$\mathfrak{b}_1$-Verma $\mathfrak{b}_2$-dual Verma supermodules

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We show that if a module M over a basic classical Lie superalgebra of type type I is simultaneously a Verma module with respect to some Borel $\mathfrak{b}_1$ and a dual Verma module with respect to Borel $\mathfrak{b}_2$, then M is isomorphic to a Verma module with respect to either distinguished or an anti-distinguished Borel. Our method proceeds by analyzing edge contractions of the finite Young lattice that controls the combinatorics of odd reflections. In principle, the same strategy, for the most part, applies to all basic classical Lie superalgebras.

Contents

1 Introduction

Basic classical Lie superalgebras are widely regarded as a natural generalization of semisimple Lie algebras. Unlike the semisimple Lie algebra case, basic classical Lie superalgebras admit genuinely different (non-conjugate) Borel subalgebras. Their root systems admit genuinely different choices of simple systems; the associated simple reflections come in two types—*even* and *odd*—and together they naturally form a groupoid. Since these are special cases of Weyl groupoids in the sense of Heckeberger and Yamane[4, 23, 24], and Weyl groupoids play a crucial role in the classification of Nichols algebras of diagonal type [1, 2], it is natural and compelling to study these Lie superalgebra cases and their attendant representation theory.

In representation theory from this viewpoint, a central object is the $\mathfrak{b}$-Verma module with respect to a Borel subalgebra $\mathfrak{b}$: for $\lambda \in \mathfrak{h}^*$, $M^{\mathfrak{b}}(\lambda) := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}\, k_\lambda$, which has simple top $L^{\mathfrak{b}}(\lambda)$. For each Borel subalgebra $\mathfrak{b}$, there is a BGG category $\mathcal{O}$ that serves as the natural home for $\mathfrak{b}$-Verma modules; it has been extensively studied[18–20, 30, 31, 33], in particular for $\mathfrak{gl}(m|n)$[5, 6, 8, 11, 14, 15, 27]. This category $\mathcal{O}$ depends only on the even part $\mathfrak{b}_{\bar 0}$, so $\mathcal{O}$ mixes Verma modules attached to essentially different Borels. In particular, for each $\mathfrak{b}$ the category $\mathcal{O}$ carries a different highest weight structure.

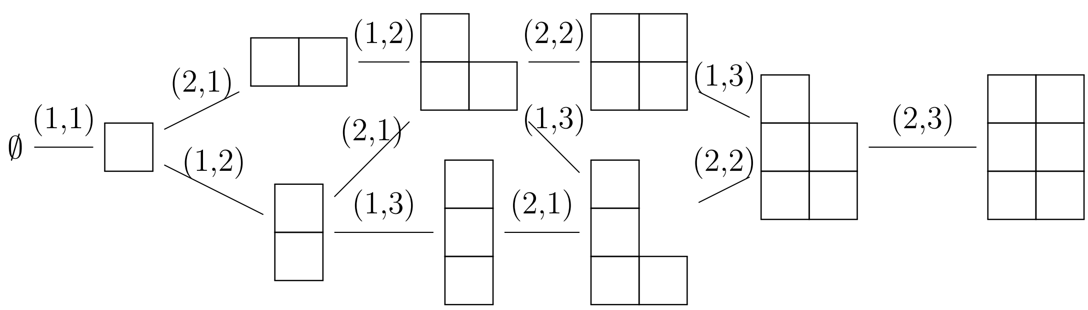
The aim of this paper is to analyze, in the prototypical case of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, the relationship between changes of Borel subalgebras and representation theory. In particular, we answer the basic question of when the contragredient dual of a Verma module is again a Verma module. In the classical (non-super) BGG category $\mathcal{O}$, this is well known: it happens exactly when the Verma module is antidominant and simple. Our generalization to $\mathfrak{gl}(m|n)$ setting is summarized below.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ and λ be an integral weight and let $\mathfrak{b}_1$ be a Borel subalgebra. If $M^{\mathfrak{b}_1}(\lambda) \cong M^{\mathfrak{b}_2}(\mu)^\vee$ for some μ and some Borel $\mathfrak{b}_2$, then $M^{\mathfrak{b}_1}(\lambda)$ is isomorphic to a Verma module with respect to either distinguished Borel $()$ or anti-distinguished Borel (n^m), and $L^{\mathfrak{b}_1}(\lambda)$ is antidominant. Conversely, if λ is antidominant, then $M^{()}(\lambda) \cong M^{(n^m)}(\lambda - 2\rho_{\bar 1})^\vee$.*

Note that an atypical Verma module is always not simple.

Our main technical work toward Theorem 1.1 is a detailed analysis, for $\mathfrak{gl}(m|n)$, of the finite Young lattice $L(m, n)$ and of its appropriate edge contractions, as illustrated in the examples below. The finite Young lattice $L(m, n)$ is a well-studied object in combinatorics in its own right[17, 34].

Example 1.2. Let $\mathfrak{g} = \mathfrak{gl}(3|2)$. $L(3, 2)$ is the following edge-colored graph.



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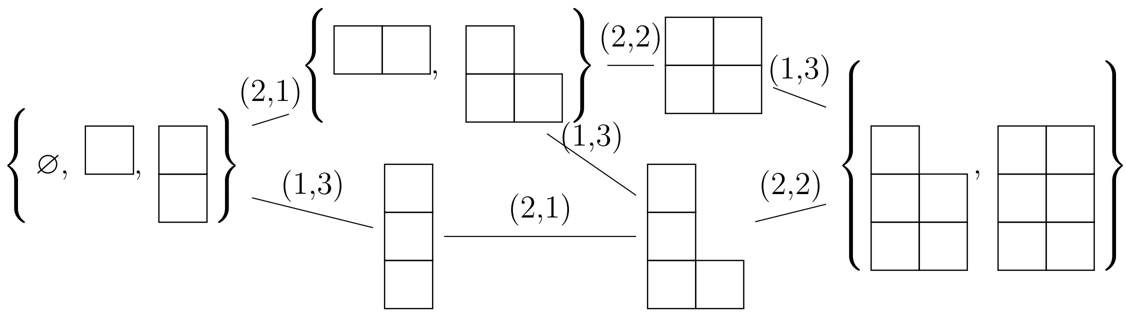
Let $\lambda = \varepsilon_1 - \delta_2$. Then the edge contraction is $L(3, 2)_\lambda \;\cong\; L(3, 2)/\{(1, 1), (1, 2), (2, 3)\}$. The resulting quotient is:



In the finite Young lattice $L(m,n)$, vertices correspond to all Borel subalgebras of $\mathfrak{gl}(m|n)$ containing the fixed even Borel of $\mathfrak{g}_{\bar{0}}$, and edges correspond to odd reflections between such Borels (coloreded by the reflected isotropic odd simple root.) Furthermore, for a fixed weight λ we consider the edge contraction $L(m,n) \rightsquigarrow L(m,n)_\lambda$ (we call it odd reflection graph of Verma modules $\{\, M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}})\,\}_{\mathfrak{b}}$, here, $\rho^{\mathfrak{b}}$ is $\mathfrak{b}$-Weyl vector). The vertex set of $L(m,n)_\lambda$ can be identified with the isomorphism classes of Verma modules having the same character, $\{\, M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}})\,\}_{\mathfrak{b}}$, and walks in this graph correspond to compositions of homomorphisms among these modules. In prior work [26], these homomorphisms are described; Theorem 1.1 follow by applying that description. Note that, for the type I Lie superalgebra $\mathfrak{osp}(2 \mid 2n)$, the corresponding graph is always a line segment, so Theorem 1.1 is immediate. In Section 5, we reinterpret the class of *totally disconnected* dominant weights (as introduced in Su and Zhang[35] ,see also [16]) in terms of changes of Borel.

1.1 Acknowledgments

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2 Well known basics

In this section, we summarize basic facts about $\mathfrak{gl}(m|n)$ and its representation theory. All material in this section is well known.

