ORTHOGONAL ROOTS, QUANTUM HAFNIANS, AND GENERALIZED ROTHE DIAGRAMS

R.M. GREEN AND TIANYUAN XU

Abstract. Let U be a set of positive roots of type ADE, and let Ω_U be the set of all maximum cardinality orthogonal subsets of U. For each element $R \in \Omega_U$, we define a generalized Rothe diagram whose cardinality we call the level, $\rho(R)$, of R. We define the generalized quantum Hafnian of U to be the generating function of ρ, regarded as a q-polynomial in U. Several widely studied algebraic and combinatorial objects arise as special cases of these constructions, and in many cases, Ω_U has the structure of a graded partially ordered set with rank function ρ. A motivating example of the construction involves a certain set of k^2 roots in type D_{2k}, where the elements of Ω_U correspond to permutations in S_k, the generalized Rothe diagrams are the traditional Rothe diagrams associated to permutations, the level of a permutation is its length, the generalized quantum Hafnian is the q-permanent, and the partial order is the Bruhat order. We exhibit many other natural examples of this construction, including one involving perfect matchings, two involving labelled Fano planes, and one involving the invariant cubic form in type E_6.

1. Introduction

Let U be a subset of the positive roots of a Weyl group of type ADE, and let Ω_U be the set of orthogonal subsets of U of maximal cardinality. For each $R \in \Omega_U$, we define a *residue of R with respect to U* to be an element $\gamma \in U$ such that $s_\beta(\gamma)$ is positive for all $\beta \in R$, where s_β denotes the reflection associated to the vector β. The *generalized Rothe diagram* of R is the set of residues of R in U, which we denote by $\mathrm{Res}_U(R)$. We call $\rho(R) := |\mathrm{Res}_U(R)|$ the *level* of R, and we define the *generalized quantum Hafnian* of U to be the element of $\mathbb{Z}[U][q]$ given by

$$\sum_{R \in \Omega_U} q^{\rho(R)} \prod_{\beta_i \in R} x_{\beta_i},$$

where the x_β are commuting indeterminates indexed by the set U.

Although it is not *a priori* clear why one might want to study these definitions, which are new to the best of our knowledge, we will show that many choices for U give rise to concise constructions of known algebraic and combinatorial objects, in some cases endowing them with additional structure. For example, we will use our methods to associate a generalized Rothe diagram in a canonical way to each of the 45 terms of the invariant cubic polynomial in type E_6, which gives a nonrecursive combinatorial interpretation of the coefficients of the polynomial.

One of the main motivating examples of this phenomenon (Theorem 3.13) is the case where $U = \{\varepsilon_i + \varepsilon_{k+j} : 1 \leq i, j \leq k\}$ is a set of k^2 roots of type D_{2k}, corresponding to the positions (i, j) on a $k \times k$ grid. Here, the set Ω_U corresponds to the set of configurations of k nonattacking rooks, or equivalently, the elements of the symmetric group S_k. If we place the origin at the top left, then a vacant position (a, b) is a residue for $R \in \Omega_U$ unless it is attacked from above or from its left; see Figure 1 and Remark 3.14 (i). The set of such positions is a well-studied object known as the Rothe diagram of the permutation τ associated to R, and it is a well-known result that the cardinality of the Rothe diagram is equal to the length of τ. In this case, the level function is the

2020 *Mathematics Subject Classification.* Primary: 17B22; Secondary: 05E18, 20F55.

Key words and phrases. root system, Weyl group, Rothe diagram, quantum Hafnian, quasiparabolic set.

length of τ, the quantum Hafnian is the q-permanent of a $k \times k$ matrix A, and specializing q to $-q$ gives the q-determinant of A.

	1	2	3	4
1	★	●		
2	★		★	●
3	●			
4			●	

FIGURE 1. The Rothe diagram of $\underline{3142} \in S_4$, with residues marked by stars

Another motivating example (Theorem 3.10) is the case where $U = \{\varepsilon_i + \varepsilon_j : 1 \leq i < j < 2k\}$ is a set of $\binom{2k}{2}$ roots of type D_{2k} corresponding to the 2-subsets of the set $[2k] := \{1, 2, \ldots, 2k\}$. Here, the set Ω_U corresponds to the set of perfect matchings of the set $[2k]$, and a pair $\{a, b\}$ (where $1 \leq a < b \leq 2k$) is a residue of $R \in \Omega_U$ if and only if we have both $\tau(a) > b$ and $\tau(b) > a$, where τ is the fixed point free involution corresponding to the perfect matching. The residues in this case are the Rothe diagrams of fixed point free involutions as defined by Hamaker–Marberg–Pawlowski [9, (3.3)], the level function is a well-known statistic on perfect matchings, and the quantum Hafnian is precisely the q-Hafnian defined by Jing–Zhang [13]. Specializing q to $-q$ gives the quantum Pfaffian defined by Strickland [16], and specializing q to $+1$ and to -1 gives the Hafnian and Pfaffian of a matrix, respectively. Recall that the Pfaffian, $\mathrm{Pf}(A)$, of a skew-symmetric $2k \times 2k$ matrix A is a polynomial in the entries of A with the property that $(\mathrm{Pf}(A))^2 = \det(A)$.

The two examples just mentioned have the additional property that the level function ρ endows Ω_U with the structure of a quasiparabolic set in the sense of Rains–Vazirani [14]. This equips Ω_U with a W_I-action for some standard parabolic subgroup W_I of the Weyl group W, and with a graded partial order whose rank function is ρ. In the first example, this produces the Bruhat order on S_k.

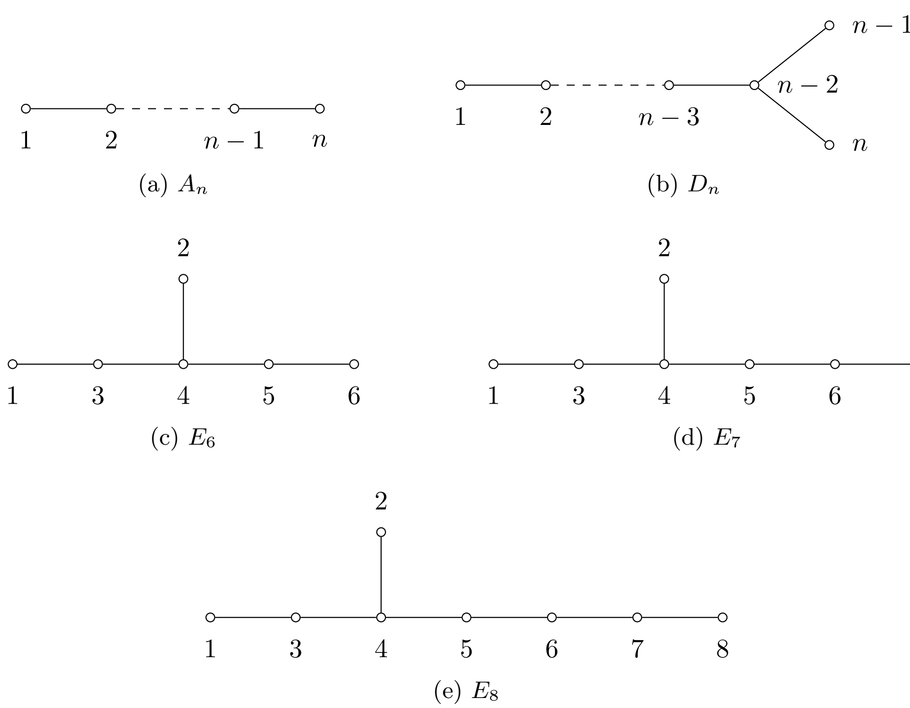
In Section 4, we construct many other examples of sets of roots U whose level function gives Ω_U the structure of a quasiparabolic set. Our construction seems to be an unreasonably effective method of producing a quasiparabolic set from a set of roots. For root systems of types D_{2k}, E_7, or E_8, we can form a quasiparabolic set by taking $U = \Phi_+$ to be the set of all positive roots (Theorem 4.3). We also discuss an abundance of examples of quasiparabolic sets Ω_U where $U = U(\Gamma, p)$ is a set of roots associated to a suitable simple root α_p in a simply laced root system. These include examples corresponding to every simple root in every root system of type A or D (Example 4.7, Remark 4.8 (i)), as well as a total of 12 examples in type E (Example 4.9 (i)). The residue sets for the examples of type A and D have interesting combinatorial interpretations in terms of rook configurations, generalizing the aforementioned motivating examples. One of the examples in type E_7, associated to the set $U = U(E_7, 2)$, corresponds in a precise way to the 30 inequivalent 7-labellings of the Fano plane, and one example in type E_8, $U = U(E_8, 2)$, corresponds to the 240 packings of the projective space $PG(3, 2)$.

In Section 5, we focus on two important examples arising from the construction of Section 4 in type E. The set U in these cases are given by a certain set $U(E_8, 8)$ of 56 roots in type E_8 and a certain set $U(E_7, 7)$ of 27 roots in type E_7, both of which appear naturally as the set of weights of a suitable minuscule representation. The set $U = U(E_7, 7)$ gives rise to a quasiparabolic set Ω_U whose elements correspond to tritangent planes on a del Pezzo surface of degree 3 and whose Pfaffian gives the well-known invariant cubic polynomial in type E_6 (Theorem 5.19). The quasiparabolic order on Ω_U endows the tritangent planes with a natural partial order and yields a non-recursive interpretation of the coefficients of the invariant polynomial, both of which are new to our knowledge.

The paper is organized as follows. Section 2 recalls the necessary background regarding the combinatorics of root systems. In Section 3, we introduce the key definitions and summarize the theory of quasiparabolic sets developed by Rains–Vazirani, and we give details of the two motivating examples. Section 4 exhibits many other examples of our construction, including the cases where U is the set of all positive roots of type E_7, E_8, or D_n for n even. Section 5 is an in-depth study of the two examples in type E mentioned in the last paragraph. Section 6 discusses some possible directions for further research.

2. Orthogonal roots

2.1. **Review of root systems.** We recall the necessary background concerning root systems in this subsection. While many definitions in this paper work more generally for all (finite) root systems, we will focus on irreducible simply laced root systems, i.e., on the root systems of types ADE. We label the vertices in the Dynkin diagrams of root systems as shown in Figure 2 until Section 5.2, where we will switch to a different convention in type E for reasons we will explain.



(a) A_n (b) D_n (c) E_6 (d) E_7 (e) E_8

Figure 2. Dynkin diagrams of irreducible simply laced Weyl groups

Let $\Phi = \Phi(\Gamma)$ be a simply laced root system with Dynkin diagram Γ. The vertices of Γ index the *simple roots* $\Pi = \{\alpha_i : i \in \Gamma\}$, and the *rank* of Φ is $\mathrm{rank}(\Phi) := |\Pi|$. The *root lattice* $\mathbb{Z}\Pi$ is the free $\mathbb{Z}$-module on Π, and it carries a symmetric $\mathbb{Z}$-bilinear form B given by

$$B(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } \ i = j; \\ -1 & \text{if } \ i \text{ and } j \text{ are adjacent in } \Gamma; \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

If $\alpha_i \in \Pi$, then the *simple reflection* s_i is the self-inverse $\mathbb{Z}$-linear operator $\mathbb{Z}\Pi \to \mathbb{Z}\Pi$ given by

$$s_i(\beta) = \beta - B(\alpha_i, \beta)\alpha_i. \tag{2.2}$$

The *Weyl group* $W = W(\Gamma)$ is the finite group generated by the set $S = S(\Gamma) = \{s_i : i \in \Gamma\}$, and we have $\Phi = \{w(\alpha_i) : \alpha_i \in \Pi, w \in W\}$. Each element of Φ is called a *root*, and each root gives rise to a *reflection*, which is the self-inverse linear map $s_\alpha : \mathbb{Z}\Pi \to \mathbb{Z}\Pi$ given by the formula

$$s_\alpha(\beta) = \beta - B(\alpha, \beta)\alpha, \tag{2.3}$$

generalizing Equation (2.2). We have $s_\alpha \in W$ for all roots $\alpha \in \Phi$. Moreover, the Weyl group acts transitively on the roots and the reflections $\{s_\alpha : \alpha \in \Phi\}$ form a single conjugacy class in W. The *reflection representation* of W is the $\mathbb{Q}W$-module $V = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}\Pi$, where each reflection acts by Equation (2.3). The simple roots form a basis of V, so that we have $\dim(V) = \mathrm{rank}(\Phi)$.

The pair (W, S) is a Coxeter system, and the set of reflections $\{s_\alpha : \alpha \in \Phi\}$ coincides with the set $T = \{wsw^{-1} : s \in S, w \in W\}$. For each subset $I \subseteq S$, the *standard parabolic subgroup* $W_I \leq W$ is the subgroup $\langle I \rangle$ generated by I.

Every root is of the form $\alpha = \sum_{i\in\Gamma} c_i\alpha_i$ for some integers c_i that are either all all nonnegative or all nonpositive. We say α is *positive* or *negative* in these cases, respectively. The sum of the coefficients $\sum_{i\in\Gamma} c_i$ is called the *height* of α. Given any $\alpha \in \Phi$, the only multiples of α in Φ are α and $-\alpha$.

There is a partial order on the root lattice $\mathbb{Z}\Pi$ defined by the condition that $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of simple roots, so that a root α is positive if and only if $\alpha > 0$ in this order. With respect to (the restriction of) this partial order, Φ has a unique maximal element, θ, called the *highest root*. We denote the set of positive roots $\{\alpha \in \Phi : \alpha > 0\}$ by Φ_+.

A *subsystem* of a root system Φ is a subset $\Psi \subseteq \Phi$ that is itself a root system. Equivalently, a subsystem of Φ is a nonempty subset that is closed under the operations $\{s_\alpha : \alpha \in \Psi\}$. The positive roots of Φ naturally induce a *positive system*, Ψ_+, and a *simple system*, Π_Ψ, in Ψ (in the sense of [10, Section 1.3]; see also [8, Section 2.2]). We have $\Psi_+ = \Psi \cap \Phi_+$, and Π_Ψ can be characterized as the set the elements of Ψ_+ that cannot be expressed as nonnegative linear combinations of other elements of Ψ_+. The elements of Π_Ψ may or may not be elements of Π, although it is true that $\Pi \cap \Psi \subseteq \Pi_\Psi$. We say that Ψ is of type Γ' if Ψ is isomorphic to a root system of type Γ'.

The bilinear form B on $\mathbb{Z}\Pi$ is nondegenerate. We have $B(\alpha, \beta) \in \{2, 1, 0, -1, -2\}$ for all roots $\alpha, \beta \in \Phi$, with $B(\alpha, \beta) = 2$ if and only if $\alpha = \beta$ ([10, §6.6]). We say two roots α and β are *orthogonal* if $B(\alpha, \beta) = 0$. The roots orthogonal to a fixed root in any root system Φ naturally form a subsystem, and it is known that when Φ has type E_8, E_7, E_6, and D_n for $n > 4$, such a subsystem has type E_7, D_6, A_5, and $A_1 + D_{n-2}$, respectively, for all roots in Φ ([7, Theorem 2.5]). The form B is W-invariant in the sense that $B(\alpha, \beta) = B(w(\alpha), w(\beta))$ for all $w \in W$ and all $\alpha, \beta \in \Phi$. In particular, the action of W on Φ preserves orthogonality.

The root systems of types A, D, and E have the following well-known coordinate representations.

Let $\varepsilon_1, \ldots, \varepsilon_n$ be the usual standard basis of the Euclidean space $\mathbb{R}^n$. The vectors $\{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}$ form a root system of type A_{n-1}. The simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. A root $\varepsilon_i - \varepsilon_j$ is positive if $i < j$ and negative if $i > j$. The highest root is $\varepsilon_1 - \varepsilon_n$. The Weyl group is isomorphic to the symmetric group S_n and acts by permuting the basis $\varepsilon_1, \ldots, \varepsilon_n$. The bilinear form B is the usual dot product. Two roots are orthogonal if and only if they have disjoint support, where by the support of any element $v = \sum_{i=1}^n c_i\varepsilon_i \in \mathbb{R}^n$ we mean the set $\{i : c_i \neq 0\}$.

The vectors $\{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$ form a root system of type D_n. The simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$ and $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$. If $i < j$ then the roots $\varepsilon_i \pm \varepsilon_j$ are positive, and the roots $-\varepsilon_i \pm \varepsilon_j$ are negative. The highest root is $\varepsilon_1 + \varepsilon_2$. The

Weyl group has order $2^{n-1}n!$ and acts by signed permutations of the orthonormal basis, with the restriction that each element effects an even number of sign changes. The bilinear form B is the usual dot product. Two roots α and β are orthogonal if and only if either (a) α and β have disjoint support or (b) α and β have the same support and $\alpha \neq \pm\beta$.

Note that the root system of type A_{n-1} may be naturally identified as the subsystem $\Phi(D_n) \cap \mathbb{Z}(\Pi \setminus \{\alpha_n\})$ of $\Phi(D_n)$, and $W(A_{n-1})$ as the parabolic subgroup W_I of the Weyl group $W = W(D_n)$ where $I = S(D_n) \setminus \{s_n\} = \{s_1, \ldots, s_{n-1}\}$.

The root system of type E_8 may be regarded as a subset of $\mathbb{R}^8$ as follows. There are 112 roots of the form $\pm 2(\varepsilon_i \pm \varepsilon_j)$ where $1 \leq i \neq j \leq 8$, and there are 128 roots of the form $\sum_{i=1}^{8} \pm\varepsilon_i$, where the signs are chosen so that the total number of $-$ is even. The simple roots are $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_8\}$, where

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8,$$

$\alpha_2 = 2(\varepsilon_1 + \varepsilon_2)$, and $\alpha_i = 2(\varepsilon_{i-1} - \varepsilon_{i-2})$ for all $3 \leq i \leq 8$. If k is the largest integer such that ε_k appears in α with nonzero coefficient c, then α is positive if and only if $c > 0$. The highest root, θ_8, is $2(\varepsilon_7 + \varepsilon_8)$, and it decomposes as the sum

$$\theta_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8. \tag{2.4}$$

The bilinear form B is 1/4 of the usual dot product. The highest root θ_8 has the property that $B(\theta_8, \alpha_8) = 1$ and $B(\theta_8, \alpha_i) = 0$ for all $1 \leq i \leq 7$.

