Star Quasiconvexity: an Unified Approach for Linear Convergence of First-Order Methods Beyond Convexity

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Abstract

We introduce a class of generalized convex functions, termed star quasiconvexity, to ensure the linear convergence of gradient and proximal point methods. This class encompasses convex, star-convex, quasiconvex, and quasar-convex functions. We establish that a function is star quasiconvex if and only if all its sublevel sets are star-shaped with respect to the set of its minimizers. Furthermore, we provide several characterizations of this class, including nonsmooth and differentiable cases, and derive key properties that facilitate the implementation of first-order methods. Finally, we prove that the proximal point algorithm converges linearly to the unique solution when applied to strongly star quasiconvex functions defined over closed, star-shaped sets, which are not necessarily convex.

Keywords: Nonconvex optimization; Gradient methods; Proximal point methods; Linear convergence; Star-shaped functions

1 Introduction

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be a proper function. The most classical problem in optimization is the unconstrained minimization problem:

$$\min_{x \in \mathbb{R}^n} h(x). \tag{P}$$

This paper investigates a generalized notion of convexity that ensures the linear convergence of first-order methods for solving the minimization problem in (P).

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The standard assumption for achieving linear convergence in gradient and proximal point methods is that the objective function h is strongly convex. This property is highly desirable as it guarantees the existence and uniqueness of a solution, implies quadratic growth conditions, and ensures the Polyak-Łojasiewicz (PL) property, among other benefits.

Motivated by these properties, several authors have introduced broader classes of functions that retain these convergence guarantees. Notable examples include:

Strongly quasiconvex functions: introduced by Polyak in 1966 as a particular case of uniformly quasiconvex functions [37], this was one of the first such generalizations. Although not immediately proven, it was later established that this class satisfies PL property and quadratic growth, leading to linear convergence rates for gradient-type methods [17, 26] and proximal point methods [25].

Strongly star-convex functions: defined by Nesterov and Polyak in 2006 [33], this class also ensures linear convergence for both gradient and proximal point methods, as demonstrated in [20] and [8], respectively.

Strongly quasar-convex functions: introduced under various names in [14, 15, 20] (among others), this class has been shown to yield linear convergence for gradient-type methods [14, 15, 19, 20] and, recently, for the proximal point method in [8].

The relationship between these classes is summarized below:

$$\begin{array}{ccccc} \text{strongly quasiconvex} & \Longleftarrow & \text{strongly convex} & \Longrightarrow & \text{strongly star} - \text{convex} \\ & & & & \Downarrow \\ & & & & \text{strongly quasar} - \text{convex} \end{array}$$

Since all the previous notions includes strongly convex functions and they provide linear convergence for first-order methods, the authors in [17, 26, 27] studied the relation between (strongly) quasiconvex and (strongly) quasar-convex functions in the differentiable case. It was showed in [17, Subsection 3.1] that both classes are not related each other and, as a consequence, the following natural question arises:

What is the class of generalized (strongly) convex functions that guarantees global linear convergence rate for first-order methods and includes both (strongly) quasiconvex and (strongly) quasar-convex functions?.

Our Contribution: We introduce the notion of *(strong) star quasiconvexity* for solving the question above by proving that both (strongly) quasiconvex and (strongly) quasar-convex functions are (strongly) star quasiconvex, an inclusion which is proper since we show examples of strongly star quasiconvex functions which are neither quasiconvex nor quasar-convex (see Example 8). Second, we provide four different characterizations for (strongly) star quasiconvex functions; (i) by characterizing star quasiconvex functions via the star-shapedness of its sublevel sets in Theorem 9; (ii) by showing that (strongly) star quasiconvex functions are (strongly) quasiconvex functions along rays starting from its unique minimizer in Theorem 12; (iii) by finding a stronger property for the continuous case in Corollary 16; and, (iv) by characterizing the behaviour of its gradient in the differentiable case in Proposition 24. Third, several interesting properties for smooth and nonsmooth strongly star quasiconvex functions are derived from the previous analysis as: quadratic growth, Polyak-Łojasiewicz property, and also 2-supercoercivity as well as relationships with the proximity operator on star-shaped sets among others. Notably, some of these properties are even novel for strong quasar-convexity and/or strong quasiconvexity, too. Fourth, we observed that the gradient method and its heavy ball and Nesterov accelerations achieves linear convergence rate, extending the results from [14, 15, 17, 19, 20, 26] to strongly star quasiconvex functions. Sixth, we ensure linear convergence of the proximal point algorithm (PPA henceforth) for strongly star quasiconvex on star-shaped set, extending the results from [8, 25] as well as presenting the first (as far as we know) convergence result of PPA on star-shaped sets. Finally, we present conclusions and future research lines.

The structure of the paper is as follows: In Section 2 we present notation and basic definitions on generalized convexity. In Section 3, we present the motivation for introducing (strong) star quasiconvex functions, four different characterizations for this class of functions as well as interesting properties for developing first-order type algorithms. In Section 4, we present the linear convergence of the gradient method as well as for its heavy ball and Nesterov accelerations. In Section 5, we ensures linear convergence for PPA on star-shaped sets. Conclusions and future research directions are described in Section 6.

2 Preliminaries

The inner product in $\mathbb{R}^n$ and the Euclidean norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We denote $\mathbb{R}_+ := [0, +\infty[$, $\mathbb{R}_{++} :=]0, +\infty[$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $K \subseteq \mathbb{R}^n$. Its asymptotic (recession) cone by

$$K^{\infty} := \left\{ u \in \mathbb{R}^n : \ \exists \ t_k \to +\infty, \ \exists \ x_k \in K, \ \frac{x_k}{t_k} \to u \right\}.$$

Furthermore, for any set $K \subset \mathbb{R}^n$, it follows from [4, Proposition 2.1.2] that $K^{\infty} = \{0\}$ if and only if K is bounded.

Given any $x, y, z \in \mathbb{R}^n$ and any $\beta \in \mathbb{R}$, the following relations hold:

$$\|\beta x + (1-\beta) y\|^2 = \beta \|x\|^2 + (1-\beta)\|y\|^2 - \beta(1-\beta)\|x-y\|^2. \tag{1}$$

$$\langle x - z, y - x \rangle = \frac{1}{2}\|z-y\|^2 - \frac{1}{2}\|x-z\|^2 - \frac{1}{2}\|y-x\|^2, \tag{2}$$

Let $K \subseteq \mathbb{R}^n$ be nonempty. Then given $x_0 \in K$, the set K is said to be star-shaped at the point x_0 if for every $x \in K$, the segment $[\overline{x}, x] \subseteq K$.

Given any extended-valued function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$, the effective domain of h is defined by $\operatorname{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\}$. It is said that h is proper if $\operatorname{dom} h$ is nonempty and $h(x) > -\infty$ for all $x \in \mathbb{R}^n$. The notion of properness is important when dealing with minimization problems.

It is indicated by $\operatorname{epi} h := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : h(x) \leq t\}$ the epigraph of h, by $S_\delta(h) := \{x \in \mathbb{R}^n : h(x) \leq \delta\}$ the sublevel set of h at the height $\delta \in \mathbb{R}$ and by $\operatorname{argmin}_{\mathbb{R}^n} h$ the set of all minimal points of h. A function h is lower semicontinuous (lsc henceforth) at $\overline{x} \in \mathbb{R}^n$ if for any sequence $\{x_k\}_k \subseteq \mathbb{R}^n$ with $x_k \to \overline{x}$, $h(\overline{x}) \leq \liminf_{k \to +\infty} h(x_k)$. Furthermore, the current convention $\sup_{\emptyset} h := -\infty$ and $\inf_{\emptyset} h := +\infty$ is adopted.

A function h with convex domain is said to be:

(*a*) (strongly) convex on $\operatorname{dom} h$, if there exists $\gamma \geq 0$ such that if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0,1]$, we have

$$h(\lambda y + (1-\lambda)x) \leq \lambda h(y) + (1-\lambda)h(x) - \lambda(1-\lambda)\frac{\gamma}{2}\|x-y\|^2, \tag{3}$$

(*b*) (strongly) quasiconvex on $\operatorname{dom} h$, if there exists $\gamma \geq 0$ such that if for all $x, y \in \operatorname{dom} h$ and all $\lambda \in [0,1]$, we have

$$h(\lambda y + (1-\lambda)x) \leq \max\{h(y), h(x)\} - \lambda(1-\lambda)\frac{\gamma}{2}\|x-y\|^2. \tag{4}$$

Every (strongly) convex function is (strongly) quasiconvex, while the reverse statement does not holds (see [10, 16, 25]). Furthermore, when we say that a function is *strongly convex (resp. quasiconvex) with modulus* $\gamma \geq 0$ we refer to both strongly convex (resp. quasiconvex) when $\gamma > 0$ and convex (resp. quasiconvex) when $\gamma = 0$.

