BOUNDED RATIOS FOR LORENTZIAN MATRICES

DAOJI HUANG, JUNE HUH, DANIEL SOSKIN, AND BOTONG WANG

ABSTRACT. We study multiplicative inequalities among entries of Lorentzian matrices, referred to as *bounded ratios*. These inequalities can be viewed as generalizations of the classical Alexandrov–Fenchel inequalities for mixed volumes. Our main structural result identifies the cone of all bounded ratios on Lorentzian matrices with the dual of the *cut cone*, a finitely generated integral polyhedral cone extensively studied in metric geometry and graph theory. We examine in detail the *pentagonal ratio*, which first appears for Lorentzian matrices of size at least five. For Lorentzian matrices of size three, we determine the optimal bounding constants across the entire cone of bounded ratios, obtaining an explicit entropy-like formula. We conjecture that any normalized bounded ratio is, in fact, bounded by 2.

1. INTRODUCTION

A *Lorentzian matrix* is a symmetric matrix with nonnegative real entries that has at most one positive eigenvalue. We denote by $\mathbb{L}_n$ the set of $n \times n$ Lorentzian matrices. These matrices appear prominently in space-time geometry and convex geometry. In this work, we study *multiplicative inequalities* for Lorentzian matrices—inequalities of the form

$$\prod_{i \leqslant j} p_{ij}^{\alpha_{ij}} \leqslant c \prod_{i \leqslant j} p_{ij}^{\beta_{ij}} \quad \text{for all } (p_{ij}) \in \mathbb{L}_n,$$

where α_{ij} and β_{ij} are nonnegative integers, and c is a positive constant.

For a collection of convex bodies $K = (K_1, \ldots, K_d)$ in $\mathbb{R}^d$, the *mixed volume* of K is defined to be the normalized coefficient of $t_1 \cdots t_d$ in the associated volume polynomial:

$$V(K_1, \ldots, K_d) \coloneqq \frac{1}{d!} \partial_1 \cdots \partial_d \operatorname{Vol}_n(t_1 K_1 + \ldots t_d K_d).$$

The celebrated *Alexandrov–Fenchel inequality* states that, for any collection of convex bodies

$$P_1, \ldots, P_n \text{ and } K_1, \ldots, K_{d-2} \text{ in } \mathbb{R}^d,$$

the symmetric $n \times n$ matrix of d-dimensional mixed volumes (p_{ij}) given by

$$p_{ij} = V(P_i, P_j, K_1, \ldots, K_{d-2})$$

is a Lorentzian matrix [Sch14, Section 7.3]. In particular, any of its 2×2 principal minor is nonpositive, that is,

$$p_{ii} p_{jj} \leqslant p_{ij}^2 \quad \text{for all } i \text{ and } j.$$

The Alexandrov–Fenchel inequality above is one of the first examples of bounded ratios on Lorentzian matrices, which we now define. Let $\mathrm{Sym}_n(\mathbb{R}_{>0})$ denote the set of $n \times n$ symmetric matrices with positive entries, and let $X \subseteq \mathrm{Sym}_n(\mathbb{R}_{>0})$ be a nonempty subset. Given a real vector $\alpha = (\alpha_{ij})_{1 \leqslant i \leqslant j \leqslant n}$, we write

$$\alpha = \sum_{i \leqslant j} \alpha_{ij} \mathbf{e}_{ij},$$

where $\{\mathbf{e}_{ij}\}_{i \leqslant j}$ is the standard basis of the space of all such vectors.

Definition 1.1. The set of *bounded ratios* on X, denoted $\mathrm{BR}(X)$, is the set of all real vectors $\alpha = (\alpha_{ij})_{1 \leqslant i \leqslant j \leqslant n}$ that satisfy the following condition:

$$\text{There is a positive constant } c \text{ such that } \prod_{1 \leqslant i \leqslant j \leqslant n} p_{ij}^{\alpha_{ij}} \leqslant c \text{ for all matrices } (p_{ij}) \text{ in } X.$$

We define the *optimal bounding constant* of a bounded ratio α on X, denoted $f(\alpha) = f_X(\alpha)$, to be the infimum of all possible constants c satisfying the displayed inequality for all (p_{ij}) in X.

It is straightforward to check that $\mathrm{BR}(X)$ is a convex cone in $\mathbb{R}^{\binom{n+1}{2}}$ and $f(\alpha)$ is a log-convex homogeneous function of degree 1 on the cone of bounded ratios. When X is the set of $n \times n$ positive semidefinite matrices with positive entries PSD_n^+, Yu shows in [Yu15] that the cone of bounded ratios on X is a finitely generated integral polyhedral cone with $\binom{n}{2}$ extremal rays corresponding to the 2×2 principal minors:

$$\mathrm{BR}(\mathrm{PSD}_n^+) = \mathrm{Cone}(-\mathbf{e}_{ii} - \mathbf{e}_{jj} + 2\mathbf{e}_{ij})_{1 \leqslant i < j \leqslant n}.$$

In contrast, bounded ratios for Lorentzian matrices, as we will see, exhibit more intricate behaviors. For related notions of bounded ratios in the study of cluster algebras and total positivity, see [FJ00, FGJ03, GGS, SG25].

Definition 1.2. A bounded ratio α is *integral* if all the entries α_{ij} are integers. An integral bounded ratio on X is said to be *primitive* if it cannot be expressed as the sum of two nonzero integral bounded ratios on X.

Let $\mathbb{L}_n^+$ be the set of $n \times n$ Lorentzian matrices with only positive entries. Since any Lorentzian matrix is a limit of Lorentzian matrices with positive entries [BH20, Section 2], the multiplicative inequalities for $\mathbb{L}_n$ are obtained by clearing the denominator in $p^\alpha \leqslant c$ for a bounded ratio α on $\mathbb{L}_n^+$. Our main result, Theorem B below, implies that $\mathrm{BR}(\mathbb{L}_n^+)$ is a finitely generated integral polyhedral cone.

Example 1.3. The bounded ratios on $\mathbb{L}_2^+$ are exactly the nonnegative multiples of

$$\alpha^{12} \coloneqq \mathbf{e}_{11} + \mathbf{e}_{22} - 2\mathbf{e}_{12}.$$

In other words, $\mathrm{BR}(\mathbb{L}_2^+) = \mathrm{Cone}(\alpha^{12})$. The optimal bounding constants are determined by the condition $f(\alpha^{12}) = 1$. We refer to the bounded ratio α^{12} as the *Alexandrov–Fenchel type*.

Example 1.4. Theorem B shows that there are precisely three primitive bounded ratios on $\mathbb{L}_3^+$, corresponding to the inequalities

$$\frac{p_{23}p_{11}}{p_{12}p_{13}} \leqslant 2, \quad \frac{p_{13}p_{22}}{p_{12}p_{23}} \leqslant 2, \quad \frac{p_{12}p_{33}}{p_{13}p_{23}} \leqslant 2 \quad \text{for all } (p_{ij}) \in \mathbb{L}_3^+.$$

The cone of bounded ratios $\mathrm{BR}(\mathbb{L}_3^+)$ is the three-dimensional simplicial cone in $\mathbb{R}^6$ generated by

$$\alpha^{23|1} \coloneqq \mathbf{e}_{23} + \mathbf{e}_{11} - \mathbf{e}_{12} - \mathbf{e}_{13}, \quad \alpha^{13|2} \coloneqq \mathbf{e}_{13} + \mathbf{e}_{22} - \mathbf{e}_{12} - \mathbf{e}_{23}, \quad \alpha^{12|3} \coloneqq \mathbf{e}_{12} + \mathbf{e}_{33} - \mathbf{e}_{13} - \mathbf{e}_{23}.$$

We refer to the bounded ratios $\alpha^{ij|k}$ as the *triangular ratios*. The Alexandrov–Fenchel type ratios can be expressed as the pairwise sums

$$\alpha^{12} = \alpha^{23|1} + \alpha^{13|2}, \quad \alpha^{13} = \alpha^{23|1} + \alpha^{12|3}, \quad \alpha^{23} = \alpha^{13|2} + \alpha^{12|3},$$

which are bounded but not primitive on $\mathbb{L}_3^+$. To see the validity of the displayed inequalities on $\mathbb{L}_3^+$, we use the implication

$$\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \text{ is Lorentzian} \Longrightarrow \begin{pmatrix} 0 & p_{12} & p_{13} \\ p_{12} & 0 & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} \text{ is Lorentzian.}$$

The nonnegativity of the determinant of the latter matrix gives the optimal bounding constant $f(\alpha^{12|3}) = 2$, which is witnessed by the Lorentzian matrices

$$\begin{pmatrix} \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathbb{L}_3^+ \text{ for sufficiently small } \epsilon > 0.$$

Note that $0 = \log f(\alpha^{12|3} + \alpha^{13|2}) \leqslant \log f(\alpha^{12|3}) + \log f(\alpha^{13|2}) = 2\log 2$. In Theorem C, we explicitly describe the function $f(\alpha)$ on $\mathrm{BR}(\mathbb{L}_3^+)$.

