

Universal Limits on Quantum Correlations

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The fundamental limits of quantum correlations set the foundation of quantum mechanics and quantum information science. Exact bounds – the Cramér–Rao inequality, the Heisenberg limit, the Lieb–Robinson bound – have served to anchor entire fields, yet each applies only to a narrow class of systems or observables. Here, we introduce a general framework from which all known correlation limits, as well as some new ones, can be derived from a single geometric principle: the positivity of quantum state space. This intrinsic positive geometry defines a unique determinant-ratio invariant, denoted χ , which quantifies the combinatorial structure of correlations in any quantum system. Every measure of nonclassical correlation is bounded by a simple function of χ , yielding universal, model-independent floors and ceilings valid for arbitrary architectures. For systems with Lie-group symmetries, the bounds acquire compact closed forms. We recover the Heisenberg and Cramér–Rao limits and uncover previously unknown constraints, including an exact entanglement floor in multimode squeezing networks and a universal Fisher-information ceiling in fully connected spin ensembles—demonstrating that even all-to-all connectivity cannot exceed the positivity-imposed *light-cone* in state-space. Finally, we show that every correlation bound, old or new, exhibits local catastrophe-theoretic structure, with universal critical exponents classifying its approach to saturation. Positivity geometry thus provides a unified, first-principles theory of quantum limits.

I. INTRODUCTION

The most fundamental limits of quantum mechanics, how strongly systems can correlate, how precisely they can encode information, and how rapidly they can evolve; define the quantitative structure of the quantum world. These limits are not merely technical results, but rather represent the foundational essence of quantum theory itself: the Cramér–Rao bound [1, 2] and the Heisenberg limit [3] set the scale of metrological precision, while Lieb–Robinson bounds impose causal light cones on many-body dynamics [4, 5]. Yet, much remains unknown, with each of the few known exact correlation bounds applying only to particular classes of systems. In this work, we take a significant step towards completing the picture, showing that these pillars of quantum theory represent fragments of a deeper principle, which has thus far remained hidden: that each of these bounds, whether constraining dynamics [4–6] or measurement precision [2, 7], can be viewed as a different projection of a single geometric constraint. Specifically, we show that every bound on quantum correlations is encoded in a single geometric constraint: the geometric positivity of the quantum state space. From this intrinsic positive geometry [8–13] we identify a unique invariant, denoted χ , that measures how far a state lies from the positivity boundary and encodes the combinatorial structure of its correlations. All correlation measures, from entanglement to Fisher information, are constrained by universal inequalities that depend only on χ . For systems with simple symmetries these inequalities become closed-form “floors” and “ceilings,” reproducing known results such as the Heisenberg and Cramér–Rao limits and revealing new ones, including an exact entanglement floor and a universal Fisher-information ceiling even in fully connected spin networks [14–16]. Positivity geom-

etry therefore provides the first unified theory of the fundamental limits of quantum correlations.

Positive Geometry and Quantum Mechanics – Quantum mechanics can be regarded as a geometry of states. Every physical state is represented by a positive semidefinite density operator, and every measurable covariance or Gram matrix is likewise positive. This positivity condition defines a convex cone in the space of Hermitian operators—the *quantum state space*. All observable correlations, variances, and information metrics are therefore restricted by the geometry of this cone. Despite its centrality, this geometric constraint is rarely treated as a dynamical or quantitative principle. In most formulations, positivity is assumed as a consistency condition rather than elevated to a source of predictive structure.

Recent developments across physics suggest that positivity may play a more fundamental role. In high-energy theory, the discovery of positive geometries such as the amplituhedron has shown that physical consistency can be encoded directly as geometric positivity. In condensed-matter and quantum-information contexts, positive semidefiniteness of reduced density matrices underlies entanglement criteria, uncertainty relations, and majorization hierarchies. Yet these manifestations remain scattered and measure-specific. What has been missing is a unifying framework that derives all such inequalities from the single postulate that the global correlation matrix of a quantum system is positive.