Quasiconvexity is the mathematical formulation of the assumption *tendency to diversification* on the consumers in consumer's preference theory (see [12]). Furthermore, the following geometric characterization is well-known:

$$h \text{ is convex} \iff \operatorname{epi} h \text{ is a convex set.}$$
$$h \text{ is quasiconvex} \iff S_\delta(h) \text{ is a convex set for all } \delta \in \mathbb{R}.$$

A proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be:

(*i*) 2-supercoercive, if

$$\liminf_{\|x\| \to +\infty} \frac{h(x)}{\|x\|^2} > 0, \tag{5}$$

(*ii*) coercive, if

$$\lim_{\|x\| \to +\infty} h(x) = +\infty. \tag{6}$$

or equivalently, if $S_\delta(h)$ is bounded for all $\delta \in \mathbb{R}$.

Clearly, (*i*) $\Rightarrow$ (*ii*), but the converse statements does not hold as the function $h(x) = |x|$ shows.

We known by [25, Theorem 1] that strongly quasiconvex functions are 2-supercoercive, as a consequence, lsc strongly quasiconvex functions has an unique minimizer on closed convex sets as we recall next.

Lemma 1. ([25, Corollary 3]) *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, lsc, and strongly quasiconvex function on $K \subseteq \operatorname{dom} h$ with modulus $\gamma > 0$. Then,* $\operatorname{argmin}_K h$ *is a singleton.*

Furthermore, the unique minimizer $\overline{x} \in K$ of a strongly quasiconvex function h (with modulus $\gamma > 0$) satisfies a quadratic growth condition (see [18, Corollary 9]):

$$h(\overline{x}) + \frac{\gamma}{4}\|y - \overline{x}\|^2 \leq h(y), \ \forall\, y \in K. \tag{7}$$

When a minimizer $\overline{x} \in K$ satisfies relation (7), it is denoted by $\frac{\gamma}{4}$-$\operatorname{argmin}_K h$.

A function $h : \mathbb{R}^n \to \mathbb{R}$ its said to be L-smooth on $K \subseteq \mathbb{R}^n$ if it is differentiable on K and

$$\|\nabla h(x) - \nabla h(y)\| \leq L\|x - y\|, \ \forall\, x, y \in K. \tag{8}$$

Let $K \subseteq \mathbb{R}^n$ be a convex set and $h : K \to \mathbb{R}$ be a differentiable function. Then h is strongly convex on K with modulus $\gamma \geq 0$ if and only if

$$h(x) \geq h(y) + \langle \nabla h(y), x - y\rangle + \frac{\gamma}{2}\|y - x\|^2, \ \forall\, x, y \in K. \tag{9}$$

For differentiable strongly quasiconvex functions, we have the following result [41] (see also [18]). Let $K \subseteq \mathbb{R}^n$ be a convex and open set and $h : K \to \mathbb{R}$ be differentiable function. If h is strongly quasiconvex with modulus $\gamma \geq 0$, then for every $x, y \in K$, we have

$$h(x) \leq h(y) \implies \langle \nabla h(y), x - y\rangle \leq -\frac{\gamma}{2}\|y - x\|^2. \tag{10}$$

Conversely, if (10) holds, then h is strongly quasaiconvex with modulus $\frac{\gamma}{2}$ (see [41, Theorems 1 and 6] and [18, Corollary 8]).

The previous result extends the well-known characterization for differentiable quasiconvex functions given by Arrow-Enthoven [2] ($\gamma = 0$).

Another generalized convexity notion, mainly motivated by its interesting algorithmic properties, is the notion of star-convexity (see [33]).

Definition 2. (see [33]) *A proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $\overline{x} \in$* $\operatorname{argmin}_{\mathbb{R}^n} h$ *is said to be (strongly) star-convex if there exists $\gamma \geq 0$ such that for every $y \in \operatorname{dom} h$ and every $\lambda \in [0, 1]$, we have*

$$h(\lambda\overline{x} + (1 - \lambda)y) \leq \lambda h(\overline{x}) + (1 - \lambda)h(y) - \lambda(1 - \lambda)\frac{\gamma}{2}\|y - \overline{x}\|^2. \tag{11}$$

The following generalized convexity notion has been used in several applications from machine learning theory in virtue of its accelerated properties for the convergence of the gradient method (see [15, 19, 20, 27, 43] and references therein). As in [8], we define as follows:

Definition 3. (see [20] and also [8]) *A proper function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ with $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$ is said to be (strongly) quasar-convex if there exists $\beta \in]0,1]$ and $\gamma \geq 0$ such that for every $y \in \operatorname{dom} h$, we have*

$$h(\lambda\overline{x} + (1-\lambda)y) \leq \lambda\beta h(\overline{x}) + (1-\lambda\beta)h(y) - \lambda\left(1 - \frac{\lambda}{2-\beta}\right)\frac{\beta\gamma}{2}\|y - \overline{x}\|^2.$$

It follows from Definition 3 that $\operatorname{argmin}_{\mathbb{R}^n} h$ is a singleton when h is strongly quasar-convex ($\gamma > 0$). Furthermore, if $\beta = 1$, then quasar-convex functions reduces to strongly star-convex functions ($\gamma > 0$) and star-convex functions ($\gamma = 0$).

In the differentiable case, we have the following useful characterization for (strongly) quasar-convex functions. Let $K \subseteq \mathbb{R}^n$ be a convex and open set and $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with $\operatorname{argmin}_K h \neq \emptyset$. Then h is (strongly) quasar-convex with modulus $\beta \in]0,1]$ and $\gamma \geq 0$ on K if and only if (see [20, Lemma 10])

$$h(\overline{x}) \geq h(y) + \frac{1}{\beta}\langle \nabla h(y), \overline{x} - y\rangle + \frac{\gamma}{2}\|y - \overline{x}\|^2, \ \forall \ y \in K. \tag{12}$$

Hence, quasi-strongly convex functions [30] are related to strongly quasar-convex. Furthermore, differentiable quasar-convex functions ($\gamma = 0$) are the functions for which:

$$\langle \nabla h(y), \overline{x} - y\rangle \leq \beta(h(\overline{x}) - h(y)), \ \forall \ y \in K. \tag{13}$$

Functions which satisfies relation (13) has been studied in the last years in different fields, specially for accelerating the convergence of gradient type methods (see [5, 9] and references therein).

Finally, given a nonempty closed set $K \subseteq \mathbb{R}^n$, the proximity operator on K of parameter $\beta > 0$ of a proper lsc function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ at $x \in \mathbb{R}^n$ is defined as the operator $\operatorname{Prox}_{\beta h}(K, \cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ for which

$$\operatorname{Prox}_{\beta h}(K, x) = \operatorname{argmin}_{y \in K}\left\{h(y) + \frac{1}{2\beta}\|y - x\|^2\right\}. \tag{14}$$

When $K = \mathbb{R}^n$, we simple write $\operatorname{Prox}_{\beta h}(z) := \operatorname{Prox}_{\beta h}(\mathbb{R}^n, z)$. If h is proper, lsc and convex, then $\operatorname{Prox}_{\beta h}$ turns out to be a single-valued operator (see, for instance, [7, Proposition 12.15]).

For a further study on star convexity, quasiconvexity, quasar-convexity and its applications in economics and engineering among others, we refer to [5, 8, 9, 10, 16, 20, 25, 27, 33, 41, 43] and references therein.

3 An Unifying Approach for Generalized Convex Functions

3.1 Motivation

As mentioned in the introduction, the following definition (see [34]) includes (strongly) convex, (strongly) quasiconvex, (strongly) star-convex and (strongly) quasar-convex functions.

Definition 4. *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function and $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$. Then h is (strongly) star quasiconvex with respect to $\overline{x}$ if there exists $\gamma \geq 0$ such that for every $y \in \operatorname{dom} h$ and every $\lambda \in [0, 1]$, we have*

$$h(\lambda\overline{x} + (1-\lambda)y) \leq h(y) - \lambda(1-\lambda)\frac{\gamma}{2}\|y - \overline{x}\|^2. \tag{15}$$

It is said that h is strongly star quasiconvex when $\gamma > 0$ and it is star quasiconvex when $\gamma = 0$, in both cases, with respect to a specific minimizer $\overline{x}$.

