfrak{u}$ integrates uniquely an algebraic action of $\mathbf{U}$ via the exponential series, i.e., we define $u \cdot m := \sum_{n \geq 0} \frac{\mathfrak{x}^n}{n!} m$ where $u = \exp(\mathfrak{x}) \in \mathbf{U}(\overline{K})$, By untwisting, i.e. by considering the tensor product $M \otimes K_{-\eta} \otimes K_{\hat{\eta}} = M$ we get the desired result. □

Let $\mathrm{Rep}^{\infty,adm}(U_0)$ be the category of smooth admissible U_0-representation and fix an object V in here. Note that V is finite-dimensional since U_0 is compact. Denote by $D(\mathfrak{g}, U_0) \subset D(G_0)$ the subalgebra generated by $\mathfrak{g}$ and $D(U_0)$. Then we get for any $M \in \mathcal{M}_\eta$ by the proposition above a locally analytic U_0-action on $M \otimes_K V'$ and thus via the trivial action $\mathfrak{g}$ on V a $D(\mathfrak{g}, U_0)$-module structure on $M \otimes_K V'$. We set

$$X_{\hat{\eta}}(M, V) = D(G_0) \otimes_{D(\mathfrak{g},U_0)} M \otimes_K V'$$

which is a separately continuous $D(G_0)$-module on a nuclear Fréchet space. Hence its topological dual $X_{\hat{\eta}}(M,V)'$ is a locally analytic G_0-representation of compact type. We put

$$\mathcal{F}^G_{\hat{\eta}}(M,V) := \mathrm{c}-\mathrm{Ind}^G_{G_0} X_{\hat{\eta}}(M,V)'$$

which is thus a locally analytic G-representation. This construction is functorial in each entry. Hence for a fixed locally analytic character $\hat{\eta}$ we get thus a bifunctor

$$\mathcal{F}^G_{\hat{\eta}} : \mathcal{M}_\eta \times \mathrm{Rep}_K^{\infty,adm}(U_0) \to \mathrm{Rep}_K^{la}(G).$$

Proposition 4.2. *The functor $\mathcal{F}^G_{\hat{\eta}}$ is bi-exact.*

Proof. Since $\mathrm{c}-\mathrm{Ind}^G_{G_0}$ is an exact functor it is enough to see that the expression $X_{\hat{\eta}}(M,V)$ defines an exact functor in M and V. As for the smooth entry, we choose a locally analytic section s of the projection $G_0 \to G_0/U_0$ and set $\mathcal{H} = s(G_0/U_0)$. Then $X(M,V) = D(\mathcal{H}) \otimes_{U(\mathfrak{g})} M \otimes_K V'$ as vector spaces. As for the exactness in M this follows from the result [1, Corollary 7.8.7] by Agrawal and Strauch. $\square$

We can give as in [15] another description of the above representation. We consider for an object M of $\mathcal{M}_\eta$, a finite-dimensional $U(\mathfrak{u})$-subspace W which generates M as a $U(\mathfrak{g})$-module and which is equipped with a locally analytic U_0-action. Let $\mathfrak{d} = \ker(U(\mathfrak{g}) \otimes_{U(\mathfrak{u})} W \to M)$ be the kernel of the natural morphism. Let V be a smooth admissible U_0-representation. Then there is similarly as in [15] the non-degenerate pairing

$$\begin{array}{lccc} \langle\cdot,\cdot\rangle : & \big(D(G_0) \otimes_{D(U_0)} W \otimes V'\big) \otimes_K \mathrm{Ind}^{G_0}_{U_0}(W' \otimes V) & \longrightarrow & C^{an}(G_0,V). \\ & (\delta \otimes w) \otimes f & \mapsto & \Big[g \mapsto \big(\delta \cdot_r (f(\cdot)(w))\big)(g)\Big] \end{array}$$

Here, by definition, we have $\big(\delta \cdot_r (f(\cdot)(w))\big)(g) = \delta(x \mapsto f(gx)(w))$.

Then we may identify similarly to [15] our representation $\mathcal{F}^G_{\hat{\eta}}(M,V)$ with the closed subrepresentation

$$c-\mathrm{Ind}^G_{U_0}(W' \otimes V)^{\mathfrak{d}} := \{f \in c-\mathrm{Ind}^G_{U_0}(W' \otimes V) \mid \langle x, f\rangle = 0 \, \forall x \in \mathfrak{d}\}$$

of $c-\mathrm{Ind}^G_{U_0}(W' \otimes V)$.

Example 4.3. Let $M = N_{\Theta,\eta}$ be as above. Then

$$\mathcal{F}^G_{\hat{\eta}}(N_{\Theta,\eta}) = c-\mathrm{Ind}^G_{U_0}(K'_{\hat{\eta}})^{Z(\mathfrak{g})=\Theta}$$

is the closed subspace of $c-\mathrm{Ind}^G_{U_0}(K'_{\hat{\eta}})$ where $Z(\mathfrak{g})$ acts by ψ'_Θ.

If $V = \mathbf{1}^\infty$ is the trivial representation of U_0, then we simply omit it from the input in our functor as in [15]. E.g., we write $\mathcal{F}^G_{\hat{\eta}}(M)$ for $\mathcal{F}^G_{\hat{\eta}}(M,\mathbf{1}^\infty)$.

Proposition 4.4. *Let $M \in \mathcal{M}_\eta$ and let $V \in \mathrm{Rep}_K^\infty(U_0)$. Suppose that V is admissible. Then the representation $X_{\hat{\eta}}(M,V)'$ is strongly admissible and $\mathcal{F}^G_{\hat{\eta}}(M,V)$ is ind-admissible.*

Proof. The Lie algebra representation M is a finitely generated $U(\mathfrak{g})$-module. As V is finite-dimensional the module $X_{\hat{\eta}}(M,V)$ is finitely generated as a $D(G_0)$-module, as well. Thus by definition $X_{\hat{\eta}}(M,V)'$ is strongly admissible .

The proof of the ind-admissibility is the same as in [12] which we recall for convenience. Let $S \subset G$ be a set of representatives for the double cosets $G_0\backslash G/G_0$. For each $s \in S$, let $R_s \subset G_0$ be a set of representatives for the finite set G_0sG_0/G_0. As for representations of finite groups [20] we have

$$\begin{aligned} c - \mathrm{Ind}_{G_0}^{G}(X_{\hat{\eta}}(M,V)')_{|G_0} &= \bigoplus_{s\in S} s\cdot\big(\bigoplus_{t\in R_s} t\cdot X_{\hat{\eta}}(M,V)'\big) \\ &= \bigoplus_{s\in S} s\cdot\big(\mathrm{Ind}_{G_0\cap sG_0s^{-1}}^{G_0} X_{\hat{\eta}}(M,V)'\big) \\ &= \bigoplus_{s\in S} s\cdot\big((\mathrm{Ind}_{G_0\cap sG_0s^{-1}}^{G_0}\mathbf{1})\otimes X_{\hat{\eta}}(M,V)'\big). \end{aligned}$$

Now G is second countable hence $G_0\backslash G/G_0$ is countable. Since any countable locally convex sum is a strict inductive limit and $\mathrm{Ind}_{G_0\cap sG_0s^{-1}}^{G_0}\mathbf{1}$ is finite-dimensional the claim follows. $\square$

To the end of this section we want to specialize our functor to the subcategory $\mathcal{N}_\eta$.