The geometric framework – We begin with the observation that for any bipartition $A|B$, the total correlation matrix

$$K = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{pmatrix} \succ 0$$

is positive definite. From the Schur-complement property

of positive matrices, $K_{AA} - K_{AB}K_{BB}^{-1}K_{BA} \succ 0$, one immediately obtains determinant inequalities of Hadamard–Fischer type, $\det K \leq \det K_{AA} \det K_{BB}$. We use this geometric inequality to define a dimensionless invariant,

$$\chi(A|B) = 1 - \frac{\det K}{\det K_{AA} \det K_{BB}},$$

which measures the fractional deficit of the joint volume relative to the product of marginal volumes. This scalar χ , bounded between 0 and 1, quantifies how far the state lies from the positivity boundary: $\chi = 0$ for uncorrelated (block-diagonal) states and $\chi \rightarrow 1$ as the positivity constraint is saturated. Because the ratio is invariant under independent linear transformations on A and B , χ captures the intrinsic correlation geometry of the bipartition and is independent of representation.

In Gaussian covariance language, χ can be expressed in terms of the singular values $\{\tau_i\}$ of the inter-block “transfer operator” $T = A^{-1/2}CB^{-1/2}$ as $1 - \chi = \prod_i (1 - \tau_i^2)$. This spectral form shows that χ aggregates the complete correlation spectrum into a single invariant parameter. In this sense χ is the “combinatorial” invariant of positivity geometry—the scalar that measures the effective curvature of state space induced by correlations.

From Invariant to Bounds – The central result of this work is that all quantitative measures of quantum correlation are bounded by functions of χ . Any correlation measure $M(A|B)$ that is monotonic under positivity-preserving maps obeys an inequality of the form

$$M(A|B) \bowtie f(\chi(A|B)),$$

where \bowtie denotes either \geq or \leq depending on whether M increases or decreases with correlation strength. The invariant χ thus acts as a *universal generator of bounds*, producing entanglement *floors* and metrological *ceilings* as two opposite manifestations of the same geometric constraint.

To illustrate this duality we analyze two canonical measures. For the logarithmic negativity [17–19] E_N we obtain an exact lower bound, $E_N(A|B) \geq -\frac{1}{2} \log[1 - \chi(A|B)]$, which is tight for Gaussian normal forms and defines a universal entanglement floor. For the quantum Fisher information (QFI) [7, 20, 21] F_Q we find an upper bound, $F_Q \leq 4[J^2(1 - s^2) + J]$ for SU(2) systems and analogous expressions for SU(1,1) networks, establishing a metrological ceiling even in all-to-all connected ensembles. These bounds are derived solely from positivity geometry and therefore hold independently of microscopic details or dynamical assumptions.

Beyond their specific forms, the derived inequalities share a striking structural feature: their approach to saturation follows universal scaling exponents classified by Thom–Arnold catastrophe theory. Bound saturation in compact systems manifests as a *fold* (quadratic) singularity, while noncompact or symmetry-protected degeneracies exhibit *cusplike* behavior with critical exponent

4/3. This universality provides a geometric taxonomy of how quantum systems reach their fundamental limits.

The framework introduced here unifies all known quantum correlation bounds and generates new ones from a single invariant of positivity geometry. It shows that the same mathematical structure that constrains scattering amplitudes and kinematic configurations also constrains the space of quantum states. Positivity geometry thereby extends from high-energy theory to the foundations of quantum information, establishing a geometric “light cone” in state space that delimits all attainable quantum correlations.

II. POSITIVITY GEOMETRY AND THE DETERMINANT–RATIO INVARIANT

The central object of the framework is the scalar invariant $\chi(A|B)$, which encodes the geometric content of Gram positivity for any bipartition of a quantum system. This section develops the invariant formally, beginning from basic properties of positive matrices, deriving χ from Schur complements, and interpreting it as a measure of distance from the positivity boundary. Although the derivation is purely linear-algebraic, the result applies to all quantum systems, independent of dimension, statistics, or dynamical model.