Let the base field k be an algebraically closed field of characteristic 0. Let **Vec** denote the category of vector spaces, and **sVec** the category of supervector spaces. Let $\Pi : \mathbf{sVec} \to \mathbf{sVec}$ be the *parity–shift functor*. Let $F : \mathbf{sVec} \to \mathbf{Vec}$ be the monoidal functor that forgets the $\mathbb{Z}/2\mathbb{Z}$–grading. Note that F is *not* a symmetric monoidal functor. In what follows, whenever we refer to the dimension of a supervector space, we mean this total (ordinary) dimension:

$$\dim V := \dim_k F(V).$$

From now on, we will denote by $\mathfrak{g}$ a finite dimensional Lie superalgebra. We consider the category $\mathfrak{g}$-sMod, where morphisms respects $\mathbb{Z}/2\mathbb{Z}$-grading. (This is the module category of a monoid object in $\mathfrak{g}$-sMod in the sense of [21].)

2.1 Root systems

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space, where $V_{\bar{0}}$ (the even part) is spanned by $v_1, \ldots, v_m$ and $V_{\bar{1}}$ (the odd part) is spanned by $v_{m+1}, \ldots, v_{m+n}$.

The space $\mathrm{End}(V)$ is spanned by basis elements E_{ij}, defined by: $E_{ij} \cdot v_k = \delta_{jk} v_i$.

The general linear Lie superalgebra $\mathfrak{gl}(m|n)$ is defined as the Lie superalgebra spanned by all E_{ij} with $1 \leq i, j \leq m+n$, under the supercommutator: $[E_{ij}, E_{kl}] = E_{ij}E_{kl} - (-1)^{|E_{ij}||E_{kl}|}E_{kl}E_{ij},)$ where $|E_{ij}| = \bar{0}$ if E_{ij} acts within $V_{\bar{0}}$ or $V_{\bar{1}}$ (even), and $|E_{ij}| = \bar{1}$ if it maps between $V_{\bar{0}}$ and $V_{\bar{1}}$ (odd).

In the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$, the even part is given by $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

We fix the standard Cartan subalgebra $\mathfrak{h} := \bigoplus_{1 \leq i \leq m+n} kE_{ii}$. Define linear functionals $\varepsilon_1, \ldots, \varepsilon_{n+m} \in \mathfrak{h}^*$ by requiring that $\varepsilon_i(E_{jj}) = \delta_{ij}$ for $1 \leq i, j \leq n+m$. Define $\delta_i = \varepsilon_{m+i}$ for $1 \leq i \leq n$. The non-degenerate symmetric bilinear form $(\, , \,)$ on is defined as follows: $(\varepsilon_i, \varepsilon_j) = (-1)^{|v_i|}\delta_{i,j}$

The root space $\mathfrak{g}_\alpha$ associated with $\alpha \in \mathfrak{h}^*$ is defined as $\mathfrak{g}_\alpha := \{ \, x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h} \, \}$. Then we have $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = kE_{ij}$.

The set of roots Δ is defined as $\Delta := \{ \, \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0 \, \} \setminus \{0\}$.

We have a root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \text{and} \quad \mathfrak{g}_0 = \mathfrak{h}.$$

we define The sets of all even roots and odd roots as follows: $\Delta_{\bar{0}} = \{ \, \varepsilon_i - \varepsilon_j, \; \delta_i - \delta_j \mid i \neq j \, \}$, $\Delta_{\bar{1}} = \{ \, \varepsilon_i - \delta_j \mid 1 \leq i \leq m, \; 1 \leq j \leq n \, \}$. Note that for any odd root $\alpha \in \Delta_{\bar{1}}$ we have $(\alpha, \alpha) = 0$; i.e., every odd root is isotropic .

Definition 2.1. A subset $\Delta^+ \subset \Delta$ is called a *positive system* if it satisfies:

1. $\Delta = \Delta^+ \cup (-\Delta^+)$ (disjoint union),

2. For any $\alpha, \beta \in \Delta^+$, if $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta^+$.

Given a positive system Δ^+, the associated *fundamental system* is defined by $\Pi := \{\alpha \in \Delta^+ \mid \alpha \text{ cannot be written as } \alpha = \beta + \gamma \text{ with } \beta, \gamma \in \Delta^+\}$. Elements of Π are called *simple roots*.

Let $\Delta^+ \subset \Delta$ be a positive system. The corresponding *Borel subalgebra* $\mathfrak{b} \subset \mathfrak{g}$ is defined as $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root space corresponding to $\alpha \in \Delta$. Choose natural root vectors $e_\alpha \in \mathfrak{g}_\alpha$, $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ Then for any $\lambda \in \mathfrak{h}^*$, $\lambda([e_\alpha, e_{-\alpha}]) = (\lambda, \alpha)$.

For a Borel subalgebra $\mathfrak{b}$, we express the triangular decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{n}^{\mathfrak{b}-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{\mathfrak{b}+}$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^{\mathfrak{b}+}$.

The sets of positive roots, even positive roots, and odd positive roots corresponding to $\mathfrak{b}$ are denoted by $\Delta^{\mathfrak{b},+}$, $\Delta_{\bar{0}}^{\mathfrak{b},+}$, and $\Delta_{\bar{1}}^{\mathfrak{b},+}$, respectively. Let $\Pi^{\mathfrak{b}}$ denote the set of all simple roots associated with the fixed Borel subalgebra $\mathfrak{b}$, and write $\Pi^{\mathfrak{b}} = \Pi^{\mathfrak{b}}_{\bar{0}} \sqcup \Pi^{\mathfrak{b}}_{\bar{1}}$ for its decomposition into even and odd simple roots.

It is clear that the root system of $\mathfrak{gl}(m|n)$, after forgetting the parity of roots, identifies with that of $\mathfrak{gl}(m+n)$. In particular, it is useful to consider the Weyl group S_{m+n} of $\mathfrak{gl}(m+n)$. This phenomenon is specific to $\mathfrak{gl}(m|n)$ and does not occur for other basic classical Lie superalgebras.

All positive root systems of $\mathfrak{gl}(m|n)$ are classified by total orders on the labeled set $\{\varepsilon_1, \dots, \varepsilon_{m+n}\}$, equivalently by permutations $\tau \in S_{m+n}$. Given τ, set

$$\Delta^+(\tau) = \{ \ \varepsilon_i - \varepsilon_j \mid \tau(i) < \tau(j) \ \}$$

yielding a bijection $\tau \leftrightarrow \Delta^+(\tau)$; hence there are $(m+n)!$ positive systems.

The Weyl group $S_m \times S_n$ of $\mathfrak{g}_{\bar{0}} = \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ acts on $\mathfrak{h}^*$ by permuting the ε_i's and the δ_j's separately. In particular, the even positive root systems are all conjugate under $S_m \times S_n$. We therefore fix the standard even Borel $\mathfrak{b}_{\bar{0}}^{\mathrm{st}}$ (block upper triangular in $\mathfrak{g}_{\bar{0}}$), with even positive root system

$$\Delta_{\bar{0}}^{\mathrm{st},+} = \{ \ \varepsilon_i - \varepsilon_j \mid 1 \le i < j \le m \ \} \ \cup \ \{ \ \delta_p - \delta_q \mid 1 \le p < q \le n \ \}.$$

With this choice fixed, positive systems whose even part equals $\Delta_{\bar{0}}^{\mathrm{st},+}$ are classified by $\varepsilon\delta$–sequences, equivalently by (mn)-shuffle

$$\tau \in \mathrm{Sh}(m|n) := \big\{ \, w \in S_{m+n} \, \big| \, w(1) < \cdots < w(m) \text{ and } w(m{+}1) < \cdots < w(m{+}n) \, \big\}.$$

Given τ, set

$$\Delta^+(\tau) = \{ \ \varepsilon_i - \varepsilon_j \mid \tau(i) < \tau(j) \ \}$$

yielding a bijection $\tau \leftrightarrow \Delta^+(\tau)$; hence there are $\frac{(m+n)!}{m!n!}$ positive systems with standard even one.

Definition 2.2. A partition is a weakly decreasing finite sequence of positive integers; we allow the empty partition $()$. For example, we write $5+3+3+1$ as (53^21). We identify partitions with Young diagrams in French notation (rows increase downward). A box has coordinates (i, j) with i the column (from the left) and j the row (from the bottom).

Each $\varepsilon\delta$–sequences with n symbols ε and m symbols δ determines a lattice path from $(n, 0)$ to $(0, m)$ (left step for ε, up step for δ); the region weakly southeast of the path inside (n^m) is a Young diagram fitting in the $m \times n$ rectangle (n^m), and this gives a bijection between $\varepsilon\delta$–sequences and the set of Young diagrams fitting in the $m \times n$ rectangle (n^m).