If $\Phi = \Phi(E_8)$ is the root system of type E_8 constructed above, then the root system $\Phi(E_7)$ of type E_7 may be obtained from Φ as a subsystem, namely, as the set $\Phi \cap \mathbb{Z}(\Pi \setminus \{\alpha_8\})$; the highest root in $\Phi(E_7)$ is $2(\varepsilon_8 - \varepsilon_7)$, and it decomposes as the sum

$$\theta_7 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7. \tag{2.5}$$

Similarly, the root system $\Phi(E_6)$ of type E_6 can be identified as the set $\Phi \cap \mathbb{Z}(\Pi \setminus \{\alpha_8, \alpha_7\})$.

Definition 2.1. Let Φ be any root system with Dynkin diagram and let $\Pi = \{\alpha_i : i \in \Gamma\}$ be the simple roots of Φ. For each root $\alpha = \sum_{i \in \Gamma} c_i \alpha_i$ and each $i \in \Gamma$, we define the *i-height*, denoted $\mathrm{ht}_i(\alpha)$, to be the coefficient c_i. For all $i \in \Gamma$ and $j \in \mathbb{Z}_{\geq 0}$, we define

$$\Phi_{i,j} = \{\alpha \in \Phi_+ : \mathrm{ht}_i(\alpha) = j\}.$$

2.2. **Orthogonal sets of a fixed size.** We introduce the notion of k-roots and a natural Weyl group action on them in this subsection.

Definition 2.2. Let Φ be a root system with Weyl group W, and let k be a positive integer.

(i) For each root $\alpha \in \Phi$, we define $|\alpha|$ to be the positive root in the set $\{\alpha, -\alpha\}$.

(ii) We define a *k-root* of Φ (or of W) to be a set of k mutually orthogonal roots in Φ, and we say a k-root is *positive* if all its elements are positive. We denote the set of all k-roots by Φ^k and the set of all positive k-roots by Φ^k_+. More generally, for any $U \subseteq \Phi$ we define U^k and U^k_+ to be the sets of k-roots and positive k-roots whose elements all lie in U, respectively.

(iii) We define

$$|R| = \{|\alpha| : \alpha \in R\}$$

for each k-root $R \in \Phi^k$, and we define the *standard action* of W on the positive k-root to be the W-action given by

$$w \cdot R = |\{w \cdot \alpha : \alpha \in R\}| = \{|w \cdot \alpha| : \alpha \in R\} \tag{2.6}$$

for all $w \in W$ and $R \in \Phi^k_+$.

Note that Equation (2.6) indeed defines a W-action on Φ_+^k. Also note that since orthogonal sets in R are linearly independent in the reflection representation V, k-roots cannot exist for $k > \dim(V) = \mathrm{rank}(\Phi)$. The notion of k-roots generalize the notions of 2-roots and n-roots studied in [7, 8], where $n = \mathrm{rank}(\Phi)$; see [7, Remark 3.7] and [8, Section 1].

Lemma 2.3. *Let Φ be an irreducible simply laced root system of rank n, and let W be the Weyl group of Φ.*

(i) *There exists an n-root of Φ (i.e., the set Φ^n is nonempty) if and only Φ is of type E_7, E_8, or D_n for some even integer n. When this is the case, every set of mutually orthogonal roots of Φ can be extended to an n-root.*

(ii) *If Φ has type E_7, E_8, or D_n for n even, then the element-wise action of W on Φ^n given by*

$$w \cdot \{\alpha_1, \dots, \alpha_n\} = \{w \cdot \alpha_1, \dots, w \cdot \alpha_n\}$$

is transitive, and so is the standard action of W on the positive n-roots Φ_+^n given by Equation (2.6).

Remark 2.4. (i) By a perfect matching of the set $[n] := \{1, 2, \dots, n\}$ for an even integer $n = 2k$, we mean a partition of the form $M = \{\{i_1, j_1\}, \dots, \{i_k, j_k\}\}$ where $\{i_1, j_1, \dots, i_k, j_k\} = \{1, 2, \dots, 2k\}$.

(ii) From now on, whenever we speak of "n-roots" of a root system Φ we will assume that Φ is a root system of rank n and type E_7, E_8, or D_n for n even. By Lemma 2.3 (i), in these cases the n-roots coincide with the maximal sets (with respect to set inclusion) of pairwise orthogonal roots, as well as with the bases of the reflection representation V that consist of pairwise orthogonal roots. We will thus use the terms "n-roots", "maximal orthogonal set", and "orthogonal basis" interchangeably.

(iii) The fact that any general k-root can be extended to an n-root makes n-roots very useful, as will be evident in Sections 4 and 5.

Proof of Lemma 2.3. Part (i) is well known, and can be proved similarly to [8, Lemma 3.2] by an inductive argument. The first assertion of (ii) is [8, Lemma 3.2], and the second assertion then follows immediately. ☐

The following example on the 4-roots in type D_4 turns out to be crucial for the general theory of n-roots.

Example 2.5. The root system $\Phi = \Phi(D_4)$ of type D_4 has exactly three positive 4-roots, namely, the sets

$$Q_1 = \{\varepsilon_1 - \varepsilon_2,\ \varepsilon_1 + \varepsilon_2,\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_3 + \varepsilon_4\}, \quad Q_2 = \{\varepsilon_1 - \varepsilon_3,\ \varepsilon_1 + \varepsilon_3,\ \varepsilon_2 - \varepsilon_4,\ \varepsilon_2 + \varepsilon_4\},$$

and

$$Q_3 = \{\varepsilon_1 - \varepsilon_4,\ \varepsilon_1 + \varepsilon_4,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_2 + \varepsilon_3\}.$$

They correspond to the perfect matchings $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$, respectively. The reflection s_α for the root $\alpha = \varepsilon_2 - \varepsilon_3$ acts on roots as the transposition $(2, 3)$ on indices, so it interchanges Q_1 and Q_2 while fixing Q_3 in the standard action of the Weyl group on Φ_+^4.

Note that the 4-roots Q_1, Q_2, Q_3 form a partition of Φ_+. Also note that decomposing each root as a sum of simple roots yields

$$Q_1 = \{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}, \quad Q_2 = \{\alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_4,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\},$$

and

$$Q_3 = \{\alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_4,\ \alpha_2 + \alpha_3 + \alpha_4\}.$$

We will use these facts in Section 2.3.

2.3. **Coplanar quadruples and D_4-subsystems.** Let Φ be a root system of type E_7, E_8, or D_n for n even, so that n-roots exist for Φ, and let W be the Weyl group of Φ. In [8], we showed that certain quadruples of orthogonal roots called coplanar quadruples are key to understanding the n-roots of Φ and the action of W on them. These quadruples remain useful for understanding the more general k-roots in this paper, so we further develop their properties in this subsection.

Definition 2.6. Let Φ be a root system of type E_7, E_8, or D_n for n even. We define a set of four distinct positive roots $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\} \subseteq \Phi$ to be a *coplanar quadruple* if $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\beta$ for some root $\beta \in \Phi$.

Example 2.7. The 4-roots Q_1, Q_2 and Q_3 of type D_4 from Example 2.5 all sum to twice a root, namely, twice the roots $\beta_1 = \varepsilon_1 + \varepsilon_3, \beta_2 = \varepsilon_1 + \varepsilon_2$, and $\beta_3 = \varepsilon_1 + \varepsilon_2$, respectively, so they are all coplanar quadruples.

Although every orthogonal quadruple in type D_4 is coplanar, this is not true in general. For example, the four orthogonal simple roots $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$ in type D_6 do not form a coplanar quadruple.

Coplanar quadruples are intimately connected to D_4-subsystems of Φ. To make this precise, it is helpful to have the following definition.

Definition 2.8. Let Φ be a root system of type E_7, E_8, or D_n for n even. Let $R \subset \Phi$ be a set of mutually orthogonal positive roots of Φ. For each integer $4 \leq k \leq |R|$, we define $D_k(R)$ to be the set

$$D_k(R) = \{H \subseteq R : |H| = k \text{ and } \mathrm{Span}(H) \cap \Phi \text{ is a subsystem of type } D_k \text{ in } \Phi\}.$$

We define $\mathcal{Q}(R)$ to be the set of all D_4-subsystems Ψ of Φ such that $\Psi \cap R \in D_4(R)$.

The following proposition expands [8, Proposition 3.3] and details many useful properties of coplanar quadruples.

Proposition 2.9. *Let Φ be a root system of type E_7, E_8, or D_n for $n \geq 4$ even, let V be the reflection representation of the Weyl group of Φ, and let $R \in \Phi_+^n$ be an orthogonal basis of V.*

(i) *If $\beta \in \Phi_+ \setminus R$, then β is orthogonal to all but four elements $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ of R. Moreover, in this case we have $\beta = (\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)/2$ for suitable choices of sign; the set*

$$\Psi_Q := Q \cup (-Q) \cup \{(\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4)/2\} \tag{2.7}$$

is a subset of Φ that forms a subsystem of type D_4; and Ψ_Q is the smallest subsystem of Φ containing both Q and β.

(ii) *The following are equivalent for a subset $Q \subseteq R$:*
  (a) *Q is a coplanar quadruple, i.e., we have $\sum_{\beta \in Q} \beta = 2\alpha$ for some root α;*
  (b) *Q is the support of some root $\alpha \in \Phi \backslash (R \cup -R) \subseteq V$ with respect to the basis R;*
  (c) *$Q \in D_4(R)$;*
  (d) *$Q = \Psi \cap R$ for some $\Psi \in \mathcal{Q}(R)$.*

(iii) *The maps*

$$\phi : D_4(R) \to \mathcal{Q}(R),\ Q \mapsto \mathrm{Span}(Q) \cap R$$

and

$$\psi : \mathcal{Q}(R) \to D_4(R),\ \Psi \mapsto \Psi \cap R$$

are mutually inverse maps. Every root $\alpha \in \Psi \setminus (R \cup -R)$ lies in $\mathrm{Span}(Q)$ for a unique element $Q \in D_4(R)$, namely, the support of α.

Remark 2.10. For each maximal orthogonal set of roots R, part (ii) of Proposition 2.9 shows that the elements of $D_4(R)$ are precisely the coplanar quadruples in R, and part (iii) establishes natural bijections between the coplanar quadruples and the D_4-subsystems of Φ intersecting R in an element of $D_4(R)$. Part (ii) also shows that each positive root α not in R naturally gives rise to a coplanar quadruple in R, namely, the support of α with respect to R.

Proof of Proposition 2.9. Part (i) follows from [8, Proposition 3.8] and the first paragraph of its proof.

To prove (ii), first note that (a) immediately implies (b). Suppose that α and Q satisfy the conditions of (b). Since R is orthogonal, for each root $\gamma = \sum_{\beta_i \in R} c_i\beta_i \in \Phi$ we have $c_i = B(\gamma, \beta_i)/2$, so the support of γ with respect to R consists precisely of the elements in R not orthogonal to R. It follows from (i) that Q is a quadruple $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$, and that the D_4-subsystem Ψ_Q defined in Equation (2.7) is contained in $\mathrm{Span}(Q) \cap \Phi$. Conversely, for any root $\gamma = c_1\beta_1 + c_2\beta_2 + c_3\beta_3 + c_4\beta_4 \in \mathrm{Span}(Q)$, we have

$$2 = B(\gamma, \gamma) = 2c_1^2 + 2c_2^2 + 2c_3^2 + 2c_4^2$$

and

$$c_i = B(\gamma, \beta_i)/2 \in \{0, \pm 1, \pm(1/2)\}$$

for each i by Section 2.2, so either we have $c_i = 1$ and $\gamma = \pm\beta_i$ for some i, or we have $c_i = \pm(1/2)$ for all i. It follows that $\mathrm{Span}(Q) \cap \Phi = \Psi_Q$, which proves (c). The same argument also shows that if a root γ not in R lies in $\mathrm{Span}(Q)$ for a quadruple $Q \subseteq R$, then Q is necessarily the support of γ, which proves the last sentence in (iii).

Now assume the hypothesis of (c), so that $\Psi = \mathrm{Span}(Q) \cap \Phi$ is a D_4-subsystem of Φ. This implies that $Q \subseteq \Psi \cap R$, and that $\mathrm{Span}(\Psi)$ is a 4-dimensional subspace of V. Since R is an orthogonal basis for V, we must have $|\Psi \cap R| \leq 4 = |Q|$, proving (d).

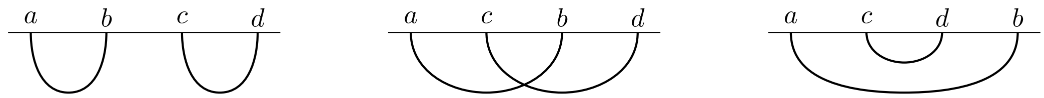
Finally, assume the hypothesis of (d), so that $Q = \Psi \cap R$. Any orthogonal quadruple $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ in a D_4-subsystem has the property that $\gamma = (\beta_1 + \beta_2 + \beta_3 + \beta_4)/2$ is a root, which proves (a) and completes the proof of (ii).

The map ψ sends $\mathcal{Q}(R)$ to $D_4(R)$ since (d) implies (c) in (ii), and the proof of the fact that (c) implies (d) shows both that ϕ sends $D_4(R)$ to $\mathcal{Q}(R)$ and that $\psi \circ \phi$ is the identity map. For each $\Psi \in \mathcal{Q}(R)$ and the quadruple $Q = \Psi \cap R$, the proof of the fact that (b) implies (c) shows that any positive root $\alpha \in \Psi \setminus Q$ must satisfy $B(\alpha, \beta) = \pm 1/2$ for all β and have Q as its support, and that $\phi(Q) = \mathrm{Span}(Q) \cap \Phi = \Psi_Q = \Psi$, where the last equality holds by (i). It follows that $\phi \circ \psi$ is the identity map on $\mathcal{Q}(R)$, and therefore ϕ and ψ are mutual inverses. $\square$

As in [8], it will be useful in this paper to further categorize coplanar quadruples into three mutually exclusive types. The definition of these types come from the combinatorics of perfect matchings.

Definition 2.11. For any two blocks $\{a, b\}$ and $\{c, d\}$ in a perfect matching M such that $a < b$ and $c < d$, we call the pair $P = \{\{a, b\}, \{c, d\}\}$ an *alignment* if $a < b < c < d$, a *crossing* if $a < c < b < d$, and a *nesting* if $a < c < d < b$; see Figure 3. We also call each alignment, crossing, or nesting a *feature*.

It follows from Definition 2.11 that any pair of blocks in a perfect matching forms exactly one type of feature. For example, the matchings $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$ corresponding to the coplanar quadruples Q_1, Q_2, Q_3 in the root system of type D_4 from Example 2.7 form an alignment, crossing, and nesting, respectively. Note that by Example 2.5, the quadruples Q_1, Q_2 and Q_3 can be distinguished by the numbers of simple roots in them, because they contain $3, 0$, and 1 simple root(s), respectively. This motivates the following definition, which extends the notions of alignments, crossings, and nestings to the context of orthogonal roots.



(a) alignment (b) crossing (c) nesting

FIGURE 3. Features formed by two blocks $\{a, b\}, \{c, d\}$ in a perfect matching

Definition 2.12. Let Q be a coplanar quadruple in Φ and let $\Psi = \mathrm{Span}(Q) \cap \Phi$ be the corresponding subsytem of type D_4. Let Π_Ψ be the induced simple system of Ψ. We call Q an *alignment* if $|Q \cap \Pi_\Psi| = 3$, a *crossing* if $|Q \cap \Pi_\Psi| = 0$, and a *nesting* if $|Q \cap \Pi_\Psi| = 1$. We also call each alignment, crossing, or nesting a *feature*.

Remark 2.13. Definition 2.12 resembles but differs from [8, Definition 3.9], where the classification of coplanar quadruples into the three types of possible features is characterized in a less concise way. However, the definitions are equivalent: in the notation of Example 2.7, each coplanar quadruple is necessarily one of Q_1, Q_2 and Q_3, and both definitions classify Q_1 as an alignment, Q_2 as a crossing, and Q_3 as an alignment. In other words, both definitions guarantee that each coplanar quadruple Q is an alignment, crossing, or nesting if and only if the perfect matching corresponding to it is a feature of the same type under the usual coordinates for the D_4-subsystem $\Psi = \mathrm{Span}(Q) \cap \Phi$ associated to Q.

The next result discusses how coplanar quadruple may interact in a fixed n-root. For any integer $n > 4$, we define two positive roots $\alpha, \beta \in \Phi_+$ in a root system of type D_n to be a *couple* if any $\gamma \in \Phi_+ \backslash \{\alpha, \beta\}$ is orthogonal to α if and only if γ is orthogonal to β. In terms of coordinates, two positive roots form a couple if and only if they are of the form $\{\varepsilon_i + \varepsilon_j,\ \varepsilon_i - \varepsilon_j\}$ for some $1 \leq i < j \leq n$.

Lemma 2.14. *Let Φ be a root system of type E_7, E_8, or D_n for n even. Let $R \in \Phi_+^n$ be a positive n-root, let Q and Q' be features in R such that $Q \neq Q'$ and $Q \cap Q' \neq \emptyset$, and define $H = Q \cup Q'$.*

- (i) *We have $|H| = 6$, and $\Xi := \mathrm{Span}(H) \cap \Phi$ is a subsystem of type D_6. In other words, we have $H \in D_6(R)$.*
- (ii) *The sets Q, Q', $Q \cap Q'$, and $Q \cup Q'$ all consist of couples in Ξ.*
- (iii) *If Φ has type E_8, then $R \setminus Q$ is also a coplanar quadruple.*

Proof. Part (i) and (ii) a restatement of [8, Proposition 3.20]. Part (iii) follows from [8, Lemma 3.18], which states that the features of R form the blocks of a Steiner quadruple system $S(3, 4, 8)$, meaning that every 3-element subset of R is contained in a unique feature. More precisely, there is a unique Steiner quadruple system on 8 points ([8, Remark 3.16]) and its blocks are closed under taking complements, so (iii) follows. □

3. Residues and quasiparabolic sets

In Section 3.1, we establish the key new definitions of residues, Rothe diagrams, the level statistic, as well as generalized Hafnians and Pfaffians and their q-analogues. We also recall the notion of a quasiparabolic set, due to Rains and Vazirani. Then, in Section 3.2, we give details of two of the motivating examples of residues: one based on perfect matchings of a set of size $2k$ (Theorem 3.10), and one based on permutations of k objects (Theorem 3.13).