Let Z be a finite-dimensional $Z(\mathfrak{g})$-representation. Then $I_\eta(Z) = U(\mathfrak{g})\otimes_{Z(\mathfrak{g})U(\mathfrak{u})} Z\otimes K_\eta \in \mathcal{N}_\eta$ and we have a locally analytic U_0-action on $I_\eta(Z)$. Thus we get via the trivial action $\mathfrak{g}$ on V a $D(\mathfrak{g},U_0)$-module structure on $I_\eta(Z)\otimes_K V'$. We set

$$\tilde{\mathcal{F}}_{\hat{\eta}}^G(Z,V) := \mathcal{F}_{\hat{\eta}}^G(I_\eta(Z),V).$$

This construction is functorial in each entry. Hence for a fixed $\hat{\eta}$ we get thus a bifunctor

$$\tilde{\mathcal{F}}_{\hat{\eta}}^G : \mathrm{Rep}_K^{fd}(Z(\mathfrak{g}))\times \mathrm{Rep}_K^{\infty,adm}(U_0) \to Rep_K^{la}(G).$$

Proposition 4.5. *The functor $\tilde{\mathcal{F}}_{\hat{\eta}}^G$ is bi-exact.*

Proof. This follows from Proposition 4.2 and the exactness of the functor I_η. $\square$

5. Irreducibility

The first part of this chapter concerns the following irreducibility result. Here we fix as in the previous section a maximal ideal Θ of $Z(\mathfrak{g})$ and let η be a non-degenerate character with locally analytic lift $\hat{\eta}$.

Theorem 5.1. *The locally analytic G-representation $\mathcal{F}_{\hat{\eta}}^G(N_{\Theta,\eta})$ is topologically irreducible.*

Proof. We abbreviate $N_{\Theta,\eta}$ by N. The proof is at the beginning verbatim the same as in [15, 12] by replacing P_0 by U_0 and with the obvious (notational) changes. We first prove that $X = X_{\hat{\eta}}(N)$ is a simple $D(G_0)$-module and consider for an arbitrary open normal subgroup H of G_0 the decomposition

$$X = \bigoplus_{g\in G_0/H} \delta_g \star \big(D(HU_0) \otimes_{D(\mathfrak{g},U_0)} N\big) .$$

Here we recall that for a $D(H)$-module N and $g \in G_0$, we denote by $\delta_g \star N$ the space N, equipped with the structure of a $D(H)$-module given by $\delta \cdot_g n = (\delta_{g^{-1}}\delta\delta_g)n$, where $n \in N$, $\delta \in D(H)$, and the product $\delta_{g^{-1}}\delta\delta_g \in D(G_0)$ is contained in $D(H)$.

We arrive at the following modified version of Thm. 5.5 loc.cit. in a reduced version which proves that $X_{\hat{\eta}}(N)$ is a simple $D(G_0)$-module.

Theorem 5.2. *Let H be an open normal subgroup of G_0, and let $g, g_1, g_2 \in G_0$. Then*

(i) The $D(H)$-module $\delta_g \star \big(D(HU_0) \otimes_{D(\mathfrak{g},U_0)} N\big)$ is simple.

(ii) The $D(H)$-modules $\delta_{g_1} \star \big(D(HU_0) \otimes_{D(\mathfrak{g},U_0)} N\big)$ and $\delta_{g_2} \star \big(D(HU_0) \otimes_{D(\mathfrak{g},U_0)} N\big)$ are isomorphic if and only if $g_1HU_0 = g_2HU_0$.

Proof. The proof passes through several reduction steps. The first step is to reduce to the case of a suitable subgroup $H_0 \subset H$ which is normal in G_0 and uniform pro-p.

Step 1: reduction to H_0. Let $\mathrm{Lie}(\mathbf{G}_0)$, $\mathrm{Lie}(\mathbf{U}_0)$ etc. be the Lie algebras of G_0, U_0 etc. These are O_L-lattices in $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{u} = \mathrm{Lie}(U)$ etc. respectively. Moreover, $\mathrm{Lie}(\mathbf{G}_0)$, $\mathrm{Lie}(\mathbf{U}_0)$ etc. are $\mathbb{Z}_p$-Lie algebras, and we have as usual a decomposition $\mathrm{Lie}(\mathbf{G}_0) = \mathrm{Lie}(\mathbf{U}_0) \oplus \mathrm{Lie}(\mathbf{T}_0) \oplus \mathrm{Lie}(\mathbf{U}_0^-)$. For $m_0 \geq 1$ ($m_0 \geq 2$ if $p = 2$) the O_L-lattices $p^{m_0}\mathrm{Lie}(\mathbf{G}_0)$, $p^{m_0}\mathrm{Lie}(\mathbf{U}_0)$ etc. are powerful $\mathbb{Z}_p$-Lie algebras, cf. [4, sec. 9.4], and hence $\exp_G : \mathfrak{g} \dashrightarrow G$ converges on these O_L-lattices. We set

$$H_0 = \exp_G\Big(p^{m_0}\mathrm{Lie}(\mathbf{G}_0)\Big)\ ,\ \ H_0^0 = \exp_G\Big(p^{m_0}\mathrm{Lie}(\mathbf{T}_0)\Big), \tag{5.1}$$

$$H_0^+ = \exp_G\Big(p^{m_0}\mathrm{Lie}(\mathbf{U}_0)\Big), H_0^- = \exp_G\Big(p^{m_0}\mathrm{Lie}(\mathbf{U}_0^-)\Big),$$

which are uniform pro-p groups. Moreover H_0 is normal in G_0 and H_0^+ in U_0, respectively. We choose m_0 large enough such that H_0 is contained H. With the same argument as in [15] we may assume from now on that $H_0 = H$.

Step 2: passage to $D_r(H)$. We put

$$\kappa = \begin{cases} 1 & , \quad p > 2 \\ 2 & , \quad p = 2 \end{cases}$$

and denote by r a real number in $(0, 1) \cap p^{\mathbb{Q}}$ with the property that there is $m \in \mathbb{Z}_{\geq 0}$ such that $s = r^{p^m}$ satisfies

$$\max\{\frac{1}{p}, |\pi|p^{-1/(p-1)}\} < s \text{ and } s^\kappa < p^{-1/(p-1)} \tag{5.2}$$

where $\pi \in O_L$ is an uniformizer. We let $\|\cdot\|_r$ denote the norm on $D(H)$ associated to the canonical p-valuation. Here $D_r(H)$ is the corresponding Banach space completion. This is a noetherian Banach algebra, and $D(H) = \varprojlim_{r<1} D_r(H)$. Set $\mathbf{N} := D(HU_0) \otimes_{D(\mathfrak{g},U_0)} N$. As explained in loc.cit. it is enough to show that

$$\mathbf{N}_r := D_r(H) \otimes_{D(H)} \mathbf{N} = D_r(HU_0) \otimes_{D(\mathfrak{g},U_0)} N$$

are simple $D_r(H)$-modules for a sequence of r's converging to 1. From now on we assume that, in addition to (5.2), that r is chosen such that $\mathbf{N}_r \neq 0$, and we consider N as being contained in $\mathbf{N}_r$.

