

# Group Representations of Lorentz Transformations Extended to Superluminal Observers

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## Abstract

We construct an extension of the proper orthochronous Lorentz subgroup  $\text{SO}(3,1)_{\uparrow}^+$  that includes space-time transformations for observers moving with superluminal relative velocities in arbitrary direction. This extension is generated by the following realization of the Klein four group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  depending on polar and azimuthal angles identifying a spatial direction:

$$Z_{(\theta,\phi)} = \{I, -I, \Lambda_{\infty}(\theta, \phi), -\Lambda_{\infty}(\theta, \phi)\},$$

where  $\Lambda_{\infty}(\theta, \phi)$  are involutive matrices arising as infinite-velocity limits of superluminal boosts and satisfy  $\Lambda_{\infty}^2 = I$ ,  $\det \Lambda_{\infty} = -1$ . The extension is:

$$\mathcal{L}_{\text{ext}} \cong (\text{SO}(3,1)_{\uparrow}^+ \rtimes_{\text{Ad}_{\Lambda_{\infty}}} \mathbb{Z}_2) \times \mathbb{Z}_2$$

where  $\mathbb{Z}_2$  is  $\{-1,1\}$ , the nontrivial automorphism  $\text{Ad}_{\Lambda_{\infty}}$  is conjugation by  $\Lambda_{\infty}(\theta, \phi)$  and conjugation with matrices corresponding to different values  $(\theta', \phi')$  produces equivalent group extensions.  $\mathcal{L}_{\text{ext}}$  has the same identity component  $\text{SO}(3,1)_{\uparrow}^+$  as the usual Lorentz group but components  $\pm\Lambda_{\infty}$  in place of parity and time-reversal of  $\text{O}(3,1)$ . Each transformation in this extended group preserves the Minkowski form up to sign and can exchange time-like and spacelike directions. Passing to the extended Poincaré group  $\mathcal{P}_{\text{ext}} = \mathbb{R}^4 \rtimes \mathcal{L}_{\text{ext}}$ , we classify unitary irreducible representations (UIRs). The action of  $\mathcal{L}_{\text{ext}}$  on momentum space merges the timelike and spacelike  $\text{SO}(3,1)_{\uparrow}^+$ -orbits with invariants  $p^{\mu}p_{\mu} = \pm M^2$  into a single orbit  $|p^{\mu}p_{\mu}| = M^2$ , while merges light-like  $\text{SO}(3,1)_{\uparrow}^+$ -orbits with  $p_0 < 0$  and  $p_0 > 0$ . The resulting induced UIRs for  $M \neq 0$  restrict to the ordinary Poincaré subgroup as a multiplicity-one direct sum of a massive forward ( $p_0 > 0, p^{\mu}p_{\mu} > 0$ ), a massive backward ( $p_0 < 0, p^{\mu}p_{\mu} > 0$ ) and a tachyonic ( $p^{\mu}p_{\mu} < 0$ ) Wigner's UIR, while for  $M = 0$  as direct sum of forward and backward massless Wigner's UIRs depending on a two valued parameter. We then derive wave equations corresponding to solutions of the Casimir eigenvalue problem of Poincaré algebra obtained differentiating the above representations. This set of equations contains all the wave equations known to date in quantum field theory together with new wave equations describing tachyonic behaviour and a new class of massless representations. We finally show that tachyonic wave functions provide a relevant representation theoretic tool for interpretation of parity-violation phenomena in quantum field theory

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# 1 Introduction

A first attempt to formally understand superluminal Lorentz transformations from a group theoretical perspective is, in author's knowledge, [16], where it is given a representation of subluminal and superluminal Lorentz transformations employing complex space-time coordinates. A first study of the same kind involving real space-time coordinates appears in [3] however the authors of this paper only classify real representations of superluminal boosts with respect to a static observer but do not classify representations of boosts between subluminal and superluminal observers. In more recent years in [5] a first attempt to represent boosts between subluminal and superluminal observers in real space-time coordinates has been done. In that paper it is defined a transformation between a subluminal and a superluminal observer and is called "superflip". In this paper, based on the above works, we define, for the first time in author's knowledge a full classification of subluminal and superluminal Lorentz transformations in 4 dimensions employing real coordinates. Here we achieve this exploiting a symmetry of matrices representing Lorentz boosts with respect to the following conformal transformation of the boost speed vector  $v$ :

$$v \rightarrow c^2 \left( \frac{v_x}{|v|^2}, \frac{v_y}{|v|^2}, \frac{v_z}{|v|^2} \right)$$

The above is a particular case of a transformation known in geometry as "6 Spheres Coordinates" and is obtained by inverting the 3D Cartesian coordinates of the speed vector across the 2-sphere of radius  $c$ . Exploiting the symmetry of Lorentz transformations with respect to the above we are able to represent Lorentz boost for  $v$  in  $\mathbb{R}^3/\{c\}$  and not only in  $(-c, c)$  in real space-time coordinates. We show that the set of matrices defined in this way form a group of space-time transformations  $\{\Lambda\}$  such that:

$$|\Lambda v|^2 = \pm |v|^2 \quad |v|^2 = v^T \mu v$$

where  $\mu$  is Minkowski metric and  $v$  a 4 vector. This group of transformations can send space-like vectors into time-like vectors and vice-versa.

A quantum mechanical description of Tachyons has been subject of theoretical research since the sixties. The first notable attempts in author's opinion are [8] [4]. These attempts, though very accurate studies, assume that unitary representations of tachyons coincide with those of massive particles and are unitary representations of  $SO(3)$ . In [17] it is recognized that those representations should be different and indeed tachyonic wavefunctions should come from unitary representations of  $SO(2, 1)$  and not  $SO(3)$  like standard particles. This conclusion is in line with the original work done by Wigner in [21] but still lacks to give a complete representation of tachyonic wavefunctions. We here show that by including involutive "infinite-velocity" limits of boosts, labelled by a spatial direction  $(\theta, \phi)$ , one obtains a finite semidirect-product extension of the proper orthochronous group:

$$\mathcal{L}_{\text{ext}} \cong (\text{SO}(3, 1)_{\dagger}^{\dagger} \rtimes_{\text{Ad}_{\Lambda_{\infty}}} \mathbb{Z}_2) \times \mathbb{Z}_2,$$

where  $\Lambda_{\infty}(\theta, \phi)$  are involutive matrices with  $\det \Lambda_{\infty} = -1$  and  $\Lambda_{\infty}^2 = \mathbb{I}$ . This construction respects local compactness and semidirect-product structure as discussed in harmonic-analysis texts [11, 9].

We then proceed to the extended Poincaré group

$$\mathcal{P}_{\text{ext}} = \mathbb{R}^4 \rtimes \mathcal{L}_{\text{ext}},$$

and classify its unitary irreducible representations (UIRs) by applying Mackey induction techniques in the presence of a finite extension. In doing so we find that the timelike and spacelike orbits under the standard Poincaré group become unified under  $\mathcal{L}_{\text{ext}}$  into a single orbit with invariant  $|p^{\mu} p_{\mu}| = M^2$ . The corresponding induced UIRs restrict to the usual Poincaré UIRs (massive positive, massive negative, tachyonic) as a direct sum of multiplicity-one components.

Next, by differentiating the representation action we the generators of the extended Poincaré algebra, including momentum and Lorentz operators, and thereby derives wave equations associated to the Casimir eigenvalue problems. These wave equations include the usual Klein–Gordon, Dirac, Maxwell, as well as novel equations for tachyonic and new massless sectors permitted by the extension. Finally, we demonstrate an application of the tachyonic wave functions to reinterpret parity-violation phenomena in the weak interaction via representations of  $SO(2, 1)$  rather than  $SO(3)$ .

## 2 Extension of Proper Orthochronous Group Beyond the Light Speed Limit

### 2.1 Proper Orthochronous group

Lorentz transformations in special relativity are defined as a class of real  $4 \times 4$  matrices  $L_s(v, \Theta)$ , depending on a relative velocity vector  $v \in \mathbb{R}^3$  and a set of spatial axis orientations  $\Theta$ , which satisfy the following [6]:

- All entries of  $L_s(v, \Theta)$  are real;
- The matrix preserves the Minkowski metric  $\mu = \text{diag}(-1, 1, 1, 1)$ :

$$\mu = L_s(v, \Theta)\mu L_s(v, \Theta)^{-1}; \quad (1)$$

- The component  $(L_s)_{00} > 0$ ;
- $\det L_s = 1$ .

The general form of such transformations is

$$L_s(v, \Theta) = \Lambda_s(v)R(\Theta), \quad (2)$$

where  $\Lambda_s(v)$  denotes a collinear Lorentz boost and  $R(\Theta)$  a spatial rotation matrix. In Cartesian coordinates,  $\Lambda_s(v)$  is given by

$$\Lambda_s(v) = \begin{bmatrix} \gamma_s & -\gamma_s \frac{v_x}{c} & -\gamma_s \frac{v_y}{c} & -\gamma_s \frac{v_z}{c} \\ -\gamma_s \frac{v_x}{c} & 1 + K_s \frac{v_x^2}{|v|^2} & K_s \frac{v_x v_y}{|v|^2} & K_s \frac{v_x v_z}{|v|^2} \\ -\gamma_s \frac{v_y}{c} & K_s \frac{v_x v_y}{|v|^2} & 1 + K_s \frac{v_y^2}{|v|^2} & K_s \frac{v_y v_z}{|v|^2} \\ -\gamma_s \frac{v_z}{c} & K_s \frac{v_x v_z}{|v|^2} & K_s \frac{v_y v_z}{|v|^2} & 1 + K_s \frac{v_z^2}{|v|^2} \end{bmatrix}, \quad (3)$$

with

$$\gamma_s = \frac{1}{\sqrt{1 - \frac{|v|^2}{c^2}}}, \quad K_s = \gamma_s - 1. \quad (4)$$

The rotation matrix  $R(\Theta)$  can be defined as

$$R(\Theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(x', x) & \cos(x', y) & \cos(x', z) \\ 0 & \cos(y', x) & \cos(y', y) & \cos(y', z) \\ 0 & \cos(z', x) & \cos(z', y) & \cos(z', z) \end{bmatrix}, \quad (5)$$

in terms of the full set of cosines. As is well known every rotation matrix needs only three parameters in order to be fully specified, and using Euler angles parametrization with Z–Y–Z convention may be written as:

$$R(\Theta = \alpha, \beta, \chi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha \cos \beta \cos \chi - \sin \alpha \sin \chi & -\cos \alpha \cos \beta \sin \chi - \sin \alpha \cos \chi & \cos \alpha \sin \beta \\ 0 & \sin \alpha \cos \beta \cos \chi + \cos \alpha \sin \chi & -\sin \alpha \cos \beta \sin \chi + \cos \alpha \cos \chi & \sin \alpha \sin \beta \\ 0 & -\sin \beta \cos \chi & \sin \beta \sin \chi & \cos \beta \end{bmatrix}$$

Here,  $\alpha \in [0, 2\pi)$ ,  $\beta \in [0, \pi]$ , and  $\chi \in [0, 2\pi)$  are the Euler angles.

This class of transformations forms the proper orthochronous Lorentz group  $\text{SO}(3, 1)_\uparrow^+$ .

## 2.2 Definition of Superluminal Lorentz Boosts

We now define superluminal boosts by extending this representation to velocities  $V \in \mathbb{R}^3$  with  $|V| > c$ . The corresponding matrices take the form

$$\Lambda_S(V) = \begin{bmatrix} \gamma_S & -\gamma_S \frac{V_x}{c} & -\gamma_S \frac{V_y}{c} & -\gamma_S \frac{V_z}{c} \\ -\gamma_S \frac{V_x}{c} & 1 + K_S \frac{V_x^2}{|V|^2} & K_S \frac{V_x V_y}{|V|^2} & K_S \frac{V_x V_z}{|V|^2} \\ -\gamma_S \frac{V_y}{c} & K_S \frac{V_x V_y}{|V|^2} & 1 + K_S \frac{V_y^2}{|V|^2} & K_S \frac{V_y V_z}{|V|^2} \\ -\gamma_S \frac{V_z}{c} & K_S \frac{V_x V_z}{|V|^2} & K_S \frac{V_y V_z}{|V|^2} & 1 + K_S \frac{V_z^2}{|V|^2} \end{bmatrix}, \quad (6)$$

with

$$\gamma_S = \frac{1}{\sqrt{\frac{|V|^2}{c^2} - 1}}, \quad K_S = \gamma_S - 1. \quad (7)$$

These matrices satisfy:

- All entries are real;
- They reverse the sign of the Minkowski metric:

$$-\mu = \Lambda_S(V) \mu \Lambda_S(V)^{-1}; \quad (8)$$

- The component  $(\Lambda_S)_{00} \in \mathbb{R}$ ;
- $\det \Lambda_S = -1$ .