3.1. Key definitions.

The following definition introduces the key notion of residues, which applies to all root systems. This notion is new to the best of our knowledge.

Definition 3.1. Let Φ be an arbitrary root system, and let $U \subseteq \Phi_+$.

(i) We define Ω_U to be the set of all κ-roots of U, where κ is the maximum cardinality of a pairwise orthogonal subset of U.

(ii) For any orthogonal set $R \in \Omega_U$, we define the set of *residues of R with respect to U* to be

$$\mathrm{Res}_U(R) = \{\gamma \in U : s_\beta(\gamma) > 0 \text{ for all } \beta \in R\},$$

and we will also refer to $\mathrm{Res}_U(R)$ as the *(generalized) Rothe diagram of R with respect to U*. We define $\rho_U(R) := |\mathrm{Res}_U(R)|$ to be the *level of R with respect to U*.

(iii) We identify an element $R = \{\beta_1, \beta_2, \ldots, \beta_k\} \in \Omega_U$ with the monomial $\prod_{\beta \in R} x_\beta$, where $\{x_\beta : \beta \in U\}$ is a set of commuting indeterminates. The *(generalized) quantum Hafnian of U*, $\mathrm{QHf}(U) = \mathrm{QHf}(U, q)$, is the element of $\mathbb{Z}[U][q]$ given by

$$\sum_{R \in \Omega_U} q^{\rho(R)} \prod_{\beta_i \in R} x_{\beta_i}.$$

The specializations $\mathrm{QPf}(U) := \mathrm{QHf}(U, -q)$, $\mathrm{Hf}(U) := \mathrm{QHf}(U, 1)$, and $\mathrm{Pf}(U) := \mathrm{QHf}(U, -1)$ are called the *(generalized) quantum Pfaffian*, the *(generalized) Hafnian*, and the *(generalized) Pfaffian* of U, respectively.

(iv) We define the *Poincaré polynomial* of U to be the polynomial

$$PS_U(q) := \sum_{R \in \Omega_U} q^{\rho(R)} \in \mathbb{Z}[q]$$

obtained from $\mathrm{QHf}(U)$ by specializing all the indeterminates x_β to 1. If $PS_U(q)$ has the form $\prod_{d \in D} [d]_q$ for some multiset D of elements from $\mathbb{N} \backslash \{0, 1\}$, where $[d]_q$ is the quantum integer

$$[d]_q = \frac{q^d - 1}{q - 1} = 1 + q + q^2 + \ldots + q^{d-1},$$

then we call the elements of D the *degrees* of U.

Remark 3.2. (i) Since $s_\gamma(\gamma) = -\gamma < 0$ for any positive root γ in a root system, it follows that $\mathrm{Res}_U(R)$ cannot contain any element of R in the setting of Definition 3.1.

(ii) Factorization of a polynomial in $\mathbb{Z}[q]$ into quantum integers is unique up to order when possible, so the multiset D in Definition 3.1 (iv) is well-defined.

Example 3.3. Let Φ be the root system of type D_4, whose highest root is $\theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, and define $U = \Phi_+$. There are three maximal orthogonal subsets of positive roots in U, namely,

$$\begin{aligned} Q_1 &= \{\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}, \\ Q_2 &= \{\alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_4,\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}, \quad \text{and} \\ Q_3 &= \{\alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_4,\ \alpha_2 + \alpha_3 + \alpha_4\}. \end{aligned}$$

Direct computation shows that

$$\mathrm{Res}_U(Q_1) = \emptyset, \quad \mathrm{Res}_U(Q_2) = \{\theta\} = \{\varepsilon_1 + \varepsilon_2\}, \quad \text{and} \quad \mathrm{Res}_U(Q_3) = \{\theta - \alpha_2,\ \theta\} = \{\varepsilon_1 + \varepsilon_2,\ \varepsilon_1 + \varepsilon_3\},$$

from which it follows that $\rho_U(Q_1) = 0$, $\rho_U(Q_2) = 1$, and $\rho_U(Q_3) = 2$.

Using the coordinate system for roots of type D_4, and writing u_{ij} as shorthand for $x_{\varepsilon_i + \varepsilon_j} x_{\varepsilon_i - \varepsilon_j}$, we have

$$\begin{aligned} \mathrm{QHf}(U) &= u_{12}u_{34} + q u_{13}u_{24} + q^2 u_{14}u_{23}, \\ \mathrm{QPf}(U) &= u_{12}u_{34} - q u_{13}u_{24} + q^2 u_{14}u_{23}, \\ \mathrm{Hf}(U) &= u_{12}u_{34} + u_{13}u_{24} + u_{14}u_{23}, \quad \text{and} \\ \mathrm{Pf}(U) &= u_{12}u_{34} - u_{13}u_{24} + u_{14}u_{23}. \end{aligned}$$

It follows that $PS_{\Phi_+}(q) = 1 + q + q^2$, so that 3 is the only degree of U.

Note that the sets Q_i for $1 \leq i \leq 3$ are exactly the three coplanar quadruples partitioning Φ_+ from Example 2.7, with Q_1 being an alignment, Q_2 a crossing, and Q_3 a nesting. The highest root θ and the second highest root $\theta - \alpha_2$ are the only roots that may appear in $\mathrm{Res}_U(R)$ for any $R \in \Omega_U$. Note also that if α is a residue and $\alpha \in \{\theta, \theta - \alpha_2\}$, then α has full support with respect to any of the orthogonal bases $Q_i = \{\beta_1, \beta_2, \beta_3, \beta_4\}$.

The next definition considers an important special case of Definition 3.1 (ii). Note that by Section 2.1, if R is a k-root of a root system Φ, then the set $\Psi := \mathrm{Span}(R) \cap \Phi$ is a subsystem of Φ, and the positive roots Φ_+ of Φ induce a positive system $\Psi_+ = \Psi \cap \Phi_+ = \mathrm{Span}(R) \cap \Phi_+$ of Ψ.

Definition 3.4. Let Φ be an arbitrary root system, let $R \subseteq \Phi_+$ be a k-root of Φ, and let $\Psi_R = \mathrm{Span}(R) \cap \Phi$ be the subsystem of Φ induced by R. We define the set of *residues of R* to be the set

$$\mathrm{Res}(R) = \mathrm{Res}_{\Psi_+}(R),$$

where $\Psi_+ = \mathrm{Span}(R) \cap \Phi_+$ is the positive system of Ψ induced by Φ_+. We refer to $\mathrm{Res}(R)$ as the *(generalized) Rothe diagram of R*, and we define $\rho(R) := |\mathrm{Res}(R)|$ to be the *level of R*.

Remark 3.5. (i) It follows from Definitions 3.1 (ii) and 3.4 that for any k-root R of a root system Φ, we have $\mathrm{Res}(R) = \mathrm{Res}_{\Phi_+}(R) \cap \mathrm{Span}(R)$. In particular, if R is an n-root where $n = \mathrm{rank}(\Phi)$, then $\mathrm{Res}(R) = \mathrm{Res}_{\Phi_+}(R)$ and $\rho(R) = \rho_{\Phi_+}(R)$.

(ii) In the setting of Example 3.3, we have $\mathrm{Res}(Q_i) = \mathrm{Res}_{\Phi_+}(Q_i) = \mathrm{Res}_U(Q_i)$ and $\rho(Q_i) = \rho_{\Phi_+}(Q_i) = \rho_U(Q_i)$ for each i. Since the calculation of the residue sets in this example depends only on the choice of a simple system for a type D_4 root system, it follows that whenever Q is a feature in a root system of type E_7, E_8 or D_n for n even, we have $\rho(Q) = 0$ if Q is an alignment, $\rho(Q) = 1$ if Q is a crossing, and $\rho(Q) = 2$ if Q is a nesting.

Quasiparabolic sets were introduced by Rains and Vazirani for a general Coxeter system as follows.

Definition 3.6. [14, Section 2, Section 5] Let (W, S) be a Coxeter system with set of reflections T. A *scaled W-set* is a pair (X, λ) where X is a W-set and $\lambda : X \to \mathbb{Z}$ is a function satisfying $|\lambda(sx) - \lambda(x)| \leq 1$ for all $s \in S$. An element $x \in X$ is *W-minimal* if $\lambda(sx) \geq \lambda(x)$ for all $s \in S$ and is *W-maximal* if $\lambda(sx) \leq \lambda(x)$ for all $s \in S$.

A *quasiparabolic set for W*, or *quasiparabolic W-set*, is a scaled W-set X satisfying the following two properties:

(QP1) for any $r \in T$ and $x \in X$, if $\lambda(rx) = \lambda(x)$, then $rx = x$;
(QP2) for any $r \in T$, $x \in X$, and $s \in S$, if $\lambda(rx) > \lambda(x)$ and $\lambda(srx) < \lambda(sx)$, then $rx = sx$.

For a quasiparabolic set X, we define the *quasiparabolic order* on X, $\leq_Q$, to be the weakest partial order such that $x \leq_Q rx$ whenever $x \in X$, $r \in T$, and $\lambda(x) \leq \lambda(rx)$.

Rains and Vazirani call $\lambda(x)$ the *height* of x and $\leq_Q$ the *Bruhat order*, but we will refer to them as the *level* of x and the *quasiparabolic order* because of the potential for confusion in the context of this paper.

For any quasiparabolic W-set (X, λ), it follows from [14, Corollary 2.10] that each W-orbit in X contains at most one W-minimal element, so if X is a finite and transitive W-set then X has a unique W-minimal element, x_0. Moreover, when this is the case, the level of any element $x \in X$ equals the minimum length of the elements $w \in W$ such that $w(x_0) = x$ by [14, Corollary 2.13].

We may associate a quantum Hafnian and a Poincaré polynomial to a general quasiparabolic set by defining $\mathrm{QHf}(X) = \sum_{x \in X} q^{\rho(R)} x$ and $PS_X(q) = \sum_{x \in X} q^{\rho(x)}$. However, we will not require this level of generality, because our focus for the rest of the paper will be on quasiparabolic sets that arise from sets of orthogonal roots via the following construction.

Definition 3.7. Let W be a finite Weyl group with root system Φ and generating set S, let $I \subseteq S$, and let $U \subseteq \Phi_+$. We say that (W, I, U) is a *quasiparabolic triple*, or *QP-triple*, if the set Ω_U is a quasiparabolic set for W_I under the restriction of the standard action (Definition 2.2) and level function $\lambda(R) := \rho_U(R)$. A QP-triple (W, I, U) is *transitive* if Ω_U is a transitive quasiparabolic W_I-set.

3.2. **Motivating examples.** We now present the two motivating examples of residues: one based on perfect matchings, and the other based on permutations. The following definition will be helpful in the context of the perfect matching example.

Definition 3.8. Let $M = \{\{i_1, j_1\}, \ldots, \{i_k, j_k\}\}$ be a perfect matching of the set $[2k]$, where we have $k \geq 2$ and $i_1 < \ldots < i_k$ and $i_t < j_t$ for all $1 \leq t \leq k$. Following [12, §2], we define $\sigma_M \in S_{2k}$ to be the permutation whose two-line notation is given by

$$\sigma_M = \begin{bmatrix} 1 & 2 & \ldots & 2k-1 & 2k \\ i_1 & j_1 & \ldots & i_k & j_k \end{bmatrix}.$$

We define $\tau_M \in S_{2k}$ to be the fixed point free involution whose disjoint cycle notation is given by

$$\tau_M = (i_1, j_1)(i_2, j_2) \ldots (i_k, j_k).$$

Let

$$I_M = \{(i, j) : 1 \leq i < j \leq 2k \text{ and } \sigma_M(i) > \sigma_M(j)\}$$

be the set of inversions of σ_M, and define

$$D_M = \{(i, j) : 1 \leq i < j \leq 2k \text{ and } \tau_M(i) > j \text{ and } \tau_M(j) > i\}.$$

Lemma 3.9. *Maintain the notation of Definition 3.8. There is a bijection $\psi_M : I_M \to D_M$ given by*

$$\psi_M((i, j)) = (\tau_M \sigma_M(i), \sigma_M(j)).$$

Proof. For any integer $1 \leq e \leq k$, the definitions of σ_M and I_M imply that $(2e-1, 2e) \notin I_M$, and the definitions of τ_M and D_M imply that $(\sigma_M(2e-1), \sigma_M(2e)) \notin D_M$. It follows that every pair $(i, j) \in I_M$ corresponds to two distinct integers $1 \leq e < f \leq k$ such that $i \in \{2e-1, 2e\}$ and $j \in \{2f-1, 2f\}$, and every pair $(i, j) \in I_M$ consists of elements from two blocks $\{\sigma_M(2e-1), \sigma_M(2e)\}$ and $\{\sigma_M(2f-1), \sigma_M(2f)\}$ for such two distinct integers $1 \leq e < f \leq k$.

Suppose we have $1 \leq e < f \leq k$, and let $a := \sigma_M(2e-1)$, $b := \sigma_M(2e)$, $c := \sigma_M(2f-1)$ and $d := \sigma_M(2f)$. It follows from the definition of σ_M that we have $a < c < d$ and $a < b$. There are three possibilities for the relative order of a, b, c, and d: $a < c < d < b$, $a < c < b < d$, and $a < b < c < d$. These correspond respectively to nestings, crossings, and alignments of the matching M.

Of the three cases just described, the first case gives two inversions of σ_M, namely $(2e, 2f-1)$ and $(2e, 2f)$, and two elements of D_M, namely (a, d) and (a, c). The second case gives one inversion of σ_M, namely $(2e, 2f)$, and one element of D_M, namely (a, c). The third case does not give rise to any inversions of σ_M or any elements of D_M. The map ψ_M matches the inversions with the elements of D_M in each case, which completes the proof. $\square$

Theorem 3.10. *Let (W, S) be a Weyl group of type D_n, where $n = 2k \geq 4$ is even, and let $I = S \backslash \{s_{2k}\}$, so that $W_I \cong S_{2k}$. Define U to be the set of the $\binom{2k}{2}$ positive roots given by*

$$U = \{\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq 2k\},$$

and write $u_{ij} = u_{ji} := \varepsilon_i + \varepsilon_j$ for all $1 \leq i < j \leq 2k$. Maintain the notation of Definition 3.8, and let $\mathcal{M}$ be the set of perfect matchings of $[2n]$.

(i) *The triple (W, I, U) is a QP-triple.*

(ii) *The set Ω_U has $(2k-1)!!$ elements, each of which has the form*

$$R(M) = \{u_{i_1j_1}, u_{i_2j_2}, \dots, u_{i_kj_k}\}$$

for some perfect matching $M \in \mathcal{M}$.

(iii) *For each $M \in \mathcal{M}$, we have*

$$\mathrm{Res}_U(R(M)) = \{u_{ij} : 1 \leq i < j \leq 2k, \tau_M(i) > j, \textit{and } \tau_M(j) > i\}.$$

(iv) *For each $M \in \mathcal{M}$, the set*

$$\{(i,j) : j < i \textit{ and } u_{ji} \in \mathrm{Res}_U(R(M))\} = \{(i,j) : (j,i) \in D_M\}$$

is the Rothe diagram $\hat{D}_{\mathtt{FPF}}(\tau_M)$ of the fixed point free involution τ_M in the sense of Hamaker–Marberg–Pawlowski [9, (3.3)].

(v) *For each $M \in \mathcal{M}$, we have*

$$\rho_U(R(M)) = \ell(\sigma_M) = (\ell(\tau_M) - k)/2,$$

where ℓ denotes the length of a permutation.

(vi) *The generalized quantum Hafnian* $\mathrm{QHf}(U)$ *is the q-Hafnian in the sense of Jing–Zhang* [13, Proposition 3.6]*, and the generalized quantum Pfaffian* $\mathrm{QPf}(U)$ *is the q-Pfaffian in the sense of Strickland* [16] *and Jing–Zhang* [13, Proposition 3.4].

(vii) *The Hafnian* $\mathrm{Hf}(U)$ *is the Hafnian of a $2k \times 2k$ symmetric matrix (a_{ij}) satisfying $a_{ij} = u_{ij}$ for all $1 \leq i < j \leq 2k$, and the Pfaffian* $\mathrm{Pf}(U)$ *is the Pfaffian of a $2k \times 2k$ skew-symmetric matrix (a_{ij}) satisfying $a_{ij} = u_{ij}$ for all $1 \leq i < j \leq 2k$.*

(viii) *The Poincaré polynomial of U is $\prod_{i=2}^{k}[2i-1]_q$, and its degrees are $3, 5, 7, \dots, 2k-1$.*

Proof. Part (ii) follows from the observation that two elements of U are orthogonal if and only if they have disjoint support with respect to the ε-basis.

If $\beta, u_{ij} \in U$ satisfy $s_\beta(u_{ij}) < 0$, then we must either have $\beta = u_{aj}$ for some $a \leq i$, or $\beta = u_{ib}$ for some $b \leq j$. It follows that u_{ij} is a residue of $R(M)$ if and only if we have both $\tau_M(i) > j$ and $\tau_M(j) > i$, proving (iii).

The set

$$\{(j,i) : j < i, \tau(i) > j \text{ and } \tau(j) > i\}$$

is the Rothe diagram $\hat{D}_{\mathtt{FPF}}(\tau)$ of the fixed point free involution τ as defined by Hamaker–Marberg–Pawlowski [9, (3.3)]. Part (iv) follows by applying (iii) to this definition.