Step 3: passage to $U_r(\mathfrak{g})$. Let $U_r(\mathfrak{g}) = U_r(\mathfrak{g}, H)$ be the topological closure of $U(\mathfrak{g})$ in $D_r(H)$. Then

(5.3) $\quad U(\mathfrak{g})$ is dense in $D_r(H)$ if $r^\kappa < p^{-\frac{1}{p-1}}$ and $U_r(\mathfrak{g}) = D_r(H)$.

Let $(P_m(H))_{m\geq 1}$ be the lower p-series of H, cf. [4, 1.15]. Note that $P_1(H) = H$. For $m \geq 0$ put $H^m := P_{m+1}(H)$ so that $H^0 = H$. Then H^m is a uniform pro-p group with $\mathbb{Z}_p$-Lie algebra equal to $p^m \mathrm{Lie}_{\mathbb{Z}_p}(H)$. Thus our subgroups above have the following description. Let $\mathfrak{g}_{\mathbb{Z}}$ be a $\mathbb{Z}$-form of $\mathfrak{g}$, i.e. $\mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} L = \mathfrak{g}$. We fix a Chevalley basis $(x_\gamma, y_\gamma, h_\alpha \mid \gamma \in \Phi^+, \alpha \in \Delta)$ of $[\mathfrak{g}_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}}]$. We have $x_\gamma \in \mathfrak{g}_\gamma$, $y_\gamma \in \mathfrak{g}_{-\gamma}$, and $h_\alpha = [x_\alpha, y_\alpha] \in \mathfrak{t}$, for $\alpha \in \Delta$. Then

$$\mathrm{Lie}_{\mathbb{Z}_p}(H^-) = p^{m_0}\mathrm{Lie}(\mathbf{U}_0^-) = \bigoplus_{\beta\in\Phi^+} O_L y_\beta^{(0)} ,$$

where $y_\beta^{(0)} = p^{m_0} y_\beta$. Moreover, the $\mathbb{Z}_p$-Lie algebra of $H^{-,m} := H^m \cap H^-$ is

$$\mathrm{Lie}_{\mathbb{Z}_p}(H^{-,m}) = p^m \mathrm{Lie}(H^-) = \bigoplus_{\beta\in\Phi^+} O_L y_\beta^{(m)} ,$$

where $y_\beta^{(m)} = p^{m_0+m} y_\beta$. In the same way

$$\mathrm{Lie}_{\mathbb{Z}_p}(H^{+,m}) = p^m \mathrm{Lie}(H^+) = \bigoplus_{\beta\in\Phi^+} O_L x_\beta^{(m)} ,$$

where $x_\beta^{(m)} = p^{m_0+m} x_\beta$ and

$$\mathrm{Lie}_{\mathbb{Z}_p}(H^{0,m}) = p^m \mathrm{Lie}(H^0) = \bigoplus_{\beta\in\Phi^+} O_L h_\beta^{(m)} ,$$

where $h_\beta^{(m)} = p^{m_0+m} h_\beta$.

Let $s = r^{p^m}$ be as in (5.2). Denote by $\|\cdot\|_s^{(m)}$ the norm on $D(H^m)$ induced by the canonical p-valuation on H^m. Then, by [16, 6.2, 6.4], the restriction of $\|\cdot\|_r$ on $D(H)$ to $D(H^m)$ is equivalent to $\|\cdot\|_s^{(m)}$, and $D_r(H)$ is a finite and free $D_s(H^m)$-module on a basis any set of coset representatives for H/H^m. By (5.3) we can conclude:

(5.4) $\quad$ If $s = r^{p^m}$ is as in (5.2), then $U(\mathfrak{g})$ is dense in $D_s(H^m)$, hence $U_r(\mathfrak{g}, H) = D_s(H^m)$.

In particular, $U_r(\mathfrak{g}) \cap H = H^m$ is an open subgroup of H. Let

$$\mathfrak{n}_r := U_r(\mathfrak{g})N$$

be the $U_r(\mathfrak{g})$-submodule of $\mathbf{N}_r$ generated by N. The module N is dense in $\mathfrak{n}_r$ with respect to this topology. Now N is essentially a finite direct sum of copies of $U(\mathfrak{u}^-)$ by the Harish-Chandra isomorphism (see also identity (5.8) below). Hence it follows from [5, 1.3.12] or [14, 3.4.8] that $\mathfrak{n}_r$ is a simple $U_r(\mathfrak{g})$-module, and in particular a simple $D_r(\mathfrak{g}, U_0)$-module. Thus for every $g \in G_0$ the $U_r(\mathfrak{g})$-module $\delta_g \star \mathfrak{n}_r$ (which is defined as above by composing the natural action with conjugation with g) is simple. As in [15] we have for $U_{0,r} := HU_0 \cap D_r(\mathfrak{g}, U_0)$

(i) $U_{0,r} = H^m U_0 = U_0 H^{0,m} H^{-,m}$.

(ii) $D_r(HU_0) = \bigoplus_{g \in HU_0/U_{0,r}} \delta_g D_r(\mathfrak{g}, U_0)$.

By (ii) we deduce that $\mathfrak{n}_r = D_r(\mathfrak{g}, U_0) \otimes_{D(\mathfrak{g},U_0)} N \subset D_r(HU_0) \otimes_{D(\mathfrak{g},U_0)} N = \mathbf{N}_r$ and

$$\mathbf{N}_r = D_r(HU_0) \otimes_{D(\mathfrak{g},U_0)} N = D_r(HU_0) \otimes_{D_r(\mathfrak{g},U_0)} \mathfrak{n}_r = \bigoplus_{g \in HU_0/U_{0,r}} \delta_g \mathfrak{n}_r \,. \tag{5.5}$$

Here the action of $U_r(\mathfrak{g})$ on $\delta_g \mathfrak{n}_r$ is the same as on $\delta_g \star \mathfrak{n}_r$. Thus it suffices to show the theorem below. Here we only consider $\mathfrak{n}_r$ as a module over $U_r(\mathfrak{g})$. Thus for proving Theorem 5.2 it suffices to apply the next theorem.