In what follows, we represent a Borel subalgebrawith sandard even Borel subalgebra by a partition. In particular, we call $()$ the *distinguished Borel subalgebra*, and (n^m) the *anti-distinguished Borel subalgebra.*

2.2 Finite Young lattices $L(m, n)$

Definition 2.3. We define the edge colored graph $\mathrm{Cay}(S_{m+n})$ as follows:

- **Vertex set:** the set of all positive root systems. (The set of all Borel subalgebras).
- **Color set:** the quotient $\Delta/(\pm 1)$, i.e. pairs of opposite roots.
- **Edges:** for each color $\alpha \in \Delta/(\pm 1)$, connect two vertices if they are related by the simple reflection corresponding to the root α .

This is precisely the Caylay graph of Weyl groupoid of $\mathfrak{gl}(m|n)$ in the sense of Heckenberger–Yamane. In this framework, the same graph construction extends to regular symmetrizable Kac–Moody Lie superalgebras and to Nichols algebras of diagonal type.

We write $r_\alpha\mathfrak{b}$ for the Borel obtained from $\mathfrak{b}$ by the reflection at α.

Note that the definition of $\mathrm{Cay}(S_{m+n})$ is formulated purely in the language of $\mathfrak{gl}(m+n)$ (via its root data and reflections), independent of the super decomposition of $\mathfrak{gl}(m|n)$. However, recalling parity, clearly there are two kinds of edges. We call the edge associated to an even root an even reflection, and the edge associated to an odd root an odd reflection. It is therefore natural to consider the following graph.

Definition 2.4. The edge colored graph $OR(\mathfrak{g})$ is defined as follows:

- **Vertex set:** the set of positive root systems whose even part is the standard positive even root system $\Delta_{\bar{0}}^{\mathrm{st}+}$.
- **Color set:** the quotient $\Delta_{\bar{1}}/(\pm 1)$, i.e. pairs of opposite odd roots.
- **Edges:** for each color $\alpha \in \Delta_{\bar{1}}/(\pm 1)$, connect two vertices if they are related by the odd reflection corresponding to the odd root α.

Definition 2.5. [Edge-colored finite Young's Lattice] Define an edge-colored graph $L(m, n)$ as follows:

- **Vertex set:** the set of Young diagrams fitting in the $m \times n$ rectangle ;
- **Color set:** $C := \{1, 2, ..., n\} \times \{1, 2, ..., m\}$;
- **Edges:** there is an edge of color (i, j) between vertices x and y if and only if y is obtained from x by adding or removing a box at coordinate (i, j), while all other boxes remain fixed.

From the above discussion, the following observation is immediate.

Proposition 2.6. *we obtain a natural edge-colored graph isomorphism:*

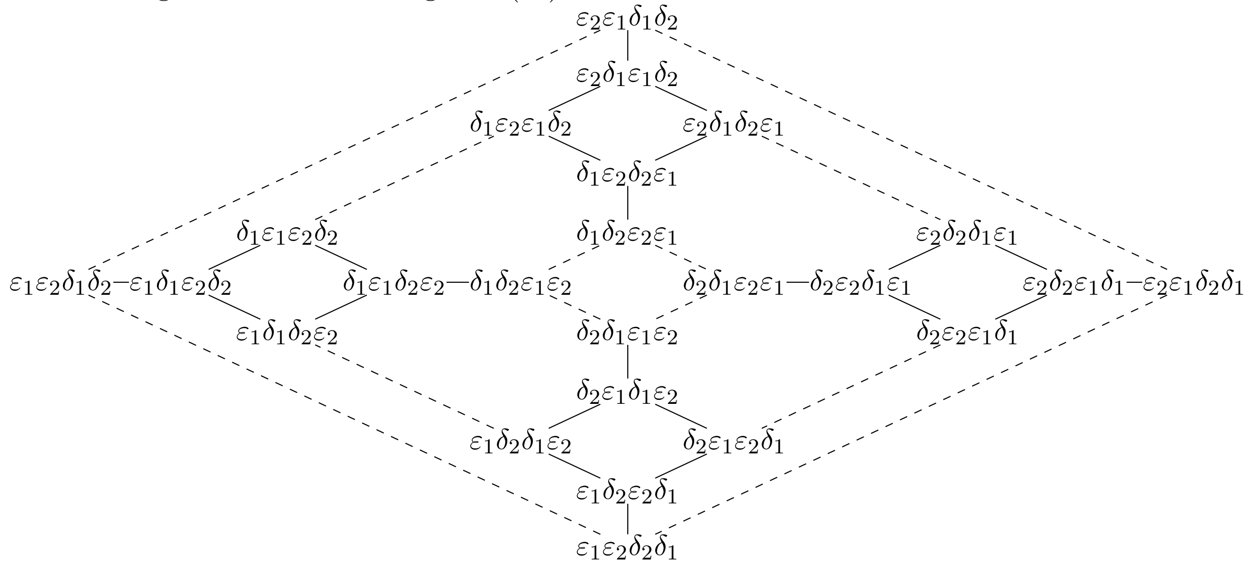
$$OR(\mathfrak{gl}(m|n)) \cong L(m, n)$$

The color set of $OR(\mathfrak{gl}(m|n))$ can naturally be identified with the collection of coordinates of boxes in an $m \times n$ rectangle By assigning the color $\varepsilon_i - \delta_j$ to the box at coordinates $(j, m + 1 - i)$

Corollary 2.7. *Two Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ share the same even part if and only if they are related by a sequence of odd reflections. In particular, the graph $L(m, n)$ is isomorphic, as edge colored graph, to any connected component obtained from* $\mathrm{Cay}(S_{m+n})$ *by deleting all edges corresponding to even roots.*

Example 2.8. In $\mathrm{Cay}(S_{2+2}) = \mathrm{Cay}(S_4)$, label each vertex by the subscripted $\varepsilon\delta$–sequence corresponding to its positive root system, and draw the *even reflections* (i.e., those in $S_2 \times S_2$ swapping two ε's or two δ's) as dashed edges. Deleting all even-reflection (dashed) edges yields a disjoint union of $|S_2 \times S_2| = 4$ connected components, each (as a colored graph) isomorphic to $L(2, 2)$. Here the sequence

$\varepsilon_1\,\delta_1\,\varepsilon_2\,\delta_2$ encodes the distinguished Borel subalgebra (), whereas $\delta_1\,\delta_2\,\varepsilon_1\,\varepsilon_2$ encodes the anti-distinguished Borel subalgebra (2^2).



2.3 Contragredient duality

Definition 2.9. We denote by $s\mathcal{W}$ the category of locally finite $\mathfrak{h}$-semisimple $\mathfrak{g}$-modules (i.e. weight modules with finite dimensional weight spaces).

A weight module M admits a weight space decomposition :

$$M = \bigoplus_{\lambda\in\mathfrak{h}^*} M_\lambda, \qquad M_\lambda := \{v \in M \mid h\cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

Lemma 2.10 (See also [5] Lemma 2.2, [9] Proposition 2.2.3)**.** *We can choose $p \in \mathrm{Map}(\mathfrak{h}^*, \mathbb{Z}/2\mathbb{Z})$ such that*

$$\mathcal{W} := \{M \in s\mathcal{W} \mid \deg M_\lambda = p(\lambda) \textit{ for } \lambda\in\mathfrak{h}^*, M_\lambda \neq 0\}$$

forms a Serre subcategory, $\mathcal{W}$ contains the trivial module and $s\mathcal{W} = \mathcal{W} \oplus \Pi\mathcal{W}$.

Similarly, for $\mathfrak{g}_{\bar 0}$ we introduce the category $s\mathcal{W}_{\bar 0}$, and define the category $\mathcal{W}_{\bar 0}$ so as to be compatible with the restriction functor $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$.

Lemma 2.11 ([3], Th. 2.2; [18], §6.1)**.** *Let* $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} : \mathfrak{g}\text{-}sMod \to \mathfrak{g}_{\bar 0}\text{-}sMod$*,* $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} : \mathfrak{g}_{\bar 0}\text{-}sMod \to \mathfrak{g}\text{-}sMod$*, and* $\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} : \mathfrak{g}_{\bar 0}\text{-}sMod \to \mathfrak{g}\text{-}sMod$ *denote restriction, induction, and coinduction, respectively; thus* $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \dashv \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \dashv \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$.