It follows from [9, Proposition 3.6] that the level, $\rho_U(R(M)) = |\hat{D}_{\mathtt{FPF}}(\tau_M)|$, of R is equal to $(\ell(\tau_M) - k)/2$. Lemma 3.9 (iii) implies that $\rho_U(R(M)) = \ell(\sigma_M)$, and this completes the proof of (v).

Rains–Vazirani [14, §4] proved that the function $(\ell(\tau_M) - k)/2$ is a quasiparabolic level function for the set of fixed point free involutions of S_{2k}, regarded as an S_{2k}-set in the natural way. Combining this result with part (v) proves (i).

It follows from (v) that we have

$$\mathrm{QHf}(U) = \sum_M q^{\ell(\sigma_M)} u_{i_1j_1} u_{i_2j_2} \dots u_{i_kj_k},$$

where the sum is over all perfect matchings of $[2k]$. This agrees with Jing–Zhang's construction of the quantum Hafnian from [13, Proposition 3.6] All the assertions of (vi) and (vii) now follow from the relevant definitions.

Part (viii) is a standard result that has been rediscovered several times; see [8, Proposition 7.1]. ∎

Example 3.11. In the case $n = 6$ of Theorem 3.10, we have

$$\begin{aligned}\mathrm{QHf}(U) = \ & u_{12}u_{34}u_{56} + qu_{13}u_{24}u_{56} + qu_{12}u_{35}u_{46} + q^2u_{13}u_{25}u_{46} + q^2u_{14}u_{23}u_{56} \\ & + q^2u_{12}u_{36}u_{45} + q^3u_{14}u_{25}u_{36} + q^3u_{15}u_{23}u_{46} + q^3u_{13}u_{45}u_{26} + q^4u_{16}u_{23}u_{45} \\ & + q^4u_{15}u_{24}u_{36} + q^4u_{14}u_{35}u_{26} + q^5u_{15}u_{26}u_{34} + q^5u_{16}u_{24}u_{35} + q^6u_{16}u_{25}u_{34}.\end{aligned}$$

Remark 3.12. In certain cases, the product of the residues of a perfect matching, when regarded as a polynomial in the ε_i, has an interpretation in terms of Schubert polynomials; see [9, Theorem 1.3].

Theorem 3.13. *Let (W, S) be a Weyl group of type D_n, where $n = 2k \geq 4$ is even, and let $I = S\backslash\{s_k, s_{2k}\}$, so that $W_I \cong S_k \times S_k$. Define U to be the set of k^2 positive roots given by*

$$U = \{\varepsilon_i + \varepsilon_{j+k} : i, j \in [k]\},$$

write $v_{ij} := \varepsilon_i + \varepsilon_{j+k}$ for all $1 \leq i, j \leq k$.

(i) *The triple (W, I, U) is a QP-triple.*

(ii) *The set Ω_U has $k!$ elements, each of which has the form*

$$R(\pi) := \{v_{1\pi(1)}, v_{2\pi(2)}, \ldots, v_{k\pi(k)}\}$$

for some permutation $\pi \in S_k$.

(iii) *For each $\pi \in S_k$, we have*

$$\mathrm{Res}_U(R(\pi)) = \{v_{ij} : i, j \in [k], \pi(i) > j, \textit{and } \pi^{-1}(j) > i\}.$$

(iv) *The set*

$$\{(i, j) : v_{ij} \in \mathrm{Res}_U(R(\pi))\}$$

is the Rothe diagram $D(\pi)$ of the permutation π.

(v) *For each permutation $\pi \in S_k$, we have*

$$\rho_U(R(M)) = \ell(\pi),$$

where ℓ denotes the length of a permutation.

(vi) *The generalized quantum Hafnian $\mathrm{QHf}(U)$ is the q-permanent of the matrix $A = (v_{ij})$ in the sense of* [12, (2.7)], *and the generalized quantum Pfaffian $\mathrm{QHf}(U)$ is the quantum row permanent $\mathrm{per}_q(A)$ of A in the sense of* [13, (3.18)].

(vii) *The Hafnian $\mathrm{Hf}(U)$ and the Pfaffian $\mathrm{Pf}(U)$ are the permanent $\mathrm{per}(A)$, and the determinant $\det(A)$, respectively.*

(viii) *The Poincaré polynomial of U is $\prod_{i=2}^{k}[i]_q$, and its degrees are $2, 3, 4, ..., k$.*

Proof. Parts (ii) and (iii) follow by restriction from Theorem 3.10 (ii) and (iii), with each k-root $R(\pi)$ defined in (ii) corresponding to the perfect matching $M\{\{i, \pi(i)+k\} : i \in [k]\}$. Part (iv) then follows from the definition of the Rothe diagram $D(\pi)$ [9, (2.7)]. Part (v), which is well known, follows from the fact that there is a bijection between the inversions of π and the Rothe diagram of π given by $(i, j) \mapsto (i, \pi(j))$.

If we identify Ω_U with S_k and W_I with $S_k \times S_k$, then the W_I-action on Ω_U is given by

$$(w_1, w_2) \cdot w = w_1 w (w_2)^{-1}.$$

By [14, Theorem 3.1], this makes S_k into a quasiparabolic set with height function ℓ, proving (i).

Parts (vi) and (vii) follow from the relevant definitions, and (viii) follows from the well-known identity $\sum_{\pi \in S_k} q^{\ell(\pi)} = [k]_q!$. □

Remark 3.14. (i) It is sometimes helpful to identify U as the positions on a $k \times k$ board and Ω_U with configurations of k nonattacking rooks on the board, with each root $v_{ij} = \varepsilon_i \pm \varepsilon_{j+k}$ corresponding to the (i, j)-th position. A position on the board then corresponds to a residue if and only if it is vacant and not attacked either from the left or from above. Plotting a k-root and its residues on the board yields the original Rothe diagram of the permutation correspoding to the rook configuration; see Figure 1.

(ii) Instead of regarding Ω_U as an $S_k \times S_k$-set, we may regard it as an (S_k, S_k)-biset, as in the proof of part (v) of the theorem, and from this perspective, Ω_U has a left S_k-action permuting the rows, and a right S_k-action permuting the columns. The actions commute with each other, and Ω_U is a quasiparaboic S_k-set under each action.

4. Further examples of QP-triples

In this section, we give details of many other examples of QP-triples. Section 4.1 studies the root systems of rank n where n-roots exist, and we show that for these root systems we can form a QP-triple by taking U to be the entire set of positive roots (Theorem 4.3). In Section 4.2, we show that it is often possible to construct examples of QP-triples by selecting those positive roots in which particular simple roots appear with odd coefficients.

4.1. Orthogonal bases of roots.

Recall from Lemma 2.3 that a Weyl group W of rank n and type ADE has a set of n orthogonal roots if and only if W has type E_7, E_8, or D_n for n even, and we will show that (W, S, Φ_+) is a QP-triple in these cases. These examples are particularly useful because many other examples of QP-triples can be constructed from them by restriction. The next result develops some key properties of residues of n-roots in these cases. Recall from Remark 3.5 (i) that for each n-root R of Φ, we have $\mathrm{Res}(R) = \mathrm{Res}_{\Phi_+}(R)$ and $\rho(R) = \rho_{\Phi_+}(R)$.

Proposition 4.1. *Let Φ be a root system of type E_7, E_8, or D_n for n even, and let R be a orthogonal set of n positive roots in Φ.*

(i) *The set of residues of R can be expressed as the disjoint union*

$$\mathrm{Res}(R) = \dot{\bigcup}_{Q \in D_4(R)} \mathrm{Res}(Q).$$

(ii) *If Φ is of type E_8, then with respect to each fixed feature $Q \in D_4(R)$, we have*

$$\mathrm{Res}(R) = \mathrm{Res}(Q) \quad \dot{\cup} \quad \dot{\bigcup}_{Q \subset H \in D_6(R)} (\mathrm{Res}(H) \backslash \mathrm{Res}(Q)) \quad \dot{\cup} \quad \mathrm{Res}(R \backslash Q).$$

Proof. Remark 3.2 (i) and Proposition 2.9 (ii) imply that every $\gamma \in \mathrm{Res}(R)$ lies in the support of a unique $Q \in D_4(R)$. It follows that $\gamma \in \mathrm{Res}(Q)$ in this case, which proves $\mathrm{Res}(R) \subseteq \dot{\bigcup}_{Q \in D_4(R)} \mathrm{Res}(Q)$. Conversely, any element $\gamma \in \mathrm{Res}(Q)$ is fixed by all reflections s_β such that $\beta \in R \backslash Q$, which implies that $\gamma \in \mathrm{Res}(R)$ and establishes the reverse containment. The disjointness assertion follows from Proposition 2.9 (iii), which completes the proof of (i).

Suppose Φ has type E_8, and let $Q \in D_4(R)$. The set $R \setminus Q$ is a feature by Lemma 2.14 (iii), and it is the only element of $D_4(R)$ that is disjoint from Q by cardinality considerations. For any $Q' \in D_4(R) \setminus \{Q, R \setminus Q\}$, Lemma 2.14 (i) and (ii) imply that the set $H = Q \cup Q'$ lies in $D_6(R)$ and contains three elements of $D_4(R)$, namely, Q, Q', and their symmetric difference, $Q'' = Q \,\Delta\, Q'$. Applying part (i) to the type D_6 subsystem $\Xi = \mathrm{Span}(H) \cap \Phi$ yields

$$\mathrm{Res}(H) = \mathrm{Res}(Q) \,\dot{\cup}\, \mathrm{Res}(Q') \,\dot{\cup}\, \mathrm{Res}(Q'')$$

and thus

$$\mathrm{Res}(H) \backslash \mathrm{Res}(Q) = \mathrm{Res}(Q') \,\dot{\cup}\, \mathrm{Res}(Q'').$$

The conclusions of (ii) now follow from (i). □

Corollary 4.2. *Let Φ be a root system of type E_7, E_8, or D_n for n even. If R is a maximal orthogonal set of positive roots in Φ, then $\rho(R) = C(R) + 2N(R)$, where $C(R)$ and $N(R)$ are the numbers of crossings and nestings in R, respectively.*

Proof. This is immediate from Proposition 4.1 (i) and Remark 3.5 (ii). □

Theorem 4.3. *Let W be a Weyl group of type E_7, E_8, or D_n for $n = 2k \geq 4$, with root system Φ and generating set S. The triple (W, S, Φ_+) is a transitive QP-triple.*

Proof. By Lemma 2.3 and the paragraph preceding it, the set Ω_{Φ_+} consists of the positive n-roots of Φ, and the standard action of $W_S = W$ on Ω_{Φ_+} is transitive. It is known [8, Theorem 4.5] that Ω_{Φ_+} is a quasiparabolic set for W under the standard action and level function $C+2N$, where C and N count the numbers of crossings and nestings in each n-root. Since $C(R)+2N(R) = \rho(R) = \rho_{\Phi_+}(R)$ for all $R \in \Omega_{\Phi_+}$ by Corollary 4.2, it follows that (W, S, Φ_+) is a transitive QP-triple, which completes the proof. □

Corollary 4.4. *Maintain the assumptions of Theorem 4.3. Let R be a positive n-root of Φ, let $\alpha \in \Phi_+ \backslash R$, and let Q be the support of α with respect to R. If $\rho(s_\alpha Q) > \rho(Q)$, then the number*

$$d := (\rho(s_\alpha R) - \rho(R)) - (\rho(s_\alpha Q) - \rho(Q))$$

is a nonnegative even number.

Proof. This follows from Remark 3.12 and Proposition 4.7 of [8]. By that remark and Remark 3.5 (ii), if $\rho(s_\alpha Q) > \rho(Q)$ then we have (a) Q is an alignment and $s_\alpha Q$ is a crossing, or (b) Q is a crossing and $s_\alpha Q$ is a nesting, or (c) Q is an alignment and $s_\alpha Q$ is a nesting. In these cases, the value of $\rho(s_\alpha(Q)) - \rho(Q)$ is 1, 1, and 2, respectively, and d is a nonnegative even number by parts (ii), (iii), and (iv) of [8, Proposition 4.7], respectively. □

Remark 4.5. The paper [8] details many interesting properties of the quasiparabolic sets produced by Theorem 4.3. We highlight some of them that are directly related to this paper below.

(i) In type D_{2k}, the quasiparabolic set Ω_{Φ_+} for the Weyl group $W = W(D_{2k})$ is essentially the same as the quasiparabolic set Ω_U for the parabolic subgroup $W_I \cong S_{2k}$ from Theorem 3.10 (where $I = S(D_{2k} \setminus \{s_{2k}\})$, in the sense that there is a natural bijection from Ω_U to Ω_{Φ_+} that is W_I-equivariant and preserves level, namely, the map

$$\phi : \Omega_U \to \Omega_{\Phi_+},\ \{\varepsilon_{i_1} + \varepsilon_{j_1}, \dots, \varepsilon_{i_k} + \varepsilon_{j_k}\} \mapsto \{\varepsilon_{i_1} + \varepsilon_{j_1}, \varepsilon_{i_1} - \varepsilon_{j_1}, \dots, \varepsilon_{i_k} + \varepsilon_{j_k}, \varepsilon_{i_k} - \varepsilon_{j_k}\}.$$

(ii) The Poincaré polynomial of $PS_{\Phi_+}(q)$ factors into quantum integers, with the multiset of degrees being $\{3, 5, \dots, 2k-1\}$ in type D_{2k}, $\{3, 5, 9\}$ in type E_7, and $\{3, 5, 9, 15\}$ in type E_8 ([8, Proposition 7.1 (i)]). In particular, the cardinality of Ω_{Φ_+} is $3 \cdot 5 \cdot 9 = 135$ in type E_7 and $3 \cdot 5 \cdot 9 \cdot 15 = 2025$ in type E_8.

(iii) The unique W-minimal element in type E_8 is the set

$$R = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \theta_4, \theta_6, \theta_7, \theta_8\},$$

where θ_4, θ_6, θ_7, and θ_8 are the highest roots in the standard parabolic subsystems of types D_4, D_6, E_7, and E_8 of Φ, respectively. This follows from [8, §6.3], and can also be checked directly using the definition of residues. Since the lowest-degree term of $PS_{\Phi_+}(q)$ is the constant 1 by (ii), the 8-root R has no residues with respect to Φ_+.

4.2. **An effective construction.** The constructions of the following definition turn out to provide extremely effective methods for constructing QP-triples from the root systems of type ADE, and many of the resulting QP-triples provide constructions of known objects, in some cases providing extra insight into their structure. We will record the examples without giving any detailed proofs in this section, but we note that the examples in types A and D can all be interpreted in terms of the combinatorics of rook configurations, and we will study the examples $T(E_8, 8)$ and $T(E_7, 7)$ in full detail in Section 5.

Definition 4.6. Let W be a Weyl group of type ADE with Dynkin diagram Γ, generating set S, and root system Φ. Let α_p be a simple root of Φ. We define $U(\Gamma, p) \subseteq \Phi_+$ to be the set of all roots in which α_p appears with odd positive coefficient. We define $\Omega(\Gamma, p) := \Omega_{U(\Gamma,p)}$ and define $T(\Gamma, p)$ to be the triple

$$(W, S\backslash\{s_p\}, U(\Gamma, p)).$$

Example 4.7. (i) The triple (W, I, U) appearing in Theorem 3.10 is $T(D_{2k}, 2k)$. In type D_n, a simple root can only appear with coefficient ± 2, ± 1, or 0, so $U(D_{2k}, 2k)$ is the set of roots in which α_{2k} appears with coefficient 1.

(ii) The triples $T(D_{2k}, 2k-1), T(D_{2k-1}, 2k-1)$, and $T(D_{2k-1}, 2k-2)\}$ are also QP-triples.

The above three triples all give rise to essentially the same quasiparolic set as $T(D_{2k}, 2k)$, in the sense that their associated quasiparabolic sets $\Omega_{U(\Gamma,p)}$ all admit canonical level-preserving bijections to the quasiparabolic set $\Omega(D_{2k}, 2k)$ from (i). This bijection is induced by the symmetry of the Dynkin diagram for $\Omega(D_{2k}, 2k-1)$. For the case of $\Omega = \Omega(D_{2k-1}, 2k-1)$, each element of Ω corresponds to a (non-perfect) matching of $\{1, 2, \ldots, 2k-1\}$ into $(k-1)$ pairs and one unmatched point, and the canonical bijection from Ω to $\Omega(D_{2k}, 2k)$ sends such an element to the $2k$-root in $\Omega(D_{2k}, 2k)$ corresponding to the unique perfect matching containing those $(k-1)$ pairs as blocks.

(iii) The triple $T(A_n, p)$ is a QP-triple for all integers $n \geq 1$ and $1 \leq p \leq n$.

The residue sets arising from the triple $T(A_n, p)$ resemble the Rothe diagrams mentioned in Remark 3.14 (i). More specifically, the set $U(A_n, p)$ consists of the roots of the form $\varepsilon_i - \varepsilon_j$ where $1 \leq i \leq p < j \leq n+1$, so we may identify U with the positions on a $p \times (n+1-p)$ board with rows labeled by 1 to p from top to bottom and columns labelled by $p+1$ to $n+1-p$ from left to right. Each element of $R \in \Omega_{U(A_n,p)}$ consists of $\min(p, n+1-p)$ orthogonal roots, and if we view R as an involution in the natural way, as the product $w_R \in S_{n+1}$ of the 2-cycles (i, j) for which $\varepsilon_i - \varepsilon_j \in R$, then a root $\varepsilon_a - \varepsilon_b \in U(A_n, p)$ is a residue of R with respect to $U(A_n, p)$ if and only if we have both $w_R(a) < b$ and $w_R(b) > a$. Equivalently, a position (a, b) on the board is a residue R if and only if it is not attacked by a rook to its right or above it in the rook configuration corresponding to R, as shown in Figure 4, where the residues are marked by stars as in Figure 1.