Theorem 5.3. *Assume that all the above assumptions are satisfied. Then for any $g, h \in G_0$ with $gU_{0,r} \neq hU_{0,r}$ the $U_r(\mathfrak{g})$-modules $\delta_g \star \mathfrak{n}_r$ and $\delta_h \star \mathfrak{n}_r$ are not isomorphic.*

Proof. We start with the following observation. By our assumption on r and s we have $U_r(\mathfrak{u}^-) = U_r(\mathfrak{u}^-, H^-) = D_s(H^{-,m})$. Elements in $U_r(\mathfrak{u}^-)$ thus have a description as power series in $(y_\beta^{(m)})_{\beta \in \Phi^+}$:

$$U_r(\mathfrak{u}^-) = \left\{ \sum_{n=(n_\beta)} d_n (y^{(m)})^n \mid \lim_{|n| \to \infty} |d_n| s^{\kappa |n|} = 0 \right\},$$

where $(y^{(m)})^n$ is the product of the $(y_\beta^{(m)})^{n_\beta}$ over all $\beta \in \Phi^+$, taken in some fixed order. Let $\| \cdot \|_s^{(m)}$ be the norm on $D_s(H^{-,m})$ induced by the canonical p-valuation on $H^{-,m}$. Then we have for any generator $y_\beta^{(m)}$

$$\| y_\beta^{(m)} \|_s^{(m)} = s^\kappa \,. \tag{5.6}$$

Now let $\phi : \delta_g \star \mathfrak{n}_r \xrightarrow{\simeq} \delta_h \star \mathfrak{n}_r$ be an isomorphism of $U_r(\mathfrak{g})$-modules. We may assume $h = 1$. Let $I \subset G_0$ be the Iwahori subgroup. Using the Bruhat decomposition

$$G_0 = \coprod_{w \in W} I w B_0$$

we may write $g = k_1wk_2$ with $k_1 \in I$, $w \in W$ and $k_2 \in B_0$. Since $T_0 \subset I \cap B_0$ and w normalizes T_0 we may suppose that $k_2 \in U_0$. As $\delta_u \star \mathfrak{n}_r = \mathfrak{n}_r$ for $u \in U_0$ we may then suppose that $k_2 = e$. By the Iwahori decomposition

$$I = (I \cap B_0^-) \cdot (I \cap U_0)$$

we may write $k_1 = hu^+$ with $u^+ \in I \cap U_0$ and $h \in I \cap B_0^-$. By shifting the factors from the left to the right it suffices to consider an isomorphism

$$\phi : \delta_w \star \mathfrak{n}_r \xrightarrow{\simeq} \delta_h \star \mathfrak{n}_r \, .$$

with $h \in B_0^-$.

Suppose that $w \neq 1$. Then there is a positive root $\beta \in \Phi^+$ such that $w^{-1}\beta < 0$, hence $w^{-1}(-\beta) > 0$, cf. [3, 2.3]. Consider a non-zero element element $y \in \mathfrak{g}_{-\beta}$, and let $v^+ \in N$ be a generator (the unique Whittaker vector) of N. Then we have the following identity in $\delta_w \star \mathfrak{n}_r$,

$$y \cdot_w v^+ = \mathrm{Ad}(w^{-1})(y) \cdot v^+ = \eta(\mathrm{Ad}(w^{-1})(y))v^+.$$

On the other hand, we have $\phi(v^+) = v$ for some nonzero $v \in \mathfrak{n}_r$. And therefore

$$\eta(\mathrm{Ad}(w^{-1})(y))v = \phi(y \cdot_w v^+) = y \cdot_h \phi(v^+) = y \cdot_h v = \mathrm{Ad}(h^{-1})(y) \cdot v \, .$$

Since $h \in B_0^-$ we see that $\mathrm{Ad}(h^{-1})(y) \in \mathfrak{u}^-$. But $U(\mathfrak{u}^-)$ acts freely on N since $U(\mathfrak{g})$ is free over $Z(\mathfrak{g}) \otimes U(\mathfrak{u})$, cf. Theorem 3.5. It follows that $U(\mathfrak{u}^-)$ acts freely on $\mathfrak{n}_r$, as well. Thus $\mathrm{Ad}(h^{-1})(y) \cdot v$ cannot be a scalar multiple of v.

So we may suppose that $w = 1$ and that there is an isomorphism

$$\phi : \mathfrak{n}_r \xrightarrow{\simeq} \delta_h \star \mathfrak{n}_r$$

with $h \in B_0^-$. Write $h = tu$ with $t \in T_0$ and $u \in U_0^-$. By shifting t to the left hand side we get an isomorphism

$$\phi : \delta_t \star \mathfrak{n}_r \xrightarrow{\simeq} \delta_u \star \mathfrak{n}_r \, .$$

Let again $v = \phi(v^+)$. For $x \in \mathfrak{u}$ we get

$$\eta(\mathrm{Ad}(t^{-1})(x))v = \phi(x \cdot_t v^+) = x \cdot_u \phi(v^+) = x \cdot_u v = \mathrm{Ad}(u^{-1})(x) \cdot v \, . \tag{5.7}$$

Thus v is an eigenvector for the action of $\mathrm{Ad}(u^{-1})(x)$. Next we consider the formal completion $\hat{N}$ of N. In contrast to loc.cit., i.e. for objects in the BGG-category $\mathcal{O}$, it is defined as follows.

Let $y_1, \ldots, y_t$ be basis of $\mathfrak{u}^-$. Then $\hat{U}(\mathfrak{u}^-) = \prod_{\underline{n} \in \mathbb{N}_0^t} K y_1^{n_1} \cdots y_r^{n_r}$ is the formal completion of $U(\mathfrak{u}^-)$. This in an obvious way a module for $U(\mathfrak{u}^-)$ and the module structure extends to a representation of U^-. Indeed we consider the exponential $\exp_{U^-} : \mathfrak{u}^- \dashrightarrow U^-$ and the inverse map $\log_{U^-}$. We define for $u \in U^-$ and $v = \sum_{\underline{n}} v_{\underline{n}} \in \hat{U}(\mathfrak{u}^-)$:

$$\delta_u \cdot v = \sum_{\underline{n}} \sum_{i \geq 0} \frac{1}{i!} \log_{U^-}(u)^i \cdot v_{\underline{n}} \, .$$

Note that this sum is well-defined in $\hat{U}(\mathfrak{u}^-)$, because $\log_{U^-}(u)$ is in $\mathfrak{u}^-$, and there are hence only finitely many terms of a given homogeneous degree in this sum. Moreover, it gives an action of U^- on $\hat{U}(\mathfrak{u}^-)$ because $\exp(\log_{U^-}(u_1)) \cdot \exp(\log_{U^-}(u_2)) = \exp\big(\mathcal{H}(\log_{U^-}(u_1), \log_{U^-}(u_2))\big) = \exp(\log_{U^-}(u_1u_2))$ where $\mathcal{H}(X,Y) = \log(\exp(X)\exp(Y))$ is the Baker-Campbell-Hausdorff series (which converges on $\mathfrak{u}^-$, as $\mathfrak{u}^-$ is nilpotent). It is then immediate that this action is compatible with the action of $U(\mathfrak{u}^-)$.

Since $U(\mathfrak{g})$ is finite free over $Z(\mathfrak{g})U(\mathfrak{u})$ we find by the Harish-Chandra isomorphism $h_1, \ldots, h_r \in U(\mathfrak{t})$ such that the elements of $U(\mathfrak{u}^-) \oplus U(\mathfrak{u}^-)h_1 \oplus \cdots \oplus U(\mathfrak{u}^-)h_r$ form a basis. Then

$$\hat{N} = \hat{U}(\mathfrak{u}^-) \times \hat{U}(\mathfrak{u}^-)t_1 \cdots \times \hat{U}(\mathfrak{u}^-)t_r \otimes K_{\Theta,\eta}. \tag{5.8}$$

Here we have again an action of U^- which is given component wise.