A bijective correspondence between subluminal and superluminal boosts is realized via the map:

$$V = \frac{c^2}{|v|^2} (v_x, v_y, v_z), \quad |v| < c, \quad (9)$$

yielding the relations:

$$\gamma_S = \gamma_s \frac{|v|}{c} = \gamma_s \frac{c}{|V|} \quad (10)$$

$$\frac{V_k V_l}{|V|^2} = \frac{v_k v_l}{|v|^2}, \quad k, l = x, y, z. \quad (11)$$

Taking the limit  $\Lambda_\infty = \lim_{|V| \rightarrow \infty} \Lambda_S(V)$ , we obtain the matrix:

$$\Lambda_S(\infty) := \begin{bmatrix} 0 & -\frac{V_x}{|V|} & -\frac{V_y}{|V|} & -\frac{V_z}{|V|} \\ -\frac{V_x}{|V|} & 1 - \frac{V_x^2}{|V|^2} & -\frac{V_x V_y}{|V|^2} & -\frac{V_x V_z}{|V|^2} \\ -\frac{V_y}{|V|} & -\frac{V_x V_y}{|V|^2} & 1 - \frac{V_y^2}{|V|^2} & -\frac{V_y V_z}{|V|^2} \\ -\frac{V_z}{|V|} & -\frac{V_x V_z}{|V|^2} & -\frac{V_y V_z}{|V|^2} & 1 - \frac{V_z^2}{|V|^2} \end{bmatrix} \quad (12)$$

This limit matrix depends only on the angular direction of  $\mathbf{V}$ , and can be expressed in terms of polar and azimuthal angle the speed vector forms in three space exploiting spherical coordinates. We first recall:

$$\begin{aligned} \hat{V}_x &= \sin \theta \cos \phi \\ \hat{V}_y &= \sin \theta \sin \phi \end{aligned}$$

$$\hat{V}_z = \cos \theta$$

and then substitute the above relations into (12):

$$\Lambda_\infty(\theta, \phi) := \begin{bmatrix} 0 & -\sin \theta \cos \phi & -\sin \theta \sin \phi & -\cos \theta \\ -\sin \theta \cos \phi & 1 - \sin^2 \theta \cos^2 \phi & -\sin^2 \theta \cos \phi \sin \phi & -\sin \theta \cos \theta \cos \phi \\ -\sin \theta \sin \phi & -\sin^2 \theta \cos \phi \sin \phi & 1 - \sin^2 \theta \sin^2 \phi & -\sin \theta \cos \theta \sin \phi \\ -\cos \theta & -\sin \theta \cos \theta \cos \phi & -\sin \theta \cos \theta \sin \phi & 1 - \cos^2 \theta \end{bmatrix} \quad (13)$$

A similar construction yields the matrix  $\Lambda_{-\infty}(\theta, \phi) = \lim_{|V| \rightarrow -\infty} \Lambda_S(V)$ :

$$\Lambda_S(-\infty) := \begin{bmatrix} 0 & \frac{V_x}{|V|} & \frac{V_y}{|V|} & \frac{V_z}{|V|} \\ \frac{V_x}{|V|} & 1 - \frac{V_x^2}{|V|^2} & -\frac{V_x V_y}{|V|^2} & -\frac{V_x V_z}{|V|^2} \\ \frac{V_y}{|V|} & -\frac{V_x V_y}{|V|^2} & 1 - \frac{V_y^2}{|V|^2} & -\frac{V_y V_z}{|V|^2} \\ \frac{V_z}{|V|} & -\frac{V_x V_z}{|V|^2} & -\frac{V_y V_z}{|V|^2} & 1 - \frac{V_z^2}{|V|^2} \end{bmatrix} \quad (14)$$

and consequently:

$$\Lambda_{-\infty}(\theta, \phi) := \begin{bmatrix} 0 & \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \sin \theta \cos \phi & 1 - \sin^2 \theta \cos^2 \phi & -\sin^2 \theta \cos \phi \sin \phi & \sin \theta \cos \theta \cos \phi \\ \sin \theta \sin \phi & -\sin^2 \theta \cos \phi \sin \phi & 1 - \sin^2 \theta \sin^2 \phi & \sin \theta \cos \theta \sin \phi \\ \cos \theta & -\sin \theta \cos \theta \cos \phi & -\sin \theta \cos \theta \sin \phi & 1 - \cos^2 \theta \end{bmatrix} \quad (15)$$

we observe:

$$\Lambda_\infty(\theta, \phi) = \Lambda_\infty(\theta, \phi)^{-1} = -\Lambda_{-\infty}(\theta, \phi). \quad (16)$$

Thus, in particular,  $\Lambda_\infty(\theta, \phi)$  is an involution and satisfies:

$$\Lambda_\infty(\theta, \phi)^2 = \mathbb{I} \quad (17)$$

### 2.3 Extension of Relativistic Frames Transformations

We are now going to study the multiplicative structure of the matrices defined so far. The set of matrices, defined given a pair of azimuthal and polar angles  $(\theta, \phi)$ ,

$$Z_{(\theta, \phi)} := \{\mathbb{I}, -\mathbb{I}, \Lambda_\infty, -\Lambda_\infty\} \quad (18)$$

forms a discrete abelian group isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  where  $\mathbb{Z}_2 = \{\pm 1\}$ . In appendix A.1 it is showed that:

$$\Lambda_S(V) = \Lambda_S(v)\Lambda_S(\infty), \quad (19)$$

where the subluminal velocity  $v$  and the superluminal velocity  $V$  are related via the involutive map (9). Equation (19) expresses the fact that an observer moving with velocity  $v$  relative to a rest frame is related, via an infinite-speed boost, to an observer moving at velocity  $V = c^2/v$  along the same direction. From (14), we note that the inverse of  $\Lambda_S(\infty)$  is not  $\Lambda_S(-\infty)$ , but rather  $-\Lambda_S(-\infty)$ . This is in contrast to ordinary Lorentz boosts, where the inverse of a boost is given by reversing the sign of the velocity. Indeed, from (16), we obtain

$$-\Lambda_S(v) = \Lambda_S(V)\Lambda_S(-\infty), \quad (20)$$

which shows that  $\Lambda_S(-\infty)$  maps  $\Lambda_S(V)$  to  $-\Lambda_S(v)$ . Consequently, the negative identity matrix  $-\mathbb{I}$  is also an element of the closure of this set:

$$-\mathbb{I} = \Lambda_S(\infty)\Lambda_S(-\infty). \quad (21)$$

This observation was previously noted in [16] for a restricted subgroup (corresponding to  $\theta = 0$  in (13) expressed in complexified spacetime coordinates.

We now derive the multiplication table involving products of superluminal and subluminal boosts. Let the following subluminal boost with velocity  $v = (0, 0, v_z)$

$$\Lambda_s(v) = \begin{bmatrix} \gamma_s & 0 & 0 & -\gamma_s \frac{v_z}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_s \frac{v_z}{c} & 0 & 0 & \gamma_s \end{bmatrix},$$

For a superluminal boost along  $z$  with speed vector parameter related to  $v$  by (9)

$$\Lambda_S(V) = \begin{bmatrix} \gamma_S & 0 & 0 & -\gamma_S \frac{V_z}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_S \frac{V_z}{c} & 0 & 0 & \gamma_S \end{bmatrix},$$

As already mentioned  $\Lambda_S$  exchanges timelike and spacelike vectors along the chosen axis, sending a four vector  $x^\mu$  such that  $x^\mu x_\mu > 0$  to  $x'^\mu$  such that  $x'^\mu x'_\mu < 0$ .

The infinite-speed limit along  $z$  is the involution

$$\Lambda_\infty(\theta = 0, \phi = 0) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix swaps the  $t$ - and  $z$ -axes and leaves the  $x$ - and  $y$ -axes fixed. By direct computation we have, using 10:

$$\Lambda_S(V) = \Lambda_\infty(0, 0) \Lambda_s(v) = \Lambda_s(v) \Lambda_\infty(0, 0) \quad (22)$$

Let  $v, v'$  be subluminal collinear parameters along direction  $z$ , with sum  $v''$  given by the 1D Einstein velocity addition [7]. By (22) and (19)

$$\begin{aligned} \Lambda_S(V) \Lambda_S(V') &= \Lambda_s(v) \Lambda_\infty \Lambda_s(v') \Lambda_\infty = \Lambda_s(v) \Lambda_s(v') \Lambda_\infty^2 = \Lambda_s(v''), \\ \Lambda_s(v) \Lambda_S(V') &= \Lambda_s(v) \Lambda_s(v') \Lambda_\infty = \Lambda_S(V''), \\ \Lambda_S(V') \Lambda_s(v) &= \Lambda_s(v') \Lambda_\infty \Lambda_s(v) = \Lambda_s(v) \Lambda_s(v') \Lambda_\infty = \Lambda_S(V''), \end{aligned}$$

Including the element  $-I$  for group closure define

$$\mathbf{s} := \Lambda_s(\cdot), \quad \mathbf{S} := \Lambda_S(\cdot), \quad -\mathbf{s} := -\Lambda_s(\cdot), \quad -\mathbf{S} := -\Lambda_S(\cdot),$$

which multiply as:

$\cdot$	$\mathbf{s}$	$\mathbf{S}$	$-\mathbf{s}$	$-\mathbf{S}$
$\mathbf{s}$	$\mathbf{s}$	$\mathbf{S}$	$-\mathbf{s}$	$-\mathbf{S}$
$\mathbf{S}$	$\mathbf{S}$	$\mathbf{s}$	$-\mathbf{S}$	$-\mathbf{s}$
$-\mathbf{s}$	$-\mathbf{s}$	$-\mathbf{S}$	$\mathbf{s}$	$\mathbf{S}$
$-\mathbf{S}$	$-\mathbf{S}$	$-\mathbf{s}$	$\mathbf{S}$	$\mathbf{s}$

the block ( $\mathbf{s}$  vs  $\mathbf{S}$ ) is determined by the parity of the number of  $\mathbf{S}$  factors and the overall sign multiplies. The above multiplication table has been derived for a given choice of  $\theta, \phi$  but it holds true independently of this choice. Indeed for arbitrary values of  $\theta, \phi$  pick a spatial

rotation  $R(\theta, \phi) \in SO(3)$  such that  $R(\theta, \phi)z = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta)$  where  $z = (0, 0, 1)$ . Define  $\hat{R} := \text{diag}(1, R(\theta, \phi))$  and

$$\begin{aligned}\Lambda_s(v') &:= \hat{R}, \Lambda_s(v) \hat{R}^{-1} \\ \Lambda_S(V') &:= \hat{R}, \Lambda_S(V) \hat{R}^{-1} \\ \Lambda_\infty(\theta, \phi) &:= \hat{R} \Lambda_\infty(0, 0) \hat{R}^{-1}.\end{aligned}$$

Then all identities above are preserved under conjugation by  $\hat{R}$ , yielding the multiplication table for  $\Lambda_s(v'), \Lambda_S(V'), \Lambda_\infty(\theta, \phi)$  for collinear boosts along arbitrary directions, where vectors  $v, V$  and  $v', V'$  are related through  $v' = \hat{R}v, V' = \hat{R}V, |V| = \frac{c}{|v|}, |V'| = \frac{c}{|v'|}$

For boosts along different directions one recovers the usual Thomas precession [12]: the product of two boosts (subluminal or superluminal) in different directions is a boost composed with a spatial rotation  $R \in SO(3)$  (see (2)). We thus can drop the subluminal/superluminal distinction and write a generic transformation in this group generalizing (2) as:

$$L(v, \Theta) = \Lambda(v)R(\Theta) \quad |v| \in \mathbb{R}, \quad (23)$$

The matrix group defined is freely generated by multiplication of matrices  $H = SO(3, 1)_\dagger^+, \Lambda_\infty, -I$ . We indicate it as:

$$\mathcal{L}_{\text{ext}} := \langle H, \Lambda_\infty, -I \rangle \quad (24)$$

From a group theoretic perspective the extended Lorentz group defined in (24) is a realization of a semidirect product extension [9] of  $H = SO(3, 1)_\dagger^+$  via the Klein's four group  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . In appendix A.2 it is showed:

$$\mathcal{L}_{\text{ext}} \cong (SO(3, 1)_\dagger^+ \rtimes_{\text{Ad}_{\Lambda_\infty}} \mathbb{Z}_2) \times \mathbb{Z}_2, \quad (25)$$

Where  $\rtimes_{\text{Ad}_{\Lambda_\infty}}$  denotes extension by semidirect product group where non trivial automorphism  $\text{Ad}_{\Lambda_\infty}$  denotes conjugation with  $\Lambda_\infty$  and is composed with extension with the first  $\mathbb{Z}_2$  factor while the other extension factor acts trivially at the conjugation class level.