	4	5	6	7	8
1			●	★	★
2				●	★
3		●			★

FIGURE 4. The generalized Rothe diagram of $\{\varepsilon_1 - \varepsilon_6, \varepsilon_2 - \varepsilon_7, \varepsilon_3 - \varepsilon_5\} \in \Omega_{U(A_7,3)}$

Using the rook configuration model, it is possible to deduce that the Poincaré polynomial of $\Omega(A_n, p)$ factors into quantum integers, as the product $PS_{U(A_n,p)} = \prod_D [d]$ where D is the set (with no repetition) containing each of the $\min(p, n+1-p)$ consecutive integers leading up to (and including) $\max(p, n+1-p)$. The row–column symmetry of the model also implies that there is a natural level-preserving bijection between the quasiparabolic sets $\Omega(A_n, p)$ and $\Omega(A_n, n+1-p)$; in particular, the Poincaré polynomials of $U(A_n, p)$ and $U(A_n, n+1-p)$ agree.

(iv) For all integers $n \geq 4$ and $1 \leq p \leq n$, the triple $T(D_n, p)$ is a QP-triple whenever we have $1 \leq p \leq \lfloor n/2 \rfloor$ (where $\lfloor \cdot \rfloor$ denotes the floor function), but not when $\lfloor n/2 \rfloor < p < n-2$. However, if we relax the setting of Definition 4.6 and consider the set $U'_p = U(D_n, p) \cap U(D_n, n)$, then the triple $T'(\Gamma, P) = (W(D_n), S(D_n) \setminus \{s_p\}, U'_p)$ forms a QP-triple for all $1 \leq p \leq n-2$. Note that the QP-triple (W, I, U) from Theorem 3.13 is a special case of the triple $T'(\Gamma, p)$ from (v), namely, $T'(D_{2k}, k)$.

We note that for all the triples $T(D_n, p)$ mentioned in the last paragraph, the elements $R \in \Omega_{U(D_n,p)}$ and their residues can be interpreted in terms of rook configurations on a $p \times (n-p)$ board as in (iii), with the caveat that a position on the board may now correspond to either one of two roots $\varepsilon_i \pm \varepsilon_j$. The fact that $T(D_n, p)$ fails to be a QP-triple when $\lfloor n/2 \rfloor < p < n-2$ has to do with this caveat and the fact that in these cases R does not contribute a rook to every row.

Remark 4.8. (i) To summarize Example 4.7, every simple root α_p in a root system of type A_n or D_n gives rise to one or two QP-triples: $T(A_n, p)$ in type A_n, $T'(D_n, p)$ in type D_n, and $T(D_n, p)$ if $p \leq \lfloor n/2 \rfloor$.

(ii) The rook configuration model described in Example 4.7 (iii) is inspired by a similar model used by Garsia and Remmel in [3] to study a q-analogue of the Stirling numbers of the second kind. Garsia and Remmel place rooks on a general Ferrers board, and our model corresponds to the special case where the board is a $a \times b$ rectangle containing $\min(a, b)$ non-attacking rooks, with the residues in our construction corresponding to the positions marked with a circle on the board; see [3, Figure (I.7)].

(iii) The rook configuration models mentioned in Example 4.7 allow combinatorial proofs of the Poincaré polynomials of the quasiparabolic triples. In particular, it follows from the models that for all the quasiparabolic sets mentioned in the example, the Poincaré polynomials factor into quantum integers (that have combinatorial interpretations). For example, the quasiparabolic sets associated to $T(A_n, p)$ and $T(A_n, n+1-p)$ have the same Poincaré polynomial for any $1 \leq p \leq \lfloor n+1-p \rfloor$, namely, $\prod_{d=n-2p+2}^{n+1-p} [d]_q$.

Example 4.9. (i) Table 1 gives the complete list of cases for which $T(E_n, p)$ is a QP-triple, where $n \in \{6, 7, 8\}, 1 \leq p \leq n$, and κ stands for the cardinality of the elements in Ω_U in each case. We have verified that this is the complete list by computation, and we will give a detailed treatment of $T(E_8, 8)$ and $T(E_7, 7)$, two examples that have natural connections to minuscule representations and del Pezzo surfaces, but we do not have a combinatorial model or a conceptual proof of the quasiparabolic structure for all the examples. We also note that by our computation, in the cases where $T(E_n, p)$ is not a QP-triple, it always happens that $T(E_n, p)$ is not even a scaled W_I-set.

(ii) The structures of the QP-triple $T(E_7, 2)$ and the associated quasiparolic set $\Omega(E_7, 2)$ are known by [8, Section 6.2]: the parabolic subgroup W_I has type A_6, the set $\Omega(E_7, 2)$ coincides with the set of positive 7-roots of $\Phi(E_7)$ containing no alignments, and $W_I \cong S_7$ acts transitively on $\Omega(E_7, 2)$ in the standard action. The set $\Omega(E_7, 2)$ is also in a natural bijection with the 30 inequivalent labellings of the Fano plane. Under the correspondence between the 7-roots and labelled Fano planes, the generalized Rothe diagram of a labelled Fano plane has a simple combinatorial description: a residue is a triple of non-collinear points such that each point has a higher label than the point collinear with the other two. For example, the unique W_I-minimal labelling

$$\{136, 145, 127, 235, 246, 347, 567\}$$

is the only labelling with no residues, and the unique W_I-maximal labelling

$$\{123, 145, 246, 257, 347, 356, 167\}$$

$T(\Gamma, p)$	$\lvert U\rvert$	$\lvert\Omega_U\rvert$	κ	degrees
$T(E_6, 1)$	16	40	2	5, 8
$T(E_6, 2)$	20	30	4	2, 3, 5
$T(E_6, 6)$	16	40	2	5, 8
$T(E_7, 1)$	32	120	4	3, 5, 8
$T(E_7, 2)$	35	30	7	2, 3, 5
$T(E_7, 5)$	35	30	7	2, 3, 5
$T(E_7, 7)$	27	45	3	5, 9
$T(E_8, 1)$	64	240	8	2, 3, 5, 8
$T(E_8, 2)$	64	240	8	2, 3, 5, 8
$T(E_8, 5)$	64	240	8	2, 3, 5, 8
$T(E_8, 6)$	64	240	8	2, 3, 5, 8
$T(E_8, 8)$	56	630	4	5, 9, 14

Table 1. Table of QP-triples of the form $T(E_n, p)$

has seven residues: 267, 357, 367, 456, 457, 467, and 567. The generalized Rothe diagram of each labelling can be interpreted as the set of non-positions in the context of games arising from the Steiner system $S(2, 3, 7)$; see [11].

(iii) For the triple $T(E_8, 2)$, the results of [8, Section 6.3] show that the corresponding subgroup W_I has type A_7, the set $\Omega(E_8, 2)$ coincides with the set of positive 8-roots of $\Phi(E_8)$, and $W_I \cong S_8$ acts transitively on $\Omega(E_8, 2)$ in the standard action. The set $\Omega(E_8, 2)$ is in a natural bijection with the set of 240 inequivalent labellings of the Fano plane by the set $\{1, 2, \ldots, 8\}$, as well as with the 240 packings of the projective space $PG(3, 2)$. The generalized Rothe diagram of each 8-root in $\Omega(E_8, 2)$ has a characterization similar to the one in (ii), although the details are more complicated. This particularly intricate example is the subject of the paper [6].

5. The QP-triples $T(E_8, 8)$ and $T(E_7, 7)$

This section studies the QP-triples

$$T(E_8, 8) = (W(E_8), S(E_8) \setminus \{s_8\}, U(E_8, 8)) \quad \text{and} \quad T(E_7, 7) = (W(E_7), S(E_7) \setminus \{s_7\}, U(E_7, 7))$$

from Example 4.9 in detail. The sets $U(E_8, 8)$ and $U(E_7, 7)$ have interesting natural connections to minuscule representations and del Pezzo surfaces. We give a non-computational proof that $T(E_8, 8)$ is a QP-triple in Section 5.1, and we use this result to deduce that $T(E_7, 7)$ is a QP-triple in Section 5.2. In Section 5.3, we show that the Pfaffian of $T(E_7, 7)$ coincides with the invariant polynomial associated to a 27-dimensional minuscule representation of type E_6, and we also describe the generalized Rothe diagrams of the elements of $\Omega(E_7, 7)$, which we expect may have applications to Schubert varieties or del Pezzo surfaces.

5.1. $T(E_8, 8)$**.** Let Φ be the root system of type E_8 throughout this subsection. Let θ be the highest root of Φ, and let $U_i = \Phi_{8,i} \subseteq \Phi$ be the set of positive roots of 8-height i for each $i \in \{0, 1, 2\}$. It follows from Section 2.1 that θ is the only element of U_2, so we have $U(E_8, 8) = U_1$, and to prove that $T(E_8, 8)$ is a QP-triple it suffices to show that Ω_{U_1} is a quasiparabolic set for $W_I \cong W(E_7)$, where $I = S(E_8) \setminus \{s_8\}$.

As we will see in Corollary 5.4, the elements of Ω_U turn out to be precisely the features in Φ whose elements all come from U_1, and they can also be characterized as the intersections of the form $Q = R \cap U_1$ for the 8-roots R of Φ that do not contain θ. Our strategy for proving the quasiparabolic structure of Ω_{U_1} is to leverage the quasiparabolic structure of the set Ω_{Φ_+} of the

8-roots, so inclusions of the form $Q \subseteq R$ where $Q \in \Omega_{U_1}$ and $R \in \Omega_{\Phi_+}$ will play an important role in our arguments, as will chains of the form $Q \subseteq H \subseteq R$ where R is an 8-root, $H \in D_6(R)$, and Q is a feature in R. More specifically, it will be useful to study the relationship between the residues of Q, H, and R with various 8-heights in such a chain. We will do so in lemmas 5.2–5.3 and 5.5–5.8.

Definition 5.1. Let Φ be the root system of type E_8, and let R be a positive 8-root of Φ. Suppose that $H \in D_4(R)$, or $H \in D_6(R)$, or $H = R$. For each $i \in \{0, 1, 2\}$, we define

$$\mathrm{Res}^{(i)}(H) := \mathrm{Res}(H) \cap U_i \quad \text{and} \quad \rho^{(i)}(H) = |\mathrm{Res}^{(i)}(H)|.$$

Recall from Section 2.1 that for any $H \in D_6(R)$ where R is an 8-root, the D_6 subsystem $\Xi := \mathrm{Span}(H) \cap \Phi$ has positive roots $\{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 6\}$, simple roots $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_5 - \varepsilon_6, \varepsilon_5 + \varepsilon_6\}$, and highest root $\varepsilon_1 + \varepsilon_2$ in the coordinate realization of Ξ. A simple root of Φ is necessarily also simple in Ξ. By Lemma 2.14, every $Q \subseteq H$ consists of two couples and has the form $\{\varepsilon_a \pm \varepsilon_b, \varepsilon_c \pm \varepsilon_d\}$ for distinct elements $a, b, c, d \in [6]$, and the set $H \setminus Q$ consists of the couple $\{\varepsilon_e \pm \varepsilon_f\}$ where $\{a, b, c, d, e, f\} = [6]$. Also recall from Section 2.1 that the highest root θ has the property that $B(\theta, \alpha_8) = 1$ while $B(\theta, \alpha_i) = 0$ for all the simple roots α_i where $1 \leq i \leq 7$, so a root $\gamma \in \Phi_+$ lies in U_i if and only if $B(\theta, \gamma) = i$. Finally, note that for any roots $\beta \in U_i$ and $\gamma \in U_j$ where $0 \leq i < j$, the root $s_\beta(\gamma)$ must be a positive root because it has positive 8-height. We will frequently use the facts mentioned in this paragraph without further comment in the proofs of this subsection.

Lemma 5.2. *Let R be a positive 8-root of Φ such that $\theta \notin R$. Suppose that $H \in D_4(R)$ or $H \in D_6(R)$, and let $\Xi = \mathrm{Span}(H) \cap \Phi$ be the D_4 or D_6 subsystem of Φ corresponding to H.*

(i) *We have $\mathrm{Res}^{(2)}(R) = \{\theta\}$ and $\theta \in \mathrm{Res}_{\Phi_+}(H)$.*

(ii) *If $|H \cap U_0| < 4$, then $\mathrm{Res}^{(0)}(H) = \emptyset$.*

(iii) *If $|H \cap U_1| < 4$, then $\theta \notin \Xi$ and $\mathrm{Res}^{(2)}(H) = \emptyset$.*

Proof. Part (i) holds because θ has 8-height 2 while all other positive roots have strictly smaller heights.

We prove (ii) by proving its contrapositive. Suppose that $\mathrm{Res}^{(0)}(H)$ contains a root γ. Applying Proposition 4.1 (i) to $\Xi = \mathrm{Span}(H) \cap \Phi$ shows that γ is the highest or second highest root in a D_4 subsystem $\Psi \subseteq \Xi$ corresponding to some feature $Q \subseteq H$. Each simple root in a D_4 subsystem appears with nonzero coefficient in each of the two highest roots, so it follows from the assumption $\gamma \in U_0$ that $\Psi \subseteq U_0$. In particular, the four elements of Q lie in U_0, so $|H \cap U_0| \geq 4$. The conclusion of (ii) now follows.

Suppose $|H \cap U_1| < 4$, and suppose for a contradiction that $\theta = \varepsilon_1 + \varepsilon_2 \in \Xi$. The assumption $\theta \notin R$ implies that the perfect matching corresponding to H contains two distinct blocks $\{1, j\}$ and $\{2, k\}$. We then have

$$2\theta = (\varepsilon_1 + \varepsilon_j) + (\varepsilon_1 - \varepsilon_j) + (\varepsilon_2 + \varepsilon_k) + (\varepsilon_2 - \varepsilon_k)$$

in Ξ, which implies that each of the four roots on the right hand side has 8-height 1. This contradicts the assumption that $|H \cap U_1| < 4$, so we have $\theta \notin \Xi$. It follows that $\theta \notin \mathrm{Res}(H)$, so we have $\mathrm{Res}^{(2)}(H) = \emptyset$, proving (iii). □

Lemma 5.3. *Let R be a positive 8-root of Φ such that $\theta \notin R$.*

(i) *We have $R = Q_1 \,\dot\cup\, Q_0$, where Q_i is a feature consisting of roots in U_i for each i.*

(ii) *In the setting of (i), we have $\theta \in \mathrm{Res}(Q_1)$. We also have*

$$\mathrm{Res}^{(2)}(R) = \{\theta\}, \quad \mathrm{Res}^{(1)}(R) = \mathrm{Res}_{U_1}(Q_1), \text{ and } \quad \mathrm{Res}^{(0)}(R) = \mathrm{Res}(Q_0).$$

Proof. Since $\theta \notin R$, every element of R has 8-height 0 or 1. By Proposition 2.9, the support of θ with respect to R is a feature $Q_1 := \{\beta_1, \beta_2, \beta_3, \beta_4\}$ such that $\pm\beta_1 \pm \beta_2 \pm \beta_3 \pm \beta_4 = 2\theta$, and θ is orthogonal to the other four elements of the set $Q_0 := R \setminus Q_1$. Since θ has 8-height 2, it must be the case that $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\theta$, so Q_1 is a feature and we have $Q_1 \subseteq U_1$. Lemma 2.14 (iii) implies that Q_0 is a feature because $Q_0 = R \setminus Q_1$, and we have $Q_0 \subseteq U_0$ because $B(\theta, \gamma) = 0$ for all $\gamma \in Q_0$. This proves (i).

The last paragraph shows that θ lies in the D_4-subsystem $\Phi \cap \mathrm{Span}(Q_1)$. Since $\theta \in \mathrm{Res}_{\Phi_+}(Q_1)$ by Lemma 5.2 (i), it follows that $\theta \in \mathrm{Res}(Q_1)$. This proves the first assertion of (ii).

The equation $\mathrm{Res}^{(2)}(R) = \{\theta\}$ holds by Lemma 5.2 (ii). The fact that $Q_1 \subseteq R$ implies that $\mathrm{Res}^{(1)}(R) = \mathrm{Res}_{\Phi_+}(R) \cap U_1 \subseteq \mathrm{Res}_{U_1}(Q_1)$. Conversely, we have $\mathrm{Res}_{U_1}(Q_1) \subseteq \mathrm{Res}_{\Phi_+}(R) \cap U_1$ because $s_\beta(\gamma) > 0$ for all $\beta \in Q_0 \subseteq U_0$ and $\gamma \in U_1$, so we have $\mathrm{Res}^{(1)}(R) = \mathrm{Res}_{U_1}(Q_1)$. Finally, we have $\mathrm{Res}(R) = \bigcup_{Q \in D_4(R)} \mathrm{Res}(Q)$ by Proposition 4.1 (i) and therefore $\mathrm{Res}^{(0)}(R) = \bigcup_{Q \in D_4(R)} \mathrm{Res}^{(0)}(Q)$. Every feature Q other than Q_0 appearing in the union satisfies $|Q \cap U_0| < 4$ and thus $\mathrm{Res}^{(0)}(Q) = \emptyset$ by Lemma 5.2 (ii), so it follows that $\mathrm{Res}^{(0)}(R) = \mathrm{Res}^{(0)}(Q_0)$. This completes the proof of (ii). $\square$

Corollary 5.4. *We have* $\Omega_{U_1} = \{R \cap U_1 : R \in \Omega_{\Phi_+}, \theta \notin R\} = \{Q \subseteq U_1 : Q \text{ is a feature}\}$.

Proof. We have $B(\gamma, \theta) = 1 \neq 0$ for all $\gamma \in U_1$, so it follows from Lemma 2.3 that every element $Q \in \Omega_{U_1}$ can be extended to an 8-root R not containing θ. The asserted equalities now follow from Lemma 5.3 (i). $\square$

Lemma 5.5. *Let R be a positive 8-root of Φ such that $\theta \notin R$. Let $H \in D_6(R)$, let $\Xi = \mathrm{Span}(H) \cap \Phi$ be the associated D_6 subsystem of Φ, and suppose that we have $|H \cap U_1| = 2$ and $|H \cap U_0| = 4$.*

(i) *We have $\theta \notin \Xi$.*
(ii) *The highest root $\psi := \varepsilon_1 + \varepsilon_2$ of Ξ has 8-height 1.*
(iii) *The simple root $\beta_1 := \varepsilon_1 - \varepsilon_2$ of Ξ has 8-height 1; the other simple roots of Ξ have 8-height 0.*
(iv) *A root $\gamma \in \Xi_+$ has 8-height 1 if $\gamma = \varepsilon_1 \pm \varepsilon_i$ for some $i > 1$, and has 8-height 0 otherwise.*

Proof. Part (i) follows from Lemma 5.2 (iii) and the assumption that $|H \cap U_1| = 2 < 4$.