The inequalities for the primitive bounded ratios $\alpha^{ij|k}$ appear in the work of André Weil on the proof of the Riemann hypothesis for algebraic curves over finite fields, where they are referred to as the *Castelnuovo–Severi inequality* [Mil16, Theorem 1.5]. This is the first special case of the *reverse Khovanskii–Teissier inequality* [LX17, Theorem 5.7]. The inequalities for $\alpha^{ij|k}$ also appear in early works on Brunn–Minkowski theory; see [Fro15, Page 396] and [BF87, Section 51].[1]

It turns out that there are no essentially new bounded ratios on $\mathbb{L}_4^+$:

$$\mathrm{BR}(\mathbb{L}_4^+) = \mathrm{Cone}(\alpha^{ij|k})_{1\leqslant i,j,k\leqslant 4}, \quad \text{where } \alpha^{ij|k} \coloneqq \mathbf{e}_{ij} + \mathbf{e}_{kk} - \mathbf{e}_{ik} - \mathbf{e}_{jk}.$$

Exactly one new type of bounded ratios appear for $\mathbb{L}_5^+$, which we refer to as the *pentagonal ratios*:

$$\alpha^{ijk|lm} \coloneqq \mathbf{e}_{ij} + \mathbf{e}_{ik} + \mathbf{e}_{jk} + \mathbf{e}_{ll} + \mathbf{e}_{lm} + \mathbf{e}_{mm} - \mathbf{e}_{il} - \mathbf{e}_{jl} - \mathbf{e}_{kl} - \mathbf{e}_{im} - \mathbf{e}_{jm} - \mathbf{e}_{km}.$$

[1] We thank Ramon van Handel for pointing out these references.

Theorem A. The *pentagonal inequality* holds for any $(p_{ij}) \in \mathbb{L}_n$:

$$p_{ij}p_{ik}p_{jk}p_{ll}p_{lm}p_{mm} \leqslant 4p_{il}p_{jl}p_{kl}p_{im}p_{jm}p_{km} \text{ for any } i,j,k,l,m.$$

The constant 4 is optimal when all five indices are distinct, and

$$\mathrm{BR}(\mathbb{L}_5^+) = \mathrm{Cone}(\alpha^{ijk|lm})_{1\leqslant i,j,k,l,m\leqslant 5}.$$

Note that the triangular ratios appear as degenerate cases of the pentagonal ratios:

$$\alpha^{ijk|kk} = \alpha^{ij|k} \text{ for any } i,j,k.$$

The Alexandrov–Fenchel type ratio α^{ij} appears as a degenerate triangular ratio $\alpha^{ii|j}$, which is bounded but not extremal in $\mathrm{BR}(\mathbb{L}_5^+)$. The cone of bounded ratios $\mathrm{BR}(\mathbb{L}_5^+)$ has forty extremal rays, corresponding to the ten nondegenerate pentagonal ratios and the thirty nondegenerate triangular ratios. We have a factorization of the pentagonal ratio into a product of three triangular ratios and one reciprocal:

$$\frac{p_{12}p_{13}p_{23}p_{44}p_{45}p_{55}}{p_{14}p_{15}p_{24}p_{25}p_{34}p_{35}} = \frac{p_{12}p_{44}}{p_{14}p_{24}} \cdot \frac{p_{13}p_{55}}{p_{15}p_{35}} \cdot \frac{p_{22}p_{45}}{p_{24}p_{25}} \cdot \left(\frac{p_{22}p_{34}}{p_{23}p_{24}}\right)^{-1}.$$

Even though the reciprocal on the right-hand side is unbounded on its own, the pentagonal ratio on the left-hand side is bounded on $\mathbb{L}_n^+$. Factorizations of this type are not unique.

In general, we show that $\mathrm{BR}(\mathbb{L}_n^+)$ is a finitely generated polyhedral cone whose extremal rays are generated by the primitive bounded ratios. Up to S_n-symmetry, the numbers of primitive bounded ratios for $n = 3, 4, 5, 6, 7, 8$ are 1, 1, 2, 4, 36, and 2169, respectively. The two new types of primitive bounded ratios for $\mathbb{L}_6^+$ correspond to the inequalities of the form

$$\frac{p_{11}p_{12}p_{12}p_{13}p_{13}p_{23}p_{44}p_{45}p_{46}p_{55}p_{56}p_{66}}{p_{14}p_{14}p_{15}p_{15}p_{16}p_{16}p_{24}p_{25}p_{26}p_{34}p_{35}p_{36}} \leqslant c, \quad \frac{p_{11}p_{11}p_{11}p_{12}p_{12}p_{22}p_{34}p_{35}p_{36}p_{45}p_{46}p_{56}}{p_{13}p_{13}p_{14}p_{14}p_{15}p_{15}p_{16}p_{16}p_{23}p_{24}p_{25}p_{26}} \leqslant c,$$

although we do not know the optimal bounding constants for these inequalities; see Conjecture 1.9. To prove the general statement, we relate $\mathrm{BR}(\mathbb{L}_n^+)$ to the *cut cone*, a finitely generated integral polyhedral cone extensively studied in metric geometry and graph theory. For a comprehensive introduction to the geometry and combinatorics of the cut cone, we refer to the book [DL10].

Definition 1.5. For any subset $S \subseteq [n]$, define the *cut vector* $\delta(S) = (\delta(S)_{ij})_{1\leqslant i<j\leqslant n}$ by

$$\delta(S)_{ij} = \begin{cases} 1, & \text{if } S \text{ contains exactly one of } i \text{ and } j, \\ 0, & \text{if otherwise.} \end{cases}$$

The *cut cone* Cut_n is the polyhedral cone in $\mathbb{R}^{\binom{n}{2}}$ generated by all the cut vectors $\delta(S)$ for $S \subseteq [n]$.

The extremal rays of Cut_n are generated by the cut vectors of nonempty proper subsets of $[n]$. The cut cone can be identified with the space of semimetrics on n points that are isometrically embeddable in ℓ_1-space [DL10, Section 1.1].

We consider the *scaling action* of $(\mathbb{R}_{>0})^n$ on $\mathrm{Sym}_n(\mathbb{R}_{>0})$ defined by the formula

$$(c_1, \ldots, c_n) \cdot (p_{ij}) := (c_i c_j p_{ij}).$$

If $X \subseteq \mathrm{Sym}_n(\mathbb{R}_{>0})$ is stable under the scaling action, then the orbit space of X can be identified with the set $\underline{X}$ of symmetric matrices in X with diagonal entries 1. We say that such matrices in $\underline{X} \subseteq X$ are *normalized*, and define the cone of *reduced bounded ratios* on X by

$$\underline{\mathrm{BR}}(X) := \left\{ (\alpha_{ij})_{1 \leqslant i < j \leqslant n} \;\middle|\; \text{there is } c > 0 \text{ such that } \prod_{1 \leqslant i < j \leqslant n} p_{ij}^{\alpha_{ij}} \leqslant c \text{ for all } (p_{ij}) \in \underline{X} \right\}.$$

As before, the optimal bounding constants $\underline{f}(\alpha)$ for reduced bounded ratios are defined to be the infimum of all possible constants c satisfying the displayed inequality for all (p_{ij}) in $\underline{X}$. It is straightforward to check that $\underline{\mathrm{BR}}(X)$ is a convex cone in $\mathbb{R}^{\binom{n}{2}}$ and $\underline{f}(\alpha)$ is a log-convex homogeneous function of degree 1 on the cone of reduced bounded ratios.

If α is a bounded ratio on X, then the corresponding monomial p^α must be invariant under the scaling action of $(\mathbb{R}_{>0})^n$. In other words, if $\alpha \in \mathrm{BR}(X)$, then

$$2\alpha_{ii} = -\sum_{j \neq i} \alpha_{ij} \;\text{ for all } i.$$

Therefore, the projection $\pi : \mathbb{R}^{\binom{n+1}{2}} \to \mathbb{R}^{\binom{n}{2}}$ that omits the diagonal entries restricts to a bijection between $\mathrm{BR}(X)$ and $\underline{\mathrm{BR}}(X)$. Moreover, we have $f(\alpha) = \underline{f}(\pi(\alpha))$ for any bounded ratio α.

Theorem B. The cone of reduced bounded ratios on $\mathbb{L}_n^+$ is dual of the cut cone:

$$\underline{\mathrm{BR}}(\mathbb{L}_n^+) = \left\{ (\alpha_{ij})_{1 \leqslant i < j \leqslant n} \;\middle|\; \sum_{1 \leqslant i < j \leqslant n} \alpha_{ij}\beta_{ij} \leqslant 0 \;\text{ for all } (\beta_{ij})_{1 \leqslant i < j \leqslant n} \in \mathrm{Cut}_n \right\} \subseteq \mathbb{R}^{\binom{n}{2}}.$$

In particular, the cone of bounded ratios on $\mathbb{L}_n^+$ is a finitely generated integral polyhedral cone.

Thus, the primitive bounded ratios on $\mathbb{L}_n^+$ is in bijection with the facets of Cut_n. Karp and Papadimitriou showed in [KP82] that there is no computationally tractable description of all facets of the cut cone unless NP $=$ coNP. Consequently, obtaining a complete and explicit description of the primitive bounded ratios on $\mathbb{L}_n^+$ is likely to be a challenging problem.

Nevertheless, several interesting classes of facets of Cut_n are known, the first among which is given by the *hypermetric inequalities* [DL10, Chapter 28]: For any vector $h = (h_i)_{1 \leqslant i \leqslant n}$ satisfying $\sum_{i=1}^n h_i = 1$, we have

$$\sum_{1 \leqslant i < j \leqslant n} h_i h_j \beta_{ij} \leqslant 0 \;\text{ for all } (\beta_{ij})_{1 \leqslant i < j \leqslant n} \text{ in } \mathrm{Cut}_n.$$

Thus, we may deduce the following statement from Theorem B.

Corollary 1.6. If $\sum_{i=1}^n h_i = 1$, the vector $(h_i h_j)_{1 \leqslant i < j \leqslant n}$ is a reduced bounded ratio on $\mathbb{L}_n^+$.

Bounded ratios of this type are called *hypermetric*. Determining the optimal bounding constants for hypermetric bounded ratios as a function of h is an interesting and challenging problem. The hypermetric inequality for $h = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ gives the triangular ratio $\alpha^{12|3}$, and the hypermetric inequality for $h = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5$ gives the pentagonal ratio $\alpha^{123|45}$. Up to S_n-symmetry, two additional hypermetric inequalities are needed to describe all facets of Cut_6 [DL10, Remark 15.2.11]. These hypermetric inequalities correspond to

$$h = 2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6 \text{ and } h = -2\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6.$$

For $n \geqslant 7$, the cut cone has facets that do not correspond to any hypermetric inequality; see [DL10, Section 30.6] for a complete list of facets of Cut_7. For a comprehensive discussion of the facets of the cut cone, we refer the reader to Deza and Laurent's monograph [DL10, Part V].