The four connected components correspond to the four cosets  $SO(3, 1)_\dagger^+, \Lambda_\infty SO(3, 1)_\dagger^+, (-I)SO(3, 1)_\dagger^+, (-I)\Lambda_\infty SO(3, 1)_\dagger^+$  and may be identified with labels in rows and columns of the product table of extended group boosts above. A different  $\Lambda_\infty(\theta', \phi')$  produces an equivalent extension as it is related to the one with  $\Lambda_\infty(\theta, \phi)$  by conjugation with an element of the group (namely of the subgroup  $SO(3)$ , i.e. a spatial rotation).

## 3 Unitary Representations of Extended Poincaré Group

### 3.1 Extended Poincaré group

Let  $H = SO(3, 1)_\dagger^+, \mathcal{T} \cong \mathbb{R}^4$  and  $Z = \{\mathbb{I}, -\mathbb{I}, \Lambda_\infty, -\Lambda_\infty\}$ . The *extended Poincaré group* is the semidirect product

$$\mathcal{P}_{\text{ext}} = \mathcal{T} \rtimes \mathcal{L}_{\text{ext}} = \mathcal{T} \rtimes (H \rtimes_{\text{Ad}_{\Lambda_\infty}} Z) \quad (26)$$

with multiplication

$$(h, a) \cdot (h', a') = (hh', a + ha'),$$

where  $\mathcal{T} \cong \mathbb{R}^4$  is translation group,  $h \in \mathcal{L}_{\text{ext}}$  acts on  $\mathcal{T}$  by the standard linear action on  $\mathbb{R}^4$ .

With the subspace topology from  $GL(4, \mathbb{R})$  on  $\mathcal{L}_{\text{ext}}$  and the Euclidean topology on  $\mathcal{T}$  we have:

1.  $\mathcal{L}_{\text{ext}}$  is a locally compact, separable Lie group (closed in  $GL(4, \mathbb{R})$ ).
2.  $\mathcal{T} \cong \mathbb{R}^4$  is locally compact and separable.

Local compactness and separability of  $\mathcal{P}_{\text{ext}}$  are inherited from  $\mathcal{L}_{\text{ext}}$  and  $\mathcal{T}$  as it is defined via semidirect product extension from those subgroups. The ordinary Poincaré group is

$$\mathcal{P}_0 := \mathcal{T} \rtimes H \quad (27)$$

which is itself a locally compact and separable topological Lie group.

In appendix A.3 it is showed that:

$$\mathcal{P}_{\text{ext}} \cong \mathcal{P}_0 \rtimes_{\alpha} Z \quad (28)$$

where  $\alpha$  is an automorphism of  $(h, a) \in \mathcal{P}_{\text{ext}}$  defined by:

$$\alpha_z(h, a) = (zhz^{-1}, z \cdot a), \quad z \in Z \quad (29)$$

We will use this fact in next section to derive unitary irreducible representations of extended Poincaré group.

### 3.2 Classification of UIRs and Relationship with Wigner's Representations

Let  $N \equiv \mathcal{P}_0 = (\mathcal{T} \rtimes SO(3, 1)_{\uparrow}^+)$ , in (28). Define:

$$\hat{N} = \{\text{Wigner's UIRs of } N\} \quad (30)$$

An element  $z \in Z$  acts on  $N$  via automorphisms, namely its action on a given element  $n = (h, a) \in N$  with  $h \in SO(3, 1)_{\uparrow}^+$  of the Poincaré group:

$$n' = z \cdot n \quad (31)$$

is such that it exists an automorphism of  $N$ ,  $\alpha_z(n)$  such that

$$n' = \alpha_z(n) \quad (32)$$

where  $\alpha_z$  is defined in (29). This induces an action on the set of characters (i.e. UIRs) of  $\mathcal{P}_0$ :

$$z \cdot \pi(n) = \pi(\alpha_z(n)) \quad \forall n \in N \quad (33)$$

The set  $\hat{N}$  is constituted by UIRs of Poincaré group which have been classified by Wigner in [21]. UIRs of Poincaré group are classified by the value of the four momentum vector  $p = p_{\mu}$  together with the value of the invariant length  $|L_s \cdot p|^2$  associated to the orbit

$$O_p = \{L_s \cdot p | L_s \in SO(3, 1)_{\uparrow}^+\} \quad (34)$$

$p$  is called the representative of the induced UIR of Poincaré group. Given choice of  $p_{\mu}$  an  $SO(3, 1)_{\uparrow}^+$ -orbit representative in Wigner's classification, its invariant could be  $p^{\mu}p_{\mu} \neq 0$  or  $p^{\mu}p_{\mu} = 0$ . Choosing a non massless ( $p^{\mu}p_{\mu} \neq 0$ ) representative  $p_{\mu}$  we have:

- Massive Representations: Unitary irreps of  $SO(3)$  for time-like orbits

$$\pi_{\text{mass}}^+ := \text{UIRs of } \{ L \in SO(3, 1)_{\uparrow}^+ | L_{\nu}^{\mu}p_{\mu} = p_{\nu} \quad p^{\mu}p_{\mu} > 0 \quad p_0 > 0 \} \quad (35)$$

$$\pi_{\text{mass}}^- := \text{UIRs of } \{ L \in SO(3, 1)_{\uparrow}^+ | L_{\nu}^{\mu}p_{\mu} = p_{\nu} \quad p^{\mu}p_{\mu} > 0 \quad p_0 < 0 \} \quad (36)$$

- Tachyonic Representations: Unitary irreps of  $SO(2, 1)$  for space-like orbits

$$\pi_{\text{tach}} := \text{UIRs of } \{ L \in SO(3, 1)_{\uparrow}^+ | L_{\nu}^{\mu}p_{\mu} = p_{\nu} \quad p^{\mu}p_{\mu} < 0 \} \quad (37)$$

Choosing a massless ( $p^\mu p_\mu = 0$ ) we have

- Massless Representations: Unitary irreps of ISO(2) for light-like orbits

$$\pi_{\text{massless}}^{\text{fwd}} := \text{UIRs of } \{ L \in SO(3,1)_\dagger^+ | L_\nu^\mu p_\mu = p_\nu \quad p^\mu p_\mu = 0 \quad p^0 > 0 \} \quad (38)$$

$$\pi_{\text{massless}}^{\text{bwd}} := \text{UIRs of } \{ L \in SO(3,1)_\dagger^+ | L_\nu^\mu p_\mu = p_\nu \quad p^\mu p_\mu = 0 \quad p^0 < 0 \} \quad (39)$$

Now, with reference to the action in (33) define, given a representative  $\pi \in \hat{N}$ , its  $Z$ -orbit:

$$O_\pi = \{ z\pi \mid z \in Z \} \quad (40)$$

and the stabilizers

$$Z_\pi = \{ z \in Z \mid z\pi(n) = \pi(n) \} \quad (41)$$

where  $n$  is in  $N \equiv \mathcal{P}_0$ .

We have two different possibilities [18].

If  $Z_\pi = \{e\} \quad \forall \pi(n) \in \hat{N}$  with  $e$  identity element of  $\pi$  then  $\exists U(z)$  such that

$$U(z)\pi(n)U(z) = \pi(z^{-1}nz) \quad \forall z \in Z \quad (42)$$

and the UIRs of  $\mathcal{P}_0$  (namely the set  $\hat{N}$ ) consist of UIRs of the extended group too. In this case UIRs which are inequivalent as representations of  $\mathcal{P}_0$  become equivalent for  $\mathcal{P}_{\text{ext}}$ . This is due to the fact that orbits of  $SO(3,1)_\dagger^+$  in momentum space preserve  $p^\mu p_\mu$  with its sign while in  $\mathcal{L}_{\text{ext}}$  the sign can change.

If, on the contrary,  $Z_\pi \neq \{e\}$  for some  $\pi$  then its elements may give rise to inequivalent UIRs depending on the action of  $z \in Z_\pi$  on  $\pi \in \hat{N}$ .

In what follows we are going to specialize these considerations to UIRs of the extended group corresponding to non-lightlike and lightlike orbits.

### 3.2.1 UIRs Corresponding to Non Lightlike Orbits

For a given choice of momentum  $p_\mu$ , the possible non-lightlike  $z$ -orbits of the extended group are those with  $p^\mu p_\mu \neq 0$  and give rise to the following UIRs of the Poincaré group  $\mathcal{P}_0$ :

$$\hat{N} = \{ \pi_{\text{mass}}^+, \pi_{\text{mass}}^-, \pi_{\text{tach}} \} \quad (43)$$

The stabilizers of the corresponding orbits in (40)  $H_\pi = e, \forall \pi \in \hat{N}$  and from (42) we have:

$$U(\Lambda_\infty)\pi_{\text{mass}}^+(n)U(\Lambda_\infty)^{-1} = \pi_{\text{mass}}^+(\Lambda_\infty n \Lambda_\infty^{-1}) = \pi_{\text{tach}} \quad (44)$$

$$U(\Lambda_{-\infty})\pi_{\text{mass}}^+(n)U(\Lambda_{-\infty})^{-1} = \pi_{\text{mass}}^-(\Lambda_{-\infty} n \Lambda_{-\infty}^{-1}) = \pi_{\text{tach}} \quad (45)$$

$$U(-\mathbb{I})\pi_{\text{mass}}^+(n)U(-\mathbb{I})^{-1} = \pi_{\text{mass}}^+((- \mathbb{I})n(- \mathbb{I})) = \pi_{\text{mass}}^- \quad (46)$$

We thus have that the set of UIRs in (43) is a set of equivalent irreducible representations of the extended group and to fully specify the UIR we need to explicit the intertwiners above.