We note immediately that when $\gamma > 0$, strongly star quasiconvex functions has an unique minimizer. As a consequence, in this case, we simple say that h is strongly star quasiconvex with modulus $\gamma > 0$, without reference to a point.

Strongly star quasiconvex functions are an extension of strongly quasiconvex functions since one of the points is fixed at the minimum (which always exists for lsc strongly quasiconvex functions by Lemma 1), and it is also an extension of strongly quasar-convex since the term $\lambda\beta h(\overline{x}) + (1-\lambda\beta)h(y) \leq h(y) = \max\{h(\overline{x}), h(y)\}$ for all $\lambda \in [0, 1]$ and $\beta \in]0, 1]$. The easy proof of the following result is omitted.

Proposition 5. *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function. Then the following relationship holds (qcx = quasiconvex and qconvex = quasar-convex):*

$$\begin{array}{ccccccc}
\text{strongly qcx} & \Leftarrow & \text{strongly convex} & \Rightarrow & \text{strongly star convex} & \Rightarrow & \text{strongly qconvex} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{strongly star qcx} & & \text{convex} & \overset{*}{\Rightarrow} & \text{star} - \text{convex} & \Rightarrow & \text{qconvex} \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
\text{star qcx} & \overset{*}{\Leftarrow} & \text{qcx} & \overset{*}{\Rightarrow} & \text{star qcx} & &
\end{array}$$

where "$*$" denotes that $\operatorname{argmin}_{\mathbb{R}^n} h \neq \emptyset$ is required.

Remark 6. *(i) All the reverse implications does not hold true in general. Indeed, almost all the counter-examples are well-known (see [10, 25, 33]), and its remains to show an example of a strongly star quasiconvex function without being neither (strongly) quasiconvex not (strongly) quasar-convex, this will be given in Example 8 (below).*

(ii) There is no relationship between (strongly) quasiconvex and (strongly) quasar-convex functions. Indeed, the function in [17, Example 9] is strongly quasiconvex but not quasar-convex, while the function in Example 20 (below) is strongly quasar-convex but not quasiconvex. For differentiable functions, sufficient conditions under which strong quasiconvexity implies quasar-convexity are provided in [17, 27].

In the next result, we characterize (strongly) star quasiconvex functions along segments. This is known for (strongly) quasiconvex functions (see [22, Theorem 1]) while for (strongly) quasar-convex functions, is new.

Proposition 7. *A function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is strongly star quasiconvex with modulus $\gamma \geq 0$ if and only if for $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$ and every $y \in \mathbb{R}^n \backslash \{\overline{x}\}$, the function*

$$h_y(t) := h\left(\overline{x} + t\frac{y - \overline{x}}{\|y - \overline{x}\|}\right), \tag{16}$$

with $t \in [0, \|y - \overline{x}\|]$, is strongly star quasiconvex with the same modulus $\gamma \geq 0$.

Proof. ($\Rightarrow$): Let $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$ and $y \in \mathbb{R}^n \backslash \{\overline{x}\}$. Define h_y as in (16), and note that $\operatorname{argmin}_{\mathbb{R}} h_y = \{0\}$. Hence, for $t_1 = 0$ and $t_2 \in [0, \|y - \overline{x}\|]$, we have

$$\begin{aligned}
h_y(\lambda 0 + (1-\lambda)t_2) &= h\left(\overline{x} + (\lambda 0 + (1-\lambda)t_2)\frac{y - \overline{x}}{\|y - \overline{x}\|}\right) \\
&= h\left(\lambda\,(\overline{x} + 0) + (1-\lambda)\left(\overline{x} + \frac{t_2}{\|y - \overline{x}\|}(y - \overline{x})\right)\right) \\
&\leq h\left(\overline{x} + \frac{t_2}{\|y - \overline{x}\|}(y - \overline{x})\right) - \lambda(1-\lambda)\frac{\gamma}{2}\left\|\frac{t_2}{\|y - \overline{x}\|_2}(y - \overline{x})\right\|^2 \\
&= h_y(t_2) - \lambda(1-\lambda)\frac{\gamma}{2}(t_2 - 0)^2,
\end{aligned}$$

i.e., h_y is strongly star quasiconvex on $[0, \|y - \overline{x}\|]$ with modulus $\gamma \geq 0$.

($\Leftarrow$): Assume that for $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$ and every $y \in \mathbb{R}^n \backslash \{\overline{x}\}$, h_y is strongly star quasiconvex with modulus $\gamma \geq 0$ on $[0, \|y - \overline{x}\|]$. Note that $0 \in \operatorname{argmin}_{\mathbb{R}^n} h_y$. For every $t \in [0, 1]$ we write $t\|y - \overline{x}\| = (1-t)0 + t\|y - \overline{x}\|$ and use the strongly star quasiconvexity of h_y at the points 0, $\|y - \overline{x}\|$ and $(1-t)0 + t\|y - \overline{x}\|$. Then,

$$\begin{aligned}
h_y(t\|y - \overline{x}\|) &\leq h_y(\|y - \overline{x}\|) - \frac{\gamma}{2}t(1-t)\|y - \overline{x}\|^2 \\
\Longrightarrow \;\; h((1-t)\overline{x} + ty) &\leq h(y) - \frac{\gamma}{2}t(1-t)\|y - \overline{x}\|^2,
\end{aligned}$$

and the result follows. $\square$

Now, we have all the ingredients for showing a function which is strongly star quasiconvex without being neither quasiconvex nor quasar-convex.

Example 8. (4-Leaf Clover Example) *Take $g_1 : \mathbb{R}_+ \to \mathbb{R}$ given by*

$$g_1(x) = \begin{cases} 3ex^2 - 2ex^3, & \text{if } 0 \leq x < 1, \\ e^n(1 + (e-1)(3(x-n)^2 - 2(x-n)^3)), & \text{if } n \leq x < n+1, \ \forall\, n \in \mathbb{N}. \end{cases}$$

Then, we define the function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = f(x) + g(x),$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are given by $f(x) = x^2$ and

$$g(x) = \begin{cases} g_1(-x), & \text{if } x < 0, \\ g_1(x), & \text{if } x \geq 0, \end{cases}$$

which is strongly quasiconvex with modulus $\gamma = 2$, but it is not quasar-convex as proved in [17, Example 9]). Furthermore, note that $\min_{\mathbb{R}^n} h = h(0) = 0$.

Set $\alpha := h(1)$ and $\beta := h\left(\frac{1}{\sqrt{2}}\right) < \alpha$ (because h is increasing on $\mathbb{R}_+$), and let $\|\cdot\|_p$ be the p-pseudonorm $\|(x_1, x_2)\|_p := (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ where $p := \frac{\ln 2}{\ln(\frac{2\alpha}{\beta})} < 1$ has been chosen so that $\|\left(\frac{1}{2}, \frac{1}{2}\right)\|_p = \frac{\alpha}{\beta}$.

Define $\phi : \mathbb{R}^2 \to \mathbb{R}$ by (see Figure 1)

$$\phi(x) = \begin{cases} \frac{\|x\|}{\|x\|_p} h(\|x\|), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

Here, $\operatorname{argmin}_{\mathbb{R}^n} \phi = 0$ and, for every $y \in \mathbb{R}^2 \backslash \{0\}$, we have

$$\phi_y(t) := \phi\left(t \frac{y}{\|y\|}\right) = \frac{\|y\|}{\|y\|_p} h(t),$$

so ϕ_y is strongly star quasiconvex with modulus $\frac{\|y\|}{\|y\|_p}\gamma = \frac{2\|y\|}{\|y\|_p}$. Since $\|\cdot\|_p$ and $\|\cdot\|$ are equivalent, there exists $m > 0$ such that $\frac{\|y\|}{\|y\|_p} \geq m$ for all $y \neq 0$. Hence, ϕ is strongly star quasiconvex with modulus $2m > 0$ by Proposition 7.

On the other hand, the restriction of ϕ on the x axis is not quasar-convex, i.e., ϕ is not quasar-convex (see [17]). Furthermore, ϕ is not even quasiconvex. Indeed, let us consider the points $x = (1, 0)$, $y = (0, 1)$ and $z = \frac{1}{2}x + \frac{1}{2}y$. Hence, $\phi(x) = \phi(y) = 1h(1) = \alpha$, but $\phi(z) = \frac{\|\left(\frac{1}{2}, \frac{1}{2}\right)\|_p}{\|\left(\frac{1}{2}, \frac{1}{2}\right)\|} h\left(\left\|\left(\frac{1}{2}, \frac{1}{2}\right)\right\|_p\right) = \frac{\frac{\alpha}{\beta}}{\frac{1}{\sqrt{2}}}\beta > \alpha$, i.e., ϕ is not quasiconvex.