We begin by outlining the proof techniques for our main theorem and situating them within a broader context. Brändén and the second author introduced *Lorentzian polynomials* [BH20], a class of homogeneous polynomials that unify and extend various notions of log-concavity arising in combinatorics and geometry. A quadratic Lorentzian polynomial is, by definition, a quadratic form whose Hessian matrix is Lorentzian. Meanwhile, Baker and Bowler developed a unifying algebraic framework for *matroids over tracts* [BB19], which simultaneously generalized many variants of matroids. Building on this framework, the authors of [BHKL] identified the space of Lorentzian polynomials with given support with the corresponding thin Schubert cell in the Grassmannian over the *triangular hyperfield*, up to homeomorphism. This enabled the authors to use Gromov's theorem on δ-hyperbolic spaces in [BHKL25] to derive new results on Lorentzian polynomials. Our proof is inspired by this perspective, which we now explain in concrete terms.

Let $\mathrm{Sym}_n(\mathbb{R}_{\geqslant 0})$ denote the set of $n \times n$ symmetric matrices with nonnegative entries.

Definition 1.7. Let $\Delta_n(\mathbb{T}_0)$ be the set of matrices $(p_{ij}) \in \mathrm{Sym}_n(\mathbb{R}_{\geqslant 0})$ such that

$$\text{the maximum among } p_{ij}p_{kl}, p_{ik}p_{jl}, p_{il}p_{jk} \text{ is achieved at least twice for any } i,j,k,l \in [n].$$

For a positive real number p, let $\Delta_n(\mathbb{T}_p)$ be the set of matrices $(p_{ij}) \in \mathrm{Sym}_n(\mathbb{R}_{\geqslant 0})$ such that

$$(p_{ij}p_{kl})^{1/p} \leqslant (p_{ik}p_{jl})^{1/p} + (p_{il}p_{jk})^{1/p} \text{ for any } i,j,k,l \in [n].$$

For any nonnegative real number p, let $\Delta_n^+(\mathbb{T}_p)$ denote the intersection $\Delta_n(\mathbb{T}_p) \cap \mathrm{Sym}_n(\mathbb{R}_{>0})$.

Note that $\Delta_n^+(\mathbb{T}_p)$ is invariant under the scaling action of $(\mathbb{R}_{>0})^n$. As before, we identify the orbit space with the set $\underline{\Delta}_n^+(\mathbb{T}_p)$ of matrices in $\Delta_n^+(\mathbb{T}_p)$ with diagonal entries 1. A key observation from [BHKL] is that

$$\underline{\Delta}_n^+(\mathbb{T}_0) \subseteq \underline{\mathbb{L}}_n^+ \subseteq \underline{\Delta}_n^+(\mathbb{T}_2),$$

where $\underline{\mathbb{L}}_n^+$ is the set of $n \times n$ Lorentzian matrices with diagonal entries 1 and off-diagonal entries positive. By taking the coordinatewise logarithm and omitting the diagonal coordinates, we have

$$\log \underline{\Delta}_n^+(\mathbb{T}_0) \subseteq \log \underline{\mathbb{L}}_n^+ \subseteq \log \underline{\Delta}_n^+(\mathbb{T}_2) \text{ in } \mathbb{R}^{\binom{n}{2}}.$$

We observe in Section 3 that $\log \underline{\Delta}_n^+(\mathbb{T}_2)$ parametrizes Gromov's δ-hyperbolic metrics on n points for $\delta = \log 2$, while $\log \underline{\Delta}_n^+(\mathbb{T}_0)$ parametrizes 0-hyperbolic metrics on n points, which are precisely the tree metrics. It follows from Gromov's tree approximation theorem [Gro87, Section 6.1] that the two spaces are within finite Hausdorff distance of one another. Consequently,

$$\underline{\mathrm{BR}}(\Delta_n^+(\mathbb{T}_0)) = \underline{\mathrm{BR}}(\mathbb{L}_n^+) = \underline{\mathrm{BR}}(\Delta_n^+(\mathbb{T}_p)).$$

Since Cut_n is the convex hull of the space of tree metrics on n points, we conclude that $\underline{\mathrm{BR}}(\mathbb{L}_n^+)$ and Cut_n are dual to each other; see Section 3 for further details.

Having understood the cone of bounded ratios for Lorentzian matrices, it is natural to ask how the optimal bounding constants $f(\alpha)$ behave as a function on this cone. As mentioned before,

$$\log f(\alpha+\beta) \leqslant \log f(\alpha) + \log f(\beta) \text{ and } \log f(c\alpha) = c \log f(\alpha).$$

In Theorem C below, we give an explicit description of the function $f(\alpha)$ when $n = 3$.

Recall that the cone of bounded ratios $\mathrm{BR}(\mathbb{L}_3^+)$ has three extremal rays $\alpha^{23|1}, \alpha^{13|2}, \alpha^{12|3}$ corresponding to the optimal inequalities

$$\frac{p_{23}p_{11}}{p_{12}p_{13}} \leqslant 2, \quad \frac{p_{13}p_{22}}{p_{12}p_{23}} \leqslant 2, \quad \frac{p_{12}p_{33}}{p_{13}p_{23}} \leqslant 2.$$

Since $\log f$ is a homogeneous function of degree 1, it is enough to determine

$$f(a,b,c) \coloneqq f(a \cdot \alpha^{23|1} + b \cdot \alpha^{13|2} + c \cdot \alpha^{12|3}) \text{ when } a+b+c = 1 \text{ and } a,b,c \geqslant 0.$$

Theorem C. Let a, b, c be nonnegative numbers such that $a + b + c = 1$.

(1) If $a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \leqslant 0$, then $f(a,b,c) = 1$.

(2) If $a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \geqslant 0$, $a \geqslant b$, $a \geqslant c$, then

$$f(a,b,c) = 2 \cdot a^a \cdot b^b \cdot c^c \cdot (2a-1)^{2a-1} \cdot (1-2b)^{2b-1} \cdot (1-2c)^{2c-1}.$$

See Figure 1 for an illustration.

We conclude with a general conjecture on the optimal bounding constant $f(\alpha)$.

Definition 1.8. We say that a reduced bounded ratio on $\mathbb{L}_n^+$ is *normalized* if the sum of its coordinate is -1.

For example, reduced bounded ratios of the form $\pi(\alpha^{ij|k}) = \mathbf{e}_{ij} - \mathbf{e}_{ik} - \mathbf{e}_{jk}$ are normalized. Since the set of Lorentzian matrices is preserved when a diagonal entry is replaced by 0, the coordinate sum of any nonzero reduced bounded ratio is negative. Thus, any nonzero reduced bounded ratio is uniquely a positive multiple of a normalized reduced bounded ratio.

Conjecture 1.9. The optimal bounding constant for any normalized reduced bounded ratio on $\mathbb{L}_n^+$ is at most 2.

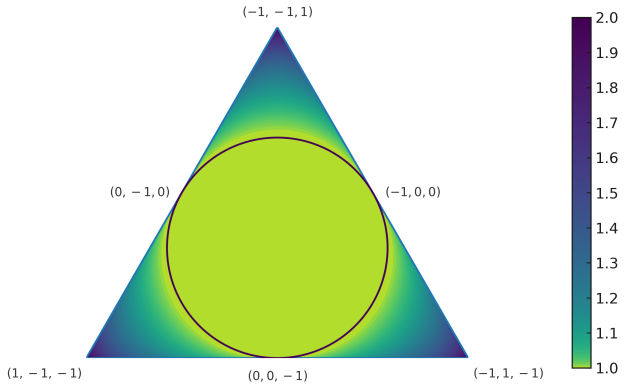

FIGURE 1. Optimal bounding constants for an equilateral triangle cross section of $\underline{\mathrm{BR}}(\mathbb{L}_3^+)$. The midpoints of the three edges correspond to the Alexandrov–Fenchel type inequalities. Inside the inscribed circle, the optimal bounding constant is 1. Outside of the inscribed circle, the optimal bounding constants are given by an entropy-like function.

Using log-convexity of the optimal bounding constants, Conjecture 1.9 can be reduced to the case of extremal bounded ratios. Thus, by Theorem A, we know that Conjecture 1.9 holds for $n \leqslant 5$. See Section 5 for a closely related conjecture for rank 2 Lorentzian matrices.

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2. THE PENTAGONAL INEQUALITY

We prove Theorem A. The primary difficulty in proving this theorem lies in determining the optimal bounding constant for the pentagonal inequality. The statement that $\mathrm{BR}(\mathbb{L}_5^+)$ is generated by the pentagonal ratios follows from Theorem B and the fact that the facets of Cut_5 are defined by the pentagonal ratios [DL10, Section 30.6].

We begin by observing that the constant 4 in the pentagonal inequality is, if valid, best possible. Let t be a nonnegative real number, and consider the matrix

$$M(t) \coloneqq \begin{pmatrix} 0 & 1 & 1 & t & 2+t \\ 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2+t & t \\ t & 2 & 2+t & 4t & 4+4t \\ 2+t & 2 & t & 4+4t & 4t \end{pmatrix}.$$

It is straightforward to check that $M(t)$ has rank 3 for $t \geqslant 0$. Since the leading 3×3 principal submatrix of M is nondegenerate and Lorentzian, we see that $M(t)$ is Lorentzian for any $t \geqslant 0$. We have

$$\lim_{t \to 0} \frac{p_{12}p_{13}p_{23}p_{44}p_{45}p_{55}}{p_{14}p_{15}p_{24}p_{25}p_{34}p_{35}} = \lim_{t \to 0} \frac{(4t)^2(4+4t)}{4t^2(2+t)^2} = 4.$$

Since any Lorentzian matrix is a limit of Lorentzian matrices with positive entries [BH20, Section 2], there is a family in $\mathbb{L}_5^+$ on which the evaluation of the pentagonal ratio limits to 4.