Let  $\bar{U}_\infty$  be the unitary operator corresponding to  $U(\Lambda_\infty)$  in (44). We look for an operator such that:

$$\bar{U}_\infty \bar{U}(a, h) \bar{U}_\infty^{-1} = \bar{U}(\Lambda_\infty a, \Lambda_\infty h \Lambda_\infty^{-1}) \quad (47)$$

Where  $\bar{U}(a, h)$  with  $a \in \mathcal{T}$  and  $h \in SO(3,1)_\dagger^+$  is an operator representing an element in  $\pi_{\text{mass}}^+$  with representative  $p_0 = (M, 0, 0, 0)$ . Define the section:

$$k_t(p) \in SO(3,1)_\dagger^+ \quad k_t(p)p_0 = p \quad (48)$$

and the corresponding Wigner rotation:

$$s(h, p) = k_t(p)^{-1} h k_t(h^{-1}p) \quad (49)$$

It can be easily verified that  $s$  applied to  $p_0$  gives  $p_0$  so  $s_t(h, p)$  is in the (little) subgroup of  $SO(3, 1)_\dagger^+$  stabilizing  $p_0$ . The operator  $\bar{U}(a, h)$  acts on wavefunctions as [20]:

$$\bar{U}(a, h) : \psi(p) \rightarrow [\bar{U}(a, h)\psi](p) \quad (50)$$

where

$$[\bar{U}(a, h)\psi](p) = e^{ip \cdot a} U(s(h, p)) \psi(h^{-1}p) \quad (51)$$

and  $U(s(h, p))$  is in  $\pi_{\text{mass}}^+$  which is an UIR of the stabilizer  $SO(3)$ . Let  $q_0 = \Lambda_\infty^{-1}p_0$  and define the section:

$$k_s(\Lambda_\infty^{-1}p) := \Lambda_\infty^{-1}k_t(p) \quad (52)$$

and the corresponding Wigner rotation:

$$s(h, q) = k_s(q)^{-1} h k_s(h^{-1}q) \quad (53)$$

it can be verified that  $s(h, q)$  applied to  $q_0 = \Lambda_\infty^{-1}p_0$  gives  $q_0$  thus  $s(h, q)$  is in the stability group of  $q_0$ . Moreover we have the relations:

$$s(h, q) = \Lambda_\infty s(\Lambda_\infty h \Lambda_\infty^{-1}, p) \Lambda_\infty^{-1} \quad (54)$$

Upon defining the action:

$$\bar{U}_\infty : \psi(p) \rightarrow [\bar{U}_\infty \psi](p) := U(\Lambda_\infty) \psi(\Lambda_\infty^{-1}p) \quad (55)$$

where  $\bar{U}_\infty$  is the operator corresponding to  $U(\Lambda_\infty)$  in (44) we have:

$$[\bar{U}_\infty \bar{U}(a, h) \bar{U}_\infty^{-1} \psi](p) = [\bar{U}_\infty \bar{U}(a, h) \phi](q) \quad (56)$$

where

$$\phi(q) = U(\Lambda_\infty) \psi(q) \quad (57)$$

Then from the above equation and (51)

$$[\bar{U}(a, h) \phi](q) = e^{iq \cdot a} U(s(h, q)) \psi(h^{-1}q) \quad (58)$$

Finally using (54)

$$[\bar{U}_\infty \bar{U}(a, h) \bar{U}_\infty^{-1} \psi](p) = e^{i\Lambda_\infty p \cdot a} U(s(\Lambda_\infty h \Lambda_\infty^{-1}, p)) \psi(\Lambda_\infty h^{-1} \Lambda_\infty^{-1} p) \quad (59)$$

Upon writing the following relations:

$$\Lambda_\infty^{-1}p \cdot a = p \cdot \Lambda_\infty a \quad \Lambda_\infty h^{-1} \Lambda_\infty^{-1} p = (\Lambda_\infty^{-1} h \Lambda_\infty)^{-1} p \quad (60)$$

we have :

$$[\bar{U}_\infty \bar{U}(a, h) \bar{U}_\infty^{-1} \psi](p) = [\bar{U}(\Lambda_\infty a, \Lambda_\infty h \Lambda_\infty^{-1}) \psi](p) \quad (61)$$

which is (47). The only other constraint on  $U(\Lambda_\infty)$  is that its square be  $\mathbb{I}$ . Thus we can set  $U(\Lambda_\infty) = \mathbb{I}$  itself and set:

$$[\bar{U}_\infty \psi](p) = \psi(\Lambda_\infty^{-1}p) \quad (62)$$

Following a similar reasoning for  $U(-I)$  and  $\bar{U}(-I)$  we have:

$$[\bar{U}(-I) \psi](p) = \psi(-p) \quad (63)$$

where  $U(-I) = \mathbb{I}$ . The induced UIR for non lightlike orbits is thus, given a representative  $p$ , the set of equivalent UIRs in (43) each occuring with multiplicity one (given that  $H_\pi = e \ \forall \pi$ ), the intertwiners  $U(\Lambda_\infty)$  and  $U(-\Lambda_\infty)$  transform an element of an UIR of a positive massive and negative massive representations into an element of a tachyonic representation (see (44), (45)) while  $U(-I)$  transforms an element of an UIR of a positive massive representation into an element of an UIR of a negative massive representation (see (46)).

### 3.2.2 UIRs Corresponding to Lightlike Orbits

We now consider the lightlike  $z$ -orbit

$$\mathcal{O}_0 = \{ Lp_0 \mid L \in SO(3,1)_\uparrow^+, p_0^2 = 0 \}, \quad (64)$$

with the standard representative chosen as

$$p_0 = (\omega, \omega \sin \theta \cos \phi, \omega \sin \theta \sin \phi, \omega \cos \theta), \quad \omega > 0. \quad (65)$$

The usual Wigner classification for the ordinary Poincaré group  $\mathcal{P}_0$  yields two massless UIRs, denoted

$$\pi^{\text{fwd}}, \pi^{\text{bwd}},$$

corresponding to forward and backward light-cones ( $p_0^0 > 0$  and  $p_0^0 < 0$ ). They are both induced from unitary irreps of the Euclidean group  $ISO(2)$  (helicity or continuous-spin class) and are inequivalent as representations of  $\mathcal{P}_0$ .

In the extended Lorentz group  $\mathcal{L}_{\text{ext}}$  the matrix  $\Lambda_{-\infty} = -\Lambda_\infty$  satisfies  $\Lambda_{-\infty}p_0 = p_0$ , hence it lies in the stability subgroup of the chosen representative. Consequently, the geometric little group at  $p_0$  in  $\mathcal{L}_{\text{ext}}$  is

$$ISO(2) \times \{I, \Lambda_{-\infty}\}. \quad (66)$$

Let  $U_0$  be a unitary representation of the little group  $ISO(2)$  associated with either helicity or continuous-spin. The induced Poincaré action on wavefunctions  $\psi : \mathcal{O}_0 \rightarrow \mathcal{H}_0$  has the usual form

$$[\bar{U}(a, h)\psi](p) = e^{ip \cdot a} U_0(s_0(h, p)) \psi(h^{-1}p), \quad (a, h) \in \mathcal{P}_0, \quad (67)$$

and this construction gives the standard massless Wigner UIRs  $\pi^{\text{fwd}}$  and  $\pi^{\text{bwd}}$  depending on the choice of representative on the forward or backward orbit.

In contrast to  $\Lambda_{-\infty}$ , the transformation  $\Lambda_\infty$  maps  $p_0$  to the *backward* representative

$$\Lambda_\infty(\theta, \phi) p_0 = -p_0, \quad (68)$$

thus interchanging the forward and backward light-cones. In particular  $\Lambda_\infty$  does not belong to the geometric stabilizer of  $p_0$ . However, at the level of massless UIRs of  $\mathcal{P}_{\text{ext}}$ ,  $\pi^{\text{fwd}}$  and  $\pi^{\text{bwd}}$  are equivalent since the unitary operator representing  $U(\Lambda_\infty)$  is thus an intertwiner between  $\pi_{\text{fwd}}$  and  $\pi_{\text{bwd}}$

$$U(\Lambda_\infty) \pi^{\text{fwd}}(n) U(\Lambda_\infty)^{-1} = \pi^{\text{fwd}}(\Lambda_\infty^{-1}n\Lambda_\infty) = \pi^{\text{bwd}}(n), \quad \forall n \in \mathcal{P}_0. \quad (69)$$

Thus, under the action of the discrete factor  $Z_{(\theta, \phi)} = \{I, -I, \Lambda_\infty, -\Lambda_\infty\}$  on the dual space  $\widehat{\mathcal{P}}_0$ , (i.e. the set of Wigner's UIRs) the set  $\Pi \equiv \{\pi^{\text{fwd}}, \pi^{\text{bwd}}\}$  constitute a single equivalence class of UIRs in which the transformation  $U(\Lambda_\infty)$  is a non trivial stabilizer, namely:

$$\Lambda_\infty \Pi = \Pi \quad (70)$$

As a consequence, the induced massless UIRs of the extended Poincaré group  $\mathcal{P}_{\text{ext}}$  do not live on a single copy of  $\mathcal{H}_0$ , but rather on the direct sum

$$\mathcal{H}_\oplus = \mathcal{H}_{\text{fwd}} \oplus \mathcal{H}_{\text{bwd}}, \quad (71)$$

where  $\mathcal{H}_{\text{fwd}}$  and  $\mathcal{H}_{\text{bwd}}$  carry  $\pi^{\text{fwd}}$  and  $\pi^{\text{bwd}}$ , respectively. Writing  $\Psi = (\psi_{\text{fwd}}, \psi_{\text{bwd}}) \in \mathcal{H}_\oplus$ , the extended action of  $(a, h) \in \mathcal{P}_0$  is block-diagonal:

$$[\bar{U}(a, h)\Psi](p) = ([\bar{U}_{\text{fwd}}(a, h)\psi_{\text{fwd}}](p), [\bar{U}_{\text{bwd}}(a, h)\psi_{\text{bwd}}](p)), \quad (72)$$

with each component governed by (67).

The discrete elements of  $\mathcal{L}_{\text{ext}}$  act as follows. For  $\Lambda_\infty$  we choose an operator  $\bar{U}(\Lambda_\infty)$  on  $\mathcal{H}_\oplus$  of the form

$$[\bar{U}(\Lambda_\infty)\Psi](p) := (U(\Lambda_\infty)\psi_{\text{bwd}}(\Lambda_\infty^{-1}p), U(\Lambda_\infty)^{-1}\psi_{\text{fwd}}(\Lambda_\infty^{-1}p)), \quad (73)$$

where  $C$  implements the intertwining between  $\pi^{\text{fwd}}$  and  $\pi^{\text{bwd}}$  as above. One checks, using a section  $k_0$  adapted as

$$k_0(\Lambda_\infty^{-1}p) = \Lambda_\infty^{-1}k_0(p), \quad (74)$$

and the covariance relation

$$s_0(\Lambda_\infty h \Lambda_\infty^{-1}, p) = \Lambda_\infty s_0(h, \Lambda_\infty^{-1}p) \Lambda_\infty^{-1}, \quad (75)$$

that

$$\bar{U}(\Lambda_\infty)\bar{U}(a, h)\bar{U}(\Lambda_\infty)^{-1} = \bar{U}(\Lambda_\infty a, \Lambda_\infty h \Lambda_\infty^{-1}), \quad (76)$$

so  $\bar{U}(\Lambda_\infty)$  correctly represents the extension.

Similarly, for  $-I$  we may choose

$$[\bar{U}(-I)\Psi](p) := (\psi_{\text{bwd}}(-p), \psi_{\text{fwd}}(-p)), \quad (77)$$

which exchanges forward and backward sectors and satisfies

$$\bar{U}(-I)\bar{U}(a, h)\bar{U}(-I)^{-1} = \bar{U}(-a, (-I)h(-I)^{-1}). \quad (78)$$

The only further constraint on  $\bar{U}(\Lambda_\infty)$  is that  $\bar{U}(\Lambda_\infty)^2 = I$ . This restricts  $U(\Lambda_\infty)$  in (73) to a unitary with  $U(\Lambda_\infty)^2 = \mathbb{I}$  on  $\mathcal{H}_0$ , so that its eigenvalues are  $\pm 1$ . Different signs for the eigenvalues of  $U(\Lambda_\infty)$  lead to two inequivalent massless UIRs of the extended group for each standard Wigner class (helicity or continuous-spin). In this sense, the lightlike orbit supports a *doublet* representation

$$\pi_\varepsilon^{\text{ext}} \sim \pi_\varepsilon^{\text{fwd}} \oplus \pi_\varepsilon^{\text{bwd}}, \quad \varepsilon = \pm 1,$$

where  $\varepsilon$  labels the choice of sign in the representation of the extended little group of lightlike momenta representatives.

## 4 Extended Poincarè Algebra

### 4.1 Restriction of Extended UIRs to Poincarè UIRs

In previous section they have been characterized all possible UIRs of the extended Poincarè group. This classification differs from Wigner's one of Poincarè group because:

- tachyonic and massive representations are inequivalent UIRs of Poincarè group and become equivalent representations of the extended Poincarè group
- massless representation split into two inequivalent UIRs of the extended Poincarè group because of the action of  $U(\Lambda_\infty(\theta, \phi))$  on the representation space of the UIR corresponding to stabilizer subgroup (or little group)  $ISO(2) \rtimes \{\Lambda_{-\infty}, I\}$  of Poincarè group.