Then: (1) each of $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$*,* $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$*,* $\mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$ *is exact and restricts between $\mathcal{W}_{\bar 0}$ and $\mathcal{W}$; (2) there is a natural isomorphism of functors* $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \cong \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$*; (3) The unit* $\mathrm{Id}_{\mathcal{O}_{\bar 0}} \to \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \circ \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}}$ *is a split monomorphism, and the counit* $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \circ \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \twoheadrightarrow \mathrm{Id}_{\mathcal{W}_{\bar 0}}$ *is a split epimorphism. (4)* $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \circ \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \cong \Lambda(\mathfrak{g}_{\bar 1}) \otimes -$ *on $\mathcal{W}_{\bar 0}$.*

Note that $\Lambda^{\dim \mathfrak{g}_{\bar 1}}(\mathfrak{g}_{\bar 1}) \cong \Lambda^{2mn}(\mathfrak{g}_{\bar 1}) \cong L_{\bar 0}(0)$ in our general linear setting, but in general, we have $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \cong \mathrm{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \circ (- \otimes \Lambda^{\dim \mathfrak{g}_{\bar 1}} \mathfrak{g}_{\bar 1})$. $L_{\bar 0}(0) = \Lambda^0(\mathfrak{g}_{\bar 1})$ and $\Lambda^{\dim \mathfrak{g}_{\bar 1}}(\mathfrak{g}_{\bar 1})$ are direct summands of $\Lambda(\mathfrak{g}_{\bar 1})$.

Definition 2.12. For a module M in the category $\mathcal{W}$, the character $\mathrm{ch}\, M$ is a formal sum that encodes the dimensions of the weight spaces of M: $\mathrm{ch}\, M := \sum_\lambda \dim M_\lambda e^\lambda$.

Lie superalgebra $\mathfrak{gl}(m|n)$ has an antiautomorphism τ given by the formula $\tau(E_{ij}) := -(-1)^{|i|(|i|+|j|)} E_{ji}$. For a weight module $M = \bigoplus_{\nu\in\mathfrak{h}^*} M_\nu$, we let $M^\vee := \bigoplus_\nu M_\nu^*$ be the restricted dual of M. We may define an action of $\mathfrak{gl}(m|n)$ on $M^\vee$ by $(g\cdot f)(x) := -f(\tau(g)x)$, $f\in M^\vee$, $g\in\mathfrak{gl}(m|n)$, $x\in M$. This defines a contravariant duality functor $(-)^\vee : \mathcal{W}\to\mathcal{W}$.

Proposition 2.13 (Properties of contragredient duality)**.** *1. $(-)^\vee$ is an exact contravariant functor.*

2. There is a natural isomorphism $M \xrightarrow{\sim} M^{\vee\vee}$ for all $M\in\mathcal{W}$.

3. $\mathrm{ch}(M^\vee) = \mathrm{ch}(M)$; in particular, the weight multiplicities are preserved.

4. Since τ is an automorphism of $\mathfrak{g}$ compatible with the inclusions of subalgebras, there is a natural isomorphism of functors $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \cong (\,\cdot\,)^\vee \circ \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \circ (\,\cdot\,)^\vee$. Consequently, $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \cong (\,\cdot\,)^\vee \circ \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar 0}} \circ (\,\cdot\,)^\vee$.

2.4 Verma modules

Let $k^{\mathfrak{b}}_\lambda$ be the one-dimensional $\mathfrak{b}$-module corresponding to $\lambda\in\mathfrak{h}^*$. The $\mathfrak{b}$-Verma module with highest weight λ is defined by $M^{\mathfrak{b}}(\lambda) = U(\mathfrak{g})\otimes_{U(\mathfrak{b})} k^{\mathfrak{b}}_\lambda$. (Here, the parity of $k^{\mathfrak{b}}_\lambda$ is chosen so that $M^{\mathfrak{b}}(\lambda)\in\mathcal{W}$.)

Its simple top is denoted by $L^{\mathfrak{b}}(\lambda)$. Similarly, for the even part $\mathfrak{g}_{\bar 0}$, the corresponding Verma module and simple module are denoted by $M_{\bar 0}(\lambda)$ and $L_{\bar 0}(\lambda)$, respectively.

We define Berezinian weight $ber := \varepsilon_1+\cdots+\varepsilon_m-(\delta_1+\cdots+\delta_n)$, A weight λ is orthogonal to all roots if and only if $\lambda = tber$ for some $t\in k$ (i.e. a scalar multiple of the Berezinian weight. Ber $:= L^{()}(ber)$. For any $t\in k$, the simple highest weight module $L^{()}(tber)$ is one-dimensional; indeed $L^{()}(tber)\cong \mathrm{Ber}^{\otimes t}$.

Definition 2.14 (integral Weyl vectors [5])**.** Define the following vectors in $\mathfrak{h}^*$

$$\rho_{\bar 0} := \frac{1}{2}\sum_{\beta\in\Delta^+_{\bar 0}}\beta,\quad \rho^{\mathfrak{b}}_{\bar 1} := \frac{1}{2}\sum_{\gamma\in\Delta^{\mathfrak{b}+}_{\bar 1}}\gamma.$$

We write $\rho_{\bar 1} := \rho^{()}_{\bar 1}$. Here integral Weyl vector $\rho := -\varepsilon_2-2\varepsilon_3-\cdots-(m-1)\varepsilon_n+(m-1)\delta_{m+1}+(m-2)\delta_{m+2}+\cdots+(m-n)\delta_{m+n}$.

One checks that $\rho = \rho_{\bar 0}-\rho^{()}_{\bar 1}-\frac{1}{2}(m+n-1)ber$

$$\rho^{\mathfrak{b}} := \rho_{\bar 0}-\rho^{\mathfrak{b}}_{\bar 1}-\frac{1}{2}(m+n-1)ber$$

It is worth noting that $\rho^{r_\alpha\mathfrak{b}} = \rho^{\mathfrak{b}}+\alpha$ for a b-simple root α. If $\alpha\in\Pi^{\mathfrak{b}}$, then we have $(\rho^{\mathfrak{b}},\alpha^{\mathfrak{b}}_i) = \frac{1}{2}(\alpha,\alpha)$. In particular, if α is isotropic, then we have $(\rho^{\mathfrak{b}},\alpha)=0$.

Lemma 2.15. *For $\mathfrak{b},\mathfrak{b}'\in L(m,n)$ and $\lambda,\lambda'\in\mathfrak{h}^*$, the following statements hold:*

1. $\mathrm{ch}\, M^{\mathfrak{b}}(\lambda-\rho^{\mathfrak{b}}) = \mathrm{ch}\, M^{\mathfrak{b}'}(\lambda-\rho^{\mathfrak{b}'})$;

2. $\dim M^{\mathfrak{b}'}(\lambda-\rho^{\mathfrak{b}'})_{\lambda-\rho^{\mathfrak{b}}} = 1$;

3. $\mathrm{ch}\, M^{\mathfrak{b}}(\lambda) = \mathrm{ch}\, M^{\mathfrak{b}'}(\lambda') \iff \lambda+\rho^{\mathfrak{b}} = \lambda'+\rho^{\mathfrak{b}'}$;

4. $\dim \operatorname{Hom}(M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}), M^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'})) = 1$*;*

5. If $M^{\mathfrak{b}}(\lambda-\rho^{\mathfrak{b}}) \ncong M^{\mathfrak{b}'}(\lambda-\rho^{\mathfrak{b}'})$, then $M^{\mathfrak{b}}(\lambda-\rho^{\mathfrak{b}})$ is not isomorphic to any $\mathfrak{b}'$-Verma module.

6. Furthermore, let $\mathfrak{b}$ and $\mathfrak{b}'$ be related by an odd reflection with respect to a $\mathfrak{b}$–simple odd root α.

(a) if $(\lambda, \alpha) \neq 0$, then we have $M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}) \cong M^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'})$, $\quad L^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}) \cong L^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'})$.

(b) If $(\lambda, \alpha) = 0$, then we have $M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}) \ncong M^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'})$, $\quad L^{\mathfrak{b}}(\lambda) \cong L^{\mathfrak{b}'}(\lambda)$.