The identity (5.7) implies then the following identity in $\hat{N}$

$$\delta_{u^{-1}} \cdot (x \cdot (\delta_u \cdot v)) = \mathrm{Ad}(u^{-1})(x) \cdot v = \eta(Ad(t^{-1})x)v \ ,$$

for all $x \in \mathfrak{u}$, and thus, multiplying both sides of the previous equation with δ_u

$$x \cdot (\delta_u \cdot v) = \eta^t(x) := \eta(Ad(t^{-1})x)\delta_u \cdot v.$$

Hence $\delta_u \cdot v \in \hat{N}$ is a Whittaker vector (since η^t defines also a non-degenerate Whittaker function) and must therefore be equal to a non-zero scalar multiple of v^+. After scaling v appropriately we have $\delta_u \cdot v = v^+$ or $v = \delta_{u^{-1}} \cdot v^+$ with

$$\delta_{u^{-1}} \cdot v^+ = \sum_{n \geq 0} \frac{1}{n!}(-\log_{U_P^-}(u))^n \cdot v^+ =: \Sigma \ . \tag{5.9}$$

As in loc.cit. we shall show that the series Σ, which is an element of $\hat{N}$, does in fact not lie in the image of $\mathfrak{n}_r$ in $\hat{N}$, if $u \notin U_{0,r}$. For this we write

$$\log_{U_P^-}(u) = \sum_{\beta \in B(u)} z_\beta \ ,$$

with a non-empty set $B(u) \subset \Phi^+$ and non-zero elements $z_\beta \in \mathfrak{g}_{-\beta}$. Put

$$B^+(u) = \{\beta \in B(u) \mid z_\beta \notin O_L y_\beta^{(m)}\} \ .$$

This is a non-empty subset of $B(u)$ since $u \notin U_{0,r}$. Put $B'(u) = B(u) \setminus B^+(u)$,

$$z^+ = \sum_{\beta \in B^+(u)} z_\beta \quad \text{ and } \quad z' = \sum_{\beta \in B'(u)} z_\beta = \log_{U^-}(u) - z^+ \ .$$

Then $z' \in \mathrm{Lie}_{\mathbb{Z}_p}(H^{-,m})$, and thus $\exp(z') \in H^{-,m}$. The element $u_1 = u\exp(z')^{-1} = u\exp(-z')$ does not lie in $H^{-,m}$, and $\delta_u \star \mathfrak{n}_r \cong \delta_{u_1} \star \mathfrak{n}_r$. We may hence replace u by $u_1 = u\exp(-z')$. By iterating this process we can assume that $B'(u) = \emptyset$, cf. loc.cit.

Next we chose as in loc.cit. an extremal element β^+ among the $\beta \in B(u) = B^+(u)$, i.e. one of the minimal generators of the cone $\sum_{\beta \in B(u)} \mathbb{R}_{\geq 0}\beta$ inside $\bigoplus_{\alpha \in \Delta} \mathbb{R}\alpha$. Then if we write $\log_{U^-}(u)) = z_{\beta^+} + z_{\beta_2} + \ldots + z_{\beta_k}$ where $B(u) = \{\beta^+, \beta_2, \ldots, \beta_k\}$ it suffices to see that the series $\frac{(-1)^n}{n!} z_{\beta^+}^n \cdot v^+$ does not converge to an element in $\mathfrak{n}_r$.

Because $B'(u) = \emptyset$, we have $z_{\beta+} = \alpha\pi^{-k} y_{\beta+}^{(m)}$ for some $k \geq 1$ and $\alpha \in O_L^*$. We then get

$$||(-1)^n \frac{z_{\beta+}^n}{n!}||_s^{(m)} = |\pi^{-k}| \frac{s^n}{|n!|} = |\frac{\pi^{-n(k-1)}}{n!}||(\frac{s}{\pi})|^n.$$

By identity (5.2), we have $|\frac{s}{\pi})| > p^{-\frac{1}{(p-1)}}$ which is the radius of convergence of the exponential series. As further $|\pi^{-1}| > 1$ we see that Σ does not converge.

Now we show that $c - \mathrm{Ind}_{G_0}^G X'$ is topologically irreducible. For this we check the levelwise Mackey criterion by Bode [12, Theorem A.23]. For r as above let $X_r := D_r(G_0) \otimes_{D(G_0)} X$ be the completion of X. For a non-trivial double coset representative $g \in G_0 \backslash G / G_0$, let $G_0^g := gG_0g^{-1}$ and $F = G_0 \cap G_0^g$. We have to show that the $D_r(F)$-modules X_r and $\delta_g \star X_r$ are disjoint i.e., that they do not have a simple constituent in common. We shall show that these modules are even as $U_r(\mathfrak{g})$-modules disjoint. For this it suffices to see that there is no non-trivial homomorphism $\delta_{h_1} \star \mathfrak{n}_r \to \delta_{g \cdot h_2} \mathfrak{n}_r$ for $h_1, h_2 \in G_0$.

By the Cartan decomposition we may suppose that $g \in T^- = \{t \in T \mid |\alpha(t)| \geq 1 \, \forall \alpha \in \Delta\}$ is an element of the torus. So suppose that $g = t \in T^-$ and that $\phi : \delta_{h_1} \star \mathfrak{n}_r \to \delta_t h_2 \star \mathfrak{n}_r$ is an isomorphism. We may assume by shifting factors that $h_2 = 1$.

Suppose first the case $h_1 = 1$. Consider the formal completion $\hat{N}$ of N. Then

$$H^0(\mathfrak{n}, N) = H^0(\mathfrak{n}, \hat{N}) = Kv^+.$$

Since $\hat{N}$ is also the formal completion of $\delta_t \star N$ and $\delta_t \star \eta$ is also a non-degenerate Whittaker function for v^+ we assume that $\phi(v^+) = v^+$. But then we have $\eta(Ad(t)x)\phi(v^+) = \phi(xv^+) = \phi(\eta(x)v^+) = \eta(x)\phi(v^+)$ for all $x \in \mathfrak{u}$. Thus $\eta(Ad(t)x) = \eta(x)$ for all $x \in \mathfrak{u}$ which is impossible since η is non-degenerate. In the case $h_1 \neq 1$ we can repeat the proof of Theorem 5.3 using the Bruhat decomposition. Observe that $\delta_t \star \mathfrak{n}$ is also Whittaker module with respect to the Whittaker function η^t.

Hence the summands of the two direct summands are pairwise not isomorphic which finishes our proof. □

A natural generalization of the above theorem is to consider also a smooth contribution V in our functor. By functoriality V has to be irreducible in order that $\mathcal{F}_{\hat{\eta}}^G(N_{\Theta,\eta}, V)$ is topologically irreducible as G-representation. This condition is also sufficient:

Theorem 5.4. *Let $\hat{\eta}$ and Θ be chosen as above. Let V be a smooth irreducible representation of U_0. Then $\mathcal{F}_{\hat{\eta}}^G(N_{\Theta,\eta}, V)$ topologically irreducible as G-representation.*

Proof. The proof follows the same lines as in [15] We present a compactified version and refer for more details to loc.cit.