Since $\theta \notin \Xi$, the highest root ψ of Ξ has 8-height at most 1. On the other hand, since $H \cap U_1 \neq \emptyset$ by assumption, H contains a root of 8-height 1 and hence so does Ξ. It follows that $\psi \in U_1$, which proves (ii).

We now prove the claim that $\beta_1 \in U_1$. The set $Q_0 := H \cap U_0$ has size 4 by assumption and is thus a feature in U_0 by Lemma 5.3 (i). Lemma 2.14 (ii) implies that Q_0 contains two couples in Ξ while $H \cap U_1 = H \setminus U_0$ consists of one couple. We have just proved that $\psi = \varepsilon_1 + \varepsilon_2$ lies in U_1, so if $\psi \in H$ then we have $H \cap U_1 = \{\varepsilon_1 \pm \varepsilon_2\}$ and thus $\beta_1 \in U_1$. If $\psi \notin H$, then the perfect matching of the set $[6]$ corresponding to H contains two distinct pairs $\{1, j\}$ and $\{2, k\}$. These four roots $\varepsilon_1 \pm \varepsilon_j, \varepsilon_2 \pm \varepsilon_k \in H$ sum to 2ψ in this case, so either $\{1, j\}$ corresponds to a pair of 8-height 1 and $\{2, k\}$ corresponds to a pair of 8-height 0, or vice versa. One the other hand, the root $\beta_1 \in \Xi_+ \subseteq \Phi_+$ has nonnegative 8-height and satisfies

$$2\beta_1 = ((\varepsilon_1 + \varepsilon_j) + (\varepsilon_1 - \varepsilon_j)) - ((\varepsilon_2 + \varepsilon_k) + (\varepsilon_2 - \varepsilon_k)).$$

It follows that $\varepsilon_1 \pm \varepsilon_j \in U_1, \varepsilon_2 \pm \varepsilon_k \in U_0$, and $\beta_1 \in U_1$, as claimed.

Since every simple root of Ξ appears in the decomposition of ψ into simple roots and ψ and β_1 have the same 8-height, it follows that all simple roots of Ξ other than β lie in U_0, proving (iii).

Part (iv) follows from (iii) by expressing γ as a linear combination of simple roots. $\square$

Lemma 5.6. *Let R be a positive 8-root in type E_8 such that $\theta \notin R$, and let $R = Q_0 \dot\cup Q_1$ be the decomposition of Lemma 5.3 (i). Suppose that Q is a feature in R such that $|Q \cap Q_0| = |Q \cap Q_1| = 2$, and let $H = Q \cup Q_0$.*

(i) *All residues of Q have 8-height 1, so that* $\mathrm{Res}(Q) = \mathrm{Res}^{(1)}(Q)$.

(ii) *We have $H \in D_6(R)$, and we also have*

$$\mathrm{Res}^{(2)}(H) = \{\theta\}, \quad \mathrm{Res}^{(1)}(H) = \mathrm{Res}_{U_1}(Q), \text{ and } \quad \mathrm{Res}^{(0)}(H) = \mathrm{Res}(Q_0).$$

Proof. The assumptions that $|Q \cap Q_0| = 2 < 4$ and $|Q \cap Q_1| = 2 < 4$ imply that $\mathrm{Res}^{(0)}(Q) = \emptyset$ and $\mathrm{Res}^{(2)}(Q) = \emptyset$ by Lemma 5.2 (ii) and (iii), respectively, and (i) follows.

The fact that $H \in D_6(R)$ follows from Lemma 2.14 (i). Let $\Xi = \mathrm{Span}(H) \cap \Phi$ be the corresponding D_6 subsystem in Φ. We have $\theta \notin \Xi$ by Lemma 5.5 (i), so $\theta \notin \mathrm{Res}(H)$ and $\mathrm{Res}^{(2)}(H) = \emptyset$, proving the first equality in (ii). The two remaining equalities follow from Proposition 4.1 (i) and Lemma 5.2 (ii) in the same way as the last two equalities in Lemma 5.3 (ii). ☐

Lemma 5.7. *Maintain the setting of Lemma 5.6, and suppose that Q is the support of a root $\alpha \in \Phi \setminus R$. Let $r = s_\alpha$ be the reflection corresponding to α.*

(i) *If $\rho(Q) < \rho(rQ)$, then we have*

$$\rho^{(1)}(rH) - \rho^{(1)}(H) \geq \rho(rQ) - \rho(Q). \tag{5.1}$$

(ii) *If α is a simple root of Φ, then we have $\rho(Q_0) = \rho(rQ_0)$.*

Proof. Let $\Xi = \mathrm{Span}(H) \cap \Phi$, as in the proof of Lemma 5.6. Lemma 5.5 (iv) implies that in the coordinate realization of Ξ we have $\Xi \cap U_1 = \{\varepsilon_1 \pm \varepsilon_i : 2 \leq i \leq 6\}$, so it follows that

$$H = \{\varepsilon_1 \pm \varepsilon_a, \varepsilon_b \pm \varepsilon_c, \varepsilon_d \pm \varepsilon_e\} \tag{5.2}$$

where $b < c, d < e$, and $\{1, a, b, c, d, e\} = [6]$, and where we may assume that $Q = \{\varepsilon_1 \pm \varepsilon_a, \varepsilon_b \pm \varepsilon_c\}$. The assumptions on α and Q implies that Q and rQ are two distinct features in the same D_4 system $\mathrm{Span}(Q) \cap \Phi$, so we have $rQ = \{\varepsilon_1 \pm \varepsilon_{a'}, \varepsilon_{b'} \pm \varepsilon_{c'}\}$ where $\{a', b', c'\} = \{a, b, c\}$ but $a \neq a'$.

Let $\beta \in \Xi \cap U_1$. If we have $\beta = \varepsilon_1 - \varepsilon_i$ for some $2 \leq i \leq 6$ or $\beta = \varepsilon_1 + \varepsilon_i$ for some $i > a$, then we have $s_\gamma(\beta) < 0$ for $\gamma = \varepsilon_1 + \varepsilon_a \in H$, so $\beta \notin \mathrm{Res}(H)$. On the other hand, if $\beta = \varepsilon_1 + \varepsilon_i$ for some $1 < i < a$ then we have $s_\gamma(\beta) > 0$ for all $\gamma \in H$ and hence $\beta \in \mathrm{Res}(H)$. It follows that $\mathrm{Res}^{(1)}(H) = \{\varepsilon_1 + \varepsilon_i : 1 < i < a\}$ and $\rho^{(1)}(H) = a - 2$. Similarly, we have $\rho^{(1)}(rH) = a' - 2$, so that $\rho^{(1)}(rH) - \rho^{(1)}(H) = a' - a$. By Remarks 2.13 and 3.5 (ii), the assumption that $\rho(rQ) > \rho(Q)$ implies that we have one of the following situations:

(1) Q is an alignment and rQ is a crossing, so that $a < b = a' < c$ and $\rho(rQ) - \rho(Q) = 1$;
(2) Q is crossing and rQ is a nesting, so that $b < a < c = a'$ and $\rho(rQ) - \rho(Q) = 1$;
(3) Q is an alignment and rQ is a nesting, so that $a < b < c = a'$ and $\rho(rQ) - \rho(Q) = 2$.

We have $\rho^{(1)}(rH) - \rho^{(1)}(H) = a' - a \geq \rho(rQ) - \rho(Q)$ in all these cases, which proves (i). Now suppose that α is a simple root of Φ. This implies that α is also a simple root of Ξ, so the reflection r induces a basic transposition $s_i := (i, i+1)$ on the perfect matching $M := \{\{1, a\}, \{b, c\}\}$ corresponding to Q, with $\{i, i+1\} \subseteq \{1, a, b, c\}$ because Q is the support of α. If $\{i, i+1\} \in M$, then $rH = H$ and (ii) follows. Otherwise, s_i must move either 1 or a to an adjacent integer, and in either case the numbers (b, c, d, e) and $(s_i(b), s_i(c), s_i(d), s_i(e))$ appearing in the matchings corresponding to Q_0 and rQ_0 must be in the same relative order. It follows that $\rho(Q_0) = \rho(rQ_0)$, proving (ii). ☐

Lemma 5.8. *Maintain the hypotheses of Lemma 5.7, and suppose further that α is a root in $\Phi(E_7) \subseteq \Phi$. If $\rho(Q) < \rho(rQ)$, then we have $\rho_{U_1}(Q_1) < \rho_{U_1}(rQ_1)$.*

Proof. We first note that the assumption that $\alpha \in \Phi(E_7)$ implies that r cannot change the 8-height of a root, so we have $\theta \notin rR$ and $rR = rQ_0 \,\dot\cup\, rQ_1$ is the decomposition of rR into a feature in U_0 and a feature in U_1 as in Lemma 5.3 (i). The conclusions of Lemma 5.6 therefore still hold if we replace Q, Q_0, H and R with rQ, rQ_0, rH and rR, respectively. Recall that $H = Q \cup Q_0$.

To prove $\rho_{U_1}(Q_1) < \rho_{U_1}(rQ_1)$, it suffices to prove that $\rho^{(1)}(R) < \rho^{(1)}(rR)$ by Lemma 5.3 (ii). We have

$$\mathrm{Res}(R) = \mathrm{Res}(Q) \,\dot\cup\, \mathrm{Res}(R \setminus Q) \,\dot\cup\, \dot{\bigcup}_{Q \subseteq H' \in D_6(R)} (\mathrm{Res}(H') \setminus \mathrm{Res}(Q))$$

by Proposition 4.1 (ii). Taking intersections with U_1 then implies that

$$\mathrm{Res}^{(1)}(R) = \mathrm{Res}^{(1)}(Q) \,\dot{\cup}\, \mathrm{Res}^{(1)}(R \setminus Q) \,\dot{\cup}\, \dot{\bigcup}_{Q \subseteq H' \in D_6(R)} (\mathrm{Res}^{(1)}(H') \setminus \mathrm{Res}(Q),$$

where we have $\mathrm{Res}^{(1)}(Q) = \mathrm{Res}(Q)$ by Lemma 5.6 (i) and $\mathrm{Res}(Q) = \mathrm{Res}^{(1)}(Q) \subseteq \mathrm{Res}^{(1)}(H')$ for all $Q \subseteq H' \in D_6(R)$ by Proposition 4.1 (i). Taking cardinalities of the sets in the last equation then yields the equation

$$\rho^{(1)}(R) = \rho(Q) + \rho^{(1)}(R \setminus Q) + (\rho^{(1)}(H) - \rho(Q)) + \sum_{Q \subseteq H' \in D_6(R) \setminus \{H\}} (\rho^{(1)}(H') - \rho(Q)). \tag{5.3}$$

By the first paragraph of this proof, we also similarly have

$$\rho^{(1)}(rR) = \rho(rQ) + \rho^{(1)}(rR \setminus rQ) + (\rho^{(1)}(rH) - \rho(rQ)) + \sum_{rQ \subseteq H' \in D_6(R) \setminus \{rH\}} (\rho^{(1)}(H') - \rho(rQ)). \tag{5.4}$$

We have $rR \setminus rQ = R \setminus Q$ since r fixes every element in $R \setminus Q$, and we have

$$\{H' : rQ \subseteq H' \in D_6(R) \setminus \{rH\}\} = \{rH' : Q \subseteq H' \in D_6(R) \setminus \{H\}\},$$

so subtracting Equation (5.3) from Equation (5.4) yields that

$$\begin{aligned} \rho^{(1)}(rR) - \rho^{(1)}(R) &= (\rho(rQ) - \rho(Q)) + ((\rho^{(1)}(rH) - \rho^{(1)}(H)) - (\rho(rQ) - \rho(Q))) \\ &\quad + \sum_{Q \subseteq H' \in D_6(R) \setminus \{H\}} \left((\rho^{(1)}(rH') - \rho^{(1)}(H')) - (\rho(rQ) - \rho(Q)) \right). \end{aligned} \tag{5.5}$$

On the right side of Equation (5.5), we have $\rho(rQ) - \rho(Q) > 0$ by assumption, and we have

$$(\rho^{(1)}(rH) - \rho^{(1)}(H)) - (\rho(rQ) - \rho(Q)) > 0$$

by Lemma 5.7 (i). For any $Q \subseteq H' \subseteq D_6(R) \setminus \{H\}$, we have $|H' \cap U_0| = |rH' \cap U_0| < 4$ and therefore that $\mathrm{Res}^{(0)}(H') = \mathrm{Res}^{(0)}(rH') = \emptyset$ by Lemma 5.2 (ii). Furthermore, we have either $H' = Q_1 \cup Q$ or $|H' \cap Q_1| < 4$. In the former case, Lemma 5.3 (ii) and Proposition 4.1 (i) imply that $\theta \in \mathrm{Res}(Q_1) \subseteq \mathrm{Res}(H')$ and similarly $\theta \in \mathrm{Res}(rQ_1) \subseteq \mathrm{Res}(rH')$, so we have $\mathrm{Res}^{(2)}(H') = \mathrm{Res}^{(2)}(rH') = \{\theta\}$. In the latter case, we have $\mathrm{Res}^{(2)}(H') = \mathrm{Res}^{(2)}(rH') = \emptyset$ by Lemma 5.2 (iii). It follows that $\rho^{(0)}(H') = \rho^{(0)}(rH')$ and $\rho^{(2)}(H') = \rho^{(2)}(H')$ in all cases, so we have

$$(\rho^{(1)}(rH') - \rho^{(1)}(H')) - (\rho(rQ) - \rho(Q)) = (\rho(rH') - \rho(H')) - (\rho(rQ) - \rho(Q)) > 0$$

for any $Q \subseteq H' \in D_6(R) \setminus \{H\}$, where the inequality holds by an application of Corollary 4.4 to the 6-root H' of the D_6 system $\mathrm{Span}(H') \cap \Phi$. Equation (5.5) now implies that $\rho^{(1)}(rR) > \rho^{(1)}(R)$, as desired. ☐

Theorem 5.9. *Let W be the Weyl group of type E_8, with generating set S and root system Φ. Let $I = S \backslash \{s_8\}$, so that $W_I \cong W(E_7)$, and let $U \subset \Phi$ be the set of all roots of 8-height 1. The triple $T(E_8, 8) = (W, I, U)$ is a QP-triple; in other words, the set Ω_U is a quasiparabolic W_I-set with respect to the level function ρ_U.*

Proof. Let $Q_1 \in \Omega_U$, and let $r = s_\beta$ be a reflection in W_I corresponding to a positive root $\beta \in \Phi(E_7) = \Phi_{8,0}$. Corollary 5.4 and Lemma 5.3 imply that we can extend Q_1 to an 8-root $R = Q_1 \,\dot{\cup}\, Q_0$ where $\theta \notin R$ and Q_i is a feature in U_i for each i. Since r cannot change the 8-heights of roots, we also have $rR = rQ_0 \,\dot{\cup}\, rQ_1$, where $\theta \notin rR$ and rQ_i is a feature in U_i for each i.

Note that if $rQ_1 \neq Q_1$, then since $rQ_1 = rR \cap U_1$ and $Q_1 = R \cap U_1$ we must have $rR \neq R$, so that $\beta \notin R$. It then follows from Proposition 2.9, Remark 2.13, and Remark 3.5 (ii) that the support of β with respect to the basis R is a feature in R, Q, such that $\rho(rQ) \neq \rho(Q)$. The assumption $rQ_1 \neq Q_1$ also implies that $Q \neq Q_0$, because if $Q = Q_0$ then r would fix every element of Q_1, so Lemma 2.14 (i) implies that either $Q = Q_1$ or Q is a feature in R such that $|Q \cap Q_0| = |Q \cap Q_1| = 2$.

Moreover, if $Q = Q_1$, then we have $rQ_0 = Q_0$, so the set equalities asserted in Lemma 5.3 (ii) imply that

$$|\rho_U(sQ_1) - \rho_U(Q_1)| = |(\rho(sR) - 1 - \rho(sQ_0)) - (\rho(R) - 1 - \rho(Q_0))| = |\rho(sR) - \rho(R)|. \tag{5.6}$$

Suppose that $s = s_\alpha$ is a simple reflection in W_I such that $sQ_1 \neq Q_1$, and let Q' be the support of α with respect to R. By the last paragraph, we have either $Q' = Q_1$ or Q' is a feature such that $|Q' \cap Q_0| = |Q' \cap Q_1| = 2$. In the former case we have $sQ_0 = Q_0$, and in the latter case we have $\rho(Q_0) = \rho(sQ_0)$ by Lemma 5.7 (ii). Lemma 5.3 (ii) implies Equation (5.6) in both these cases, so Theorem 4.3 implies that

$$|\rho_U(sQ_1) - \rho_U(Q_1)| = |\rho(sR) - \rho(R)| \leq 1.$$

It follows that Ω_U is a scaled set.

Suppose $rQ_1 \neq Q_1$, and let Q be the support of β with respect to R. By the second paragraph, we have $\rho(rQ) \neq \rho(Q)$, and we have either $Q = Q_1$ or Q is a feature such that $|Q \cap Q_0| = |Q \cap Q_1| = 2$. Replacing Q with rQ if necessary, we may assume that $\rho(Q) < \rho(rQ)$. This implies that $\rho(R) < \rho(rR)$ by Corollary 4.4. If $Q = Q_1$, then we have $\rho_U(Q_1) - \rho_U(rQ_1) = \rho(R) - \rho(rR) < 0$ by Equation (5.6). Otherwise, we have $\rho_U(Q_1) < \rho_U(rQ_1)$ by Lemma 5.8. It follows that Ω_U satisfies condition (QP1).