2.5 Category $\mathcal{O}$

Definition 2.16. Let $\mathfrak{b} \in L(m,n)$. The category $\mathcal{O}^{\mathfrak{b}}$ is defined as the Serre subcategory of $\mathcal{W}$ generated by $\{L^{\mathfrak{b}}(\lambda) \mid \lambda \in \mathfrak{h}^*\}$. According to Lemma 2.15, as an Serre subcategory, it depends only on $\mathfrak{b}_{\bar{0}}$ (however,the highest weight structure depends on $\mathfrak{b}$).

It is well known that the BGG category $\mathcal{O}$ contains all $\mathfrak{b}$–Verma modules for any $\mathfrak{b} \in L(m,n)$. Moreover, $\mathcal{O}$ is a finite–length abelian category. Similarly, by replacing $\mathfrak{g}$ with $\mathfrak{g}_{\bar{0}}$, we define $\mathcal{O}_{\bar{0}}$ as a full subcategory of $\mathcal{W}_{\bar{0}}$.

For a Borel subalgebra $\mathfrak{b}$, we call a module of the form $M^{\mathfrak{b}}(\lambda)^{\vee}$, i.e. the contragredient dual of the $\mathfrak{b}$–Verma module, a $\mathfrak{b}$*–dual Verma module.* The contragredient dual functor $(-)^{\vee}$ restricts to the BGG category, i.e. $(-)^{\vee} : \mathcal{O}^{\mathrm{op}} \to \mathcal{O}$ is exact and contravariant; in particular $M^{\mathfrak{b}}(\lambda)^{\vee}$ belongs to $\mathcal{O}$ and $L^{\mathfrak{b}}(\lambda)^{\vee} \simeq L^{\mathfrak{b}}(\lambda)$.

Since each of $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$, $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$, $\operatorname{Coind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}}$ is exact and restricts between $\mathcal{O}_{\bar{0}}$ and $\mathcal{O}$, for each choice of Borel subalgebra $\mathfrak{b}$, the category $\mathcal{O}$ carries a highest weight structure with standard objects the $\mathfrak{b}$–Verma modules $M^{\mathfrak{b}}(\lambda)$ (and costandard objects $(M^{\mathfrak{b}}(\lambda))^{\vee}$); the weight poset is $(\mathfrak{h}^*, \leq_{\mathfrak{b}})$ where $\mu \leq_{\mathfrak{b}} \lambda$ iff $\lambda - \mu$ is a sum of roots in $\Delta^{\mathfrak{b}+}$. For a weight λ, we write $P(\lambda)$ and $I(\lambda)$ for its projective cover and injective hull in $\mathcal{O}$, respectively. Similarly, we write $P_{\bar{0}}(\lambda)$, $I_{\bar{0}}(\lambda)$ for projective cover and injective hull of $L_{\bar{0}}(\lambda)$ in in $\mathcal{O}_{\bar{0}}$, respectively .

For $\mathcal{O}$ one has BGG reciprocity $\left[P^{\mathfrak{b}}(\lambda) : M^{\mathfrak{b}}(\mu)\right] = \left[M^{\mathfrak{b}}(\mu) : L^{\mathfrak{b}}(\lambda)\right]$.

For fixed $\mathfrak{b} \in B(\mathfrak{g})$, an object $M \in \mathcal{O}$ has a $\mathfrak{b}$-Verma flag if it admits a finite filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_r = M$ with subquotients $F_i/F_{i-1} \cong M^{\mathfrak{b}}(\lambda_i)$. Let $\mathcal{F}\Delta^{\mathfrak{b}} \subset \mathcal{O}$ denote the full subcategory of objects with a $\mathfrak{b}$-Verma flag. Similarly, we define the category of modules with a $\mathfrak{b}$-dual Verma flag, $\mathcal{F}\nabla^{\mathfrak{b}} \subset \mathcal{O}$, and the category of modules with an even Verma flag, $\mathcal{F}\Delta_{\bar{0}} \subset \mathcal{O}_{\bar{0}}$.

Proposition 2.17. *Let $\lambda \in \mathfrak{h}^*$. Fix a Borel subalgebra $\bar{\mathfrak{b}}$.*

1. The category $\mathcal{F}\Delta^{\bar{\mathfrak{b}}}$ is closed under direct summands. 2.$\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} M_{\bar{0}}(\lambda), \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} P_{\bar{0}}(\lambda), P^{\bar{\mathfrak{b}}}(\lambda) \in \bigcap_{\mathfrak{b} \in \mathfrak{B}} \mathcal{F}\Delta^{\mathfrak{b}}$; *3.* $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{g}_{\bar{0}}} M^{\bar{\mathfrak{b}}}(\lambda) \in \mathcal{F}\Delta_{\bar{0}}$.

2.6 Kac functors

Recall that we represent a Borel subalgebrawith sandard even Borel subalgebra by a partition. In particular, we denote () the *distinguished Borel subalgebra*, and (n^m) the *anti-distinguished Borel subalgebra.*

Define $\mathfrak{g}_1 := \mathfrak{g}_{\bar{1}} \cap ()$ $\quad \mathfrak{g}_{-1} := \mathfrak{g}_{\bar{1}} \cap (n^m)$.

The general linear Lie superalgebra $\mathfrak{gl}(m|n)$ admits a natural $\mathbb{Z}$-grading given by $\mathfrak{gl}(m|n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1$. This grading satisfies the relations $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0$. We define $\mathfrak{g}_{\geq 0} := \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_{\leq 0} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}}$.

Define the inflation functor $\mathrm{Infl}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}_{\geq 0}} \colon \mathfrak{g}_{\bar{0}}\text{-sMod} \to \mathfrak{g}_{\geq 0}\text{-sMod}$ by letting $\mathfrak{g}_1$ act trivially. Define the functor $(\,)^{\mathfrak{g}_1} \colon \mathfrak{g}_{\geq 0}\text{-sMod} \to \mathfrak{g}_{\bar{0}}\text{-sMod}$ by $M \mapsto M^{\mathfrak{g}_1} := \{\, m \in M \mid \mathfrak{g}_1 m = 0 \,\}$. Then these functors are mutually adjoint each other and exact.

The Kac functor is defined as $K_{\geq 0} := \mathrm{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} \circ \mathrm{Infl}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}_{\geq 0}} : \mathfrak{g}_{\bar{0}}\text{-sMod} \to \mathfrak{g}\text{-sMod}$, which is exact. Then $K_{\geq 0}$ is left adjoint to the exact functor $\mathrm{Res}_{\geq 0} := (\,)^{\mathfrak{g}_1} \circ \mathrm{Res}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} :$ $\mathfrak{g}\text{-sMod} \to \mathfrak{g}_{\bar{0}}\text{-sMod}$. If we denote by $\mathrm{Coind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}}$ [10] the right adjoint of $\mathrm{Res}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}}$, and define $\mathrm{Coind}_{\geq 0} := \mathrm{Coind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} \circ \mathrm{Infl}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}_{\geq 0}} : \mathfrak{g}_{\bar{0}}\text{-sMod} \to \mathfrak{g}\text{-sMod}$ then there is an isomorphism of functors $\mathrm{Coind}_{\geq 0} \circ (- \otimes L_{\bar{0}}(-2\rho_{\bar{1}})) \cong K_{\geq 0}$, see [10]. Note that $L_{\bar{0}}(-2\rho_{\bar{1}}) \cong \bigwedge^{\dim \mathfrak{g}_{-1}} \mathfrak{g}_{-1}$ as $\mathfrak{g}_{\bar{0}}$-modules.

Similarly, we define the dual Kac functor $K_{\leq 0} := \mathrm{Ind}_{\mathfrak{g}_{\leq 0}}^{\mathfrak{g}} \circ \mathrm{Infl}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}_{\leq 0}} : \mathfrak{g}_{\bar{0}}\text{-sMod} \to$ $\mathfrak{g}\text{-sMod}$, $\mathrm{Res}_{\leq 0} := \mathrm{Res}_{\mathfrak{g}_{\leq 0}}^{\mathfrak{g}} \circ (\,\cdot\,)^{\mathfrak{g}_{-1}}$ and $\mathrm{Coind}_{\leq 0}$.