The reason for this is that, as already mentioned at the end of section 3.2.1, homogenous part of Poincarè group contains transformations which preserve Minkowski metric with its sign while for extended group the sign can change (because of the action of  $\Lambda_\infty$  on  $L \in SO(3, 1)_\uparrow^+$ ). This sign change implies that homogenous part of extended Poincarè group can send time-like momenta four vectors into space-like ones and in this extended framework the distinction between space coordinates and time coordinate is observer dependent. This dependency gives rise to new representations of the wavefunctions for the enlarged group according to the principle that, given

$\psi(p)$  a wavefunction and  $(h, a) \in \mathcal{P}_{\text{ext}}$ , the corresponding element of a unitary representation  $\bar{U}(h, a)$  sends  $\psi(p)$  in

$$[\bar{U}(h, a)\psi](p) = e^{ip \cdot a} U(s(h, p)) \psi(h^{-1}p) \quad (79)$$

where  $U(s(h, p))$  is a UIR of the stability group of the momentum representative choosen and the latter wavefunction is a representation of  $\psi(p)$  in the reference frame obtained multiplying  $p$  by  $h \in \mathcal{L}_{\text{ext}}$  and thus an allowed state. The classification of possible quantum states according to this framework results by restricting UIRs of the extended group to UIRs of Poincarè group. This restricted representation in non massless case decomposes as:

$$Ind_{\mathcal{P}_0}^{\mathcal{P}_0 \times Z} \pi_{\text{non ml}} = \pi_{\text{mass}}^+ \oplus \pi_{\text{mass}}^- \oplus \pi_{\text{tach}} \quad (80)$$

where  $Ind_{\mathcal{P}_0}^{\mathcal{P}_0 \times Z}$  represents the restriction of a non massless UIR of the extended Poincarè group  $\pi_{\text{non ml}}$  to a Poincarè group representation. For massless case:

$$Ind_{\mathcal{P}_0}^{\mathcal{P}_0 \times Z} \pi_{\text{ml}} = \pi_{\text{ml } \epsilon}^{\text{fwd}} \oplus \pi_{\text{ml } \epsilon}^{\text{bwd}} \quad (81)$$

where the definition  $\pi_{\text{ml } \epsilon}^{\text{fwd bwd}}$  depends through parameter  $\epsilon$  on the action of the finite group  $\{\Lambda_\infty(\theta, \phi), \mathbb{I}\}$  on massless Poincarè UIRs.

## 4.2 Infinitesimal Restricted UIRs and Extended Poincarè Algebra

For  $h = \mathbb{I}$  in (79) we have

$$[\bar{U}(a, \mathbb{I})\psi](p) = e^{ip \cdot a} \psi(p) \quad (82)$$

Writing  $\bar{U}(a, \mathbb{I}) = \exp(ia^\mu \hat{P}_\mu)$  and differentiating at  $a = 0$  gives

$$\left. \frac{\partial}{\partial a^\mu} [\bar{U}(a, \mathbb{I})\psi](p) \right|_{a=0} = ip_\mu \psi(p) = i(\hat{P}_\mu \psi)(p), \quad (83)$$

thus

$$(\hat{P}_\mu \psi)(p) = p_\mu \psi(p) \quad (84)$$

Starting from the induced action of the homogeneous Poincaré transformations,

$$[\bar{U}(h)\psi](p) = U(s(h, p)) \psi(h^{-1}p), \quad (85)$$

where  $U(s(h, p))$  represents the little-group action for the choosen momentum representative.

Let  $h(\omega) = \exp(\frac{1}{2} \omega^{\mu\nu} m_{\mu\nu})$  be an element of the Lorentz group near the identity, with  $\omega^{\mu\nu} = -\omega^{\nu\mu}$  infinitesimal. We expand

$$\bar{U}(h(\omega)) = I + \frac{i}{2} \omega^{\mu\nu} \hat{M}_{\mu\nu} + \mathcal{O}(\omega^2), \quad (86)$$

and determine  $\hat{M}_{\mu\nu}$  by differentiating (85) with respect to  $\omega^{\mu\nu}$ .

Differentiating (85) gives

$$\frac{\partial}{\partial \omega^{\mu\nu}} [\bar{U}(h(\omega))\psi](p) = \left( \frac{\partial U(s(h(\omega), p))}{\partial \omega^{\mu\nu}} \right) \psi(h^{-1}p) + U(s(h(\omega), p)) \frac{\partial \psi(h^{-1}p)}{\partial \omega^{\mu\nu}}. \quad (87)$$

The first term in (87) depends on the variation of the little-group operator  $U(s(h, p))$  which in turn depends on the representative  $p_0$  (massless, non massless). We define

$$\Sigma_{\mu\nu}(p) := i \left. \frac{\partial U(s(h(\omega), p))}{\partial \omega^{\mu\nu}} \right|_{\omega=0}, \quad (88)$$

so that, to first order,

$$U(s(h(\omega), p)) = I - \frac{i}{2} \omega^{\mu\nu} \Sigma_{\mu\nu}(p) + \mathcal{O}(\omega^2). \quad (89)$$

The matrices  $\Sigma_{\mu\nu}(p)$  act on the indices of the wavefunctions coming from UIRs of stability group of momentum representative.

The second term in (87) encodes the change of the argument  $h^{-1}p$ . Using the infinitesimal Lorentz action on a four-vector,

$$(h^{-1}p)^\rho = p^\rho - \frac{1}{2} \omega^{\alpha\beta} (m_{\alpha\beta} \cdot p)^\rho, \quad (m_{\mu\nu} \cdot p)^\rho = \eta_\mu^\rho p_\nu - \eta_\nu^\rho p_\mu, \quad (90)$$

we find

$$\begin{aligned} \left. \frac{\partial}{\partial \omega^{\mu\nu}} \psi(h^{-1}p) \right|_{\omega=0} &= -\frac{1}{2} (m_{\mu\nu} \cdot p)^\rho \frac{\partial \psi(p)}{\partial p^\rho} \\ &= -\frac{1}{2} (p_\nu \partial_{p^\mu} - p_\mu \partial_{p^\nu}) \psi(p). \end{aligned} \quad (91)$$

Substituting (88) and (91) into (87), and comparing with (86), we obtain

$$(\hat{M}_{\mu\nu} \psi)(p) = i (p_\mu \partial_{p^\nu} - p_\nu \partial_{p^\mu}) \psi(p) + \Sigma_{\mu\nu}(p) \psi(p). \quad (92)$$

Thus, for small  $\omega^{\mu\nu}$ ,

$$[\bar{U}(h(\omega)) \psi](p) \simeq \left( 1 + \frac{i}{2} \omega^{\mu\nu} \hat{M}_{\mu\nu} \right) \psi(p), \quad (93)$$

with

$$\hat{M}_{\mu\nu} = i (p_\mu \partial_{p^\nu} - p_\nu \partial_{p^\mu}) + \Sigma_{\mu\nu}(p) \quad (94)$$

The Poincaré algebra is generated by the momentum operators  $\hat{P}_\mu$  and the Lorentz generators  $\hat{M}_{\mu\nu}$  and satisfy

$$[\hat{P}_\mu, \hat{P}_\nu] = 0, \quad (95)$$

$$[\hat{M}_{\rho\sigma}, \hat{P}_\mu] = i (\eta_{\mu\rho} \hat{P}_\sigma - \eta_{\mu\sigma} \hat{P}_\rho), \quad (96)$$

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = i \left( \eta_{\nu\rho} \hat{M}_{\mu\sigma} - \eta_{\mu\rho} \hat{M}_{\nu\sigma} + \eta_{\mu\sigma} \hat{M}_{\nu\rho} - \eta_{\nu\sigma} \hat{M}_{\mu\rho} \right). \quad (97)$$

The Pauli Lubanski vector is defined as

$$\hat{W}^\mu := \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \hat{P}_\nu \hat{M}_{\rho\sigma}, \quad (98)$$

where  $\varepsilon^{\mu\nu\rho\sigma}$  is the Levi-Civita symbol (with  $\varepsilon^{0123} = +1$  and metric  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ ).

Using the algebra (95) - (96), we compute:

$$\begin{aligned} [\hat{W}^\mu, \hat{P}_\lambda] &= \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \left( [\hat{P}_\nu, \hat{P}_\lambda] \hat{M}_{\rho\sigma} + \hat{P}_\nu [\hat{M}_{\rho\sigma}, \hat{P}_\lambda] \right) \\ &= \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \hat{P}_\nu (\eta_{\lambda\rho} \hat{P}_\sigma - \eta_{\lambda\sigma} \hat{P}_\rho). \end{aligned} \quad (99)$$

The two terms cancel because of the antisymmetry of  $\varepsilon^{\mu\nu\rho\sigma}$  under  $\rho \leftrightarrow \sigma$ , so that

$$[\hat{W}^\mu, \hat{P}_\lambda] = 0 \quad (100)$$

This implies that all components of  $\hat{W}^\mu$  commute with all  $\hat{P}_\lambda$ , and therefore the operators

$$C_1 = \hat{P}^\mu \hat{P}_\mu, \quad C_2 = \hat{W}^\mu \hat{W}_\mu,$$

commute with each other and with all generators of the algebra. They are thus Casimir invariants of the Poincaré algebra.

Because  $C_1$  and  $C_2$  commute with all  $\hat{P}_\mu$ , one can find a common eigenspace of the representation space spanned by wavefunctions  $\psi(p)$  which are simultaneous eigenfunctions of these operators:

$$\hat{C}_1\psi(p) = M^2\psi(p), \quad \hat{C}_2\psi(p) = -M^2C_{\text{little}}\psi(p), \quad (101)$$

where  $C_{\text{little}}$  is the Casimir eigenvalue of the little group algebra depending on the representative choosen (massless: two inequivalent UIRs of  $\mathfrak{iso}(2)$ ; non massless: three inequivalent UIRs, two UIRs of  $\mathfrak{so}(3)$  with  $p_0 > 0$  and  $p_0 < 0$  respectively and one UIR of  $\mathfrak{so}(2, 1)$  for tachyons).

Using the explicit form of the Lorentz generators (92) we can write  $\hat{W}^\mu$  as

$$\hat{W}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\hat{P}_\nu [i(p_\rho\partial_{p^\sigma} - p_\sigma\partial_{p^\rho}) + \Sigma_{\rho\sigma}]. \quad (102)$$

The first (orbital) part inside the brackets is proportional to  $p_\rho p_\sigma - p_\sigma p_\rho = 0$ , and therefore vanishes identically. Hence,

$$\hat{W}^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\hat{P}_\nu \Sigma_{\rho\sigma}. \quad (103)$$

The six Lorentz generators are decomposed into spatial rotations  $J_i$  and boosts  $K_i$ :

$$J_i = \frac{1}{2}\varepsilon_{ijk}\Sigma_{jk}, \quad K_i = \Sigma_{0i}. \quad (104)$$

Expliciting (103) using (104) we get:

$$W^0 = P_1\Sigma_{23} + P_2\Sigma_{31} + P_3\Sigma_{12} = P_1J_1 + P_2J_2 + P_3J_3 \quad (105)$$

$$W^1 = P_0\Sigma_{23} + P_2\Sigma_{30} + P_3\Sigma_{02} = P_0J_1 - P_2K_3 + P_3K_2, \quad (106)$$

$$W^2 = P_0\Sigma_{31} + P_3\Sigma_{10} + P_1\Sigma_{03} = P_0J_2 - P_3K_1 + P_1K_3, \quad (107)$$

$$W^3 = P_0\Sigma_{12} + P_1\Sigma_{20} + P_2\Sigma_{01} = P_0J_3 - P_1K_2 + P_2K_1. \quad (108)$$

In compact vector form:

$$W = P_0\vec{J} - \vec{P} \times \vec{K}, \quad W^0 = \vec{P} \cdot \vec{J}. \quad (109)$$

where  $\vec{V} = (V_1, V_2, V_3)$  identifies boosts and rotations generators and spatial part of momentum four vector.

We now evaluate this invariant in standard representative frames for the different classes of orbits. This gives rise to all possible quantum wave equations of the Extended Lorentz Group defined in this work.

#### 4.2.1 Massive Representations

**Particles:**  $p^\mu p_\mu > 0$ ,  $p_0 > 0$   $P^\mu = (M, 0, 0, 0)$ :

$$W^0 = 0, \quad \vec{W} = M\vec{J}, \quad (110)$$

$$C_1 = P^\mu P_\mu = -M^2, \quad (111)$$

$$C_2 = W^\mu W_\mu = -M^2|J|^2 = -M^2C_{\mathfrak{so}(3)}. \quad (112)$$

where  $\vec{V} = (V_1, V_2, V_3)$ ,  $|J|^2 = J_1^2 + J_2^2 + J_3^2$  is  $\mathfrak{so}(3)$  Casimir operator and  $J_i$  satisfy:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k. \quad (113)$$

and denotes generators of  $\mathfrak{so}(3)$  algebra. The wave equations are

$$(P^\mu P_\mu + M^2)\psi = 0, \quad (W^\mu W_\mu + M^2s(s+1))\psi = 0. \quad (114)$$

where  $s(s+1)$  denotes  $\mathfrak{so}(3)$  Casimir eigenvalue.