For any reflection $r = s_\beta \in W_I$ corresponding to a positive root $\beta \notin R$, Corollary 4.4 implies that we have $\rho(Q) < \rho(rQ)$ if and only if we have $\rho(R) < \rho(rR)$. The case discussion in the last paragraph thus implies that for any reflection $r \in W_I$ and any 8-root $R = Q_0 \dot\cup Q_1$ extending an element $Q_1 \in \Omega_U$, if we have $\rho_U(rQ_1) > \rho_U(Q_1)$ then we have $\rho(rR) > \rho(R)$.

Now suppose that $s = s_\alpha$ is a simple reflection of W_I and $r = r_\beta$ a reflection of W_I such that $\rho_U(rQ_1) > \rho_U(Q_1)$ and $\rho_U(srQ_1) < \rho_U(sQ_1)$. We have $srQ_1 \in \Omega_U$ and $sQ_1 = (srs)(srQ_1)$, where srs is also a reflection in W_I, so by the last paragraph we have $\rho(rR) > \rho(R)$ and $\rho(sR) > \rho(srR)$. This implies that $sR = rR$ by Theorem 4.3, and taking intersections with U_1 yields $sQ_1 = rQ_1$, so Ω_U satsfies condition (QP2). This completes the proof. □

Remark 5.10. (i) Using Remark 4.5 (ii) and Lemma 5.3 (i), one can show that exactly 135 of the positive 8-roots in Ω_{Φ_+} contain θ and that the remaining 1890 positive 8-roots are in a 3-to-1 correspondence with the elements of the set Ω_U from Theorem 5.9. The 3-to-1 ratio comes from the fact the each element $Q_1 \in \Omega_U$ can be extended in three ways to a positive 8-root $R = Q_1 \dot\cup Q_0$, where Q_0 is one of the three features in the D_4-subsystem given by $\Psi = \{\beta \in \Phi : B(\beta, \gamma) = 0 \text{ for all } \gamma \in Q_1\}$. Combining this fact with Lemma 5.3 (ii), Remark 3.5 (ii), and Remark 4.5 (ii) shows that the Poincaré polynomial of Ω_U is $PS_U(q) = [5]_q[9]_q[14]_q$, which agrees with Table 1. In particular, the set Ω_U contains $5 \cdot 9 \cdot 14 = 630$ triples.

(ii) Recall from Remark 4.5 (iii) that the 8-root

$$R = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \theta_4, \theta_6, \theta_7, \theta_8\}$$

is the unique $W(E_8)$-minimal element of Ω_{Φ_+} and has no residues with respect to Φ_+. It follows from Theorem 4.3 that $s_8(R)$ has a single residue, namely, θ_8. Lemma 5.3 (ii) then implies that the unique $W(E_7)$-minimal element of Ω_U is the quadruple $\{s_8(\alpha_7), s_8(\theta_6), s_8(\theta_7), s_8(\theta)\}$.

5.2. $T(E_7, 7)$**.** In this subsection, we maintain the notation of Theorem 5.9, identify the root system $\Phi(E_7)$ with the subsystem $\Phi' = \Phi_{8,0}$ of Φ in the usual way, and identify the set $U' = U(E_7, 7)$ with the subset of Φ'_+ consisting of roots of positive odd 7-height. Let $\theta' = \theta_7$ be the highest root of Φ'. The 7-height of θ' is 1 by Equation (2.5), so it follows that $U' = \Phi'_{7,1}$. We will use Theorem 5.9 to prove that $T(E_7, 7)$ is a QP-triple in this subsection. In other words, we will prove that $\Omega_{U'}$ is a quasiparabolic set for $W_{I'} \cong W(E_6)$ with level function $\rho_{U'}$, where $I' = S \setminus \{s_8, s_7\}$.

We note that the set $U = U(E_8, 8)$ has a unique maximal element, which is also the second highest root of Φ, namely, $\eta := s_8(\theta) = \theta - \alpha_8$.

Lemma 5.11. *Let $Q \in \Omega_U$ and suppose that $\eta \in Q$.*

(i) *We have $\alpha_8 \notin \mathrm{Res}_U(Q)$.*
(ii) *Every element of $\mathrm{Res}_U(Q)$ is orthogonal to η.*

Proof. We have $B(\alpha_8, \eta) = -1$, which implies that $\alpha_8 \notin Q$. The elements of Q sum to 2θ, so we have $B(\alpha_8, 2\theta) = +2$. This can only happen if every $\beta \in Q \backslash \{\eta\}$ satisfies $B(\alpha_8, \beta) = +1$, which implies that $s_\beta(\alpha_8) < 0$ for all $\beta \in Q \setminus \{\eta\}$, proving (i).

Let $\gamma \in \mathrm{Res}_U(Q)$. Remark 3.2 (i) implies that $\gamma \neq \eta$, so we have $B(\eta, \gamma) \neq 2$. If it were the case that $B(\eta, \gamma) = -1$, it would follow that $\eta + \gamma = \theta$ by considering 8-heights. This would imply that $\gamma = \alpha_8$, contradicting (i). If we had $B(\eta, \gamma) = +1$, it would follow that $s_\eta(\gamma) < 0$, and γ would not be a residue. We conclude that $B(\eta, \gamma) = 0$, and (ii) follows. ☐

Lemma 5.12. (i) *The simple reflection s_8 gives a bijection from U' to the set*

$$\eta_U^\perp := \{\beta \in U : B(\beta, \eta) = 0\}.$$

(ii) *Each element $T \in \Omega_{U'}$ has size 3, so that $\Omega_{U'}$ consists of triples of pairwise orthogonal roots from U'.*
(iii) *There is an injective $W(E_6)$-equivariant map $\iota : \Omega_{U'} \to \Omega_U$ given by*

$$\iota(\{\beta_1, \beta_2, \beta_3\}) = \{s_8(\beta_1), s_8(\beta_2), s_8(\beta_3), \eta\}.$$

(iv) *For any distinct roots $\beta, \gamma \in U'$, we have $s_\beta(\gamma) < 0$ if and only if $s_{s_8(\beta)}(s_8(\gamma)) < 0$.*
(v) *For every $T \in \Omega_{U'}$, we have a bijection $\mathrm{Res}_{U'}(T) \to \mathrm{Res}_U(\iota(T))$ sending each $\gamma \in \mathrm{Res}_{U'}(T)$ to $s_8(\gamma)$. In particular, we have $\rho_{U'}(T) = \rho_U(\iota(T))$.*

Proof. Let

$$X_{i,j} = \{\alpha \in \Phi_+ : \mathrm{ht}_7(\alpha) = i \text{ and } \mathrm{ht}_8(\alpha) = j\}$$

for all $i, j \geq 0$. Direct computation shows that $s_8(X_{i,0}) \subseteq X_{i,i}$ and $s_8(X_{i,1}) \subseteq X_{i,i-1}$ for all $i \geq 0$, and we have $B(\beta, \eta) = B(s_8(\beta), \theta) = \mathrm{ht}_8(s_8(\beta))$ for any $\beta \in \Phi$. It follows that for any $\gamma \in U' = X_{1,0}$, we have $s_8(\gamma) \in X_{1,1} \subseteq U$ and $B(s_8(\gamma), \eta) = \mathrm{ht}_8(\gamma) = 0$, so $s_8(\gamma) \in \eta_U^\perp$. It also follows that for any $\beta \in \eta_U^\perp$ the element $\gamma = s_8(\beta)$ has 8-height 0, which implies that $\gamma \in X_{1,0} = U'$. Part (i) follows.

There exist orthogonal triples in U', such as the triple $\{s_8(\alpha_7), s_8(\theta_6), s_8(\theta_7)\}$ (see Remark 5.10 (ii)). For every $\beta \in U'$, the root $s_8(\beta)$ has 8-height 1 and 7-height 1, so (i) implies that if $T \in \Omega_{U'}$ then we have $s_8(T) \cup \{\eta\} \in \Omega_U$. The set $s_8(T) \cup \{\eta\}$ has size 4 in this case by Corollary 5.4, so T has size 3, proving (ii).

The last paragraph shows that the map ι given in (iii) sends elements of $\Omega_{U'}$ to Ω_U. The map ι is injective because $T = s_8(\iota(T) \setminus \{\eta\})$ for all $T \in \Omega_U$, and it is $W(E_6)$-equivariant because every simple root in $S \setminus \{\alpha_8, \alpha_7\}$ is orthogonal to both α_8 and η. Part (iii) follows.

For any distinct roots $\beta, \gamma \in U'$, we have

$$s_{s_8(\beta)}(s_8(\gamma)) = s_8(\gamma) - B(s_8(\beta), s_8(\gamma)) s_8(\beta) = (\gamma + \alpha_8) - B(\beta, \gamma)(\beta + \alpha_8).$$

It follows that $s_\beta(\gamma) < 0$ if and only if $s_{s_8(\beta)}(s_8(\gamma)) < 0$, because both conditions hold if and only if we have both $B(\beta, \gamma) = 1$ and $\gamma < \beta$. This proves (iv).

Let $T = \{\beta_1, \beta_2, \beta_3\} \in \Omega_{U'}$ and $\gamma \in \mathrm{Res}_{U'}(T)$. Since $B(\eta, s_8(\gamma)) = B(\theta, \gamma) = 0$, we have $s_{s_8(\gamma)}(\eta) = \eta > 0$. For each $i \in \{1, 2, 3\}$, we have $\gamma \neq \beta_i$ by Remark 3.2 (i), and we have $s_{\beta_i}(\gamma) > 0$ since $\gamma \in \mathrm{Res}_{U'}(T)$, so (iv) implies that $s_{s_8(\beta_i)}(s_8(\gamma)) > 0$. It follows that $s_8(\gamma) \in \mathrm{Res}_U(\iota(T))$.

Conversely, for any $\delta \in \mathrm{Res}_U(\iota(T))$, we have $\delta \in \eta_U^\perp$ by Lemma 5.11 (ii), so we have $\delta = s_8(\gamma)$ for some $\gamma \in U'$ by (i). The assumption that $\delta \in \mathrm{Res}_U(\iota(T))$ implies that $\delta \notin \iota(T)$ and, therefore, that $\gamma \notin T$. Since $\delta = s_8(\gamma) \in \mathrm{Res}_U(\iota(T))$, it follows from (iv) that $\gamma \in \mathrm{Res}_{U'}(T)$, which proves the first assertion of (v). The second assertion follows from the first by taking cardinalities. $\square$

Theorem 5.13. *Let W be the Weyl group of type E_8, with generating set S and root system Φ. Let $I' = S\backslash\{s_7, s_8\}$, so that $W_{I'} \cong W(E_6)$, and let $U' \subset \Phi$ be the set of all roots of 8-height 0 and 7-height 1. The triple $T(E_7, 7) = (W, I', U')$ is a QP-triple; in other words, the set $\Omega_{U'}$ is a quasiparabolic $W(E_6)$-set with respect to the level function $\rho_{U'}$.*

Proof. The set $\Omega_{U'}$ is a $W_{I'}$-set because elements of I do not change 7-heights of roots. Lemma 5.12 (iii) and (v) imply that for any $T \in \Omega_{U'}$ and any reflection $r \in W_{I'}$, we have $\rho_{U'}(T) = \rho_U(\iota(T))$ and $\rho_{U'}(rT) = \rho_U(\iota(rT)) = \rho_U(r(\iota(T))$ for the element $\iota(T) \in \Omega_U$. The fact that $\Omega_{U'}$ is a quasiparabolic $W(E_6)$-set with respect to $\rho_{U'}$ now follows from Theorem 5.9 and the injectivity of ι by a routine verification. $\square$

Remark 5.14.
(i) Recall from Remark 4.5 (ii) that there are 135 positive 7-roots in type E_7, and that the Poincaré polynomial of the associated to these 7-roots is $PS_{\Phi'_+}(q) = [3]_q[5]_q[9]_q$. The 7-roots are in a 3–to–1 correspondence with the elements of Ω'_U similar to the correspondence mentioned in Remark 5.10 (i). One can thus use the correspondence to deduce that $PS_{U'}(q) = [5]_q[9]_q$ and $|\Omega_{U'}| = 45$.
(ii) Combining Remark 5.10 (ii) and Lemma 5.12 shows that the unique $W(E_6)$-minimal element of $\Omega_{U'}$ is the triple $\{\alpha_7, \theta_6, \theta_7\}$.

5.3. **The invariant cubic polynomial in type E_6.** The sets $U = U(E_8, 8)$ and $U' = U(E_7, 7)$ from Theorems 5.9 and 5.13 have representation-theoretic interpretations. We recall that the exceptional Lie algebras of types E_6, E_7, and E_8 have three minuscule representations between them: two 27-dimensional representations in type E_6, $L(E_6, \omega_1)$ and $L(E_6, \omega_6)$, and a 56-dimensional representation in type E_7, $L(E_7, \omega_7)$, where each ω_i indicates the fundamental weight that appears as the highest weight of the corresponding representation (see, for example, [5, Theorem 5.1.5]). If one restricts the adjoint representation of the simple Lie algebra $\mathfrak{g}(E_8)$ to $\mathfrak{g}(E_7)$, then the set U corresponds to the weights of $L(E_7, \omega_7)$ [5, Proposition 10.5.5]. Similarly, restricting the adjoint representation of $\mathfrak{g}(E_7)$ to $\mathfrak{g}(E_6)$ identifies U' with the set of weights of $L(E_6, \omega_1)$.

The sets U and U' also have natural connections to algebraic geometry: the roots in U are in bijection with the 56 exceptional curves on a del Pezzo surface $\mathcal{S}$ of degree 2, and the roots in U' are in bijection with the 27 lines on a del Pezzo surface $\mathcal{S}'$ of degree 3 [5, Sections 9.3 and 10.1]. Under these bijections, two roots are orthogonal if and only if the intersection number between their corresponding curves is equal to 1, so the triples of $\Omega_{U'}$ are in a bijective correspondence with the tritangent planes of $\mathcal{S}'$. The quadruples of Ω_U corresponds bijectively to the what are sometimes called "tetrads" on $\mathcal{S}$; see, for example, [4, Proposition 5.4]. Theorems 5.9 and 5.13 imply that the tritangent planes and tetrads carry natural partial orders induced by the quasiparabolic orders of $\Omega_{U'}$ and Ω_U, respectively. The existence of these induced orders are new to the best of our knowledge.

In the rest of this subsection, we discuss two applications of the quasiparabolic set $\Omega_{U'}$ arising from the QP-triple $T(E_7, 7)$. First, we give explicit descriptions of the generalized Rothe diagrams associated to the elements $\Omega_{U'}$ in Theorem 5.17. These descriptions are also new to our knowledge, and we expect they may have applications to Schubert varieties or del Pezzo surfaces. Secondly, we show in Theorem 5.19 that the Pfaffian of $\Omega_{U'}$ agrees with the well-known invariant cubic polynomial associated to the minuscule representation $L(E_6, \omega_1)$. To state the theorems, we will use the following labelling conventions for the roots of U adapted from the context of del Pezzo surfaces. We note that even though Theorems 5.17 and 5.19 both focus on properties of $\Omega_{U'}$, it it more convenient to explain these conventions via the set U.

Let $\{\varepsilon_1, \dots, \varepsilon_8\}$ be an orthonormal basis for $\mathbb{R}^8$ as usual, and let $\{\alpha_1, \dots, \alpha_8\}$ be the simple roots of $\Phi = \Phi(E_8)$, so that each element $\beta \in U$ has the form $\beta = \alpha_8 + \sum_{i=1}^{7} d_i \alpha_i$ for some $d_1, \dots, d_7 \geq 0$. In the rest of the paper, we will adopt the labelling convention for the Dynkin diagram of type E_7 shown in Figure 5, which is compatible with the realization of the root system $\Phi(E_7)$ inside $\mathbb{R}^8$ where the simple roots are $\alpha'_i := 4(\varepsilon_i - \varepsilon_{i+1})$ for $1 \leq i \leq 6$ and

$$\alpha'_7 := (-2, -2, -2, 2, 2, 2, 2, -2).$$

In this realization, the 126 roots of type E_7 consist of the 56 coordinate permutations of $4(\varepsilon_1 - \varepsilon_2)$ and the 70 coordinate permutations of α'_7; a root is positive if and only if its rightmost nonzero coefficient is negative. More details about this realization can be found in [5, Example 8.1.7].

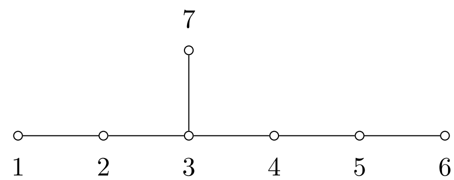


FIGURE 5. New labelling of the Dynkin diagram of type E_7

For any root $\beta = \alpha_8 + \sum_{i=1}^{7} d_i \alpha_i \in U$, we define

$$f(\beta) = (-1, -1, -1, -1, -1, -1, 3, 3) + \sum_{i=1}^{7} d_i \alpha'_{\tau(i)},$$

where $\tau(i)$ refers to the same vertex labeled by i in the Dynkin diagram in the convention of Figure 2 (d). It is known that the map f is a bijection between U and the set of vectors $\{\pm v_{i,j} : 1 \leq i < j \leq 8\}$, where $v_{i,j} = 4(\varepsilon_i + \varepsilon_j) - \sum_{k=1}^{8} \varepsilon_k$ for all i, j ([5, Example 8.1.7]). We will write X_{ij} for $v_{i,j}$ and Y_{ij} for $-v_{i,j}$. We will also identify each vector X_{ij} or Y_{ij} with the corresponding root in U from now on.

Let $a_i = X_{i7}$, $b_i = X_{i8}$, and $c_{ij} = c_{ji} = Y_{ij}$ for all $1 \leq i \neq j \leq 6$. Routine verification shows that we have

$$U' = \{a_i : 1 \leq i \leq 6\} \,\dot\cup\, \{b_i : 1 \leq i \leq 6\} \,\dot\cup\, \{c_{ij} : 1 \leq i < j \leq 6\},$$

and that three elements of U' form a triple in $\Omega_{U'}$ if and only if they are of the form $a_i c_{ij} b_j$ for some $1 \leq i \neq j \leq 6$ or of the form $c_{i_1 j_1} c_{i_2 j_2} c_{i_3, j_3}$ corresponding to a perfect matching $\{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}\}$ of $\{1, 2, \dots, 6\}$.