For a closed G-subrepresentation $W \subset \mathcal{F}_{\hat{\eta}}^G(N, V))$, we put

$$W_{sm} = \varinjlim_{H} \mathrm{Hom}_H(c - \mathrm{Ind}_{U_0}^G (K'_{\hat{\eta}})^{\mathfrak{d}}|_H, W|_H),$$

where the limit is taken over all compact open subgroups H of G. Then G acts on W_{sm} via $(g\phi)(f) = g(\phi(g^{-1}(f))$ and yields by construction a smooth representation. We consider the continuous map of G-representations

$$\Phi_W : c - \mathrm{Ind}_{U_0}^G(K'_{\hat{\eta}})^{\mathfrak{d}} \otimes_K W_{sm} \longrightarrow W\,,\ f \otimes \phi \mapsto \phi(f)\,.$$

Let $H \subset G_0$ be an open normal subgroup. We have

$$c - \mathrm{Ind}_{U_0}^G(K'_{\hat{\eta}})^{\mathfrak{d}}|_H \ = \bigoplus_{\gamma \in H\backslash G/U_0} \mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K'_{\hat{\eta}})^\gamma)^{\mathfrak{d}} \tag{5.10}$$

where $(K'_{\hat{\eta}})^\gamma$ is $K_{\hat{\eta}}$ with the twisted action.

By Theorem 5.2 all H-representations on the right hand side of (5.10) are topologically irreducible and pairwise non-isomorphic. Similarly, for the representation $c - \mathrm{Ind}_{U_0}^G(K'_{\hat{\eta}} \otimes_K V)^{\mathfrak{d}}$ we have

$$\mathrm{Ind}_{U_0}^G(K'_{\hat{\eta}} \otimes_K V)^{\mathfrak{d}}|_H = \bigoplus_{\gamma \in H\backslash G_0/U_0} \mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K'_{\hat{\eta}})^\gamma \otimes_K V^\gamma)^{\mathfrak{d}}\,.$$

Denote by $V^{H\cap U_0} \subset V$ the subspace of fix vectors for $H \cap U_0$. Then

$$\Big(\mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K'_{\hat{\eta}})^\gamma)\Big)^{\mathfrak{d}} \otimes_K V^{H\cap U_0} \cong \mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K'_{\hat{\eta}})^\gamma \otimes_K (V^{H\cap K_0})^\gamma)^{\mathfrak{d}}$$

is a subspace of $\mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K'_{\hat{\eta}})^\gamma \otimes_K V^\gamma)^{\mathfrak{d}}$. Let

$$\phi : c - \mathrm{Ind}_{U_0}^{G_0}(K'_{\hat{\eta}})^{\mathfrak{d}}|_H \to c - \mathrm{Ind}_{U_0}^{G_0}(K'_{\hat{\eta}} \otimes_K V)^{\mathfrak{d}}|_H$$

be a continuous homomorphism of H-representations. For each γ, let $f_\gamma \in \mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K')^\gamma)^{\mathfrak{d}}$ be a non-zero element. Because of the irreducibility the map ϕ is uniquely determined by the images $\phi(f_\gamma)$ which are rigid-analytic functions on the components of some covering of G_0 by 'polydiscs' Δ_i and which take values in a BH-subspace of $K' \otimes_K V$. But any such BH-subspace is finite-dimensional, hence contained in a subspace of the form $K' \otimes_K V^{H'\cap U_0}$ for some open subgroup $H' \subset H$ which is normal in G_0. Shrinking H' we may further suppose that each Δ_i is H'-stable and that each $\phi(f_\gamma)$ takes values in $K' \otimes_K V^{H'\cap U_0}$ and

$$\phi(h.f_\gamma)(g) = [h.\phi(f_\gamma)](g) = \phi(f_\gamma)(h^{-1}g) \in K' \otimes_K V^{H'\cap U_0}$$

for any $h \in H$. Because the subspace of $\mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K')^\gamma)^{\mathfrak{d}}$ generated by the functions $h.f_\gamma$, $h \in H$, is dense in $\mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K')^\gamma)^{\mathfrak{d}}$, we have $\phi(f)(g) \in K'_\eta \otimes_K V^{H'\cap U_0}$ for all $f \in \mathrm{Ind}^H_{H\cap\gamma U_0\gamma^{-1}}((K')^\gamma)^{\mathfrak{d}}$. It follows that ϕ induces a continuous map of H'-representations

$$\phi : c - \mathrm{Ind}_{U_0}^{G_0}(K')^{\mathfrak{d}}|_{H'} \to \Big[c - \mathrm{Ind}_{U_0}^{G_0}(K')^{\mathfrak{d}}\Big]\,|_{H'} \otimes_K V^{H'\cap U_0}$$

Here the irreducible H'-representation $\mathrm{Ind}^{H'}_{H'\cap\gamma U_0\gamma^{-1}}((K)^\gamma)^{\mathfrak{d}}$ contained in the left hand side is mapped via ϕ to $\Big[\mathrm{Ind}^{H'}_{H'\cap\gamma U_0\gamma^{-1}}((K')^\gamma)^{\mathfrak{d}}\Big] \otimes_K V^{H'\cap U_0}$ and the restriction of ϕ to this summand has the shape $f \mapsto f \otimes v$ for some $v \in V^{H' \cap U_0}$. As in loc.cit. one deduces from this that the canonical map

$$\begin{array}{rcl} c - \mathrm{ind}_{U_0}^G(V) & \to & \mathcal{F}_{\hat{\eta}}^G(N,V))_{sm} \\ \varphi & \mapsto & [f \mapsto \Phi(f \otimes \varphi)], \end{array}$$

is an isomorphism, where $c - \mathrm{ind}_{U_0}^G(V)$ denotes the smoothly compact induction. Moreover by the same reasoning as in loc.cit. one verifies that $W_{sm} \neq \{0\}$ for any non-zero closed G-invariant subspace $W \subset \mathcal{F}_{\hat{\eta}}^G(N,V)$.

Claim: $c - \mathrm{Ind}_{U_0}^G(K'_\eta)^{\mathfrak{d}} \otimes_K W_{sm}$ surjects onto $\mathcal{F}_{\hat{\eta}}^G(N,V)$.

For this it suffices to show that $c - \mathrm{Ind}_{U_0}^{G_0}(K'_\eta)^{\mathfrak{d}} \otimes_K W_{sm}^0$ surjects onto $X(N_{\Theta,\eta},V)'$ where $W_{sm}^0 := W_{sm} \cap \mathrm{ind}_{U_0}^{G_0}(V)$.

Consider the U_0-morphism $\Pi : W_{sm}^0 \to V$ given by $\varphi \mapsto \varphi(1)$. As $W_{sm} \neq (0)$ and therefore $W_{sm}^0 \neq (0)$ this map is surjective. Therefore, for any $v \in V$ there is some $\varphi \in W_{sm}^0$ with $\varphi(1) = v$. As W_{sm}^0 is a G_0-representation we may replace 1 by any $g \in G_0$, i.e. $\varphi(g) = v$. Then, on a neighborhood N of g, the map φ is constant with value v.