A. Gram positivity and Schur complements

Every physical correlation or covariance matrix is positive semidefinite:

$$K = \begin{pmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{pmatrix} \succ 0,$$

where the sub-blocks describe the marginal correlations within A and B and the off-diagonal blocks encode their mutual correlations. Positivity of K implies that both Schur complements,

$$K_{AA} - K_{AB}K_{BB}^{-1}K_{BA} \succ 0, \quad (1)$$

$$K_{BB} - K_{BA}K_{AA}^{-1}K_{AB} \succ 0, \quad (2)$$

are themselves positive definite. Taking determinants of Eqs. (1)–(2) produces the Hadamard–Fischer inequality,

$$\det K \leq \det K_{AA} \det K_{BB}, \quad (3)$$

with equality if and only if $K_{AB} = 0$. Equation (3) expresses a geometric fact: the phase-space volume of the joint system cannot exceed the product of the marginal volumes. The departure from equality therefore quantifies the “consumption of positivity” by inter-subsystem correlations.

B. Definition of the invariant

This observation motivates a dimensionless determinant ratio,

$$\chi(A|B) = 1 - \frac{\det K}{\det K_{AA} \det K_{BB}}, \quad (4)$$

which we call the *determinant–ratio invariant*. From (3) it follows immediately that

$$0 \leq \chi(A|B) < 1, \quad (5)$$

with $\chi = 0$ for uncorrelated (block-diagonal) states and $\chi \rightarrow 1$ as the positivity constraint is saturated. Because each determinant transforms as $\det(S_A K S_B^\top) = \det(S_A)^2 \det(S_B)^2 \det K$ under independent invertible linear changes of basis $S_A \oplus S_B$, the ratio (4) is invariant:

$$\chi(A|B) = \chi((S_A \oplus S_B)K(S_A \oplus S_B)^\top). \quad (6)$$

Hence χ depends only on the intrinsic correlation geometry of the bipartition, not on representation or coordinate choice.

C. Spectral representation and correlation spectrum

For Gaussian covariance matrices, the invariant acquires a particularly transparent form. Writing the matrix in block notation $K = \begin{pmatrix} A & C \\ C^\top & B \end{pmatrix}$, define the *transfer operator*

$$T = A^{-1/2} C B^{-1/2}, \quad (7)$$

which isolates correlations from local scales. Standard determinant identities yield

$$\frac{\det K}{\det A \det B} = \det(I - T^\top T) = \prod_i (1 - \tau_i^2), \quad (8)$$

where $\{\tau_i\}$ are the singular values of T . Substituting into (4) gives

$$\chi(A|B) = 1 - \det(I - T^\top T) = 1 - \prod_i (1 - \tau_i^2). \quad (9)$$

The set $\{\tau_i\}$ constitutes the *correlation spectrum* of the bipartition. Each mode with $\tau_i > 0$ contributes to the reduction of the joint positive volume, and the invariant χ aggregates these contributions into a single scalar quantity. If all $\tau_i = 0$ the subsystems are uncorrelated and $\chi = 0$; if any $\tau_i \rightarrow 1$, the positivity constraint approaches saturation and $\chi \rightarrow 1$.

D. Geometric interpretation

Equation (4) provides a natural geometric interpretation. In logarithmic coordinates $x_A = \log \det K_{AA}$, $x_B = \log \det K_{BB}$, and $x_{AB} = \log \det K$, the positivity condition (3) defines a convex region $x_{AB} \leq x_A + x_B$ within the space of determinant logs. This convex-region structure is a direct manifestation of standard results in convex analysis [22], which underlie all volume inequalities on positive cones. The positivity cone may therefore be viewed as a feasible set in the sense of modern convex optimization [23]. Level sets of constant χ correspond to parallel hyperplanes $x_{AB} = x_A + x_B + \log(1 - \chi)$ that foliate the interior of this region. The boundary $\chi = 0$ marks perfect factorization, while $\chi \rightarrow 1$ approaches the positivity frontier. Every physical quantum state occupies a definite χ -level surface, so χ serves as a geometric order parameter for correlations: it measures how “close to singular” the joint Gram matrix is relative to its marginals.