**Antiparticles:**  $p^\mu p_\mu > 0$ ,  $p_0 < 0$  For  $P^\mu = (-M, 0, 0, 0)$  one has

$$W^\mu = (0, -M\vec{J}), \quad C_2 = -M^2 |J|^2, \quad (115)$$

so the Casimir eigenvalues are identical; only the energy sign changes.

#### 4.2.2 Tachyonic representations

Let  $\hat{n} = (n^1, n^2, n^3)$  be a unit spatial vector parametrized by spherical angles

$$\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

We complete this to a right-handed orthonormal frame with

$$\hat{e}_1 = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \quad \hat{e}_2 = (-\sin \phi, \cos \phi, 0),$$

so that  $(\hat{e}_1, \hat{e}_2, \hat{n})$  is orthonormal and  $\hat{e}_1 \times \hat{e}_2 = \hat{n}$ .

We define the following combinations of Lorentz generators:

$$R_n = \hat{n} \cdot \vec{J} = n^1 J_1 + n^2 J_2 + n^3 J_3 \quad (116)$$

$$B_1 = \hat{e}_1 \cdot \vec{K} = e_1^1 K_1 + e_1^2 K_2 + e_1^3 K_3 \quad (117)$$

$$B_2 = \hat{e}_2 \cdot \vec{K} = e_2^1 K_1 + e_2^2 K_2 + e_2^3 K_3 \quad (118)$$

It is showed in appendix A.4 that the Poincaré algebra Casimir becomes

$$C_2 = M^2 [-(R_n)^2 + B_1^2 + B_2^2] \quad (119)$$

where we define the Casimir of  $\mathfrak{so}(2, 1)$  built from  $\{R_n, B_1, B_2\}$  to be:

$$C_{\mathfrak{so}(2,1)} = B_1^2 + B_2^2 - R_n^2. \quad (120)$$

therefore the Pauli Lubanski invariant of Poincaré algebra in the tachyonic representation is directly related to the Casimir of  $\mathfrak{so}(2, 1)$ :

$$C_2 = -M^2 C_{\mathfrak{so}(2,1)} \quad (121)$$

while

$$C_1 = M^2 \quad (122)$$

The wave equations are

$$(P^\mu P_\mu - M^2)\psi = 0, \quad (W^\mu W_\mu + M^2 t(t+1))\psi = 0. \quad (123)$$

where  $t(t+1)$  is the eigenvalue of  $C_{\mathfrak{so}(2,1)}$  which is the Casimir operator of  $\mathfrak{so}(2, 1)$  algebra.

#### 4.2.3 Massless Representations

Fix angles  $(\theta, \phi)$  and let

$$\hat{n} = \hat{n}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \quad (124)$$

Choose the massless momentum

$$p_0^\mu = (\omega, \omega \hat{n}), \quad \omega > 0, \quad (125)$$

so  $p_0^\mu p_{0\mu} = 0$ .

Complete  $\hat{n}$  to a right-handed orthonormal frame  $(\hat{e}_1, \hat{e}_2, \hat{n})$ :

$$\hat{e}_1 = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \quad \hat{e}_2 = (-\sin \phi, \cos \phi, 0).$$

Define the following linear combinations of Lorentz generators:

$$J_n := \hat{n} \cdot \vec{J}, \quad \Pi_a := \hat{e}_a \cdot (\vec{K} + \hat{n} \times \vec{J}), \quad a = 1, 2. \quad (126)$$

In appendix A.5 it is shown

$$[J_n, \Pi_1] = i \Pi_2, \quad [J_n, \Pi_2] = -i \Pi_1, \quad [\Pi_1, \Pi_2] = 0, \quad (127)$$

i.e.  $\{J_n, \Pi_1, \Pi_2\} \cong \mathfrak{iso}(2)$ .

Using  $P^\mu = p_0^\mu = (\omega, \omega \hat{n})$  and the identities above one finds for the components of the Pauli-Lubanski vector:

$$W^0 = \omega \hat{n} \cdot \vec{J} = \omega J_n, \quad (128)$$

$$\vec{W} = \omega [J_n \hat{n} + \Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2]. \quad (129)$$

Since  $P^2 = 0$  we have  $C_1 = P^\mu P_\mu = 0$ . For  $C_2 = W^\mu W_\mu$ :

$$C_2 = -(W^0)^2 + |\vec{W}|^2 = -\omega^2 J_n^2 + \omega^2 (J_n^2 + \Pi_1^2 + \Pi_2^2) \quad (130)$$

Hence

$$C_2 = \omega^2 (\Pi_1^2 + \Pi_2^2) \quad (131)$$

Thus  $C_2$  depends only on the two commuting generators  $\Pi_a$ .

For a massless representation with little group  $ISO(2) \times \{I, \Lambda_\infty\}$ , the unitary operator  $U(\Lambda_\infty)$  extending the representation to the enlarged Lorentz group must implement on the little-group representation  $U_0$  the automorphism of  $ISO(2)$  induced by  $\text{Ad}_{\Lambda_\infty}$ , namely

$$U(\Lambda_\infty) U_0(s) U(\Lambda_\infty)^{-1} = U_0(\Lambda_\infty s \Lambda_\infty^{-1}), \quad s \in ISO(2). \quad (132)$$

The conjugation by  $\Lambda_\infty$  acts on the little algebra as:

$$U(\Lambda_\infty) J_n U(\Lambda_\infty)^{-1} = J_n, \quad U(\Lambda_\infty) \Pi_a U(\Lambda_\infty)^{-1} = -\Pi_a \quad (a = 1, 2).$$

Geometrically this is a consequence of the fact that  $\Lambda_\infty$  fixes the axis  $\hat{n}$  and changes the signs of parameters in the  $\hat{e}_{1,2}$  directions.

**Helicity representations** In the helicity class of massless representations, the translation generators  $\Pi_a$  act trivially on the little group UIR space  $\mathcal{V} \simeq \mathbb{C}$ ,

$$U_0(R_\alpha) = e^{i h \alpha}, \quad U_0(T_a) = I.$$

Hence the intertwining condition (132) is automatically satisfied for any scalar operator

$$U(\Lambda_\infty) = \varepsilon I, \quad \varepsilon \in \{\pm 1\}, \quad U(\Lambda_\infty)^2 = I.$$

The two sign choices correspond to the two inequivalent one-dimensional characters of the  $\mathbb{Z}_2$  generated by  $\Lambda_\infty$  in the extended group.

**Continuous-spin representations** In the continuous-spin class, the  $\Pi_a$  act nontrivially, and the operator  $U(\Lambda_\infty)$  cannot be proportional to the identity if it is to satisfy (132). Consider the standard realization of  $U_0$  on  $\mathcal{V} = L^2(\mathbb{R}_\xi^2)$  with continuous-spin parameter  $\rho > 0$ :

$$[U_0(T_a) f](\xi) = e^{i \rho a \cdot \xi} f(\xi), \quad [U_0(R_\alpha) f](\xi) = f(R_{-\alpha} \xi). \quad (133)$$

Define the unitary reflection operator

$$[Cf](\xi) = f(-\xi). \quad (134)$$

A direct computation gives:

$$C U_0(T_a) C^{-1} f(\xi) = C(e^{i\rho a \cdot \xi} f(-\xi)) = e^{i\rho a \cdot (-\xi)} f(\xi) = U_0(T_{-a}) f(\xi), \quad (135)$$

$$C U_0(R_\alpha) C^{-1} f(\xi) = C(f(R_{-\alpha}(-\xi))) = f(-R_{-\alpha}\xi) = f(R_{-\alpha}(-\xi)) = U_0(R_\alpha) f(\xi). \quad (136)$$

Thus  $C$  realizes the required intertwining property (132):

$$C U_0(s) C^{-1} = U_0(\Lambda_\infty s \Lambda_\infty^{-1}), \quad s \in E(2),$$

and satisfies  $C^2 = \mathbf{1}$ . Differentiating gives

$$C dU_0(J_n) C^{-1} = dU_0(J_n), \quad C dU_0(\Pi_a) C^{-1} = -dU_0(\Pi_a),$$

Therefore, for continuous-spin representations the appropriate choice is

$$U(\Lambda_\infty) = \varepsilon C, \quad \varepsilon \in \{\pm 1\}, \quad (137)$$

where  $\varepsilon$  labels two inequivalent extensions of the  $ISO(2)$  representation, while  $C$  provides the nontrivial reflection of the translation generators  $\Pi_a$  required by the automorphism  $\text{Ad}_{\Lambda_\infty}$ .

## 5 Tachyonic Representations and Parity Asymmetry in Electroweak Decay

The asymmetric angular distribution of electrons observed in the celebrated experiment of C.S. Wu [22] following the proposal of Lee and Yang [14] and its analysis by Jackson, Treiman, and Wyld [13] is conventionally interpreted as evidence of parity non-conservation in weak interactions. Within the present framework, however, the same experimental pattern arises as a purely kinematical consequence of the covariance properties of wavefunctions transforming under unitary irreducible representations (UIRs) of the non-compact group  $SO(2, 1)$ .

Let  $(\theta, \phi)$  denote the spatial direction of the external magnetic field that defines the nuclear-spin quantization axis in the experiment. In the extended Poincaré framework developed in previous sections, the momentum of a superluminal (tachyonic) decay product is obtained from the timelike standard momentum  $p_0 = (M, 0, 0, 0)$  by the transformation  $q_\mu = \Lambda_\infty(\theta, \phi) p_{0\mu}$ . The subgroup of the Lorentz group leaving the direction  $(\theta, \phi)$  invariant is isomorphic to  $SO(2, 1)$ ; its UIRs therefore govern the internal degrees of freedom of tachyonic states.

Inverting the sign of magnetic field along  $(\theta, \phi)$  corresponds to the following four vector transformation:

$$B(0, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \rightarrow B(0, \sin(\theta + \pi) \cos \phi, \sin(\theta + \pi) \sin \phi, \cos(\theta + \pi)) \quad (138)$$

Assuming particles emitted in electroweak decay belong to tachyonic representations (section 4.2.2) and since

$$\Lambda_\infty(\pi + \theta, \phi) p_0 = -\Lambda_\infty(\theta, \phi) p_0 = -q_0 \quad (139)$$

where  $p_0 = (M, 0, 0, 0)$ ,  $q_0$  is the momentum of the tachyonic particle emitted before the switch and  $-q_0$  is the momentum of the particle after the switch. The associated wavefunction comes from an UIR of stability group  $SO(2, 1)$  evaluated at space-like momenta  $q$  and is invariant under transformation  $q_0 \rightarrow -q_0$ ; we thus conclude:

$$\Psi_{\text{after}}(q) = e^{i\chi} \Psi_{\text{before}}(q), \quad \chi \in \mathbb{R}, \quad (140)$$

even though the four-momentum changes sign,  $q_{\text{after}}^\mu = -q_{\text{before}}^\mu$  so that the scalar  $B_\mu q^\mu$  stays invariant. In consequence of (140) the emitted particle is seen propagating along the same spatial direction before and after magnetic field reversal in laboratory frame, reproducing the experimentally observed asymmetry

$$A = (N_\uparrow - N_\downarrow)/(N_\uparrow + N_\downarrow) \approx 1 \quad (141)$$

In this sense, what is conventionally called parity violation appears here as a manifestation of the fact that spatial inversion is not represented by an inner automorphism of the non-compact little group  $\text{SO}(2, 1)$ .