We prepare the proof of the pentagonal inequality with a few auxiliary lemmas.

Lemma 2.1. Suppose that A is a symmetric matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 1 & p_{14} & p_{15} \\ 1 & 0 & 1 & p_{24} & p_{25} \\ 1 & 1 & 0 & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & p_{44} & p_{45} \\ p_{15} & p_{25} & p_{35} & p_{45} & p_{55} \end{pmatrix}.$$

Then there exist unique matrices B and C such that $A = BCB^T$, where

$$BCB^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b_{14} & b_{24} & b_{34} & 1 & 0 \\ b_{15} & b_{25} & b_{35} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & c_{45} \\ 0 & 0 & 0 & c_{45} & c_{55} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & b_{14} & b_{15} \\ 0 & 1 & 0 & b_{24} & b_{25} \\ 0 & 0 & 1 & b_{34} & b_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. For any $p_{14}, p_{15}, p_{24}, p_{25}, p_{34}, p_{35}$, there exist unique $b_{14}, b_{15}, b_{24}, b_{25}, b_{34}, b_{35}$ such that

$$\begin{pmatrix} b_{14} & b_{24} & b_{34} \\ b_{15} & b_{25} & b_{35} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -p_{14} & -p_{24} & -p_{34} \\ -p_{15} & -p_{25} & -p_{35} \end{pmatrix}.$$

This defines the invertible lower triangular matrix B. The matrix C is uniquely determined by the condition $C = B^{-1}A(B^T)^{-1}$, and it is straightforward to check that C has the required block diagonal shape. □

Lemma 2.2. Consider the symmetric 2×2 matrices

$$M_1 = \begin{pmatrix} \mu & \nu \\ \nu & \xi \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

If M_1 is Lorentzian, M_2 is negative semidefinite, and $M_1 + M_2$ has nonnegative entries, then

$$(\mu + \alpha)(\nu + \beta)(\xi + \gamma) \leqslant \mu\nu\xi.$$

Proof. It suffices to show that $(\mu + t\alpha)(\nu + t\beta)(\xi + t\gamma)$ is a decreasing function for $0 \leqslant t \leqslant 1$. Since $M_1 + tM_2$is a Lorentzian matrix for any $0 \leqslant t \leqslant 1$, it suffices to show that

$$\frac{d}{dt}(\mu + t\alpha)(\nu + t\beta)(\xi + t\gamma)\Big|_{t=0} \leqslant 0.$$

By taking limits, we may reduce to the case when M_2 is negative definite. Since the claimed inequality is invariant under the symmetric scaling of rows and columns, we may further suppose that $\alpha = \gamma = -1$. Then $\beta \leqslant 1$, and hence

$$\frac{d}{dt}(\mu + t\alpha)(\nu + t\beta)(\xi + t\gamma)\Big|_{t=0} = \mu\nu\gamma + \mu\xi\beta + \nu\xi\alpha \leqslant -\mu\nu + \mu\xi - \nu\xi.$$

Since M_1 is Lorentzian, $\mu\xi \leqslant \nu^2$, so the right-hand side satisfies

$$-\mu\nu + \mu\xi - \nu\xi \leqslant -2\nu\sqrt{\mu\xi} + \mu\xi \leqslant -\mu\xi \leqslant 0. \qquad \square$$

Lemma 2.3. For positive numbers $x_1, x_2, x_3, y_1, y_2, y_3$, set

$$\begin{aligned} X &\coloneqq 2x_1x_2 + 2x_1x_3 + 2x_2x_3 - x_1^2 - x_2^2 - x_3^2, \\ Y &\coloneqq 2y_1y_2 + 2y_1y_3 + 2y_2y_3 - y_1^2 - y_2^2 - y_3^2, \\ Z &\coloneqq x_1(y_2 + y_3 - y_1) + x_2(y_1 + y_3 - y_2) + x_3(y_1 + y_2 - y_3). \end{aligned}$$

If X, Y, Z are nonnegative, then $XYZ < 32x_1x_2x_3y_1y_2y_3$.

We note that the final inequality may fail without the assumption that X, Y, Z are nonnegative, for example, when $(x_1, x_2, x_3, y_1, y_2, y_3) = (6, 1, 1, 1, 1, 6)$.

Proof. First suppose that (x_1, x_2, x_3) and (y_1, y_2, y_3) satisfy the triangle inequalities. Setting $u_i \coloneqq x_j + x_k - x_i > 0$ and $v_i \coloneqq y_j + y_k - y_i > 0$, we have

$$X = u_1u_2 + u_1u_3 + u_2u_3, \;\; Y = v_1v_2 + v_1v_3 + v_2v_3, \;\; 2Z = (u_2 + u_3)v_1 + (u_1 + u_3)v_2 + (u_1 + u_2)v_3.$$

The goal is to show that

$$\begin{aligned} (u_1u_2 + u_1u_3 + u_2u_3)(v_1v_2 + v_1v_3 + v_2v_3)((u_2 + u_3)v_1 + (u_1 + u_3)v_2 + (u_1 + u_2)v_3) \\ < (u_1 + u_2)(u_1 + u_3)(u_2 + u_3)(v_1 + v_2)(v_1 + v_3)(v_2 + v_3). \end{aligned}$$

The right-hand side minus the left-hand side can be simplified to

$$u_1^2u_2v_1^2v_2 + u_1^2u_3v_1^2v_2 + u_1^2u_2v_1^2v_3 + u_1^2u_3v_1^2v_3 + u_1u_2^2v_1v_2^2 + u_2^2u_3v_1v_2^2 + u_1u_2^2v_2^2v_3 + u_2^2u_3v_2^2v_3 + u_1u_3^2v_2v_3^2 + u_2u_3^2v_2v_3^2 + u_1u_3^2v_1v_3^2 + u_2u_3^2v_1v_3^2 - 2u_1u_2u_3v_1v_2v_3.$$

Notice that at most one of the nonnegative numbers u_1, u_2, u_3 is zero, and similarly for v_1, v_2, v_3. Thus, the sum of the positive coefficient terms is strictly positive. If any of the u_i is zero, then the difference is positive. If u_1, u_2, u_3 are all positive, then the only negative term $-2u_1u_2u_3v_1v_2v_3$ can be grouped with the terms $u_1^2u_2v_1^2v_2$ and $u_2u_3^2v_2v_3^2$ to form a square, and the remaining sum is positive.

Thus, without loss of generality, it is sufficient to justify the inequality under the assumption that $x_1 > x_2 + x_3$ and $x_2 \geqslant x_3$. If $y_3 < y_2$, then $(y_1 + y_3 - y_2) < (y_1 + y_2 - y_3)$. In this case,

$$x_2(y_1 + y_3 - y_2) + x_3(y_1 + y_2 - y_3) < x_2(y_1 + y_2 - y_3) + x_3(y_1 + y_3 - y_2) \text{ and}$$

$$x_1(y_2+y_3-y_1)+x_2(y_1+y_3-y_2)+x_3(y_1+y_2-y_3) < x_1(y_2+y_3-y_1)+x_2(y_1+y_2-y_3)+x_3(y_1+y_3-y_2).$$

Thus, swapping y_2 and y_3 makes Z larger without changing X, Y, and $x_1x_2x_3y_1y_2y_3$. Therefore, it is enough to consider the case when $x_1 > x_2 + x_3$ and $x_2 \geqslant x_3$ and $y_3 \geqslant y_2$.

Since $X = -(x_1-x_2-x_3)^2+4x_2x_3$, replacing x_1 by x_2+x_3 makes X larger without changing Y. We check that the same replacement makes the ratio Z/x_1 larger:

$$(y_2 + y_3 - y_1) + (x_2(y_1 + y_3 - y_2) + x_3(y_1 + y_2 - y_3))x_1^{-1} < (y_2 + y_3 - y_1) + (x_2(y_1 + y_3 - y_2) + x_3(y_1 + y_2 - y_3))(x_2 + x_3)^{-1}.$$

In fact, we have $x_1 > x_2 + x_3$ and

$$x_2(y_1 + y_3 - y_2) + x_3(y_1 + y_2 - y_3) = (x_2 + x_3)y_1 + (x_2 - x_3)(y_3 - y_2) \geqslant (x_2 + x_3)y_1 > 0.$$

Since

$$Z = y_1(x_2 + x_3 - x_1) + y_2(x_1 + x_3 - x_2) + y_3(x_1 + x_2 - x_3),$$

replacing x_1 by $x_2 + x_3$ preserves the nonnegativity of Z, this reduces the problem to the case when (x_1, x_2, x_3) satisfies the triangle inequalities. Repeating the argument for (y_1, y_2, y_3), we reduce the problem to the case where both (x_1, x_2, x_3) and (y_1, y_2, y_3) satisfy the triangle inequalities, which was considered before. □

Proof of Theorem A. To show the pentagonal ratio is at most 4, it suffices to consider Lorentzian matrices of the form

$$A = \begin{pmatrix} 0 & 1 & 1 & p_{14} & p_{15} \\ 1 & 0 & 1 & p_{24} & p_{25} \\ 1 & 1 & 0 & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & p_{44} & p_{45} \\ p_{15} & p_{25} & p_{35} & p_{45} & p_{55} \end{pmatrix}.$$