E. Universality and invariance under composition

The determinant–ratio invariant has two crucial structural properties. First, it is *universal*: any positive operator, density matrix, or covariance matrix admits a well-defined $\chi(A|B)$ regardless of Hilbert-space dimension or operator algebra. Second, it obeys a multiplicative composition rule for independent subsystems,

$$1 - \chi(A_1 A_2 | B_1 B_2) = (1 - \chi(A_1 | B_1))(1 - \chi(A_2 | B_2)), \quad (10)$$

which follows from the block-determinant identity. Equation (10) implies that χ plays the role of an “additive measure” for correlation strength under tensor products: $\log(1 - \chi)$ is extensive across independent components.

These algebraic and geometric properties establish χ as the unique scalar invariant of positivity geometry for bipartite systems. The next section shows that any monotone measure of quantum correlation must be bounded by a function of this invariant, leading to a universal hierarchy of entanglement floors and metrological ceilings.

III. THE UNIVERSAL χ -INEQUALITY

Having established the determinant–ratio invariant $\chi(A|B)$ as the unique scalar that captures the geometry of positivity for any bipartition, we now derive the general inequality that links χ to all quantitative measures of quantum correlation. The result may be stated as a theorem.

A. Theorem: the universal χ -inequality

Theorem 1. Let $M(A|B)$ be any real-valued correlation measure that is *monotonic under positivity-preserving maps* on the bipartition $A|B$. That is, M increases (or decreases) monotonically as the inter-block correlations K_{AB} strengthen, while the marginals K_{AA} and K_{BB} remain fixed. Then there exists a monotone function $f_M : [0, 1) \rightarrow \mathbb{R}$ such that

$$M(A|B) \bowtie f_M(\chi(A|B)), \quad (11)$$

where the inequality symbol \bowtie is “ \geq ” for increasing measures and “ \leq ” for decreasing measures. The function f_M is determined entirely by the normalization of M and by its dependence on the correlation spectrum $\{\tau_i\}$.

Equation (11) is the *universal χ -inequality*. It expresses the fact that positivity restricts every correlation measure to lie within a χ -dependent domain, producing either a *floor* or a *ceiling* according to the measure’s monotonicity.

B. Proof sketch

For any measure M that depends only on the positive matrix K through its principal minors or the singular values $\{\tau_i\}$ of the transfer operator $T = A^{-1/2}CB^{-1/2}$, Gram positivity implies $0 \leq \tau_i < 1$. The determinant identity $\det(I - T^T T) = \prod_i (1 - \tau_i^2) = 1 - \chi$ fully characterizes the admissible domain of $\{\tau_i\}$ at fixed χ . Hence M can be regarded as a monotone function of χ : $M(A|B) = M(\{\tau_i\}) = f_M(\chi)$. The sign of monotonicity follows from $\partial M / \partial \tau_i$. For increasing measures, $\partial M / \partial \tau_i > 0$ and M attains its minimum at $\tau_i = 0$, giving $M \geq f_M(\chi)$. For decreasing measures, the inequality reverses. This reasoning depends only on the monotonicity of M and the constraint $0 \leq \tau_i < 1$ imposed by positivity, not on the detailed structure of the system. \square

C. Floors and ceilings from monotonicity

The universal χ -inequality divides all correlation measures into two dual classes:

- **Increasing measures** (*entanglement-type*): quantities that grow with correlation strength, such as the logarithmic negativity E_N , mutual information $I(A:B)$, or Rényi entropies. These measures are bounded *below*:

$$M(A|B) \geq f_M(\chi), \quad (\text{entanglement floors}). \quad (12)$$

- **Decreasing measures** (*information-capacity-type*): quantities that decline as correlations grow, such as the quantum Fisher information F_Q , the quantum variance, or the purity $\text{Tr} \rho^2$ of reduced subsystems. These measures are bounded *above*:

$$M(A|B) \leq f_M(\chi), \quad (\text{metrological ceilings}). \quad (13)$$

The duality between (12) and (13) reflects the opposing curvature of their level surfaces within the positive cone. Entanglement-type measures vanish in the interior and increase toward the boundary; information-type measures peak in the interior and decline toward the boundary. Both are governed by the same geometric parameter χ .

D. Functional form of f_M

The explicit function $f_M(\chi)$ depends on the definition and units of M . For measures expressible as additive functions of the correlation spectrum, $M = \sum_i g(\tau_i)$ with monotone $g(\tau)$, the invariant relation (9) implies

$$f_M(\chi) = \sum_i g\left(\sqrt{1 - (1 - \chi)^{1/n}}\right), \quad (14)$$

where n is the number of active correlation modes. The precise exponent is unimportant for the qualitative structure: to leading order in small χ ,

$$f_M(\chi) \approx c_1 \chi + c_2 \chi^2 + \dots, \quad (15)$$

with coefficients c_k determined by the expansion of g . Thus all measures share the same initial slope in χ and differ only in higher-order corrections.

E. Universality and scope

Equation (11) is completely general. It applies to finite- or infinite-dimensional Hilbert spaces, discrete or continuous variables, and arbitrary physical architectures. Because the proof relies only on positivity and monotonicity, the resulting bounds are *state-space universal*: they remain valid under any dynamics or control protocol that preserves positivity.

In the following sections we instantiate the two canonical cases. Section IV analyzes the *entanglement floor* implied by (12) for the logarithmic negativity, while Section V develops the corresponding *metrological ceiling* for the quantum Fisher information. Together they exemplify the dual faces of positivity geometry and the power of the universal χ -inequality.

IV. ENTANGLEMENT FLOORS: THE LOGARITHMIC NEGATIVITY

Among all monotone measures of quantum correlation, the logarithmic negativity E_N provides a direct and experimentally accessible quantifier of bipartite entanglement. In the language of positivity geometry, E_N belongs to the “increasing” class of measures defined by Eq. (12), and therefore acquires a universal *lower bound* (floor) as a function of the invariant $\chi(A|B)$.

A. Logarithmic negativity and the positivity constraint

For a bipartite Gaussian state or for any state characterized by a positive covariance matrix V , the logarithmic negativity is defined as

$$E_N(A|B) = \sum_i \max[0, -\log \tilde{\nu}_{i,-}], \quad (16)$$

where $\{\tilde{\nu}_{i,-}\}$ are the symplectic eigenvalues of the partially transposed covariance matrix \tilde{V} . The smaller $\tilde{\nu}_{i,-}$, the greater the bipartite entanglement. Partial transposition preserves positivity only within a restricted subset of state space; the boundary at which $\tilde{\nu}_{i,-} = 1$ marks separability, and $\tilde{\nu}_{i,-} < 1$ signals entanglement.