UIRs of  $\text{SO}(2, 1)$  are conveniently realized as square-integrable functions on the two-sheeted hyperboloid  $x_0^2 - x_1^2 - x_2^2 = 1$ , with coordinates

$$x_0 = \cosh \rho, \quad x_1 = \sinh \rho \cos \theta, \quad x_2 = \sinh \rho \sin \theta.$$

In this model the Casimir operator acts as the Laplace operator, and the basis functions take the form

$$\Psi_{\lambda m}(\rho, \theta) = R_{\lambda m}(\rho) e^{im\theta}, \quad m \in \mathbb{Z}, \quad (142)$$

where  $R_{\lambda m}(\rho)$  are hypergeometric radial functions satisfying the eigenvalue equation  $C_2 \Psi_{\lambda m} = \lambda \Psi_{\lambda m}$  with  $\lambda = t(t+1)$  [1, 15, 19, 10, 2]. The integer  $m$  labels the eigenvalues of the compact generator  $J_3 = -i\partial_\theta$  (denoted  $R_n$  in (116)).

For small  $\rho$  (corresponding to low transverse momentum), the lowest harmonics  $m = 0, \pm 1$  dominate the expansion of the wavefunction, whereas higher  $|m|$  correspond to higher multipoles suppressed in the decay amplitude. These first harmonics behave as

$$\Psi_0 \sim R_0(\rho), \quad \Psi_{\pm 1} \sim R_1(\rho) e^{\pm i\theta},$$

where the hyperbolic radial functions satisfy  $R_1(\rho) \propto \sinh \rho$  near the origin [19, 2].

Writing the decay amplitude as the coherent superposition

$$\mathcal{A}(\theta) = a_0 R_0(\rho) + a_1 R_1(\rho) e^{i\theta} + a_{-1} R_1(\rho) e^{-i\theta}, \quad (143)$$

and expanding to lowest order in  $\sinh \rho$ , one finds for the differential counting rate

$$\frac{dN}{d\Omega} \propto |\mathcal{A}(\theta)|^2 \simeq A_0 [1 + \alpha \cos \theta], \quad \alpha = \frac{4 \Re(a_0^* a_1 R_0^* R_1)}{|a_0 R_0|^2 + 2|a_1 R_1|^2}. \quad (144)$$

Thus the interference between the isotropic ( $m = 0$ ) and dipole ( $m = \pm 1$ ) components of the  $\text{SO}(2, 1)$  wavefunction generates a linear  $\cos \theta$  term in the emission probability,

$$\frac{dN}{d \cos \theta} \propto 1 + \alpha \cos \theta, \quad (145)$$

in quantitative agreement with the empirical law found in Refs. [22, 13]. The emergence of this term is a general feature of the discrete series representations of  $\text{SO}(2, 1)$ , whose angular harmonics obey the addition theorem  $\Psi_m(\theta) \Psi_{m'}(\theta) \propto \cos[(m - m')\theta]$ , and hence naturally reproduce first-order angular asymmetries [19, 2].

In a compact  $\text{SO}(3)$  representation parity requires the angular distribution to be an even function of  $\theta$ , excluding linear  $\cos \theta$  terms. For the non-compact  $\text{SO}(2, 1)$  symmetry appropriate to tachyonic representations, no such constraint exists: inversion with respect to the quantization axis is not an inner operation of the little group, so that even and odd harmonics can interfere. The observed parity asymmetry therefore emerges here not from an explicit violation of Lorentz invariance, but from the intrinsic representation structure of the extended Lorentz group, in which both ordinary and tachyonic sectors are unified.

Hence, within this group-theoretic framework, the characteristic  $\cos \theta$  angular dependence of the emitted electrons is a direct signature of the  $\text{SO}(2, 1)$  covariance of the decay wavefunction, and the so-called parity violation reflects the geometric properties of superluminal representations rather than a fundamental asymmetry of the weak interaction.

## 6 Conclusions

In this article we have introduced and analysed a new framework for relativity and quantum field theory, by extending the proper orthochronous Lorentz group to include superluminal transformations in arbitrary directions. The extended group  $\mathcal{L}_{\text{ext}}$  and its associated extended Poincaré group  $\mathcal{P}_{\text{ext}}$  remain locally compact and admit a well-defined induced representation theory. We classified all unitary irreducible representations of  $\mathcal{P}_{\text{ext}}$  by adopting Mackey induction in the presence of the discrete extension, and we showed how each such UIRs restricts to a direct sum of standard Poincaré UIRs.

Next we derived the generator-form of the extended Poincaré algebra via differentiation of the representation action, and we solved the Casimir eigenvalue problems to obtain wave equations. These include all familiar relativistic wave equations (Klein–Gordon, Dirac, Maxwell, Poca, etc.) and in addition provide new wave-equations describing tachyonic representations and a novel class of massless representations implied by the extension. Furthermore, we proposed an interpretation of parity–asymmetric phenomena in electroweak decays through the representation theory derived from this new framework in which  $\text{SO}(2,1)$  rather than usual  $\text{SO}(3)$  is the group underlying symmetry of the wavefunctions, thereby providing a purely kinematical group-theoretic alternative to parity violation.

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## Conflict of interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Ethical statement

This article does not contain any studies with human participants or animals performed by the author. No ethical approval was required for the theoretical and mathematical work presented in this paper.

## Informed consent

Not applicable. This work involves no human participants, and therefore informed consent was not required.

## Data availability

No datasets were generated or analysed during the current study. All results are obtained by analytical calculations and are fully contained in the article. Any additional intermediate calculations are available from the author upon reasonable request.

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## A Appendix

### A.1 Proof of (19)

To establish the identity (19), we verify the matrix equality component-wise, utilizing the symmetry properties of Lorentz boosts and the transformation rule (9), together with the identities (10) and (11). We begin by verifying the diagonal entries, followed by the off-diagonal ones.

#### Diagonal Element (0, 0)

We first show:

$$\Lambda_S(V)_{00} = [\Lambda_s(v)\Lambda_S(\infty)]_{00}. \quad (146)$$

By direct computation:

$$[\Lambda_s(v)\Lambda_S(\infty)]_{00} = \frac{\gamma_s}{c|V|}(v_x V_x + v_y V_y + v_z V_z) = \gamma_s \frac{c}{|V|} = \gamma_S, \quad (147)$$

where in the first equality we use the explicit form of the composition, in the second we use the identity  $v \cdot V = c^2$ , and in the third the relation given in (10).

#### Spatial Diagonal Elements ( $l, l$ ), with $l = 1, 2, 3$

To simplify notation, we adopt the following index conventions:

- $l = 1$ :  $i = x, j = y, k = z$
- $l = 2$ :  $i = y, j = x, k = z$
- $l = 3$ :  $i = z, j = x, k = y$

We aim to verify:

$$\Lambda_S(V)_{ll} = [\Lambda_s(v)\Lambda_S(\infty)]_{ll}. \quad (148)$$

The right-hand side expands as:

$$\begin{aligned} \Lambda_S(V)_{ll} &= \gamma_s \frac{v_i V_i}{c|V|} + \left(1 - \frac{V_i^2}{|V|^2}\right) \left[1 + (\gamma_s - 1) \frac{v_i^2}{|v|^2}\right] \\ &\quad - (\gamma_s - 1) \frac{v_i v_j}{|v|^2} \frac{V_i V_j}{|V|^2} - (\gamma_s - 1) \frac{v_i v_k}{|v|^2} \frac{V_i V_k}{|V|^2}. \end{aligned} \quad (149)$$

Substituting expressions from (10) and (11), and regrouping terms, we obtain:

$$\begin{aligned} \Lambda_S(V)_{ll} &= \gamma_S \frac{V_i^2}{|V|^2} + 1 + \frac{V_i^2}{|V|^2} \left[ -(\gamma_s - 1)c^2 \frac{V_i^2}{|V|^2} - 1 + (\gamma_s - 1)c^2 \right. \\ &\quad \left. - (\gamma_s - 1)c^2 \frac{V_j^2}{|V|^2} - (\gamma_s - 1)c^2 \frac{V_k^2}{|V|^2} \right]. \end{aligned} \quad (150)$$

Using the identity  $V_i^2 + V_j^2 + V_k^2 = |V|^2$ , the terms simplify to:

$$\Lambda_S(V)_{ll} = \gamma_S \frac{V_i^2}{|V|^2} + 1 - \frac{V_i^2}{|V|^2} = 1 + (\gamma_S - 1) \frac{V_i^2}{|V|^2}. \quad (151)$$

### Off-Diagonal Terms $(0, l)$ and $(l, 0)$ , with $l = 1, 2, 3$

We now verify the symmetry relation:

$$\Lambda_S(V)_{0l} = \Lambda_S(V)_{l0} = [\Lambda_s(v)\Lambda_S(\infty)]_{0l}. \quad (152)$$

Expanding the matrix product and using the previously adopted index conventions:

$$\begin{aligned} \Lambda_S(V)_{0l} &= -\gamma_s \frac{V_i}{|V|} - \gamma_s \frac{v_i}{c} \left(1 - \frac{V_i^2}{|V|^2}\right) + \gamma_s \frac{v_j}{c} \frac{V_i V_j}{|V|^2} + \gamma_s \frac{v_k}{c} \frac{V_i V_k}{|V|^2} \\ &= -\gamma_s \frac{V_i}{|V|} + \gamma_s c \frac{V_i}{|V|} \left[-1 + \frac{V_i^2 + V_j^2 + V_k^2}{|V|^2}\right] \\ &= -\gamma_s \frac{V_i}{|V|} + \gamma_s c \frac{V_i}{|V|} (0) = -\gamma_s \frac{V_i}{c}, \end{aligned} \quad (153)$$

where in the last step we used the identity from (10) and the fact that  $V_i^2 + V_j^2 + V_k^2 = |V|^2$ .

### Spatial Off-Diagonal Terms $(l, m)$ , $l \neq m \in \{1, 2, 3\}$

Finally, we address the remaining components:

$$\Lambda_S(V)_{lm} = [\Lambda_s(v)\Lambda_S(\infty)]_{lm}. \quad (154)$$

We adopt the following index assignments:

- $(l, m) = (1, 2)$ :  $i = x, j = y, k = z$
- $(l, m) = (1, 3)$ :  $i = x, j = z, k = y$
- $(l, m) = (2, 3)$ :  $i = y, j = z, k = x$

The corresponding component reads:

$$\begin{aligned} \Lambda_S(V)_{lm} &= \gamma_s \frac{v_i}{c} \frac{V_j}{|V|} - \left[1 + (\gamma_s - 1) \frac{v_i^2}{|v|^2}\right] \frac{V_i V_j}{|V|^2} \\ &\quad + (\gamma_s - 1) \frac{v_i v_j}{|v|^2} \left(1 - \frac{V_j^2}{|V|^2}\right) - (\gamma_s - 1) \frac{v_i v_k}{|v|^2} \frac{V_j V_k}{|V|^2}. \end{aligned} \quad (155)$$

Using the identities (10), (11), and simplifying:

$$\begin{aligned} \Lambda_S(V)_{lm} &= \gamma_s \frac{V_i V_j}{|V|^2} - \frac{V_i V_j}{|V|^2} - (\gamma_s - 1) \frac{V_i V_j}{|V|^2} \left[\frac{V_i^2}{|V|^2} - 1 + \frac{V_j^2}{|V|^2} + \frac{V_k^2}{|V|^2}\right] \\ &= (\gamma_s - 1) \frac{V_i V_j}{|V|^2}. \end{aligned} \quad (156)$$

This completes the verification of all matrix components and thereby concludes the proof.

## A.2 Proof of (25)

First we provide a definition and few facts regarding semidirect product extensions for locally compact group (see also [11]).

Let  $H$  and  $Q$  be locally compact groups and let  $\alpha : Q \rightarrow \text{Aut}(H)$  be a continuous homomorphism. The *semidirect product* of  $H$  by  $Q$  with respect to  $\alpha$ , denoted  $G = H \rtimes_{\alpha} Q$ , is the locally compact group whose underlying manifold is  $H \times Q$  with the product topology and group law

$$(h_1, q_1)(h_2, q_2) = (h_1 \alpha_{q_1}(h_2), q_1 q_2), \quad (157)$$

inverse

$$(h, q)^{-1} = (\alpha_{q^{-1}}(h^{-1}), q^{-1}), \quad (158)$$

and identity element  $(e_H, e_Q)$ .

Then  $H$  is a closed normal subgroup of  $G$ , and the quotient  $G/H$  is isomorphic to  $Q$ . If  $Q$  is finite,  $G$  is locally compact if and only if  $H$  is.