Indeed, we can set the first three diagonal entries to be zero, since decreasing diagonal entries only decreases eigenvalues. We can then rescale rows and columns using the scaling action to

obtain matrix A. Both operations do not affect the value of the pentagonal ratio. We need to show that

$$\frac{p_{44}p_{45}p_{55}}{p_{14}p_{24}p_{34}p_{15}p_{25}p_{35}} \leqslant 4.$$

By Lemma 2.1, there is a unique factorization

$$A = B \cdot C \cdot B^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b_{14} & b_{24} & b_{34} & 1 & 0 \\ b_{15} & b_{25} & b_{35} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & c_{45} \\ 0 & 0 & 0 & c_{45} & c_{55} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & b_{14} & b_{15} \\ 0 & 1 & 0 & b_{24} & b_{25} \\ 0 & 0 & 1 & b_{34} & b_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since A is Lorentzian, C has at most one positive eigenvalue. The upper-left block of C has exactly one positive eigenvalue, it follows that the lower-right block of C is negative semidefinite. For $0 \leqslant t \leqslant 1$, we consider the matrices $A(t)$ and $C(t)$ defined by

$$A(t) = B \cdot C(t) \cdot B^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ b_{14} & b_{24} & b_{34} & 1 & 0 \\ b_{15} & b_{25} & b_{35} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & tc_{44} & tc_{45} \\ 0 & 0 & 0 & tc_{45} & tc_{55} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & b_{14} & b_{15} \\ 0 & 1 & 0 & b_{24} & b_{25} \\ 0 & 0 & 1 & b_{34} & b_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Matrices B and B^T correspond to sequences of row and column operations respectively, so

$$A(t) = \begin{pmatrix} 0 & 1 & 1 & p_{14} & p_{15} \\ 1 & 0 & 1 & p_{24} & p_{25} \\ 1 & 1 & 0 & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & X + tc_{44} & Y + tc_{45} \\ p_{15} & p_{25} & p_{35} & Y + tc_{45} & Z + tc_{55} \end{pmatrix}$$

for some X, Y, Z independent of t. The rank of $C(0)$ is 3, and hence the rank of $A(0)$ is 3. Thus, all the 4×4 minors of $A(0)$ are zero, which implies that

$$X = \frac{2p_{14}p_{24} + 2p_{14}p_{34} + 2p_{24}p_{34} - p_{14}^2 - p_{24}^2 - p_{34}^2}{2},$$
$$Z = \frac{2p_{15}p_{25} + 2p_{15}p_{35} + 2p_{25}p_{35} - p_{15}^2 - p_{25}^2 - p_{35}^2}{2},$$
$$Y = \frac{p_{14}p_{25} + p_{14}p_{35} + p_{24}p_{15} + p_{24}p_{35} + p_{34}p_{15} + p_{34}p_{25} - p_{14}p_{15} - p_{24}p_{25} - p_{34}p_{35}}{2}.$$

Since c_{44} and c_{55} are nonpositive and $p_{44} = X + c_{44}$ and $p_{55} = Z + c_{55}$ are nonnegative,

$$X + tc_{44} \geqslant 0 \text{ and } Z + tc_{55} \geqslant 0 \text{ for all } 0 \leqslant t \leqslant 1.$$

In particular, X and Z are nonnegative.

We now reduce to the case when Y is nonnegative as well. Since $C(t)$ has at most one positive eigenvalue for all $0 \leqslant t \leqslant 1$, the same holds for the matrix $A(t)$, and hence

$$\det \begin{pmatrix} X + tc_{44} & Y + tc_{45} \\ Y + tc_{45} & Z + tc_{55} \end{pmatrix} \leqslant 0.$$

Suppose Y is negative. Since $Y + c_{45} \geqslant 0$, we must have $c_{45} > 0$ and there is a unique positive $t_1 \leqslant 1$ such that $Y + t_1 c_{45} = 0$. The determinantal inequality above says that

$$X + t_1 c_{44} = 0 \text{ or } Z + t_1 c_{55} = 0.$$

If $t_1 < 1$, this implies that $c_{44} = 0$ or $c_{55} = 0$, and hence $c_{45} = 0$ by the negative semidefiniteness of the lower-right block of C, reaching a contradiction. Thus

$$p_{45} = Y + c_{45} = 0,$$

in which case the pentagonal inequality holds trivially.

The remaining case to consider is when $X, Y, Z \geqslant 0$, that is, when the matrix $A(0)$ is Lorentzian. We apply Lemma 2.2 to the sum

$$\begin{pmatrix} X & Y \\ Y & Z \end{pmatrix} + \begin{pmatrix} c_{44} & c_{45} \\ c_{45} & c_{55} \end{pmatrix} = \begin{pmatrix} p_{44} & p_{45} \\ p_{45} & p_{55} \end{pmatrix},$$

and conclude that $p_{44}p_{45}p_{55} \leqslant XYZ$. Thus, the pentagonal ratio of $A(0)$ is greater or equal to the pentagonal ratio of A. Thus, it is enough to show the pentagonal inequality for $A(0)$, which reads

$$XYZ \leqslant 4p_{14}p_{24}p_{34}p_{15}p_{25}p_{35}.$$

This is precisely the content of Lemma 2.3. □

It is interesting to compare the above analysis with that for the pentagonal ratio on $\Delta_n^+(\mathbb{T}_p)$, which is much simpler.

Proposition 2.4. Let nonnegative p and any matrix $(p_{ij}) \in \Delta_n^+(\mathbb{T}_p)$, we have

$$\frac{p_{12}p_{13}p_{23}p_{45}p_{44}p_{55}}{p_{14}p_{15}p_{24}p_{25}p_{34}p_{35}} \leqslant 8^p.$$

The equality is achieved by the matrix

$$\begin{pmatrix} 0 & 2^{\frac{p}{2}} & 2^{\frac{p}{2}} & 1 & 1 \\ 2^{\frac{p}{2}} & 0 & 2^{\frac{p}{2}} & 1 & 1 \\ 2^{\frac{p}{2}} & 2^{\frac{p}{2}} & 0 & 1 & 1 \\ 1 & 1 & 1 & 2^{\frac{p}{2}} & 2^{\frac{p}{2}} \\ 1 & 1 & 1 & 2^{\frac{p}{2}} & 2^{\frac{p}{2}} \end{pmatrix} \in \Delta_n(\mathbb{T}_p).$$

Proof. We first consider the case when p is positive. We have

$$(p_{44}p_{13})^{1/p} \leqslant 2(p_{14}p_{34})^{1/p},\ (p_{55}p_{23})^{1/p} \leqslant 2(p_{25}p_{35})^{1/p},\ \text{so } (p_{44}p_{55}p_{13}p_{23})^{1/p} \leqslant 4(p_{14}p_{34}p_{25}p_{35})^{1/p}.$$

Similarly, we have

$$(p_{44}p_{23})^{1/p} \leqslant 2(p_{24}p_{34})^{1/p},\ (p_{55}p_{13})^{1/p} \leqslant 2(p_{15}p_{35})^{1/p},\ \text{so}\ (p_{44}p_{55}p_{13}p_{23})^{1/p} \leqslant 4(p_{24}p_{34}p_{15}p_{35})^{1/p}.$$

Using the triangle inequality $(p_{12}p_{45})^{1/p} \leqslant (p_{14}p_{25})^{1/p} + (p_{15}p_{24})^{1/p}$, we get

$$\begin{aligned}(p_{12}p_{13}p_{23}p_{44}p_{55}p_{45})^{1/p} &\leqslant (p_{44}p_{55}p_{13}p_{23}p_{14}p_{25})^{1/p} + (p_{44}p_{55}p_{13}p_{23}p_{15}p_{24})^{1/p}\\ &\leqslant 8(p_{14}p_{15}p_{24}p_{25}p_{34}p_{35})^{1/p}.\end{aligned}$$

Since $\Delta_n(\mathbb{T}_0) \subseteq \Delta_n(\mathbb{T}_p)$ for all $p > 0$, the case of $p = 0$ follows from taking the limit $p \to 0$. □

3. Lorentzian matrices and δ-hyperbolic spaces

We give a detailed proof of Theorem B outlined in the introduction.

Proposition 3.1. For any $n > 0$, we have $\Delta_n(\mathbb{T}_0) \subseteq \mathbb{L}_n \subseteq \Delta_n(\mathbb{T}_2)$.

In the language of [BKH+], the proposition states that every rank 2 polymatroid over $\mathbb{T}_0$ is Lorentzian, and every Lorentzian matrix is a rank 2 polymatroid over $\mathbb{T}_2$.

Proof. The first inclusion is a special case of [BH20, Corollary 3.16] for quadratic polynomials. The second inclusion follows from [BHKL25], whose proof we reproduce here.

Suppose (p_{ij}) is a Lorentzian matrix. If i, j, k, l are distinct indices, we consider the matrix

$$\begin{pmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ p_{ij} & 0 & p_{jk} & p_{jl} \\ p_{ik} & p_{jk} & 0 & p_{kl} \\ p_{il} & p_{jl} & p_{kl} & 0 \end{pmatrix}$$

obtained from replacing the diagonal entries of a principal submatrix of (p_{ij}) by zero. This matrix is Lorentzian with

$$\begin{aligned}\det = -\left(\sqrt{p_{ij}p_{kl}} + \sqrt{p_{ik}p_{jl}} + \sqrt{p_{il}p_{jk}}\right) &\cdot \left(-\sqrt{p_{ij}p_{kl}} + \sqrt{p_{ik}p_{jl}} + \sqrt{p_{il}p_{jk}}\right)\\ &\cdot \left(\sqrt{p_{ij}p_{jl}} - \sqrt{p_{ik}p_{jl}} + \sqrt{p_{il}p_{jk}}\right) \cdot \left(\sqrt{p_{ij}p_{kl}} + \sqrt{p_{ik}p_{jl}} - \sqrt{p_{il}p_{jk}}\right) \leqslant 0,\end{aligned}$$

so $\sqrt{p_{ij}p_{kl}}$, $\sqrt{p_{ik}p_{jl}}$, $\sqrt{p_{il}p_{jk}}$ form three sides of a triangle [HHM+, Proposition 3.1].