Let $S \subset G_0$ be a compact locally analytic subset such that $S \cdot U_0 = G_0$. Let $S' \subset S$ and $C \subset U_0$ be compact open subsets such that $S' \cdot C \subset N$. Let $f : G \to K'$ be a function whose support is contained in $S' \cdot C$. Then the function $S \to K' \otimes_K V$, $s \mapsto f(s) \otimes \phi(s)$, is equal to the function $S \to K' \otimes_K V$, $s \mapsto f(s) \otimes v$.

Let $f \in \mathrm{Ind}_{U_0}^{G_0}(K'_\eta \otimes_K V)^{\mathfrak{d}}$ be any element. Then f is uniquely determined by its restriction to S. As we have seen above, the set $f(G_0)$ is contained in a finite-dimensional vector space $K'_\eta \otimes_K V_0$ with $V_0 \subset V$. Let $v_1, \ldots, v_r$ be a basis of V_0. Write $f(s) = f_1(s) \otimes v_1 + \ldots + f_r(s) \otimes v_r$ with locally analytic K'_η-valued functions f_i on S. Extend each function f_i to G_0 by $f_i(s \cdot u) = \hat{\eta}(u^{-1}) f_i(s)$. Then $f_i \in \mathrm{Ind}_{U_0}^{G_0}(K'_\eta)$ for all i. In fact, one verifies that $f_i \in \mathrm{Ind}_{U_0}^{G_0}(K'_\eta)^{\mathfrak{d}}$ for all i. For any $s \in S$ choose some $\varphi_{i,s} \in W_{sm}^0$ such that $\varphi_{i,s}(x) = v_i$ for all x in a compact open neighborhood $N_{i,s} \subset S$ of s. As S is compact, finitely many of the $N_{i,s}$ will cover S. We can then choose a (finite) disjoint refinement $(N_{i,j})_j$ of the finite covering. Then we restrict f_i to each of these $N_{i,j} \cdot U_0$, and extend it by 0 to $(S \setminus N_{i,j}) \cdot U_0$. Denote the function thus obtained by $f_{i,j}$. Again, all $f_{i,j}$ lie in $c - \mathrm{Ind}_{U_0}^{G_0}(K')^{\mathfrak{d}}$, and $x \mapsto f_{i,j}(x) \otimes \phi_{i,s(i,j)}(x) = f_{i,j}(x) \otimes v_i$, for a suitably chosen $s(i,j) \in S$. Obviously, we have: $f = \sum_{i,j} f_{i,j} \otimes \phi_{i,s(i,j)}$. Indeed, by construction, both $f|_S$ and the restriction of the sum to S coincide. And both are functions in $c - \mathrm{Ind}_{U_0}^{G_0}(K' \otimes_K V)^{\mathfrak{d}}$; therefore, they are equal. The claim follows. $\square$

6. Cuspidality

In this final section we will show that the ordinary (Hausdorff) Jacquet modules of $\mathcal{F}^G_{\hat{\eta}}(N_{\Theta,\eta}, V)$ with respect to proper parabolic subgroup vanish. However, we will see that the higher homology groups do not vanish in general. We keep the notation from the previous section and fix further a smooth representation V of U_0. For convenience we set $N = N_{\Theta,\eta}$.

Proposition 6.1. *Let $P \subset G$ be a parabolic subgroup and U_P its unipotent radical. Set $U_{P,0} = G_0 \cap U_P$. Then $H^0(U_{P,0}, X_{\hat{\eta}}(N, V)) = 0$.*

Proof. We follow the proof of [12, Thm. 6.1]. We clearly have $H^0(U_{P,0}, X_{\hat{\eta}}(N, V)) = H^0(\mathfrak{u}_P, X_{\hat{\eta}}(N, V))^{U_{P,0}}$ so that it suffices to see that $H^0(\mathfrak{u}_P, X_{\hat{\eta}}(N, V)) = 0$. Since the action of $\mathfrak{u}_P$ on V is trivial we may suppose that $V = \mathbf{1}^\infty$ is the trivial representation, cf. [13, Lemma 3.4].

Let $I \subset G_0$ be the standard Iwahori subgroup. For $w \in W$, let N^w be the $D(\mathfrak{g}, I \cap wU_0w^{-1})$-module with underlying vector space N twisted action given by composition of the given one with conjugation with w. Let

$$w^{-1}\ Iw = (U_0^- \cap w^{-1}Iw)(T_0 \cap w^{-1}Iw)(U_0 \cap w^{-1}Iw) \tag{6.1}$$

be the induced Iwahori decomposition, cf. [14, Lemma 3.3.2]. The Bruhat decomposition $G_0 = \coprod_{w\in W} IwI$ induces a decomposition

$$\begin{aligned} D(G_0) \otimes_{D(\mathfrak{g},U_0)} N &\simeq \bigoplus_{w\in W} D(I) \otimes_{D(\mathfrak{g},I\cap wU_0w^{-1})} N^w \\ &\simeq \bigoplus_{w\in W} D(w^{-1}Iw) \otimes_{D(\mathfrak{g},w^{-1}Iw\cap U_0)} N \end{aligned} \tag{6.2}$$

since $IwB_0 = IwU_0$. For each $w \in W$, we have

$$H^0(\mathfrak{u}_P, D(I) \otimes_{D(\mathfrak{g},I\cap wU_0w^{-1})} N^w) \simeq H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), D(w^{-1}Iw) \otimes_{D(\mathfrak{g},w^{-1}Iw\cap U_0)} N).$$

We can write each summand in the shape

$$\mathcal{N}^w := D(w^{-1}Iw) \otimes_{D(\mathfrak{g},w^{-1}Iw\cap U_0)} N = \varprojlim_r \mathcal{N}^w_r$$

where $\mathcal{N}^w_r = D_r(w^{-1}Iw) \otimes_{D(\mathfrak{g},w^{-1}Iw\cap U_0)} N$. If we denote by N^w_r the topological closure[1] of N in $\mathcal{N}^w_r$, we get by [15, 5.6.5] finitely many elements $u \in U_0^-$ and finitely many elements $v \in T_0$ such that

$$\mathcal{N}^w_r \simeq \bigoplus_{u,v} \delta_u\delta_v \otimes N^w_r \tag{6.3}$$

and the action of $\mathfrak{x} \in \mathrm{Ad}(w^{-1})(\mathfrak{u}_P)$ is given by

$$\mathfrak{x} \cdot \sum_{u,v} \delta_u\delta_v \otimes m_{u,v} = \sum \delta_u\delta_v \otimes \mathrm{Ad}((uv)^{-1}(\mathfrak{x}))m_{u,v}.$$

[1]here we use the notation of [13].