Example 5.15. Recall from Remark 5.14 (ii) that the unique $W(E_6)$-minimal element in $\Omega_{U'}$ is the triple $T_0 := \{\alpha_7, \theta_6, \theta_7\}$ (written in the labelling convention of Figure 2 (d)). Direct translation of notation shows that we have $T_0 = \{b_6, c_{16}, a_1\}$.

Lemma 5.16. *If β and γ are distinct elements of U', then we have $s_\gamma(\beta) < 0$ if and only if one of the following conditions holds:*

(i) *$\beta = a_i$ and $\gamma = a_j$ for some $j < i$;*
(ii) *$\beta = b_i$ and $\gamma = a_j$ for some $i \neq j$;*
(iii) *$\beta = b_i$ and $\gamma = c_{jk}$, where i, j, and k are distinct;*
(iv) *$\beta = c_{ij}$ and $\gamma = c_{kj}$, where $k > i < j$ and $k \neq j$;*
(v) *$\beta = c_{ij}$ and $\gamma = c_{il}$ for any $i < j < l$;*
(vi) *$\beta = c_{ij}$ and $\gamma = a_h$, where h, i, and j are distinct.*

Proof. This follows from a routine case by case check using the earlier description of the positive roots of type E_7 in the new coordinate realization. $\square$

Theorem 5.17. *The generalized Rothe diagrams of the triples in $\Omega_{U'}$ are given as follows.*

(i) *If $1 < i < j < 6$, then the triple $a_i c_{ij} b_j$ has level $5 + i - j$, and we have*
$$\mathrm{Res}_{U'}(a_i c_{ij} b_j) = \{a_h : 1 \le h < i\} \ \cup \ \{c_{ik} : j < k \le 6\}.$$

(ii) *If $1 < j < i < 6$, then the triple $a_i c_{ij} b_j$ has level $4 + i - j$, and we have*
$$\mathrm{Res}_{U'}(a_i c_{ij} b_j) = \{a_h : 1 \le h < i\} \ \cup \ \{c_{ki} : j < k \le 6 \text{ and } k \ne i\}.$$

(iii) *If $M := \{\{i_1, j_1\}, \{i_2, j_2\}, \{i_3, j_3\}\}$ is a perfect matching of $[6]$, then the triple $c_{i_1 j_1} c_{i_2 j_2} c_{i_3 j_3}$ has level $(\ell(\tau) + 9)/2$, where τ is the fixed point free involution associated to M and $\ell(\tau)$ is the length of τ. We have*
$$\mathrm{Res}_{U'}(c_{i_1 j_1} c_{i_2 j_2} c_{i_3 j_3}) = \{a_i : 1 \le i \le 6\} \ \cup \ \{c_{pq} : \tau(p) < q \text{ and } \tau(q) < p\}.$$

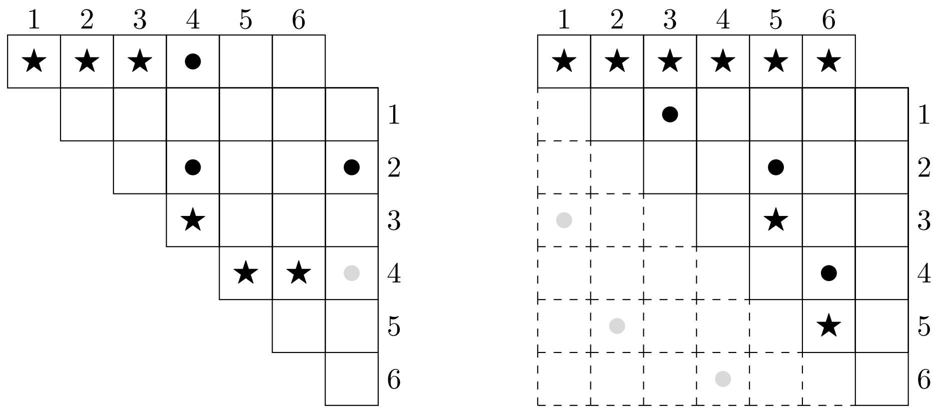
Proof. For each triple $T \in \Omega_{U'}$, we have
$$\mathrm{Res}_{U'}(T) = \{\beta \in U' : s_\gamma(\beta) > 0 \text{ for all } \gamma \in T\} = \bigcap_{\gamma \in T} (U' \setminus \{\beta \in U' : s_\gamma(\beta) < 0\}),$$
where each set appearing in the intersection can be calculated using Lemma 5.16. The descriptions of the residues follow from the lemma by straightforward computations. Taking cardinalities of the generalized Rothe diagrams yields the levels of the triples in (i) and (ii), and the claim about the level in (iii) follows from Theorem 3.10 (v). □

Example 5.18. Figure 6 depicts the generalized Rothe diagrams of the elements $T_1 = a_4 c_{24} b_2$ and $T_2 = c_{13} c_{25} c_{46}$, in the following way. First, we identify the 27 roots of U' with the 27 square cells in the array shown in Figure 6 (a) as follows: identify the cells in the top row with $a_1, a_2, \ldots,$ and a_6 from left to right, the cells in the rightmost column with $b_1, b_2, \ldots,$ and b_6 from top to bottom, and the unique cell in the (a_j, b_i)-position (i.e., the cell in the same row as b_i and same column as a_j) with $c_{ij} = c_{ji}$. If we represent each T_i by marking the elements of T with solid black dots, then the residues of T_i are the cells marked with a star. For an element of the form $T = a_i c_{ij} b_j$ where $j < i$, such as T_1, Theorem 5.17 (ii) implies that $\mathrm{Res}_{U'}(T)$ consists of the cells to the left of a_i, or below c_{ji}, or to the left of b_i (which is below b_j and not in T). For an element of the form $T = c_{i_1 j_1} c_{i_2 j_2} c_{i_3 j_3} \in \Omega_{U'}$, such as T_2, one can show using Theorem 5.17 (iii) that $\mathrm{Res}_{U'}(T)$ consists of all the six roots in the top row and the roots c_{ij} in the 27-cell grid that are neither attacked from the right nor from below by any of the roots in T or their mirror images across the diagonal consisting of the (a_k, b_k)-positions for $k \in \{1, 2, \ldots, 6\}$. We have shaded the cell $b_i = b_4$ in Figure 6 (a) and the aforementioned mirror images in Figure 6 (b).

Theorem 5.17 allows us to compute the quantum Hafnian $\mathrm{QHf}(U')$ of $\Omega_{U'}$ explicitly. We have

$$\begin{aligned}
\mathrm{QHf}(U') = \ & q^0 a_1 c_{16} b_6 + q^1 a_2 c_{26} b_6 + q^1 a_1 c_{15} b_5 + q^2 a_2 c_{25} b_5 + q^2 a_3 c_{36} b_6 + q^2 a_1 c_{14} b_4 + q^3 a_2 c_{24} b_4 \\
& + q^3 a_3 c_{35} b_5 + q^3 a_4 c_{46} b_6 + q^3 a_1 c_{13} b_3 + q^4 a_2 c_{23} b_3 + q^4 a_3 c_{34} b_4 + q^4 a_4 c_{45} b_5 + q^4 a_5 c_{56} b_6 \\
& + q^4 a_1 c_{12} b_2 + q^5 b_1 c_{12} a_2 + q^5 b_2 c_{23} a_3 + q^5 b_3 c_{34} a_4 + q^5 b_4 c_{45} a_5 + q^5 b_5 c_{56} a_6 + q^6 b_1 c_{13} a_3 \\
& + q^6 b_2 c_{24} a_4 + q^6 c_{12} c_{34} c_{56} + q^6 b_3 c_{35} a_5 + q^6 b_4 c_{46} a_6 + q^7 b_1 c_{14} a_4 + q^7 c_{13} c_{24} c_{56} + q^7 b_2 c_{25} a_5 \\
& + q^7 c_{12} c_{35} c_{46} + q^7 b_3 c_{36} a_6 + q^8 b_1 c_{15} a_5 + q^8 c_{14} c_{23} c_{56} + q^8 c_{13} c_{25} c_{46} + q^8 b_2 c_{26} a_6 + q^8 c_{12} c_{36} c_{45} \\
& + q^9 b_1 c_{16} a_6 + q^9 c_{15} c_{23} c_{46} + q^9 c_{14} c_{25} c_{36} + q^9 c_{13} c_{26} c_{45} + q^{10} c_{16} c_{23} c_{45} + q^{10} c_{15} c_{24} c_{36} \\
& + q^{10} c_{14} c_{26} c_{35} + q^{11} c_{16} c_{24} c_{35} + q^{11} c_{15} c_{26} c_{34} + q^{12} c_{16} c_{25} c_{34}.
\end{aligned}$$

We now show that the Pfaffian $\mathrm{Pf}(U') = \mathrm{QHf}(U', -1)$ associated to the QP-triple $T(E_7, 7)$ agrees with the invariant cubic polynomial associated with the Lie algebra (or Chevalley group) of type E_6 as constructed by Vavilov–Luzgarev–Pevzner [17]. This provides a nonrecursive combinatorial



(a) $\mathrm{Res}_{U'}(a_4 c_{24} b_2)$ (b) $\mathrm{Res}_{U'}(c_{13} c_{25} c_{46})$

FIGURE 6. The generalized Rothe diagrams of two elements in $\Omega_{U'}$

interpretation of the coefficients of the polynomial in terms of the parity of the cardinalities of the associated generalized Rothe diagrams.

Theorem 5.19. *The Pfaffian* $\mathrm{Pf}(U')$ *associated to the QP-triple* $T(E_7, 7)$ *is the invariant cubic polynomial* F *constructed in Vavilov–Luzgarev–Pevzner* [17, §12].

Proof. In [17, §12], Vavilov–Luzgarev–Pevzner define the minimal $W(E_6)$-element $\{b_6, c_{16}, a_1\} \in \Omega_{U'}$ from Example 5.15 to be the "distinguished triple", which they denote as $x_0 = (\lambda_0, \mu_0, \nu_0)$. They define the invariant cubic polynomial F to be the linear combination of the 45 elements in the $W(E_6)$-orbit of x_0 in which the coefficient of any monomial x_T is $(-1)^{\ell(w)}$, where w is an element of minimum length such that $w(x_0) = x_T$. As recalled in Section 3, the length of such an element is exactly the level of x_T by general properties of quasiparabolic sets, so we have $F = \mathrm{Pf}(\Omega_{U'})$. □

Remark 5.20.
- (i) Vavilov–Luzgarev–Pevzner note in [17, §12] that if we apply a simple reflection s_i on one of the monomials x_T in the invariant cubic polynomial, then either s_i fixes x_T, or the coefficients of x_T and $s_i(x_T)$ are the negatives of each other. This property also characterizes $\mathrm{Pf}(U')$ and provides an alternative proof of Theorem 5.19.
- (ii) The cubic polynomial F is written down explicitly in [17, Table 11]. This makes it possible, after translating notation, to verify Theorem 5.19 by hand.
- (iii) Let Γ be the graph whose vertices are the elements of $\Omega_{U'}$, and with adjacency given by disjointness. It can be shown that Γ is the complement of the graph $U_4(3)$ described in [1, §10.17]. In particular, Γ is strongly regular with parameters $(45, 32, 22, 24)$.

6. Concluding remarks

Remark 6.1. The original Rothe diagrams and the Rothe diagrams of fixed point free involutions are useful in the study of Schubert varieties. It would be interesting to know if this also applies to the generalized Rothe diagrams of Theorem 5.17, or to the generalized Rothe diagrams associated to the sets of all positive roots of types E_7 and E_8.

Remark 6.2. In the setting of Section 5.3, the 56 vectors X_{ij} and Y_{ij} arising from the set U can be grouped into 28 pairs $[i, j] := \{X_{ij}, Y_{ij}\}$. These pairs correspond naturally to the 28 bitangents to a plane quartic curve, and the 630 quadruples of Ω_U correspond to the 315 "tetrads of bitangents" of the quartic curve [4, Remark 5.5]. It is possible to modify the Pfaffian of Ω_U in a precise way to produce an invariant quartic polynomial known as Dickson's form (see [4, Definition 5.3] and

[2]), and this fact may be viewed as an analogue of Theorem 5.19 for Ω_U. We have chosen not to describe the modification here because it requires elaborate rescalings of coefficients for which we have no conceptual explanation. However, we note that the study of geometric objects such as the 630 tetrads corresponding to Ω_U and the 315 tetrads of bitangents has a long history dating back to the works of Cayley and Salmon (see, for example, [4, Remark 5.5] and [15, Section 262]). It would be interesting to know if the set Ω_U has further implications for these classical objects.

Remark 6.3. For any Coxeter system (W, S) and any subset $I \subseteq S$, the left cosets of the parabolic subgroup $W_I \leq W$ form a quasiparabolic W-set whose level function ℓ_I measures the lengths of shortest coset representatives, and using ℓ_I one can deform the natural action of W on the set W/W_I to construct a module for the Iwahori–Hecke algebra $\mathcal{H}$ of W. The defining properties of a quasiparabolic set X generalize the properties of W/W_I and allows a similar deformation of the W-action on X [14, Theorem 7.1]. It would be interesting to use k-roots to study the $\mathcal{H}$-modules obtained via the deformation of the quasiparabolic sets $X = \Omega_U$ constructed in this paper.

Remark 6.4. The examples of quasiparabolic sets given in this paper all share some additional structural features that are not required by the definitions or needed for the proofs, which suggests the possible existence of an undiscovered underlying structure. For example, we have no general conceptual explanation for why the Poincaré polynomial $\sum_{R \in \Omega_U} q^{\rho_U(R)}$ always factorizes as the product of quantum integers. Furthermore, the largest such integer always turns out to be equal to $|U|/\kappa$, where κ is the size of a maximal orthogonal subset of U. It is often (but not always) the case that U has a partition whose blocks are elements of Ω_U, which gives a combinatorial proof that κ divides $|U|$. In particular, such a partition always exists for the quasiparabolic sets of the form $T(E_n, p)$, except in the cases $T(E_6, 2)$, $T(E_7, 2)$, and $T(E_8, 5)$.

Remark 6.5. One can create a directed acyclic graph (DAG) from a QP-triple (W, I, U) by taking U to be the set of vertices, and installing an edge from u_1 to u_2 if $u_2 - u_1$ is a positive root of W_I. If $T \in \Omega_U$ satisfies $\rho(T) = 0$, then T is a a *kernel* of the DAG, meaning that (a) no two elements of T are joined by an edge, and (b) for all $v \in U \backslash T$, there is an edge from v to some element $u \in U$.

We can use the DAG to create a two-player game, as in Example 4.9 (ii), so that the kernel is the kernel in the sense of game theory [18, §51]. The vertices of the graph are the game positions, the directed edges are the legal moves, and playing a move that leads into the kernel constitutes a winning strategy.

References

[1] Andries E. Brouwer and Hendrik Van Maldeghem. *Strongly regular graphs*, volume 182. Cambridge University Press, 2022.

[2] L.E. Dickson. The configurations of the 27 lines on a cubic surface and the 28 bitangents to a quartic curve. *Bulletin of the American Mathematical Society*, 8(2):63–70, 1901.

[3] Adriano M. Garsia and Jeffrey B. Remmel. Q-counting rook configurations and a formula of Frobenius. *Journal of Combinatorial Theory, Series A*, 41(2):246–275, 1986.

[4] Marat Gizatullin. Two examples of affine homogeneous varieties. *European Journal of Mathematics*, 4(3):1035–1064, 2018.

[5] R.M. Green. *Combinatorics of minuscule representations*. Number 199 in Cambridge Tracts in Mathematics. Cambridge University Press, 2013.

[6] R.M. Green. A partial order on the 240 packings of $PG(3, 2)$. Preprint, 2025.

[7] R.M. Green and Tianyuan Xu. 2-roots for simply laced Weyl groups. *Transformation Groups*, 36:699–740, 2025.

[8] R.M. Green and Tianyuan Xu. Orthogonal roots, Macdonald representations, and quasiparabolic sets. *Forum of Mathematics, Sigma*, 2025. (Published online; `arXiv:2409.01948`).

[9] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski. Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures. *Journal of Combinatorial Theory, Series A*, 160:217–260, 2018.

[10] James E. Humphreys. *Reflection groups and Coxeter groups*. Number 29 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1990.

[11] Yuki Irie. Combinatorial game distributions of Steiner systems. *The Electronic Journal of Combinatorics*, pages P4–54, 2021.
[12] Naihuan Jing and Jian Zhang. Quantum Pfaffians and hyper-Pfaffians. *Advances in Mathematics*, 265:336–361, 2014.
[13] Naihuan Jing and Jian Zhang. Quantum permanents and Hafnians via Pfaffians. *Letters in Mathematical Physics*, 106(10):1451–1464, 2016.
[14] Eric M. Rains and Monica J. Vazirani. Deformations of permutation representations of Coxeter groups. *Journal of Algebraic Combinatorics*, 37(3):455–502, 2013.
[15] George Salmon. *A treatise on the higher plane curves: intended as a sequel to A treatise on conic sections.* Hodges, Foster, and Figgis, Dublin, 3rd edition, 1879. Assisted by Arthur Cayley.
[16] Elisabetta Strickland. Classical invariant theory for the quantum symplectic group. *Advances in Mathematics*, 123(1):78–90, 1996.
[17] N.A. Vavilov, A. Yu. Luzgarev, and I.M. Pevzner. Chevalley groups of type E_6 in the 27-dimensional representation. *Journal of Mathematical Sciences*, 145(1):4697–4736, 2007.
[18] John von Neumann and Oskar Morgenstern. *Theory of games and economic behavior: 60th anniversary commemorative edition*. Princeton University Press, 2007.

Department of Mathematics, University of Colorado Boulder, Campus Box 395, Boulder, Colorado, USA, 80309
Email address: `rmg@colorado.edu`

Department of Mathematics and Statistics, University of Richmond, Richmond, Virginia, USA, 23173
Email address: `tianyuan.xu@richmond.edu`