Therefore, ϕ is strongly star quasiconvex with modulus $2m > 0$, without being either quasiconvex or quasar-convex.

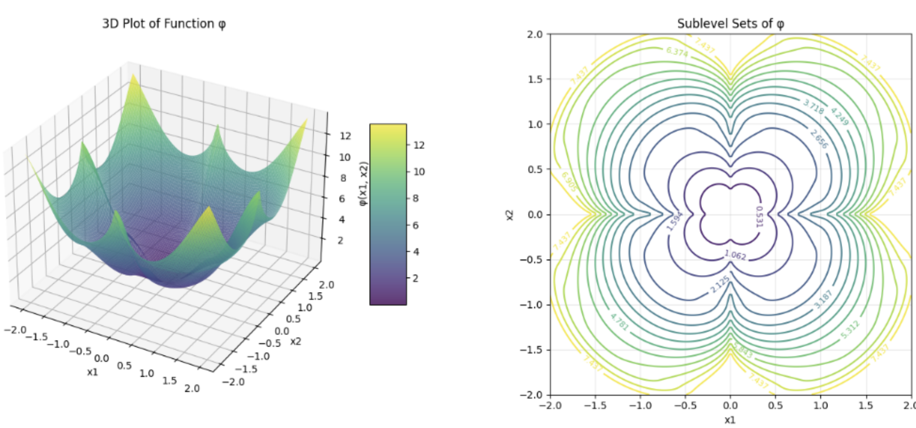

Figure 1: An illustration of the function ϕ in Example 8 (left) and its 4-leaf clover sublevel sets (right).

3.2 Star Quasiconvexity

We begin this subsection with our first main result, which provide a geometric characterization for star quasiconvex functions.

Theorem 9. (Star-shaped sublevel sets) *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function. Then, h is star quasiconvex with respect to $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$ if and only if $S_\delta(h)$ is star-shaped at $\overline{x}$ for all $\alpha \in \mathbb{R}$.*

Proof. ($\Rightarrow$) Let $y \in S_\delta(h)$. Then for every $\lambda \in [0,1]$, we define $y_\lambda := \lambda\overline{x} + (1-\lambda)y$. Since h is star quasiconvex, $h(y_\lambda) \leq h(y) \leq \delta$. Hence, $y_\lambda \in S_\delta(h)$, i.e., $[\overline{x}, y] \subset S_\delta(h)$. Then $S_\delta(h)$ is star-shaped at $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$.

($\Leftarrow$) Assume that $S_\alpha(h)$ is star-shaped at $\overline{x}$ for all $\alpha \in \mathbb{R}$.

First, we show that $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$. Indeed, suppose for contradiction that there exists $z \in \mathbb{R}^n$ such that $h(z) < h(\overline{x})$. Let $\alpha := h(z)$. Then $z \in S_\alpha(h)$, and by star-shapedness at $\overline{x}$, the segment $[\overline{x}, z] \subset S_\alpha(h)$. In particular, $\overline{x} \in S_\alpha(h)$, so $h(\overline{x}) \leq \alpha = h(z)$. This contradicts $h(z) < h(\overline{x})$. Hence, $h(\overline{x}) \leq h(z)$ for all $z \in \mathbb{R}^n$, so $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$.

Second, take any $y \in \operatorname{dom} h$ and let $\delta := h(y)$. Then $y \in S_\delta(h)$. By star-shapedness at $\overline{x}$, for every $\lambda \in [0,1]$, the point $y_\lambda := \lambda\overline{x} + (1-\lambda)y \in S_\delta(h)$, i.e., $h(y_\lambda) \leq \delta = h(y)$. Therefore, h is star quasiconvex with respect to $\overline{x}$. $\square$

A similar characterization can be obtained for star-convex functions. As far as we know, the following result is new.

Proposition 10. (Star-shaped epigraph for star-convexity) *Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function and $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$. Then, h is star-convex with respect to $\overline{x}$ if and only if $\operatorname{epi} h$ is star-shaped with respect to $(\overline{x}, h(\overline{x}))$.*

Proof. ($\Rightarrow$) Let $(y, s) \in \operatorname{epi} h$, so $s \geq h(y)$. Since h is star-convex with respect to $\overline{x}$, we have for every $\lambda \in [0,1]$ that

$$\lambda h(\overline{x}) + (1-\lambda)s \geq \lambda h(\overline{x}) + (1-\lambda)h(y) \geq h(\lambda\overline{x} + (1-\lambda)y).$$

Thus, $(\lambda\overline{x} + (1-\lambda)y, \lambda h(\overline{x}) + (1-\lambda)s) \in \operatorname{epi} h$, i.e., $\operatorname{epi} h$ is star-shaped with respect to $(\overline{x}, h(\overline{x}))$.

($\Leftarrow$) Assume that $\operatorname{epi} h$ is star-shaped with respect to $(\overline{x}, h(\overline{x}))$. Then, for any $y \in \mathbb{R}^n$, $(y, h(y)) \in \operatorname{epi} h$. By star-shapedness, we have $(\lambda\overline{x} + (1-\lambda)y, \lambda h(\overline{x}) + (1-\lambda)h(y)) \in \operatorname{epi} h$ for every $\lambda \in [0,1]$, that is,

$$\lambda h(\overline{x}) + (1-\lambda)h(y) \geq h(\lambda\overline{x} + (1-\lambda)y),$$

thus h is star-convex with respect to $\overline{x}$. $\square$

Remark 11. (*a*) *From Proposition 5 and Theorem 9 we obtain that the sublevel sets of quasar-convex and star-convex functions are star-shaped at its minimizers.*

(*b*) *Note that in Proposition 10 we are assuming that $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$. Indeed, the function $h(x) = x^2$ has a convex epigraph, i.e., a star-shaped epigraph at $(1,1)$, but $x_0 = 1$ is not its minimum.*

Theorem 9 suggest that star quasiconvex functions are quasiconvex on rays starting at the minimizer, we prove this result next. To that end, and following [18], we write (strong) quasiconvexity in the following equivalent manner: For every $x, y \in \operatorname{dom} h$ and $z \in [x, y]$, we have ($\gamma \geq 0$)

$$h(x) \leq h(y) \implies h(y) \geq h(z) + \frac{\gamma}{2}\|z - x\|\|y - z\|. \tag{17}$$

Theorem 12. (Quasiconvexity Along Rays) *A function h is (strongly) star quasiconvex with modulus $\gamma \geq 0$ with respect to $\overline{x} \in \operatorname{argmin}_{\mathbb{R}^n} h$ if and only if its restriction on any half line through $\overline{x}$ is (strongly) quasiconvex with the same modulus $\gamma \geq 0$.*

Proof. ($\Leftarrow$): It follows directly by definition.

($\Rightarrow$): Take any $x, x' \in \varepsilon(\overline{x}) := \{\overline{x} + ty : t \in \mathbb{R}_+\}$ with $y \in \operatorname{dom} h$. We assume without loss of generality that $x \in [\overline{x}, x']$. Take any $t \in [0, 1]$ and set $z := (1 - t)x + tx'$. Then $z \in [\overline{x}, x']$, so applying (17), we have

$$\begin{aligned} h(z) &\leq h(x') - \frac{\gamma}{2}\,\|x' - z\|\,\|z - \overline{x}\| \\ &\leq h(x') - \frac{\gamma}{2}\|x' - z\|\|z - x\| \\ &= h(x') - \frac{\gamma}{2}t(1 - t)\|x' - x\|^2, \end{aligned}$$

so h is (strongly) quasiconvex with modulus $\gamma \geq 0$ on $\varepsilon(\overline{x})$. □

Remark 13. *During the preparation of this work, we were pointed out about reference [34], in which the authors studied (strongly) star quasiconvex functions under a different name. However, in virtue of the reasons explained in the introduction and Theorems 9 and 12, we propose to keep the* (strongly) star quasiconvex *name.*

As a consequence of Theorem 12, we have.

Corollary 14. (Nondecreasing Along Rays) *Let $K \subseteq \mathbb{R}^n$ be a closed set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function such that $K \subseteq \operatorname{dom} h$. Suppose that $\overline{x} \in \operatorname{argmin}_K h$ and K is star-shaped at $\overline{x}$. If h is (strongly) star quasiconvex on K with modulus $\gamma \geq 0$ with respect to $\overline{x}$, then h is nondecreasing along rays through $\overline{x}$ on K.*

Conversely, if h is nondecreasing along rays through $\overline{x}$ on K, then h is star quasiconvex with respect to $\overline{x}$ (with $\gamma = 0$).