If there are exactly three distinct indices among i, j, k, l, we may suppose $k = l$. We need to show that $\sqrt{p_{ij}p_{kk}}$, $\sqrt{p_{ik}p_{jk}}$, $\sqrt{p_{ik}p_{jk}}$ form sides of a triangle, that is, $\sqrt{p_{ij}p_{kk}} \leqslant 2\sqrt{p_{ik}p_{jk}}$. This follows from the Lorentzian condition for (p_{ij}) because

$$\det \begin{pmatrix} 0 & p_{ij} & p_{ik} \\ p_{ij} & 0 & p_{jk} \\ p_{ik} & p_{jk} & p_{kk} \end{pmatrix} = p_{ij}(2p_{ik}p_{jk} - p_{ij}p_{kk}) \geqslant 0.$$

If three of i, j, k, l are equal, then the statement is trivial. Thus, the only remaining nontrivial case is when $i = k$ and $j = l$. We need to check that $\sqrt{p_{ii}p_{jj}} \leqslant 2p_{ij}$, and this follows from

$$\det \begin{pmatrix} p_{ii} & p_{ij} \\ p_{ij} & p_{jj} \end{pmatrix} = p_{ii}p_{jj} - p_{ij}^2 \leqslant 0. \qquad \square$$

We now recall the definition of tree metric in terms of phylogenetic trees, following [MS15, Section 4.3]. A *phylogenetic* tree on n leaves is a tree with n labeled leaves and no vertices of degree 2. We say that $(d_{ij})_{1 \leqslant i < j \leqslant n}$ is a *tree metric* if there is a phylogenetic tree τ with n leaves and nonnegative edge lengths ℓ on τ such that

$$\mathrm{d}_{ij} = \big(\text{the sum of all } \ell(e) \text{ over all edges } e \text{ in the unique path from } i \text{ to } j \text{ in } \tau\big).^{2}$$

Every tree metric uniquely determines a phylogenetic tree together with the assignment of edge lengths.

Lemma 3.2. The following statements hold for $\underline{\Delta}_n^+(\mathbb{T}_0)$.

(1) The set of coordinatewise logarithms $\log \underline{\Delta}_n^+(\mathbb{T}_0) \subseteq \mathbb{R}^{\binom{n}{2}}$ is equal to the space of tree metrics on n leaves.

(2) The convex hull of $\log \underline{\Delta}_n^+(\mathbb{T}_0)$ in $\mathbb{R}^{\binom{n}{2}}$ is equal to the cut cone Cut_n.

(3) The cone of reduced bounded ratios $\underline{\mathrm{BR}}(\underline{\Delta}_n^+(\mathbb{T}_0))$ is the dual of the cut cone Cut_n.

Proof. Note that the points in $\log \underline{\Delta}_n^+(\mathbb{T}_0)$ are of the form $(q_{ij}) = (\log p_{ij})$, where

$$q_{ii} = \log p_{ii} = 0 \text{ and } q_{ij} = \log p_{ij} \geqslant \log \sqrt{p_{ii}p_{jj}} = 0,$$

and in addition, for distinct indices i, j, k, l, we have

$$q_{ij} + q_{kl} = \log p_{ij}p_{kl} \leqslant \log \max(p_{ik}p_{jl}, p_{il}p_{jk}) = \max(q_{ik} + q_{jl}, q_{il} + q_{jk}).$$

Thus, the first statement is the characterization of tree metrics by the "four-point condition", which is exactly the logarithm of the condition defining $\underline{\Delta}_n^+(\mathbb{T}_0)$ [MS15, Lemma 4.3.6].

For the second statement, observe that the cut vector $\delta(S)$ is the tree metric corresponding to the phylogenetic tree that consists of a single edge of length 1 that connects the leaves in S with the leaves not in S. Therefore, the extremal rays of Cut_n are in $\log \underline{\Delta}_n^+(\mathbb{T}_0)$. For the other inclusion, consider a tree metric d given by a phylogenetic tree τ and a length function ℓ. For each edge e of τ, let $S(e)$ be the set of leaves that remain connected to the distinguished leaf 1 after removing e from τ. Then d admits the decomposition

$$d = \sum_e \ell(e)\delta(S(e)),$$

showing that d is in the cut cone. This proves the second statement.

[2] In [MS15, Section 4.3], edge lengths are allowed to be non-positive. A vector d is a tree metric in this sense if and only if the sum of d and a constant vector of large positive value is a tree metric in the more restrictive sense above.

For the third statement, notice that

$$\begin{aligned}\underline{\mathrm{BR}}\left(\underline{\Delta}_n^+(\mathbb{T}_0)\right) &= \left\{\alpha \in \mathbb{R}^{\binom{n}{2}} \;\middle|\; \exists c > 0 \text{ such that } \sum_{1\leqslant i<j\leqslant n} \alpha_{ij}q_{ij} \leqslant \log c \text{ for all } q \in \log \underline{\Delta}_n^+(\mathbb{T}_0)\right\} \\ &= \left\{\alpha \in \mathbb{R}^{\binom{n}{2}} \;\middle|\; \exists c > 0 \text{ such that } \sum_{1\leqslant i<j\leqslant n} \alpha_{ij}q_{ij} \leqslant \log c \text{ for all } q \in \mathrm{Cut}_n\right\} \\ &= \left\{\alpha \in \mathbb{R}^{\binom{n}{2}} \;\middle|\; \sum_{1\leqslant i<j\leqslant n} \alpha_{ij}q_{ij} \leqslant 0 \text{ for all } q \in \mathrm{Cut}_n\right\}.\end{aligned}$$

The first equality is the definition of $\underline{\mathrm{BR}}$, the second follows from the fact that Cut_n is the convex hull of $\log \underline{\Delta}_n^+(\mathbb{T}_0)$, and the third follows from the fact that Cut_n is a cone. □

We now review Gromov's tree approximation theorem for δ-hyperbolic spaces. Let δ be a nonnegative real number. A metric d on $[n]$ is called *δ-hyperbolic* if

$$d_{ij} + d_{kl} \leqslant \max\big(d_{ik} + d_{jl} + 2\delta, d_{il} + d_{jk} + 2\delta\big) \text{ for all } i,j,k,l \in [n].$$

As noted above, when $\delta = 0$, the condition is equivalent to d being a tree metric. We will use the following corollary of the tree approximation theorem; see [Gro87, Section 6.1] and [GdlH90, Chapitre 2].

Proposition 3.3. For any positive real number δ and a positive integer n, there is a constant c such that any δ-hyperbolic metric d on $[n]$ can be approximated by a 0-hyperbolic metric d':

$$d_{ij} - c \leqslant d'_{ij} \leqslant d_{ij} \text{ for all } i,j \in [n].$$

Corollary 3.4. For any positive integer n, there is a constant e^c such that for any $(p_{ij}) \in \underline{\Delta}_n^+(\mathbb{T}_2)$, there is $(p'_{ij}) \in \underline{\Delta}_n^+(\mathbb{T}_0)$ satisfying the condition

$$e^{-c}p_{ij} \leqslant p'_{ij} \leqslant p_{ij} \text{ for all } i,j \in [n].$$

Proof. We reformulate the statement in terms of log coordinates, where $q_{ij} = \log p_{ij}$. Since $(p_{ij}) \in \underline{\Delta}_n^+(\mathbb{T}_2)$, we have

(1) $q_{ii} = 0$ for all $i \in [n]$;

(2) $q_{ij} \leqslant q_{ik} + q_{jk} + 2\log 2$ for all distinct $i,j,k \in [n]$;

(3) $q_{ij} + q_{kl} \leqslant \max(q_{ik} + q_{jl} + 2\log 2, q_{il} + q_{jk} + 2\log 2)$ for all distinct $i,j,k,l \in [n]$,

where the last condition follows from

$$\sqrt{p_{ij}p_{kl}} \leqslant \sqrt{p_{ik}p_{jl}} + \sqrt{p_{il}p_{jk}} \leqslant \max(2\sqrt{p_{ik}p_{jl}}, 2\sqrt{p_{il}p_{jk}}).$$

Thus any such (q_{ij}) is a δ-hyperbolic metric on $[n]$ for $\delta = \log 2$. The conclusion follows from Lemma 3.2 and Proposition 3.3. □

For the following lemma, let $X \subseteq Y$ be arbitrary subsets of $n \times n$ symmetric matrices with positive entries.

Lemma 3.5. Suppose that there is a constant e^c such that, for any $(p_{ij}) \in Y$, there is $(p'_{ij}) \in X$ satisfying $e^{-c}p_{ij} \leqslant p'_{ij} \leqslant p_{ij}$ for all $i, j \in [n]$. Then $\mathrm{BR}(X) = \mathrm{BR}(Y)$.

Proof. Since $X \subseteq Y$, we have $\mathrm{BR}(Y) \subseteq \mathrm{BR}(X)$. We prove the reverse inclusion. Given $(p_{ij}) \in Y$, our assumption says that there is $(p'_{ij}) \in X$ such that

$$e^{-c}p_{ij} \leqslant p'_{ij} \leqslant p_{ij} \text{ for all } i, j \in [n].$$

If $\prod_{i \leqslant j} {p'_{ij}}^{\alpha_{ij}}$ is bounded above by a constant, then the same is true for $\prod_{i \leqslant j} {p_{ij}}^{\alpha_{ij}}$. □

Proof of Theorem B. Proposition 3.1, Corollary 3.4, and Lemma 3.5, we have

$$\underline{\mathrm{BR}}(\Delta_n^+(\mathbb{T}_0)) = \underline{\mathrm{BR}}(\mathbb{L}_n^+) = \underline{\mathrm{BR}}(\Delta_n^+(\mathbb{T}_2)).$$

Lemma 3.2 says that this cone is dual to the cut cone Cut_n. □

4. Optimal bounding constants for $\mathbb{L}_3^+$

We determine the optimal bounding constants on $\underline{\mathrm{BR}}(\mathbb{L}_3^+)$.