Lemma 2.18. *[22]* $K_{\geq 0} \cong (\,\cdot\,)^{\vee} \circ \mathrm{Coind}_{\leq 0} \circ (\,\cdot\,)^{\vee}, \qquad K_{\leq 0} \cong (\,\cdot\,)^{\vee} \circ \mathrm{Coind}_{\geq 0} \circ (\,\cdot\,)^{\vee}$. *$K_{\geq 0}$ and $K_{\leq 0}$ restricts between $\mathcal{O}_{\bar{0}}$ and $\mathcal{O}$.*

Proposition 2.19. *For every weight λ one has $K_{\geq 0}\big(M_{\bar{0}}(\lambda)\big) \cong M^{()}(\lambda)$ and $K_{\leq 0}\big(M_{\bar{0}}(\lambda)\big) \cong M^{(n^m)}(\lambda)$. Using $(\,\cdot\,)^{\vee}$, $\mathrm{Coind}_{\leq 0}\big(M_{\bar{0}}(\lambda)^{\vee}\big) \cong K_{\leq 0}\big(M_{\bar{0}}(\lambda - 2\rho_{\bar{1}})^{\vee}\big) \cong M^{()}(\lambda)^{\vee}$, and $\mathrm{Coind}_{\geq 0}\big(M_{\bar{0}}(\lambda)^{\vee}\big) \cong K_{\geq 0}\big(M_{\bar{0}}(\lambda + 2\rho_{\bar{1}})^{\vee}\big) \cong M^{(n^m)}(\lambda)^{\vee}$. We also have $K_{\geq 0}(L_{\bar{0}}(\lambda))^{\vee} \cong K_{\leq 0}(L_{\bar{0}}(\lambda - 2\rho_{\bar{1}}))$, and $\operatorname{soc} K_{\geq 0}(L_{\bar{0}}(\lambda)) \cong L^{(n^m)}(\lambda - 2\rho_{\bar{1}})$.*

Theorem 2.20. *[13] For every $M \in \mathcal{O}_{\bar{0}}$ one has $\ell(\operatorname{soc} M) = \ell(\operatorname{soc} K_{\geq 0} M) = \ell(\operatorname{soc} K_{\leq 0} M)$ and $\ell(\operatorname{top} M) = \ell(\operatorname{top} K_{\geq 0} M) = \ell(\operatorname{top} K_{\leq 0} M)$, where $\ell(-)$ denotes the length.*

Proposition 2.21. *(see [12, 30]) For a weight λ, the following are equivalent:*

1. $M_{\bar{0}}(\lambda) \cong L_{\bar{0}}(\lambda) \cong M_{\bar{0}}(\lambda)^{\vee}$.
2. $P_{\bar{0}}(\lambda) \cong I_{\bar{0}}(\lambda)$.
3. $P^{()}(\lambda) \cong I^{()}(\lambda)$.
4. *$L^{()}(\lambda)$ is the socle of some M s.t. $\mathrm{Res}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}} M \in \mathcal{F}\Delta_{\bar{0}}$..*
5. $M^{()}(\lambda) \cong M^{(n^m)}(\lambda - 2\rho_{\bar{1}})^{\vee}$.

Proof. (1)⇔(2) are classical. (2)⇔(3) follows from Lemma 2.11. (3)⇒(4) is trivial (4)⇒(3): see [28, 30]. (1)⇔(5): $M^{()}(\lambda)^{\vee} \cong K_{\leq 0}\big(M_{\bar{0}}(\lambda - 2\rho_{\bar{1}})^{\vee}\big) \cong K_{\leq 0}\big(M_{\bar{0}}(\lambda - 2\rho_{\bar{1}})\big) \cong M^{(n^m)}(\lambda - 2\rho_{\bar{1}})$. ☐

We call such a weight λ (or $L^{()}(\lambda)$) antidominant.

3 Odd reflection graph of Verma modules

This section's results hold, in principle, for any regular symmetrizable Kac-Moody Lie superalgebras [4, 32] and double bozonization of Nichols algebras of diagonal type [36].

3.1 Edge contraction

Definition 3.1 (Edge contraction in simple graphs)**.** A (finite) simple undirected graph is a pair $G = (V, E)$ with $E \subseteq \binom{V}{2} = \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ (i.e. no loops and no parallel edges).

Let $S \subseteq E$ be a set of edges. Define an equivalence relation $\sim_S$ on V by

$u \sim_S v \iff u$ and v lie in the same connected component of the spanning subgraph (V, S).

The *contraction of* S is the simple graph

$$G/S = (V', E') \quad \text{with} \quad V' := V/{\sim_S},$$

where $[v]$ denotes the $\sim_S$-class of v and

$$E' \ := \ \big\{ \ \{ [a], [b] \} \ \big| \ \{a, b\} \in E \setminus S,\ [a] \neq [b] \ \big\}.$$

Thus any loop that would arise from an edge whose endpoints merge ($[a] = [b]$) is discarded by definition, and parallel edges collapse automatically since $E' \subseteq \binom{V'}{2}$.

Remark 3.2. The edge–contraction construction does not, in general, produce a new simple edge–colored graph in the abstract setting of edge-colored graphs. Remarkably, for the finite Young lattices $L(m, n)$ (and, more generally, for the odd reflection graphs arising from regular symmetrizable Kac–Moody Lie superalgebras and from Nichols algebras of diagonal type), a special property inherited from the Weyl groupoid—what we call the "rainbow boomerang graph [25]" —ensures that the construction can be carried out in a way compatible with the coloring.

Definition 3.3 (Color quotient via edge contraction)**.** Let $L(m, n) = (V, E)$ be as defined before, with color set $\mathcal{C}$. For a subset $A \subseteq \mathcal{C}$, put

$$L(m, n)/A \ := \ L(m, n)/\{e \in E \mid c(e) \in A\},$$

and equip it with the color map $\tilde{c}$ obtained by restricting c to the surviving edges (i.e. edges from $E \setminus S$ whose endpoints do not merge).

Lemma 3.4 (Well-definedness for $L(m, n)$)**.** *For $L(m, n)$ the map $\tilde{c}$ is well defined: whenever two edges of $E \setminus S$ project to the same edge of $L(m, n)/A$, they have the same color in $\mathcal{C} \setminus A$.*

Definition 3.5. For $\lambda \in \mathfrak{h}^*$ we define an edge colored connected simple graph $L(m, n)_\lambda := L(m, n)/\{\alpha \in \Delta_{\bar{1}}/(\pm 1) \mid (\lambda, \alpha) \neq 0\}$. See Example 1.2. Note that $L(m, n)_0 := L(m, n)$. We call $L(m, n)_{\lambda + \rho^{\mathfrak{b}}}$ the odd reflection graph of $M^{\mathfrak{b}}(\lambda)$.

Corollary 3.6. *The vertex set of $L(m, n)_\lambda$ can be identified with the isomorphism classes of $\{M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}})\}_{\mathfrak{b}}$. Precisely, we have:*

$$M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}) \cong M^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'}) \iff [\mathfrak{b}] = [\mathfrak{b}'] \in L(m, n)_\lambda$$

Proof. By Lemma 2.15, whether two adjacent Verma modules are isomorphic depends only on the label α, i.e. on the color of the edge. This is exactly the claim. $\square$

Definition 3.7. Let $\lambda \in \mathfrak{h}^*$ and $\mathfrak{b} \in L(m,n)$. We define the $\mathfrak{b}$-*atypicality* of λ as

$$\mathrm{aty}^{\mathfrak{b}}(\lambda) := \max\left\{ t \;\middle|\; \begin{array}{l} \text{there exist mutually orthogonal distinct roots } \alpha_1,\dots,\alpha_t \in \Delta_{\bar{1}}/(\pm 1) \\ \text{such that } (\lambda+\rho^{\mathfrak{b}},\alpha_i)=0 \quad \text{for all } i=1,\dots,t \end{array} \right\}.$$

If $\mathrm{aty}^{\mathfrak{b}}(\lambda)=0$, then λ is called $\mathfrak{b}$-*typical*; otherwise, it is called $\mathfrak{b}$-*atypical.*

We def $\mathrm{aty}L^{\mathfrak{b}}(\lambda) := \mathrm{aty}^{\mathfrak{b}}(\lambda)$ If $\mathrm{aty}^{\mathfrak{b}}(\lambda)=0$, then $L^{\mathfrak{b}}(\lambda)$ is called *typical*; otherwise, it is called *atypical.* This definition is independent of the choice of $\mathfrak{b}$.