Proof. Suppose that h is star quasiconvex with respect to $\overline{x}$. Take any ray $\varepsilon(\overline{x}) = \{\overline{x} + tu : t \geq 0\} \cap K$ with $u \in \mathbb{R}^n$. For any $0 \leq t_1 < t_2$, let $\lambda = 1 - \frac{t_1}{t_2} \in [0, 1]$. Then, $\overline{x} + t_1 u = \lambda\overline{x} + (1 - \lambda)(\overline{x} + t_2 u)$. By star quasiconvexity,

$$h(\overline{x} + t_1 u) \leq h(\overline{x} + t_2 u) - \lambda(1 - \lambda)\frac{\gamma}{2}\|t_2 u\|^2 \leq h(\overline{x} + t_2 u),$$

Thus h is nondecreasing along rays.

Conversely, if h is nondecreasing along rays through $\overline{x}$ on K, then for any $y \in K$ and $\lambda \in [0,1]$, the point $\lambda\overline{x} + (1-\lambda)y$ lies on the ray from $\overline{x}$ to y. Since h is nondecreasing along this ray, we have

$$h(\lambda\overline{x} + (1-\lambda)y) \leq h(y),$$

i.e., h is star quasiconvex with respect to $\overline{x}$ (with $\gamma = 0$). $\square$

As a consequence, in the case of star quasiconvexity ($\gamma = 0$), we have the following characterization.

Corollary 15. *Let $K \subseteq \mathbb{R}^n$ be a closed set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function such that $K \subseteq \operatorname{dom} h$. Suppose that $\overline{x} \in \operatorname{argmin}_K h$ and K is star-shaped at $\overline{x}$. Then the following assertions are equivalent:*

(a) *h is star quasiconvex on K with respect to $\overline{x}$.*

(b) *h is quasiconvex along rays through $\overline{x}$ on K.*

(c) *h is nondecreasing along rays through $\overline{x}$ on K.*

Another consequence of Theorem 12 is given below.

Corollary 16. (Stronger Property) *Let $K \subseteq \mathbb{R}^n$ be a closed set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function such that $K \subseteq \operatorname{dom} h$. Suppose that $\overline{x} \in \operatorname{argmin}_K h$ and K is star-shaped at $\overline{x}$. If h is (strongly) star quasiconvex on K with modulus $\gamma \geq 0$ with respect to $\overline{x}$, then for every $y \in K$ and every $z = \overline{x} + t(y - \overline{x})$, with $0 < t \leq 1$, the following implication holds:*

$$h(z) \leq h(y) - \frac{\gamma}{4}(1-t^2)\|y-\overline{x}\|^2 \left(= h(y) - \frac{\gamma}{4}(\|y-\overline{x}\|^2 - \|z-\overline{x}\|^2)\right). \quad (18)$$

Conversely, if h is continuous and (18) holds for every $y \in K$, then h is (strongly) star quasiconvex on K with the same modulus $\gamma \geq 0$ with respect to $\overline{x}$.

Proof. Just take $x = \overline{x}$ in the proof of [18, Theorem 4]. $\square$

From Corollary 16, we can obtain a quadratic growth property for the unique minimizer of strongly star quasiconvex function.

Proposition 17. (Quadratic Growth Property) *Let $K \subseteq \mathbb{R}^n$ be a closed set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function such that $K \subseteq \operatorname{dom} h$. Suppose that $\overline{x} \in \operatorname{argmin}_K h$ and K is star-shaped at $\overline{x}$. If h is strongly star quasiconvex on K with modulus $\gamma > 0$, then*

$$h(\overline{x}) + \frac{\gamma}{4}\|y-\overline{x}\|^2 \leq h(y), \ \forall \ y \in K, \quad (19)$$

Proof. Exactly as the proof of [18, Corollary 9]. $\square$

The following result provides information regarding the behaviour of strongly star quasiconvex functions at infinity, a result which extends and improves [25, Theorem 1] and [8, Proposition 12] to strong star quasiconvexity and star-shaped sets.

Proposition 18. (2-Supercoercivity on Star-Shaped sets) *Let $K \subseteq \mathbb{R}^n$ be a closed set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper function such that $K \subseteq \operatorname{dom} h$. Suppose that $\overline{x} \in \operatorname{argmin}_K h$ and K is star-shaped at $\overline{x}$. If h is strongly star quasiconvex function on K with modulus $\gamma > 0$, then h is 2-supercoercive.*

Proof. Let $\{x_k\}_k \subseteq K$ be such that $\|x_k\| \to +\infty$. Then, we may assume that $\frac{x_k}{\|x_k\|} \to u \in K^{\infty}$ with $\|u\| = 1$. Since $\overline{x} \in \operatorname{argmin}_K h$, $h(\overline{x}) \leq h(x_k)$ for all $k \in \mathbb{N}$. Using relation (19), we have

$$\begin{aligned} h(x^k) \geq h(\overline{x}) + \frac{\gamma}{4}\|x_k - \overline{x}\|^2 &\Longrightarrow \frac{h(x_k)}{\|x_k\|^2} \geq \frac{h(\overline{x})}{\|x_k\|^2} + \frac{\gamma}{4}\left\|\frac{x_k}{\|x_k\|} - \frac{\overline{x}}{\|x_k\|}\right\|^2 \\ &\Longrightarrow \liminf_{k\to+\infty} \frac{h(x_k)}{\|x_k\|^2} \geq \frac{\gamma}{4} > 0, \end{aligned}$$

i.e., h is 2-supercoercive. □

The following result is useful for constructing (strongly) star quasiconvex functions with many critical points along lines which are neither global nor local minimizers (see [19, 20] for the quasar-convex case). Note that in the next proposition, differentiability of g is not needed.

Proposition 19. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $0 \in \operatorname{argmin}_{\mathbb{R}} f$ and $f(0) = 0$, and $g : \mathbb{R}^n \to \mathbb{R}$ be a function suh that $g \geq 1$. Define the function*

$$h(x) := f(\|x\|)g\left(\frac{x}{\|x\|}\right), \ \forall\, x \in \mathbb{R}^n\backslash\{0\}, \tag{20}$$

and $h(0) := 0$. If f is strongly star quasiconvex with modulus $\gamma \geq 0$, then h is strongly star quasiconvex with modulus $\gamma \geq 0$.

Proof. Let $y \in \mathbb{R}^n$ and $\overline{x} = 0 \in \operatorname{argmin}_{\mathbb{R}} f$ and $f(\overline{x}) = 0$, we have

$$h(\lambda\overline{x} + (1-\lambda)y) = h((1-\lambda)y) = f((1-\lambda)\|y\|)g\left(\frac{y}{\|y\|}\right), \ \forall\, \lambda \in [0,1].$$

Since f is strongly star quasiconvex with modulus $\gamma \geq 0$, we have

$$\begin{aligned} f((1-\lambda)\|y\|)g\left(\frac{y}{\|y\|}\right) &\leq \left(f(\|y\|) - \lambda(1-\lambda)\frac{\gamma}{2}\|y\|^2\right) g\left(\frac{y}{\|y\|}\right) \\ &= h(y) - \lambda(1-\lambda)\frac{\gamma}{2}\|y\|^2 g\left(\frac{y}{\|y\|}\right). \end{aligned}$$

Since $g \geq 1$, we have

$$h((1-\lambda)y) + \lambda(1-\lambda)\frac{\gamma}{2}\|y\|^2 \leq h(y) + \lambda(1-\lambda)\frac{\gamma}{2}\|y\|^2 \left(1 - g\left(\frac{y}{\|y\|}\right)\right)$$
$$\leq h(y), \ \forall \ \lambda \in [0,1],$$

i.e., h is strongly star quasiconvex with modulus $\gamma \geq 0$. $\square$

The following examples shows the nonconvex behaviour of a strongly star quasiconvex function of type (20).

Example 20. *Let $\alpha \in]0,1[$ and $k \in \mathbb{N}$, and the functions*

$$f(t) = \max\{|t|^{\alpha}, |t|^2 - k\}, \quad \text{and}$$
$$g(x_1, x_2) = \frac{1}{4N}\sum_{i=1}^{N}\left(a_i \sin(b_i x_1)^2 + c_i \cos(d_i x_2)^2\right),$$

with $N = 10$, a_i and c_i are independently and uniformly distributed on $[0, 20]$, while b_i and d_i are independently and uniformly distributed on $[-25, 25]$.