Proof of Theorem C. We consider the triangular section of the cone of reduced bounded ratios

$$\left\{ a \cdot \alpha^{23|1} + b \cdot \alpha^{13|2} + c \cdot \alpha^{12|3} \;\middle|\; a, b, c \text{ are nonnegative numbers that sum to } 1 \right\} \subseteq \underline{\mathrm{BR}}(\mathbb{L}_3^+).$$

The condition $a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = 0$ defines an inscribed circle in the given triangle, where the three tangent points are given by

$$(a, b, c) \;=\; (1/2, 1/2, 0),\;\; (1/2, 0, 1/2),\;\; (0, 1/2, 1/2).$$

These points are the midpoints of the sides of the triangle, corresponding to the Alexandrov–Fenchel inequalities.

The inscribed circle splits the triangle in four regions. The part within the circle is defined by

$$a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \leqslant 0.$$

Outside the circle, the connected component containing the vertex $\alpha^{23|1}$ is defined by

$$a^2 + b^2 + c^2 - 2ab - 2ac - 2bc > 0,\;\; a \geqslant b,\;\; a \geqslant c.$$

The other two regions outside of the inscribed circle are described similarly.

We may reformulate the objective as the problem of finding the global maximum of

$$R(p_{12}, p_{13}, p_{23}) := (p_{12})^{c-b-a}(p_{13})^{b-c-a}(p_{23})^{a-b-c},$$

subject to certain constrains on p_{12}, p_{13}, p_{23}. More precisely, we want to find the global maximum of $R(p_{12}, p_{13}, p_{23})$ in terms of the parameters a, b, c when $p_{12}, p_{13}, p_{23} \geqslant 1$ and

$$\det \begin{pmatrix} 1 & p_{12} & p_{13} \\ p_{12} & 1 & p_{23} \\ p_{13} & p_{23} & 1 \end{pmatrix} = 1 + 2p_{12}p_{13}p_{23} - p_{12}^2 - p_{13}^2 - p_{23}^2 \geqslant 0.$$

These inequalities are equivalent to the condition that 3×3 matrix (p_{ij}) being Lorentzian.

We apply Lagrange's method to $R(p_{12}, p_{13}, p_{23})$ separately on the three regions outside the inscribed circle. On the region containing $\alpha^{23|1}$, one finds that the interior of the domain has no critical points, and thus the maximum is attained on the boundary. The interesting case is when

$$1 + 2p_{12}p_{13}p_{23} - p_{12}^2 - p_{13}^2 - p_{23}^2 = 0.$$

Since $p_{12}, p_{13}, p_{23} \geqslant 1$, the condition is equivalent to

$$p_{23} = p_{12}p_{13} + \sqrt{(p_{12}-1)(p_{13}-1)}.$$

Thus, setting $x = p_{12}^{-2}$ and $y = p_{13}^{-2}$, the question reduces to finding the global maximum of

$$r(x,y) \coloneqq x^b y^c \left(1 + \sqrt{(1-x)(1-y)}\right)^{a-b-c} \quad \text{when } 0 \leqslant x, y \leqslant 1.$$

The unique critical point of $r(x,y)$ has coordinates

$$x_0 = 4ab/(a+b-c)^2 \text{ and } y_0 = 4ac/(a-b+c)^2,$$

and they satisfy $0 \leqslant x_0, y_0 \leqslant 1$ because $a^2+b^2+c^2-2ab-2ac-2bc > 0$. Evaluating, we get

$$r(x_0, y_0) = 2 \cdot a^a \cdot b^b \cdot c^c \cdot (2a-1)^{2a-1} \cdot (1-2b)^{2b-1} \cdot (1-2c)^{2c-1}.$$

When $a^2+b^2+c^2-2ab-2ac-2bc = 0$, we have $(a+b-c)^2 = 4ab$ and $(a-b+c)^2 = 4ac$. In this case, the critical point is $(1,1)$ and $r(x,y)$ has value 1 at the critical point. Thus, when (a,b,c) lies in the the boundary of the inscribed circle, $f(a,b,c) = 1$. By the log-convexity of $f(a,b,c)$, we conclude that $f(a,b,c) \leqslant 1$ inside the circle. On the other hand, the matrix with all entries 1 is Lorentzian, and any ratio for this matrix equals 1. Thus, $f(a,b,c) = 1$ inside the circle. □

We compare the optimal bounding constants $f(a,b,c)$ for $\mathbb{L}_3^+$ with the corresponding constants $f_p(a,b,c)$ for $\Delta_3^+(\mathbb{T}_p)$, which is piecewise log-linear.

Proposition 4.1. Consider the reduced bounded ratio $\alpha = a\cdot\alpha^{23|1} + b\cdot\alpha^{13|2} + c\cdot\alpha^{12|3}$ on $\Delta_n^+(\mathbb{T}_p)$, where a, b, c are nonnegative numbers. The optimal bounding constant for α is given by

$$f_p(a,b,c) = \begin{cases} 2^{ap} & \text{if } a > b+c, \\ 2^{bp} & \text{if } b > a+c, \\ 2^{cp} & \text{if } c > a+b, \\ 2^{\frac{p(a+b+c)}{2}} & \text{if otherwise.} \end{cases}$$

Proof. Let $q_{ij} = \log_2(p_{ij})$, where i, j are any two distinct indices from $\{1,2,3\}$. In log coordinates, the problem reduces to the maximization of the linear function

$$\ell(a,b,c) \coloneqq a(q_{23} - q_{12} - q_{13}) + b(q_{13} - q_{12} - q_{23}) + c(q_{12} - q_{13} - q_{23}),$$

where the domain given by the linear inequalities

$$\begin{aligned} q_{23} &\leqslant p + q_{12} + q_{13}, & q_{13} &\leqslant p + q_{12} + q_{23}, & q_{12} &\leqslant p + q_{13} + q_{23}, \\ 0 &\leqslant p + q_{12} + q_{12}, & 0 &\leqslant p + q_{13} + q_{13}, & 0 &\leqslant p + q_{23} + q_{23}. \end{aligned}$$

The maximum of $\ell(a,b,c)$ is attained at one of the vertices of the domain. The solution to the system contains four vertices

$$(0,-p/2,-p/2),\ \ (-p/2,0,-p/2),\ \ (-p/2,-p/2,0),\ \text{ and }\ (-p/2,-p/2,-p/2),$$

corresponding to the four cases in the displayed equation for $f_p(a,b,c)$. □

5. Rank 2 Lorentzian matrices

We show that the cone of bounded ratios on Lorentzian matrices is equal to the cone of bounded ratios on rank 2 Lorentzian matrices. The space of rank 2 Lorentzian matrices admits a natural parametrization, which allows us to formulate a stronger notion of positivity called the *subtraction-freeness*. Throughout, we work with $n \times n$ symmetric matrix M with real entries.

Lemma 5.1. The quadratic form $Q(x) = x^T M x$ factors into a product of two linear forms if and only if there is a rank at most 1 matrix B such that $M = \frac{1}{2}(B + B^T)$.

Proof. If the quadratic form $Q(x)$ factors into linear forms, we can rewrite

$$Q(x) = x^T M x = (x^T a)(b^T x) = x^T(ab^T)x = x^T(ba^T)x,$$

where a and b are column vectors. Then $B = ab^T$ is a rank at most 1 matrix and $Q(x) = x^T\left(\frac{B+B^T}{2}\right)x$. Conversely, if $M = \frac{1}{2}(B + B^T)$ for a rank at most 1 matrix B, then

$$Q(x) = x^T M x = x^T\left(\frac{B+B^T}{2}\right)x = x^T B x.$$

Since B has rank at most 1, there are real vectors a and b such that $x^T B x = x^T(ab^T)x = (x^T a)(b^T x)$. □

Lemma 5.2. There is a rank at most 1 matrix B such that $M = \frac{1}{2}(B + B^T)$ if and only if M has rank at most 2 and all 2×2 principal minors of M are nonpositive.

Proof. If $M = \frac{1}{2}(B + B^T)$ for a rank at most 1 matrix B, then, by Lemma 5.1, there is a factorization

$$x^T M x = \left(\sum_{1 \leqslant i \leqslant n} a_i x_i\right)\left(\sum_{1 \leqslant i \leqslant n} b_i x_i\right).$$

Thus, M has rank at most 2 and any 2×2 principal minor of M is nonpositive:

$$(2a_ib_i)(2a_jb_j) - (a_ib_j + a_jb_i)^2 = -a_i^2b_j^2 + 2a_ia_jb_ib_j - a_j^2b_i^2 \leqslant 0.$$

For the converse, suppose that M has rank at most 2 and that all 2×2 principal minors of M are nonpositive. We want to find a matrix $B = ab^T$ for some real vectors a, b such that $M = \frac{1}{2}(B + B^T)$. If M is rank at most 1, $M = uu^T$ for some u and we take $a = b = u$. Now suppose M is rank 2. Since M has all 2×2 principal minors nonpositive, M cannot be positive or negative semidefinite, so M must have eigenvalues $\lambda_1 > 0 > \lambda_2$. Suppose u and v are

orthonormal eigenvectors with these eigenvalues. Then $M = \lambda_1 uu^T + \lambda_2 vv^T$. It then follows that we can take $a = (\lambda_1)^{1/2}u + (-\lambda_2)^{1/2}v$ and $b = (\lambda_1)^{1/2}u - (-\lambda_2)^{1/2}v$. □

Proposition 5.3. For any nonnegative vector $(a_i)_{1\leqslant i\leqslant n}, (b_i)_{1\leqslant i\leqslant n}$, the Hessian of the quadratic form $\left(\sum_{1\leqslant i\leqslant n} a_i x_i\right)\left(\sum_{1\leqslant i\leqslant n} b_i x_i\right)$ is Lorentzian, and every Lorentzian matrix of rank at most 2 is of this form.