We define the *defect* of $\mathfrak{g}$ as $\mathrm{def}(\mathfrak{g}) := \max\{\mathrm{aty}(L) \mid L \text{ is a simple highest weight module}\}$.

It is easy to check that $\mathrm{def}(\mathfrak{gl}(m|n)) = \min(m,n)$.

Proposition 3.8. *Fix a reference Borel subalgebra $\bar{\mathfrak{b}}$. For $\lambda \in \mathfrak{h}^*$ the following are equivalent:*

1. $L^{\bar{\mathfrak{b}}}(\lambda-\rho^{\mathfrak{b}})$ *is typical;*
2. $M^{\bar{\mathfrak{b}}}(\lambda-\rho^{\mathfrak{b}}) \in \bigcap_{\mathfrak{b}\in L(m,n)} \mathcal{F}\Delta^{\mathfrak{b}}$*;*
3. *the vertex set of $L(m,n)_\lambda$ consists of a single point;*
4. $L^{\bar{\mathfrak{b}}}(\lambda-\rho^{\mathfrak{b}}) \cong L^{\mathfrak{b}}(\lambda-\rho^{\mathfrak{b}})$ *for all* $\mathfrak{b}$.

Corollary 3.9. *If $M^{\mathfrak{b}}(\lambda) \cong L^{\mathfrak{b}}(\lambda)$ or $M^{\mathfrak{b}}(\lambda) \cong P^{\mathfrak{b}}(\lambda)$, then $L^{\mathfrak{b}}(\lambda)$ is typical.*

3.2 Odd Verma's theorem

Definition 3.10. In a simple undirected graph $G=(V,E)$, a (finite) walk of length k is a sequence $v_0,\dots,v_k$ with $\{v_{i-1},v_i\} \in E$ for all $i=1,\dots,k$. A walk is called *shortest* if there is no shorter walk between the same pair of vertices. A walk is called *rainbow* [29] if all the edge colors in its sequence are distinct.

Fix a weight λ. In $L(m,n)_\lambda$, we naturally identify a length–one walk (i.e. an edge) with the unique (up to scalars) nonzero homomorphism between the corresponding two Verma modules adjacent via an odd reflection. Consequently, any walk can be identified with the composition of such homomorphisms. For example, a rainbow walk gives a nonzero homomorphism by the PBW Theorem. The following theorem provides a complete description of homomorphisms between Verma modules sharing same characters .

Theorem 3.11. *[26] Let $\lambda \in \mathfrak{h}^*$ and w a walk in $L(m,n)_\lambda$. Then,*

$$w \neq 0 \iff w \text{ is rainbow} \iff w \text{ is shortest.}$$

Corollary 3.12. *Let $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{b}_3 \in B(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$. For $i,j \in \{1,2,3\}$, let*

$$\psi_\lambda^{\mathfrak{b}_i,\mathfrak{b}_j} \in \mathrm{Hom}\left(M^{\mathfrak{b}_i}(\lambda-\rho^{\mathfrak{b}_i}),\, M^{\mathfrak{b}_j}(\lambda-\rho^{\mathfrak{b}_j})\right)$$

denote the nonzero (defined up to scalars) morphism . Then the following are equivalent:

(1) *In $L(m,n)_\lambda$, there is a shortest walk from $[\mathfrak{b}_1]$ to $[\mathfrak{b}_3]$ passing through $[\mathfrak{b}_2]$.*

(2) $\psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_1} \circ \psi_\lambda^{\mathfrak{b}_3,\mathfrak{b}_2} = \psi_\lambda^{\mathfrak{b}_3,\mathfrak{b}_1} \neq 0$

(3) $\operatorname{Im} \psi_\lambda^{\mathfrak{b}_3,\mathfrak{b}_2} \not\subset \ker \psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_1}$

(4) $\operatorname{Im} \psi_\lambda^{\mathfrak{b}_3,\mathfrak{b}_1} \subset \operatorname{Im} \psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_1}$

(5) $\operatorname{Im} \psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_1}$ *has* $L^{\mathfrak{b}_3}(\lambda-\rho^{\mathfrak{b}_3})$ *as a composition factor*

(6) $\ker \psi_\lambda^{\mathfrak{b}_3,\mathfrak{b}_2} \subset \ker \psi_\lambda^{\mathfrak{b}_3,\mathfrak{b}_1}$

(7) $\psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_3} \circ \psi_\lambda^{\mathfrak{b}_1,\mathfrak{b}_2} = \psi_\lambda^{\mathfrak{b}_1,\mathfrak{b}_3} \neq 0$

(8) $\operatorname{Im} \psi_\lambda^{\mathfrak{b}_1,\mathfrak{b}_2} \not\subset \ker \psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_3}$

(9) $\operatorname{Im} \psi_\lambda^{\mathfrak{b}_1,\mathfrak{b}_3} \subset \operatorname{Im} \psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_3}$

(10) $\operatorname{Im} \psi_\lambda^{\mathfrak{b}_2,\mathfrak{b}_3}$ *has* $L^{\mathfrak{b}_1}(\lambda-\rho^{\mathfrak{b}_1})$ *as a composition factor*

(11) $\ker \psi_\lambda^{\mathfrak{b}_1,\mathfrak{b}_2} \subset \ker \psi_\lambda^{\mathfrak{b}_1,\mathfrak{b}_3}$

4 Main results

4.1 Applying odd Verma's theorem

This subsection's results hold, in principle, for all basic classical Lie superalgebras.

Let G be a simple graph. For $A, B \in G$, we say that B is A-geodesically maximal if there is no vertex $C \in G$ with $C \neq B$ such that there is a shortest walk C to A passes through B.

Lemma 4.1. *Let* $f : M^{\mathfrak{b}_2}(\lambda-\rho^{\mathfrak{b}_2}) \to M^{\mathfrak{b}_1}(\lambda-\rho^{\mathfrak{b}_1})$ *be a homomorphism such that* $\operatorname{Im} f$ *is simple. Then:*

1. $[\mathfrak{b}_2]$ *is* $[\mathfrak{b}_1]$*-geodesically maximal in* $L(m,n)_\lambda$.
2. $L^{\mathfrak{b}_2}(\lambda-\rho^{\mathfrak{b}_2})$ *is antidominant.*
3. *If* $\operatorname{Im} f \cong \operatorname{soc} M^{\mathfrak{b}_1}(\lambda-\rho^{\mathfrak{b}_1})$*, then* $[\mathfrak{b}_2]$ *is the unique* $[\mathfrak{b}_1]$*-geodesically maximal vertex in* $L(m,n)_\lambda$.

Proof. 1. Assume the contrary. Then there exists $\mathfrak{b}_3$ such that $[\mathfrak{b}_3] \neq [\mathfrak{b}_2]$ and some shortest walk from $[\mathfrak{b}_3]$ to $[\mathfrak{b}_1]$ passes through $[\mathfrak{b}_2]$. For each edge on this path there is a nonzero homomorphism between the corresponding Verma modules; composing them yields a nonzero map $g : M^{\mathfrak{b}_3}(\lambda-\rho^{\mathfrak{b}_3}) \to M^{\mathfrak{b}_1}(\lambda-\rho^{\mathfrak{b}_1})$. By Corollary 3.12 one has $\operatorname{Im} g \subsetneq \operatorname{Im} f$, so $\operatorname{Im} f$ contains a proper nonzero submodule, contradicting simplicity. 2. follows from Proposition 2.21. 3. follows from Corollary 3.12. ☐

Corollary 4.2. *If* $M^{\mathfrak{b}_1}(\lambda) \cong \left(M^{\mathfrak{b}_2}(\mu)\right)^\vee$*, then* $[\mathfrak{b}_1]$ *and* $[\mathfrak{b}_2]$ *are unique geodesically maximal each other in* $L(m,n)_{\lambda+\rho^{\mathfrak{b}_1}}$.