Then f is strongly quasiconvex on $\mathbb{R}^n$ as the maximum of two strongly quasiconvex functions (see [13]), thus strongly star quasiconvex by Proposition 5, while g satisfies the assumptions of Proposition 19. Therefore,

$$h(x_1, x_2) = f(\|(x_1, x_2)\|)g\left(\frac{(x_1, x_2)}{\|(x_1, x_2)\|}\right),$$

is strongly star quasiconvex. Note that h is nonsmooth because f is nonsmooth. We plot function h in Figure 2, below.

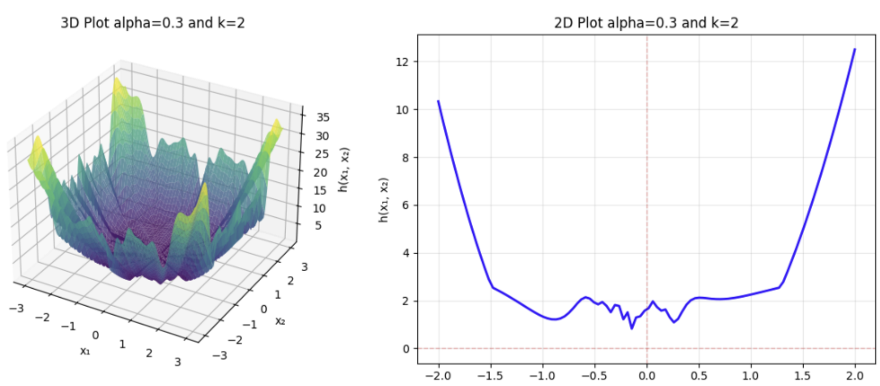

Figure 2: Function h in Example 20 when $\alpha = 0.3$ and $k = 2$. A 3D plot of h (left) and an arbitrary segment that does not contain the minimizer (right).

Now, let us study the properties of the proximity operator of a strongly star quasiconvex function h. To that end, we first observe:

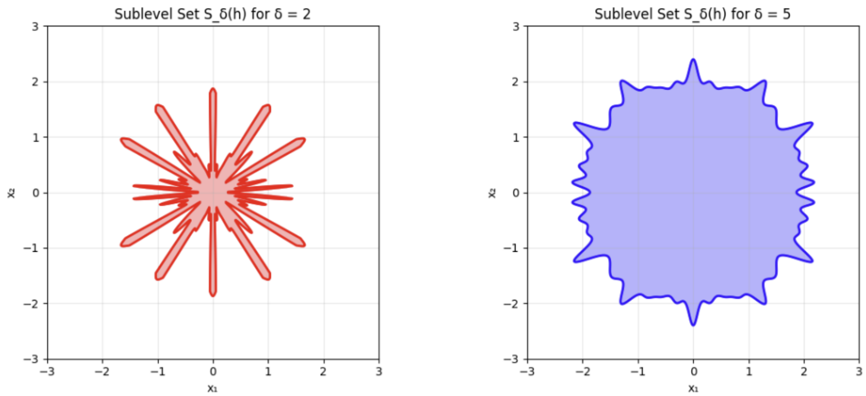

Figure 3: An illustration of the sublevel sets at height $\delta = 2$ (left) and $\delta = 5$ (right) of function h described in Example 20 when $\alpha = 0.3$ and $k = 2$.

Remark 21. *Let $\beta > 0$ in the statements below. Then,*

(i) If h is lsc and strongly star quasiconvex with $\gamma > 0$, then $\mathrm{Prox}_{\beta h} \neq \emptyset$ and compact because h is 2-supercoercive by Corollary 18.

(ii) Since strongly quasiconvex functions are strongly star quasiconvex, $\mathrm{Prox}_{\beta h}$ is not necessarily a singleton in virtue of [25, Remark 6].

(iii) If h is just star quasiconvex ($\gamma = 0$), then $\mathrm{Prox}_{\beta h}$ could be empty as for the quasiconvex function $h(x) = x^3$.

A connection between the fixed points of the proximity operator and the unique minimizer of a strongly star quasiconvex function is given below. We emphasize that in the next two results, the set K is considered star-shaped (but not necessarily convex).

Proposition 22. (Fixed Points Criteria on Star-Shaped sets) *Let $K \subseteq \mathbb{R}^n$ be a closed set, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lsc function such that $K \subseteq \mathrm{dom}\, h$, and $\beta > 0$. Suppose that $\overline{x} \in \mathrm{argmin}_K\, h$ and K is star-shaped at $\overline{x}$. If h is strongly star quasiconvex function on K with modulus $\gamma > 0$, then*

$$\mathrm{Fix}\,(\mathrm{Prox}_{\beta h}(K, \cdot)) = \frac{\gamma}{4}\text{–}\mathrm{argmin}_K\, h. \tag{21}$$

Proof. ($\supseteq$) Straightforward.

($\subseteq$) Let $x^* \in \mathrm{Prox}_{\beta h}(K, x^*)$. Then, $h(x^*) \leq h(x) + \frac{1}{2\beta}\|x^* - x\|^2$ for all $x \in K$. Since K is star-shaped at $\overline{x}$, take $x = \lambda \overline{x} + (1 - \lambda)x^* \in K$ with $\lambda \in [0, 1]$. Since

h is strongly star quasiconvex on K with modulus $\gamma > 0$, we have

$$h(x^*) \leq h(\lambda\overline{x} + (1-\lambda)x^*) + \frac{1}{2\beta}\|\lambda(\overline{x} - x^*)\|^2$$
$$\leq h(x^*) - \lambda(1-\lambda)\frac{\gamma}{2}\|x^* - \overline{x}\|^2 + \frac{\lambda^2}{2\beta}\|x^* - \overline{x}\|^2$$
$$\Longrightarrow \frac{\lambda}{2}\left(\gamma - \lambda\gamma - \frac{\lambda}{\beta}\right)\|x^* - \overline{x}\|^2 \leq 0, \ \forall \ \lambda \in]0,1].$$

Taking $0 < \lambda < \frac{\gamma\beta}{1+\gamma\beta} < 1$, we obtain $\|x^* - \overline{x}\|^2 \leq 0$, thus $x^* = \overline{x} \in \operatorname{argmin}_K h$.

Since h is strongly star quasiconvex with modulus $\gamma > 0$, the minimizer $\overline{x}$ is unique and satisfies the quadratic growth property (19), i.e., $x^* = \overline{x} \in \frac{\gamma}{4}-\operatorname{argmin}_K h$, which completes the proof. □

A relationship between the minimizer of the proximity operator and the minimizers of the function is given below

Proposition 23. (Relationship with Proximity Operator on Star-Shaped sets) *Let $K \subseteq \mathbb{R}^n$ be a closed set, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lsc function such that $K \subseteq \operatorname{dom} h$, $\beta > 0$ and $z \in K$. Suppose that $\overline{x} \in \operatorname{argmin}_K h$, K is star-shaped at $\overline{x}$ and h is strongly star quasiconvex function on K with modulus $\gamma > 0$. If $x^* \in \operatorname{Prox}_{\beta h}(K, z)$, then*

$$\frac{\gamma}{2}\|x^* - \overline{x}\|^2 \leq \frac{1}{\beta}\langle x^* - z, \overline{x} - x^*\rangle. \tag{22}$$

Proof. Since $x^* \in \operatorname{Prox}_{\beta h}(K, z)$, we have

$$h(x^*) + \frac{1}{2\beta}\|x^* - z\|^2 \leq h(x) + \frac{1}{2\beta}\|x - z\|^2, \ \forall \ x \in K. \tag{23}$$

Since K is star-shaped at $\overline{x}$ and h is strongly star quasiconvex on K with modulus $\gamma > 0$, taking $x = \lambda\overline{x} + (1-\lambda)x^* \in K$ for all $\lambda \in [0,1]$, we have

$$h(x^*) + \frac{1}{2\beta}\|x^* - z\|^2 \leq h(\lambda\overline{x} + (1-\lambda)x^*) + \frac{1}{2\beta}\|\lambda(\overline{x} - z) + (1-\lambda)(x^* - z)\|^2,$$
$$\leq h(x^*) - \lambda(1-\lambda)\frac{\kappa\gamma}{2}\|x^* - \overline{x}\|^2 + \frac{\lambda}{2\beta}\|z - \overline{x}\|^2 + \frac{(1-\lambda)}{2\beta}\|x^* - z\|^2$$
$$- \frac{\lambda(1-\lambda)}{2\beta}\|x^* - \overline{x}\|^2, \ \forall \ \lambda \in [0,1],$$

in virtue of (1). Then, it follows from (2) that

$$0 \leq \frac{\lambda}{\beta}\langle x^* - z, \overline{x} - x^*\rangle + \frac{\lambda}{2}\left(\frac{\lambda}{\beta} - \gamma + \lambda\gamma\right)\|x^* - \overline{x}\|^2,$$

and the result follows by multiplying by $\frac{1}{\lambda}$, and then taking $\lambda \downarrow 0$. □

3.3 The Differentiable Case

Under differentiability, we can said more regarding (strongly) star quasiconvex functions. As a first result, we revisited the following characterization (see [34, Theorem 3.3]).