Proof. Let M be the Hessian of the quadratic form $Q = \left(\sum_{1\leqslant i\leqslant n} a_i x_i\right)\left(\sum_{1\leqslant i\leqslant n} b_i x_i\right)$, where a, b are nonnegative real vectors. By Lemma 5.1 and Lemma 5.2, we know that M is of rank at most 2 and all 2×2 principal minors of M are nonpositive. Since a, b are nonnegative, the entries of M are nonnegative. Thus, the symmetric matrix M is a Lorentzian matrix of rank at most 2.

Conversely, for any nonzero Lorentzian matrix M of rank at most 2, by Lemma 5.1 and Lemma 5.2, the corresponding quadratic form $x^T Mx$ factors into a product

$$x^T Mx = \left(\sum_{1\leqslant i\leqslant n} a_i x_i\right)\left(\sum_{1\leqslant i\leqslant n} b_i x_i\right) \quad \text{for some real vectors } a, b.$$

It remains to show that vectors a, b can be chosen nonnegative. Since the entries of M are nonnegative, $a_i b_i \geqslant 0$ for all $1 \leqslant i \leqslant n$.

First, consider the case when $a_i b_i = 0$ for $1 \leqslant i \leqslant n$. Since M is nonzero, replacing all a_i and b_i by its negative and permuting the indices if necessary, we may suppose that $a_1 > 0$, which in turn implies that $b_1 = 0$. For any $b_i \neq 0$, since $a_1 b_i + a_i b_1 \geqslant 0$, we have $b_i > 0$. Thus, all nonzero b_i are positive. Since M is nonzero, one of the b_i must be positive. Then we apply the same argument to conclude that all nonzero a_i are positive.

When $a_i b_i > 0$ for some i, without loss of generality, we may assume that $a_1 > 0$ and $b_1 > 0$. Since $a_1 b_i + a_i b_1$ is nonnegative as well as $a_i b_i \geqslant 0$ for any i, we conclude that all a_i and b_i are nonnegative. □

Let $\mathbb{L}_n^+(2)$ denote the set of $n \times n$ Lorentzian matrices of rank at most 2. Since $\mathbb{L}_n^+(2) \subseteq \mathbb{L}_n^+$, we have $\underline{\mathrm{BR}}(\mathbb{L}_n^+) \subseteq \underline{\mathrm{BR}}(\mathbb{L}_n^+(2))$. We show that these cones coincide and they are dual to the cut cone Cut_n.

Proposition 5.4. We have $\underline{\mathrm{BR}}(\mathbb{L}_n^+) = \underline{\mathrm{BR}}(\mathbb{L}_n^+(2))$.

Proof. Theorem B says that Cut_n is dual to $\underline{\mathrm{BR}}(\mathbb{L}_n^+)$. Thus, the dual of Cut_n is contained in $\underline{\mathrm{BR}}(\mathbb{L}_n^+(2))$. We show the opposite inclusion.

Let E be the set of entry-wise exponentials of extremal rays of Cut_n, viewed as $n\times n$ symmetric matrices with diagonal entries 1. Since every element of E is the tree metric of a star tree, it follows that $E \subseteq \underline{\Delta}_n^+(\mathbb{T}_0) \subseteq \mathbb{L}_n^+$. Since every matrix in E has rank at most 2, we have $E \subseteq \mathbb{L}_n^+(2)$. Thus, any reduced bounded ratio α on $\mathbb{L}_n^+(2)$ satisfies $\alpha \cdot \beta \leqslant 0$ for all $\beta \in \log E$, and hence for all $\beta \in \mathrm{Cut}_n$. □

We close with a statement that strengthens Conjecture 1.9 for rank 2 Lorentzian matrices.

Conjecture 5.5. Let $\alpha \in \mathrm{BR}(\mathbb{L}_n^+)$ be any integral bounded ratio and $(p_{ij})_{1\leqslant i\leqslant j\leqslant n}$ be the Hessian matrix of the quadratic form $\left(\sum_{1\leqslant i\leqslant n} a_i x_i\right)\left(\sum_{1\leqslant i\leqslant n} b_i x_i\right)$. Then

$$2^{\sum_{i=1}^n \alpha_{ii}} \left(\prod_{\alpha_{ij}<0} p_{ij}^{-\alpha_{ij}}\right) - \left(\prod_{\alpha_{ij}>0} p_{ij}^{\alpha_{ij}}\right)$$

is a polynomial with nonnegative coefficients in a_i and b_i.

For example, for $(a_1x_1 + a_2x_2 + a_3x_3)(b_1x_1 + b_2x_2 + b_3x_3)$ with $a_i, b_i \geqslant 0$, we have

$$\frac{p_{11}p_{23}}{p_{12}p_{13}} = \frac{(2a_1b_1)(a_2b_3 + a_3b_2)}{(a_1b_2 + a_2b_1)(a_1b_3 + a_3b_1)} \leqslant 2,$$

and, in fact, the difference $2p_{12}p_{13} - p_{11}p_{23} = 2a_1^2b_2b_3 + 2a_2a_3b_1^2$ has nonnegative coefficients. Computations show that, for $n \leqslant 6$, every integral bounded ratio on $\mathbb{L}_n^+$ is a nonnegative integral linear combination of primitive bounded ratios on $\mathbb{L}_n^+$, and, every primitive bounded ratios on on $\mathbb{L}_n^+$ satisfy Conjecture 5.5. Since the validity of the conjecture is preserved under taking nonnegative integral linear combinations, this verifies Conjecture 5.5 for $n \leqslant 6$. For a similar subtraction-free phenomenon for bounded ratios of Plücker coordinates over totally positive Grassmannians was observed in [BF08, Conjecture 37]. For a connection to cluster algebras, see [GGS].

REFERENCES

[BB19] Matthew Baker and Nathan Bowler, *Matroids over partial hyperstructures*, Adv. Math. **343** (2019), 821–863. 6

[BF87] Tommy Bonnesen and Werner Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, ID, 1987. 3

[BF08] Adam Boocher and Bradley Froehle, *Multiplicative principal-minor inequalities for totally nonnegative matrices*, Lin. Alg. and Applications **428** (2008), no. 7, 1664–1684. 21

[BH20] Petter Brändén and June Huh, *Lorentzian polynomials*, Ann. of Math. (2) **192** (2020), no. 3, 821–891. 2, 6, 9, 14

[BHKL] Matthew Baker, June Huh, Mario Kummer, and Oliver Lorscheid, *Lorentzian polynomials and matroids over triangular hyperfields 1: Topological aspects*, arXiv:2508.02907. 6

[BHKL25] ———, *Lorentzian polynomials and matroids over triangular hyperfields 2: Analytic aspects*, 2025. 6, 14

[BKH$^+$] Matthew Baker, Donggyu Kim, June Huh, Mario Kummer, and Oliver Lorscheid, *Representation theory for polymatroids*, arXiv:2507.14718. 14

[DL10] Michel Marie Deza and Monique Laurent, *Geometry of cuts and metrics*, Algorithms and Combinatorics, vol. 15, Springer, Heidelberg, 2010. 4, 5, 6, 8

[FGJ03] Shaun Fallat, Michael Gekhtman, and Charles Johnson, *Multiplicative principal-minor inequalities for totally nonnegative matrices*, Adv. in Appl. Math. **30** (2003), no. 3, 442–470. 2

[FJ00] Shaun Fallat and Charles Johnson, *Determinantal inequalities: ancient history and recent advances*, Algebra and its applications (Athens, OH, 1999), Contemp. Math., vol. 259, Amer. Math. Soc., Providence, RI, 2000, pp. 199–212. 2

[Fro15] Georg Frobenius, *Über den gemischten flächeninhalt zweier ovale*, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin (1915), 387–404, Issued 10 June 1915; session of 3 June 1915. 3

[GdlH90] Etienne Ghys and Pierre de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988. 16

[GGS] Michael Gekhtman, Zachary Greenberg, and Daniel Soskin, *Multiplicative inequalities in cluster algebras of finite type*, arXiv:2409.06642. 2, 21

[Gro87] Mikhael Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. 7, 16

[HHM+] Daoji Huang, June Huh, Mateusz Michałek, Botong Wang, and Shouda Wang, *Realizations of homology classes and projection areas*, arXiv:2505.08881. 14

[KP82] Richard Karp and Christos Papadimitriou, *On linear characterizations of combinatorial optimization problems*, SIAM J. Comput. **11** (1982), no. 4, 620–632. 5

[LX17] Brian Lehmann and Jian Xiao, *Correspondences between convex geometry and complex geometry*, Épijournal Géom. Algébrique **1** (2017), Art. 6, 29. 3

[Mil16] James Milne, *The Riemann hypothesis over finite fields: from Weil to the present day [Reprint of 3525903]*, ICCM Not. **4** (2016), no. 2, 14–52. 3

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015. 15

[Sch14] Rolf Schneider, *Convex bodies: the Brunn-Minkowski theory*, expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014. 1

[SG25] Daniel Soskin and Michael Gekhtman, *On bounded ratios of minors of totally positive matrices*, Linear Algebra Appl. **715** (2025), 46–67. 2

[Yu15] Josephine Yu, *Tropicalizing the positive semidefinite cone*, Proc. Amer. Math. Soc. **143** (2015), no. 5, 1891–1895. 2

Daoji Huang, University of Massachusetts Amherst

Email address: daojihuang@umass.edu

June Huh, Princeton University and Korea Institute for Advanced Study

Email address: huh@princeton.edu

Daniel Soskin, University of California, Los Angeles

Email address: dsoskin@math.ucla.edu

Botong Wang, University of Wisconsin–Madison

Email address: wang@math.wisc.edu