Proof. The composition

$$M^{\mathfrak{b}_2}(\mu) \twoheadrightarrow L^{\mathfrak{b}_2}(\mu) \cong \operatorname{soc} M^{\mathfrak{b}_1}(\lambda) \hookrightarrow M^{\mathfrak{b}_1}(\lambda)$$

is a nonzero homomorphism with simple image . Note that $chM^{\mathfrak{b}_1}(\lambda) = chM^{\mathfrak{b}_2}(\mu)$. By Lemma 4.1 (the "unique" clause), $[\mathfrak{b}_2]$ is the unique farthest vertex from $[\mathfrak{b}_1]$ in $L(m,n)_{\lambda+\rho^{\mathfrak{b}_1}}$. Swapping $\mathfrak{b}_1$ and $\mathfrak{b}_2$ yields the converse. ☐

4.2 A bridge in odd reflection graphs

Definition 4.3. An edge in a graph G is called a *bridge* if removing it increases the number of connected components:

The following is trivial.

Lemma 4.4. *Suppose an edge e in a finite simple graph G is a bridge and removing e yields two components C_1 and C_2 .. Fix any $c_1 \in C_1$. Then there exists $c_2 \in C_2$ such that c_2 is c_1-geodesically maximal.*

Lemma 4.5. *For any weight λ, in $L(m,n)_\lambda$ the vertex $[(n^m)]$ is the unique $[()]$-geodesically maximal.*

Proof. Let $\mathcal{C}_\lambda := \{\alpha \in \Delta_{\bar{1}}^{()} \mid (\lambda, \alpha) = 0\}$ be the color set of $L(m,n)_\lambda$. There exists a rainbow walk from $[()]$ to $[(n^m)]$ using each color in $\mathcal{C}_\lambda$ exactly once; its length is $|\mathcal{C}_\lambda|$. Clearly this is the longest rainbow walk in $L(m,n)_\lambda$. In $L(m,n)_\lambda$, the endpoint of a rainbow walk with a fixed starting vertex is determined solely by the *set* of colors used.

□

A weight λ (or $L^{()}(\lambda)$) is *integral* if $\lambda = \sum_{i=1}^m a_i\,\varepsilon_i + \sum_{j=1}^n b_j\,\delta_j$ with $a_i, b_j \in \mathbb{Z}$ for all i, j.

It is convenient to encode an integral weight λ by the $m \mid n$-tuple $(\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n})$ of integers defined by $\lambda_i := (\lambda + \rho, \varepsilon_i)$. Note that our $\rho^{\mathfrak{b}}$ is integral. Write the set of integral weights as Λ.

For an integral weight $\lambda = (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n})$, λ (or $L^{()}(\lambda)$) is *antidominant* iff $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ and $\lambda_{m+1} \ge \lambda_{m+2} \ge \cdots \ge \lambda_{m+n}$.

We call $\lambda = (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n})$ (or $L^{()}(\lambda)$) *regular* iff $\lambda_i \neq \lambda_j$ for all $1 \le i < j \le m$ and $m+1 \le i < j \le m+n$.

Definition 4.6 (Borel relabelling of a highest weight)**.** Let $L = L^{()}(\lambda)$ be a simple module. For each Borel subalgebra $\mathfrak{b}$, define $\lambda_{\mathfrak{b}} \in \mathfrak{h}^*$ by the condition

$$L^{\mathfrak{b}}(\lambda_{\mathfrak{b}} - \rho^{\mathfrak{b}}) \cong L^{()}(\lambda).$$

In particular, $\lambda_{()} = \lambda + \rho^{()}$.

Lemma 4.7. *Let $\lambda \in \Lambda$ is (regular) antidominant, then $\lambda_{\mathfrak{b}} - \rho^{()}$ is also (regular) antidominant.*

Proof. Fix a rainbow walk $[()] \xrightarrow{c_1} [\mathfrak{b}_1] \xrightarrow{c_2} \cdots \xrightarrow{c_k} [\mathfrak{b}]$, where each color c_i is identified with an odd root $\varepsilon_{p_i} - \delta_{q_i} \in \Delta_{\bar{1}}^{()+}$. Start from $\mu^{(0)} := \lambda = (\mu_1, \ldots, \mu_m \mid \mu_{m+1}, \ldots, \mu_{m+n})$, and for $i = 1, \ldots, k$ perform the operation O_i below to obtain $\mu^{(i)}$ from $\mu^{(i-1)}$:

Operation O_i: If $\mu_{p_i}^{(i-1)} = \mu_{m+q_i}^{(i-1)}$, replace the pair by

$$\big(\mu_p^{(i-1)},\, \mu_{m+q}^{(i-1)}\big) \;\mapsto\; \big(\mu_p^{(i-1)} + 1,\, \mu_{m+q}^{(i-1)} + 1\big),$$

and leave all other entries unchanged; otherwise do nothing.

Then $\mu^{(k)} = \lambda_{\mathfrak{b}} - \rho^{()}$. This operation clearly preserves (regular) antidominance at each step. □

Theorem 4.8. *Let $\mathfrak{b} \in L(m,n)$, $\lambda \in \Lambda$ and suppose $L^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}})$ is antidominant. Then the pair $\{[()], [(n^m)]\}$ in $L(m,n)_\lambda$ is the unique pair of vertices that are unique geodesically maximal each other.*

Proof. By Lemma 4.7 $L^{()}(\lambda - \rho^{()})$ is also antidominant. By the definition of antidominance, let i be maximal with $1 \leq i \leq m$ such that there exists j with $m+1 \leq j \leq m+n$ and $\lambda_i = \lambda_j$; choose such j minimal. Then for any k with $m+1 \leq k < j$ there is no l with $1 \leq l \leq m$ and $\lambda_k = \lambda_l$. This combinatorial condition implies that $L(m,n)_\lambda$ has a bridge whose deletion isolates the singleton component $\{[()]\}$.

By Lemma 4.4 , for any vertex $c \neq [()]$, the vertex $[()]$ is of c-geodesically maximal. By Lemma 4.5 , $[()]$ and $[(n^m)]$ are mutually unique geodesically maximal, and such a pair is also unique. ☐

Combining Lemma 4.1, Corollary 4.2, and Theorem 4.8 yields a proof of Theorem 1.1.

5 Further Study of odd reflection graphs

5.1 Totally disconnected weights

Let an integral weight $\lambda = (\lambda_1, \ldots, \lambda_m \mid \lambda_{m+1}, \ldots, \lambda_{m+n})$.

We say that λ (or $L^{()}(\lambda)$) is *dominant* iff $\lambda_1 \geq \cdots \geq \lambda_m$ and $\lambda_{m+1} \leq \cdots \leq \lambda_{m+n}$.

To each regular dominant weight $\lambda \in \Lambda$ (or $L^{()}(\lambda)$) we associate, following [7], its weight diagram as following.

$I_\times(\lambda) := \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and $I_\circ(\lambda) := \{\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{m+n}\}$. The integers in $I_\times(\lambda) \cap I_\circ(\lambda)$ are labelled by $\vee$, the remaining ones in $I_\times(\lambda)$ respectively $I_\circ(\lambda)$ are labelled by $\times$ respectively $\circ$. All other integers are labelled by a $\wedge$. This labelling of the number line $\mathbb{Z}$ uniquely characterizes the weight λ.

Proposition 5.1. *Let L be a regular dominant simple module. The following are equivalent.*

1. *The weight diagram of L contains at least one $\wedge$ between any two $\vee$'s.*
2. *For every Borel $\mathfrak{b}$, the graph $L(m,n)_{\lambda_\mathfrak{b}}$ is a line segment.*

Proof. Exactly the same as the proof of 4.7, mutatis mutandis. ☐

Definition 5.2. For λ , we say that λ (or $L^{()}(\lambda)$) is *totally disconnected* if for every Borel $\mathfrak{b}$, the graph $L(m,n)_{\lambda_\mathfrak{b}}$ is a line segment.

Remark 5.3. The notion of "totally disconnected" is defined in [16, 35] in the regular dominant case as item (1) of the proposition above. Our definition should be viewed as a natural extension to arbitrary weights.

Proposition 5.4. *regular antidominant integral weight is totally disconnected.*

Proof. Exactly the same as the proof of 4.7, mutatis mutandis. ☐

Remark 5.5. There exist totally disconnected antidominant weights that are non-regular. Indeed, for $\mathfrak{g} = \mathfrak{gl}(m|1)$, every weight is totally disconnected.

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