Proposition 24. (First-order Characterization) *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and $\overline{x} \in \mathrm{argmin}_{\mathbb{R}^n} h$. Then h is (strongly) star quasiconvex with modulus $\gamma \geq 0$ with respect to $\overline{x}$ if and only if for every $y \in \mathbb{R}^n$, we have*

$$\langle \nabla h(y), y - \overline{x} \rangle \geq \frac{\gamma}{2} \|y - \overline{x}\|^2. \tag{24}$$

Proof. ($\Rightarrow$): Exactly as in [34, Theorem 3.3].

($\Leftarrow$): Let $y \in \mathbb{R}^n$ and $z := \overline{x} + t(y - \overline{x})$, with $0 < t < 1$. Take any $s \in [t, 1]$ and set $\overline{x}_s := \overline{x} + s(y - \overline{x})$, $g(s) = h(\overline{x}_s)$. Hence, by assumption, $\langle \nabla h(\overline{x}_s), \overline{x}_s - \overline{x} \rangle \geq \frac{\gamma}{2} \|\overline{x}_s - \overline{x}\|^2$. Thus, $\langle \nabla h(\overline{x}_s), y - \overline{x} \rangle \geq \frac{\gamma}{2} s \|y - \overline{x}\|^2$ or $g'(s) \geq \frac{\gamma}{2} s \|y - \overline{x}\|^2$. By integrating between t and 1, we have

$$\begin{aligned} & g(1) - g(t) \geq \frac{\gamma}{4} \|y - \overline{x}\|^2 (1 - t^2) \\ \Longleftrightarrow \quad & h(y) - h(\overline{x} + t(y - \overline{x})) \geq \frac{\gamma}{4} \|y - x\|^2 (1 - t^2). \end{aligned}$$

Hence, h satisfies (18), i.e., h is (strongly) star quasiconvex with modulus $\gamma \geq 0$ with respect to $\overline{x}$ by Corollary 16. □

Remark 25. *Observe that condition* (24) *was called Restricted Secant Condition in [44].*

Recall that a differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ satisfies the Polyak-Łojasiewicz (PL henceforth) [28, 36] (PL) property if there exists $\mu > 0$ such that

$$\|\nabla h(x)\|^2 \geq \mu(h(x) - h(\overline{x})), \ \forall \ x \in \mathbb{R}^n, \tag{25}$$

where $\overline{x} \in \mathrm{argmin}_{\mathbb{R}^n} h$. (PL) property is implied by strong convexity, but it allows for multiple minimizers and does not require any convexity assumption (see [24]).

As a consequence of Proposition 24, strongly star quasiconvex functions satisfies (PL) property when they have a Lipschitz continuous gradient. Since the proof is similar to the one in [26, Proposition 7], it is omitted.

Proposition 26. (Polyak-Łojasiewicz Property) *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable with L-Lipschitz continuous gradient ($L > 0$) and strongly star quasiconvex with modulus $\gamma > 0$. Then (PL) property holds with modulus $\mu := \frac{\gamma^2}{2L} > 0$, that is,*

$$\|\nabla h(x)\|^2 \geq \frac{\gamma^2}{2L} (h(x) - h(\overline{x})), \ \forall \ x \in \mathbb{R}^n, \tag{26}$$

where $\overline{x} = \mathrm{argmin}_{\mathbb{R}^n} h$.

4 Linear Convergence of Gradient-type Methods

Using Propositions 24 and 26, we can ensure the linear convergence of the Heavy-ball and Nesterov accelerations of the gradient method following [17, 26]. The similar proofs are omitted.

We emphasize that during the whole section, the function h is assumed to be strongly star quasiconvex with modulus $\gamma > 0$.

Theorem 27. (Linear Convergence of Heavy-Ball Acceleration) *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable strongly star quasiconvex function with modulus $\gamma > 0$ with L-Lipschitz continuous gradient and $\overline{x} = \mathrm{argmin}_{\mathbb{R}^n} h$. Suppose that*

$$\alpha \in \left[0, \frac{\sqrt{2}}{2}\right[, \quad \beta \in \left]0, \frac{1-2\alpha^2}{L}\right].$$

Then the sequence $\{x_k\}_k$, generated by the heavy ball method:

$$\begin{cases} y_k = x_k + \alpha(x_k - x_{k-1}), \\ x_{k+1} = y_k - \beta \nabla h(x_k), \end{cases} \tag{27}$$

converges linearly to $\overline{x}$ and the sequence $\{h(x_k)\}_k$ converges linearly to the optimal value $h^* = h(\overline{x})$.

Proof. Same proof than [17, Theorem 13] with $\theta = 0$. $\square$

As a direct consequence of the above theorem, we have the following convergence rate result.

Corollary 28 (Convergence rate)**.** *Under the assumptions of Theorem 27, we have the following convergence rate*

$$\begin{aligned} \frac{\gamma}{4}\|x_{k+1} - \overline{x}\|^2 &\leq h(x_{k+1}) - h^* \leq \left(1 - \frac{\rho}{\sigma}\right)^k \mathcal{E}_1, \\ \|x_{k+1} - x_k\|^2 &\leq \frac{\beta}{\alpha^2}\left(1 - \frac{\rho}{\sigma}\right)^k \mathcal{E}_1, \end{aligned}$$

where $\mathcal{E}_1 :=:= h(x_k) - h^* + \frac{\alpha^2}{\beta}\|x_k - x_{k-1}\|^2$, $\rho := \min\left\{\frac{\beta}{2}, \frac{1}{2\beta}\left(1 - \beta L - 2\alpha^2\right)\right\}$ and $\sigma := \max\left\{\frac{15}{\beta}, \frac{2L}{\gamma^2} + 15\beta\right\}$.

Remark 29. (i) *For the projected heavy-ball method for strongly star quasiconvex functions, please refer to [34].*

(ii) *Note that if $\alpha = 0$, then we obtain convergence rates for the classical gradient method.*

In the following theorem, we provide a linear convergence rate for the Nesterov acceleration.

Theorem 30. (Linear Convergence of Nesterov Acceleration) *Let* $h : \mathbb{R}^n \to \mathbb{R}$ *be a differentiable with L-Lipschitz continuous gradient and strongly quasiconvex function with modulus* $\gamma > 0$ *and* $\overline{x} = \mathrm{argmin}_{\mathbb{R}^n} h$*. Let* $\eta > 1$, $0 < \beta < \frac{\gamma}{\eta L^2}$ *and* μ_1, μ_2 *be defined by:*

$$\mu_1 := \frac{1}{1 + \beta(\gamma - \eta\beta L^2)} \in]0, 1[\quad \text{and} \quad \mu_2 := \frac{1 - \frac{1}{\eta}}{1 + \beta(\gamma - \beta\eta L^2)} > 0.$$

Let $\epsilon \in \left]0, \frac{1}{\mu_1} - 1\right[$ *and* α *be satisfying* $\alpha \leq \frac{\mu_1\mu_2(1+\epsilon)}{\mu_1\mu_2(1+\epsilon)+\mu_1+\frac{\mu_1}{\epsilon}+\mu_2}$*. Then the sequence* $\{x_k\}_k$*, generated by the Nesterov acceleration:*

$$\begin{cases} y_k = x_k + \alpha(x_k - x_{k-1}), \\ x_{k+1} = y_k - \beta \nabla h(y_k), \end{cases} \tag{28}$$

converges linearly to $\overline{x}$ *and the sequence* $\{h(x_k)\}_k$ *converges linearly to the optimal value* $h^* = h(\overline{x})$*.*

Proof. Same proof than [17, Proposition 16 and Theorem 17] with $\theta = 0$. □

As a direct consequence of the above theorem, we have the following convergence rate result for the Algorithm generated by (28).