In order to prove (25) let  $H := SO(3, 1)_\uparrow^+$  and  $Q := \mathbb{Z}_2 \times \mathbb{Z}_2 = \{s, r \mid s^2 = r^2 = 1, sr = rs\}$ . Define a homomorphism

$$\alpha : Q \longrightarrow \text{Aut}(H), \quad \alpha(s) = \text{Ad}_{\Lambda_\infty}|_H, \quad \alpha(r) = \text{id}_H.$$

Since  $\Lambda_\infty^2 = I$  and  $-I$  is central,  $\alpha$  is well defined and a group homomorphism. The underlying external product manifold  $H \times Q$  inherits a semidirect product structure:

$$H \rtimes_\alpha Q \quad \text{with multiplication} \quad (h, q) \cdot (h', q') = (h \alpha(q)(h'), qq').$$

The matrix subgroup of  $GL_4(\mathbb{R})$  defined in (24) is canonically isomorphic to  $H \rtimes_\alpha Q$  via

$$\Phi : H \rtimes_\alpha Q \rightarrow \mathcal{L}_{\text{ext}}, \quad \Phi(h, s^\varepsilon r^\delta) = (-I)^\delta \Lambda_\infty^\varepsilon h, \quad \varepsilon, \delta \in \{0, 1\}.$$

Indeed the correspondence is surjective since  $H = SO(3, 1)_\uparrow^+$  is normal in  $\mathcal{L}_{\text{ext}}$  and  $-I$  is central while  $\Lambda_\infty H \Lambda_\infty^{-1} = H$  and every element of  $\mathcal{L}_{\text{ext}}$  lies in exactly one of the four cosets  $H, \Lambda_\infty H, (-I)H, (-I)\Lambda_\infty H$ , hence has the form  $(-I)^\delta \Lambda_\infty^\varepsilon h$ . It is injective as different pairs  $(\varepsilon, \delta)$  land in different cosets, so if  $\Phi(h, s^\varepsilon r^\delta) = \Phi(h', s^{\varepsilon'} r^{\delta'})$  then  $(\varepsilon, \delta) = (\varepsilon', \delta')$  and  $\Lambda_\infty^\varepsilon h = \Lambda_\infty^{\varepsilon'} h'$ , whence  $h = h'$ . Moreover preserves group structure:

$$\begin{aligned} \Phi(h, q) \Phi(h', q') &= (-I)^\delta \Lambda_\infty^\varepsilon h \cdot (-I)^{\delta'} \Lambda_\infty^{\varepsilon'} h' \\ &= (-I)^{\delta+\delta'} \Lambda_\infty^\varepsilon (h \Lambda_\infty^{\varepsilon'} h' \Lambda_\infty^{-\varepsilon'}) \Lambda_\infty^{\varepsilon'} \\ &= (-I)^{\delta+\delta'} \Lambda_\infty^{\varepsilon+\varepsilon'} (h \alpha(s^{\varepsilon'})(h')) \\ &= \Phi\left(h \alpha(s^\varepsilon r^\delta)(h'), s^{\varepsilon+\varepsilon'} r^{\delta+\delta'}\right) \\ &= \Phi((h, q) \cdot (h', q')). \end{aligned}$$

In particular,

$$\mathcal{L}_{\text{ext}} \cong (SO(3, 1)_\uparrow^+ \rtimes_{\text{Ad}_{\Lambda_\infty}} \mathbb{Z}_2) \times \mathbb{Z}_2, \quad (159)$$

because the  $r$ -factor represented by  $\delta$  in  $\Phi$  acts trivially on  $SO(3, 1)_\uparrow^+$ .

### A.3 Proof of (28)

Let  $\mathcal{P}_0 = T \times H$  with multiplication

$$(a, h_0) \cdot (a', h'_0) = (a + h_0 a', h_0 h'_0).$$

$Z$  acts on  $\mathcal{P}_0$  by automorphisms  $\alpha_z$  defined by

$$\alpha_z(a', h'_0) = (z \cdot a', z h'_0 z^{-1}), \quad z \in Z.$$

Define

$$G_1 := (T \times H) \rtimes_\alpha Z, \quad G_2 := T \times (H \rtimes_\alpha Z),$$

where the action of  $(h_0, z) \in H \times Z$  on  $a' \in T$  is  $(h_0, z) \cdot a' := h_0(z \cdot a')$ . Then

$$\Psi : G_1 \rightarrow G_2, \quad \Psi((a, h_0), z) := (a, (h_0, z))$$

is a group isomorphism.

Indeed in  $G_1$  the product is

$$((a, h_0), z) \cdot ((a', h'_0), z') = ((a, h_0) \cdot \alpha_z(a', h'_0), zz') = \left( (a + h_0(z \cdot a'), h_0(zh'_0z^{-1})), zz' \right). \quad (160)$$

In  $G_2$  the product is

$$(a, (h_0, z)) \cdot (a', (h'_0, z')) = (a + (h_0, z) \cdot a', (h_0, z)(h'_0, z')) = \left( a + h_0(z \cdot a'), (h_0(zh'_0z^{-1}), zz') \right). \quad (161)$$

Apply  $\Psi$  to the product (160) in  $G_1$ :

$$\Psi\left(\left((a + h_0(z \cdot a'), h_0(zh'_0z^{-1})), zz'\right)\right) = \left(a + h_0(z \cdot a'), (h_0(zh'_0z^{-1}), zz')\right) \quad (162)$$

which is exactly the product (161) in  $G_2$  of  $\Psi((a, h_0), z)$  and  $\Psi((a', h'_0), z')$ . Hence  $\Psi$  is a homomorphism.

The inverse map is

$$\Psi^{-1} : G_2 \rightarrow G_1, \quad \Psi^{-1}(a, (h_0, z)) = ((a, h_0), z),$$

which is clearly two-sided inverse to  $\Psi$ . Therefore  $\Psi$  is an isomorphism.

#### A.4 Proof of (119)

From the operators defined in (116)-(118), using the Lorentz commutation relations, one computes:

$$[R_n, B_1] = (\vec{n} \times \vec{e}_1) \cdot \vec{K} = B_2 \quad (163)$$

$$[R_n, B_2] = (\vec{n} \times \vec{e}_2) \cdot \vec{K} = -B_1, \quad (164)$$

$$[B_1, B_2] = -(\vec{e}_1 \times \vec{e}_2) \cdot \vec{J} = -R_n \quad (165)$$

which is isomorphic to  $\mathfrak{so}(2, 1)$ . Now let:

$$Q^\mu = \Lambda_{\infty\mu}^\rho P_\rho = (0, M \sin \theta \cos \phi, M \sin \theta \sin \phi, M \cos \theta) = (0, M \hat{n}) \quad (166)$$

Explicit computation of the Pauli Lubanski vector components in (105) gives:

$$W^0 = Q^1 J_1 + Q^2 J_2 + Q^3 J_3 = \hat{n} \cdot \vec{J} = R_n,$$

$$W^1 = Q^2 K_3 - Q^3 K_2, \quad W^2 = Q^3 K_1 - Q^1 K_3, \quad W^3 = Q^1 K_2 - Q^2 K_1.$$

Equivalently,

$$\vec{W} = \vec{Q} \times \vec{K} = \hat{n} \times \vec{K}.$$

The Poincaré Casimir is

$$C_2 = W^\mu W_\mu = -(W^0)^2 + (W^1)^2 + (W^2)^2 + (W^3)^2.$$

From the above components,

$$(W^0)^2 = (R_n)^2$$

and

$$(W^1)^2 + (W^2)^2 + (W^3)^2 = |\hat{n} \times \vec{K}|^2 = |\vec{K}|^2 - (\hat{n} \cdot \vec{K})^2$$

Decomposing  $\vec{K}$  into components along  $(\hat{e}_1, \hat{e}_2, \hat{n})$ ,

$$\vec{K} = B_1 \hat{e}_1 + B_2 \hat{e}_2 + (\hat{n} \cdot \vec{K}) \hat{n}$$

we find

$$|\vec{K}|^2 - (\hat{n} \cdot \vec{K})^2 = B_1^2 + B_2^2$$

Thus:

$$C_2 = M^2 [-(R_n)^2 + B_1^2 + B_2^2]$$

## A.5 Proof of (127)

Let

$$\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

be a unit spatial vector, and let  $(\hat{e}_1, \hat{e}_2, \hat{n})$  be a right-handed orthonormal triad satisfying  $\hat{e}_1 \times \hat{e}_2 = \hat{n}$ . We define the following combinations of Lorentz generators:

$$J_n := \hat{n} \cdot \vec{J}, \quad \Pi_a := \hat{e}_a \cdot (\vec{K} + \hat{n} \times \vec{J}), \quad a = 1, 2. \quad (167)$$

We use the vector form of the Lorentz algebra (with  $\hbar = 1$ ):

$$[J \cdot u, J \cdot v] = i J \cdot (u \times v), \quad [J \cdot u, K \cdot v] = i K \cdot (u \times v), \quad [K \cdot u, K \cdot v] = -i J \cdot (u \times v), \quad (168)$$

for all  $u, v \in \mathbb{R}^3$ .

We also introduce the transverse-plane conventions

$$\epsilon_{12} = +1, \quad \hat{n} \times \hat{e}_a = \epsilon_{ab} \hat{e}_b, \quad \hat{e}_a \times \hat{e}_b = \epsilon_{ab} \hat{n},$$

which hold for any orthonormal triad.

From (167) and (168),

$$\begin{aligned} [J_n, \Pi_a] &= [J \cdot \hat{n}, K \cdot \hat{e}_a] + [J \cdot \hat{n}, J \cdot (\hat{n} \times \hat{e}_a)] \\ &= i K \cdot (\hat{n} \times \hat{e}_a) + i J \cdot (\hat{n} \times (\hat{n} \times \hat{e}_a)). \end{aligned} \quad (169)$$

Since  $\hat{n} \times \hat{e}_a = \epsilon_{ab} \hat{e}_b$  and  $\hat{n} \times (\hat{n} \times \hat{e}_a) = \epsilon_{ab} (\hat{n} \times \hat{e}_b)$ , we obtain

$$[J_n, \Pi_a] = i \epsilon_{ab} K \cdot \hat{e}_b + i \epsilon_{ab} J \cdot (\hat{n} \times \hat{e}_b) = i \epsilon_{ab} \Pi_b. \quad (170)$$

Hence

$$[J_n, \Pi_1] = i \Pi_2, \quad [J_n, \Pi_2] = -i \Pi_1.$$

Expanding the definition (167),

$$[\Pi_1, \Pi_2] = [K \cdot \hat{e}_1, K \cdot \hat{e}_2] + [K \cdot \hat{e}_1, J \cdot (\hat{n} \times \hat{e}_2)] + [J \cdot (\hat{n} \times \hat{e}_1), K \cdot \hat{e}_2] + [J \cdot (\hat{n} \times \hat{e}_1), J \cdot (\hat{n} \times \hat{e}_2)].$$

Each term can be evaluated using (168):

$$[K \cdot \hat{e}_1, K \cdot \hat{e}_2] = -i J \cdot (\hat{e}_1 \times \hat{e}_2) = -i J \cdot \hat{n} = -i J_n \quad (171)$$

$$[K \cdot \hat{e}_1, J \cdot (\hat{n} \times \hat{e}_2)] = -i K \cdot ((\hat{n} \times \hat{e}_2) \times \hat{e}_1) = 0 \quad (172)$$

$$[J \cdot (\hat{n} \times \hat{e}_1), K \cdot \hat{e}_2] = i K \cdot ((\hat{n} \times \hat{e}_1) \times \hat{e}_2) = 0, \quad (173)$$

$$[J \cdot (\hat{n} \times \hat{e}_1), J \cdot (\hat{n} \times \hat{e}_2)] = i J \cdot ((\hat{n} \times \hat{e}_1) \times (\hat{n} \times \hat{e}_2)) = +i J \cdot \hat{n} = +i J_n. \quad (174)$$

Summing all contributions, we find

$$[\Pi_1, \Pi_2] = (-i J_n) + 0 + 0 + (+i J_n) = 0.$$

The commutation relations of the set  $\{J_n, \Pi_1, \Pi_2\}$  are therefore

$$[J_n, \Pi_1] = i \Pi_2, \quad [J_n, \Pi_2] = -i \Pi_1, \quad [\Pi_1, \Pi_2] = 0, \quad (175)$$

which coincide with those of the Euclidean algebra  $\mathfrak{iso}(2)$ .

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