Indeed, by [14, Prop. 3.3.4] we have isomorphisms

$$D_r(w^{-1}Iw) \cong D_r(w^{-1}Iw \cap U_0^-)\hat{\otimes} D_r(w^{-1}Iw \cap T_0)\hat{\otimes} D_r(w^{-1}Iw \cap U_0)$$

which induce isomorphisms

$$D_r(\mathfrak{g}, w^{-1}Iw \cap U_0) \cong U_r(\mathfrak{u}_0^-)\hat{\otimes} U_r(\mathfrak{t})\hat{\otimes} D_r(w^{-1}Iw \cap U_0)$$

where $U_r(\mathfrak{u}_0^-)$ is the closure of $U(\mathfrak{u}_0^-)$ in $D_r(w^{-1}Iw \cap U_0^-)$ etc. Hence we get

$$\begin{aligned} & D_r(w^{-1}Iw) \otimes_{D(\mathfrak{g}, w^{-1}Iw \cap U_0)} N \\ = \;& D_r(w^{-1}Iw \cap U_0^-)\hat{\otimes} D_r(w^{-1}Iw \cap T_0) \otimes_{U_r(\mathfrak{u}_0^-)\otimes U_r(\mathfrak{t})} N_r^w. \end{aligned} \tag{6.4}$$

But by the discussion in [15, (5.5.7)- (5.5.8)] there are for the fixed open subgroup H (which is contained in $w^{-1}Iw$) finitely many elements $u \in U_0^-$ such that $D_r(H \cap U_0^-) = \bigoplus_u \delta_u U_r(\mathfrak{u}_0^-)$. On the other hand there are by the compactness of I finitely many $u \in U_0^-$ such that $D_r(w^{-1}Iw \cap U_0^-) = \bigoplus_u \delta_u D_r(H \cap U_0^-)$ which gives the above claim.

Let $w \neq 1$. Then $\mathrm{Ad}((uv)^{-1}(\mathrm{Ad}(w^{-1})\mathfrak{u}_P)$ contains an element in $\mathfrak{u}^-$. Hence this element acts injectively on N by the presentation (5.8). By a similar argument as in Step 1 of [15, Theorem 5.7] this element acts injectively on N_r^w, as well since each element in N_r^w is again a power series in weight vectors. We conclude that $H^0(\mathrm{Ad}((uv)^{-1})(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P)), N_r^w) = 0$ for all $w \neq 1$. If $w = 1$ and $u = 1$ then $\mathrm{Ad}((uv)^{-1}((\mathrm{Ad}(w^{-1})\mathfrak{u}_P)) = \mathfrak{u}_P$. But then there exists an element $x \in \mathfrak{u}_P$ attached to a simple root such that $\eta(x) \neq 0$. Hence x acts injectively, as well. Indeed one checks this by elementary computations in N or by applying [8, Th, 4.1]. By the argument above we deduce again that $H^0(\mathrm{Ad}((v)^{-1})(\mathrm{Ad}(\mathfrak{u}_P)), N_r^1) = 0$.

If $u \neq 1$ then $\mathrm{Ad}((uv)^{-1})(\mathfrak{u}_P)) \nsubseteq \mathfrak{u}_P$. Hence there is some element in it which we can write as a sum $y + x$ with $y \in \mathfrak{u}^-$. Now let $v \in N$ and identify it with an element in (5.8). Then y increases the degree whereas x decreases the degree of the corresponding polynomial expression in the y_α, $\alpha \in \Phi^+$. Hence $y + x$ acts injectively on N and we argue as above.

Hence by passing to the limit we get $H^0(\mathrm{Ad}(w^{-1})(\mathfrak{u}_P), \mathcal{N}^w) = 0$ for all $w \in W$. The claim follows. □

Corollary 6.2. *With the above notation we have* $\overline{H}_0(U_{P,0}, X_{\hat{\eta}}(N, V)) = 0$. *(where* $\overline{H}_0$ *denotes the Hausdorff completion of* H_0*).*

Proof. This follows by duality, cf. also [13, Theorem 3.5]. □

Proposition 6.3. *We have* $\overline{H}_0(U_P, \mathcal{F}^G_{\hat{\eta}}(N, V)) = 0$.

Proof. Since $c - \mathrm{Ind}^G_{G_0}(X_{\hat{\eta}}(N, V)')$ is a direct sum $\bigoplus_{g \in G/G_0} gX_{\hat{\eta}}(N, V)'$ of copies of $X_{\hat{\eta}}(N, V)'$ and $\overline{H}_0(\mathfrak{u}_p, gX_{\hat{\eta}}(N, V)') = \overline{H}_0(Ad(g)(\mathfrak{u}_P), X_{\hat{\eta}}(N, V)')$ it suffices to see that $\overline{H}_0(\mathfrak{u}_P, X_{\hat{\eta}}(N, V)') = 0$ since we consider from the very beginning arbitrary parabolic subgroups P. Now we apply the above corollary. □

In the following we discuss an example which shows that the homological vanishing criterion of Kohlhaase is not satisfied in general. For this let $G = \mathrm{GL}_2$.

Proposition 6.4. *We have* $H_1(U^-,\mathcal{F}^G_{\hat\eta}(N))\neq 0$.

Proof. By [7, Theorem 7.1] we have the identity $H_1(U^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)') = H_1(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)'))_{U^-}$. Since $\dim\mathfrak{u} = 1$ we conclude that $H_i(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)')$ is the cohomology of the complex

$$y: C^1\to C^0$$

where $C^0=C^1=\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)')$ and y denotes multiplication by $y\in\mathfrak{u}^-$.

Thus $H_1(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)'))$ coincides with $H^0(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)'))$ which is isomorphic to the topological dual of $\overline{H}_0(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N))$. But

$$\begin{aligned} H_0(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N))) &= c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N))/\mathfrak{u}^- c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)) \\ &= c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N))/\mathfrak{u}^- X_{\hat\eta}(N)) \end{aligned}$$

by the exactness of $c-\mathrm{Ind}^G_{G_0}$. Further $\mathfrak{u}^-D(G_0)=D(G_0)(\mathfrak{u}^-)^kN$ for some integer $k\geq 1$ since $\mathfrak{u}^-$ is a Ore set (cf. the proof of [9, Lemma 4.2]). But $N/(\mathfrak{u}^-)^kN$ is a finite-dimensional vector space. In particular we see that

$$D(G_0)\otimes_{U(\mathfrak{u}^-)} N/\mathfrak{u}^-D(G_0)\otimes_{U(\mathfrak{u}^-)} N = D(G_0)\otimes_{U(\mathfrak{u}^-)} (N/(\mathfrak{u}^-)^kN)$$

is a Hausdorff space. But the natural epimorphism $D(G_0)\otimes_{U(\mathfrak{u}^-)} N\to D(G_0)\otimes_{D(\mathfrak{g},U_0)} N$ is strict. It follows by the obvious diagram chasing diagram that $D(G_0)\otimes_{D(\mathfrak{g},U_0)} N/\mathfrak{u}^-D(G_0)\otimes_{D(\mathfrak{g},U_0)} N$ has to be Hausdorff, as well. Thus we conclude that $H_0(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N)) = \overline{H}_0(\mathfrak{u}^-, c-\mathrm{Ind}^G_{G_0}(X_{\hat\eta}(N))$. Moreover, the image of the Whittaker vector in $N/(\mathfrak{u}^-)^kN$ does not vanish. It follows that $D(G_0)\otimes_{D(\mathfrak{g},U_0)} N/\mathfrak{u}^-D(G_0)\otimes_{D(\mathfrak{g},U_0)} N\neq 0$. Indeed by (6.2) we may suppose that G_0 is I. Then we use the decomposition (6.4) in order to deduce the claim. This finishes the proof. □

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