Corollary 31 (Convergence rate)**.** *Under the assumptions of Theorem 30 and that* $x_0 := x_{-1}$*, we have the following convergence rate*

$$\|x_k - \overline{x}\|^2 \leq (\mu_1(1+\epsilon))^{k-1} \mathcal{E}_1,$$
$$\|x_k - x_{k-1}\|^2 \leq \left(\frac{\mu_1(1+\epsilon)}{\mu_2(1-\alpha)}\right)^{k-1} \mathcal{E}_1.$$

Moreover,

$$h(x_k) - h^* \leq \frac{L}{2} (\mu_1(1+\epsilon))^{k-1} \mathcal{E}_1,$$

where $\mathcal{E}_1 := \|x_k - \overline{x}\|^2 + \mu_2(1-\alpha)\|x_k - x_{k-1}\|^2 \geq 0$.

Remark 32. *(i) In virtue of Proposition 5, the results above includes strongly quasiconvex and strongly quasar-convex functions, simultaneously. Furthermore, the proofs are as in [17, 26] in virtue of Theorem 12.*

(ii) Observe that the convergence rate results provided for heavy ball and Nesterov accelerations for strongly star quasiconvex functions (Corollaries 28 and 31) are exactly as good as for strongly quasiconvex functions, even when we included more functions.

5 Linear Convergence of PPA on Star-Shaped Sets

As in the previous section, we emphasize that the function h is assumed to be strongly star quasiconvex with modulus $\gamma > 0$.

Let us consider the classical version of the proximal point algorithm.

Algorithm 1 PPA for Strongly Star Quasiconvex Functions (PPA-SSQC)

Step 0. Let $K \subseteq \mathbb{R}^n$ be a closed and star-shaped set at the unique minimizer of h on K. Take $x^0 \in K$, $k = 0$ and $\{\beta_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive numbers.

Step 1. Take

$$x^{k+1} \in \mathrm{Prox}_{\beta_k h}(K, x^k). \tag{29}$$

Step 2. If $x^{k+1} = x^k$, then STOP, $\{x^k\} = \frac{\gamma}{4}\text{-}\mathrm{argmin}_K\, h$. Otherwise, take $k = k + 1$ and go to Step 1.

Note that the iterates exists because h is strongly star quasiconvex (thus 2-supercoercive) and lsc, and the stopping criterio holds by Proposition 22.

In the next result, we ensure linear convergence for PPA for strongly star quasiconvex functions ($\gamma > 0$) on a closed and star-shaped set K at the minimizer.

Theorem 33. (Linear Convergence of PPA on Star-Shaped Domains) *Let $K \subseteq \mathbb{R}^n$ be a closed set and $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lsc function such that $K \subseteq \mathrm{dom}\, h$. Suppose that $\overline{x} \in \mathrm{argmin}_K\, h$, K is star-shaped at $\overline{x}$ and h is strongly star quasiconvex function on K with modulus $\gamma > 0$. Take $\beta' > 0$ and $\{\beta_k\}_{k \in \mathbb{N}}$ a sequence of positive numbers such that*

$$0 < \beta' \leq \beta_k, \ \forall \ k \in \mathbb{N}_0. \tag{30}$$

Then the sequence $\{x^k\}_{k \in \mathbb{N}}$, generated by Algorithm 1, is a minimizing sequence of h on K and converges linearly to the unique solution $\overline{x} = \mathrm{argmin}_K\, h$ with rate of at least:

$$\frac{1}{1 + \beta' \gamma} \in \,]0, 1[. \tag{31}$$

Proof. If $x_{k+1} = x_k$ for some $k \in \mathbb{N}$, then the algorithm stops and $x_k \in \mathrm{argmin}_K\, h$ by Proposition 22. So we assume that $x_{k+1} \neq x_k$ for all $k \in \mathbb{N}$.

Since $x^{k+1} \in \mathrm{Prox}_{\beta_k h}(K, x^k)$, taking $z = x^k$, $x^* = x^{k+1}$ and $x = x^k$ in (22), and since we are assuming $x^{k+1} \neq x^k$, we obtain

$$h(x^{k+1}) < h(x^{k+1}) + \frac{1}{2\beta_k}\|x^{k+1} - x^k\|^2 \leq h(x^k). \tag{32}$$

Hence, $\{h(x^k)\}_{k\in\mathbb{N}_0}$ is a decreasing sequence.

Since $x^{k+1} \in \mathrm{Prox}_{\beta_k h}(K, x^k)$, by Proposition 23, we have

$$\langle x^{k+1} - x^k, x^{k+1} - \overline{x}\rangle \leq -\frac{\gamma\beta_k}{2}\|x^{k+1} - \overline{x}\|^2, \ \forall\, y \in K. \tag{33}$$

On the other hand,

$$\begin{aligned}\|x^{k+1} - \overline{x}\|^2 &= \|x^{k+1} - x^k + x^k - \overline{x}\|^2 \\ &= \|x^{k+1} - x^k\|^2 + \|x^k - \overline{x}\|^2 + 2\langle x^{k+1} - x^k, x^k - \overline{x}\rangle \\ &= \|x^k - \overline{x}\|^2 - \|x^{k+1} - x^k\|^2 + 2\langle x^{k+1} - x^k, x^{k+1} - \overline{x}\rangle.\end{aligned}$$

Then, using (33),

$$\begin{aligned}(1+\beta_k\gamma)\|x^{k+1} - \overline{x}\|^2 &\leq \|x^k - \overline{x}\|^2 - \|x^{k+1} - x^k\|^2 \\ \implies \|x^{k+1} - \overline{x}\|^2 &\leq \frac{1}{(1+\beta'\gamma)}\|x^k - \overline{x}\|^2. \end{aligned}\tag{34}$$

Hence, $\{x^k\}_k$ converges linearly to the unique solution $\overline{x} \in \mathrm{argmin}_K\, h$, with convergence rate of at least $\frac{1}{1+\beta'\gamma} \in \,]0,1[$, and using (32) we conclude that it is a minimizing sequence, i.e., $h(x^k) \downarrow \min_{x\in K}\, h(x)$, which completes the proof. $\square$

As a corollary, we obtain linear convergence for PPA applied to strongly quasiconvex functions, a results which was not included in [25].

Corollary 34. *Let $K \subseteq \mathbb{R}^n$ be a closed and convex set, $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper, lsc and strongly quasiconvex function on $K \subseteq \mathrm{dom}\, h$ with modulus $\gamma > 0$. Let $\beta' > 0$ and $\{\beta_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that* (30) *holds. Then the sequence $\{x^k\}_{k\in\mathbb{N}}$, generated by relation* (29)*, is a minimizing sequence of h and converges linearly to the unique solution $\overline{x} = \mathrm{argmin}_K\, h$ with rate of at least:*

$$\frac{1}{1+\beta'\gamma} \in \,]0,1[.$$

The scope of Theorem 33 is analyzed below.

Remark 35. (a) *Since $\beta^{'} > 0$ in* (30)*, the linear rate in* (31) *can be as fast as we want, a situation which also happens in the strongly convex case. Furthermore, the linear rate for strongly star quasiconvex functions $\frac{1}{1+\beta'\gamma}$ is exactly as good as in the strongly convex case (see, for instance, [7, Example 23.40(ii)]).*

(b) *The function in Example 20 is strongly star quasiconvex without being neither weakly convex nor DC nor strongly quasiconvex nor strongly quasarconvex, i.e., the results from [1, 3, 6, 8, 11, 21, 25, 35, 39] and references therein can not be applied to this function.*

(c) *In contrast to strongly convex and strongly quasar-convex functions (see [38] and [8], respectively), we can not provide convergence rate for the functions values* $\{h(x^k)\}_k$. *This limitation is of course expected, since we are including strongly quasiconvex functions, a case in which the analysis of the function values is not possible.*

Remark 36. *To the best of our knowledge, Theorem 33 provides the first result guaranteeing linear convergence of the sequence generated by PPA on closed star-shaped sets, without requiring convexity. It is important to note that this is primarily a theoretical advancement. In practice, the implementation of such an algorithm remains challenging, as closed-form expressions for the proximity operator on star-shaped sets, as well as efficient subroutines for minimization over them, are largely unexplored.*

6 Conclusions and Future Work

We unify the analysis of linear convergence for first-order methods on generalized convex functions by introducing the class of (strongly) star quasiconvex functions. We provide a comprehensive study of this class, presenting several characterizations and useful properties for both nonsmooth and differentiable cases. Our theoretical results have promising implications for nonconvex optimization and open the door to further developments in computational methods on star-shaped sets.

We hope this work provides new avenues for studying the theoretical properties of functions with star-shaped sublevel sets and their applications in more sophisticated methods, such as splitting-type algorithms (see, for instance, [3, 6]). It also lays the groundwork for exploring potential connections between star quasiconvex functions and reference point theory in Prospect Theory (see [23, 40]).

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