For each normal mode i , Gram positivity of the original (untransposed) covariance matrix implies

$$\tilde{\nu}_{i,-}^2 \leq 1 - \tau_i^2, \quad (17)$$

where τ_i are the singular values of the transfer operator $T = A^{-1/2}CB^{-1/2}$ introduced in Sec. II. Equation (17) expresses the geometric fact that partial transposition can only reduce, never increase, the minimal symplectic eigenvalue, because it effectively reverses the sign of one quadrature within each correlated mode. Multiplying over all modes and using Eq. (9) gives

$$\tilde{\nu}_-^2 \leq 1 - \chi(A|B), \quad (18)$$

where $\tilde{\nu}_- = \prod_i \tilde{\nu}_{i,-}$ is the smallest collective symplectic eigenvalue. Substituting Eq. (18) into (16) yields the universal inequality

$$E_N(A|B) \geq -\frac{1}{2} \log[1 - \chi(A|B)]. \quad (19)$$

This is the *entanglement floor*. It depends solely on the positivity geometry encoded in χ and holds for all states—Gaussian or otherwise—consistent with the same second moments.

B. Tightness and saturation

The bound (19) is tight for Gaussian normal forms. For such states each correlated mode is equivalent to

a two-mode squeezed vacuum with squeezing parameter r_i related to the singular value by $\tau_i = \tanh(2r_i)$. Equation (17) is then saturated as $\tilde{\nu}_{i,-} = e^{-2r_i}$, and the inequality (19) becomes an equality. Summing over modes gives

$$E_N(A|B) = \frac{1}{2} \sum_i [-\log(1 - \tau_i^2)], \quad (20)$$

which reproduces the known analytic expression for multi-mode Gaussian entanglement and confirms that Gaussian states *minimize* the negativity at fixed χ . All non-Gaussian states with the same covariance data lie above this floor, as guaranteed by the convexity of E_N under mixtures.

C. Physical interpretation

The determinant-ratio invariant χ therefore provides an *operational entanglement witness*. Given any experimentally accessible covariance data, χ can be computed directly from block determinants, and Eq. (19) yields a certified minimum amount of entanglement without requiring full state tomography. The quantity χ is invariant under local linear transformations, so the bound is independent of local squeezing or rotations.

Equation (19) also establishes a one-to-one mapping between the geometry of state space and the resource content of the state. In this sense the positivity boundary acts as an “entanglement horizon”: as $\chi \rightarrow 1$, the joint Gram matrix approaches singularity, $\tilde{\nu}_- \rightarrow 0$, and the negativity diverges. The slope and curvature of E_N as a function of χ determine how rapidly entanglement resources accumulate near this boundary.

D. Example: two-mode squeezing network

For a single correlated pair ($n = 1$), the invariant reduces to $\chi = \tanh^2(2r)$, and Eq. (19) gives

$$E_N = -\frac{1}{2} \log[1 - \tanh^2(2r)] = \text{logcosh}(2r). \quad (21)$$

Direct evaluation of the negativity for the two-mode squeezed vacuum yields $E_N = 2r$, and indeed $E_N \geq \text{logcosh}(2r)$ for all $r > 0$, saturating in the small-squeezing limit. The bound therefore tracks the correct asymptotic scaling and provides an exact analytical envelope for all parameter regimes.

E. Example: multimode and network generalizations

For a multimode Gaussian network the transfer operator has singular values $\tau_i = \tanh(2r\sigma_i)$ determined

by the singular spectrum $\{\sigma_i\}$ of the coupling matrix. Equation (19) immediately yields

$$E_N(A|B) \geq \frac{1}{2} \sum_i [-\log(1 - \tanh^2(2r\sigma_i))]. \quad (22)$$

For rank-1 (collective) coupling, only one singular value is nonzero, $\sigma_1 = 1$, and the bound reduces to the two-mode case. For higher rank the sum in Eq. (22) shows how additional correlation channels increase the guaranteed entanglement floor. The additivity of $\log(1 - \chi)$ established in Eq. (10) implies that independent squeezing channels contribute multiplicatively to $1 - \chi$ and additively to the floor of E_N . Such additive behavior is consistent with multimode Gaussian-light statistics [24] and with continuous-variable monogamy and tangle relations in Gaussian networks [25].

F. Summary

The logarithmic negativity thus exemplifies the “entanglement-type” branch of the universal χ -inequality. Its lower bound (19) originates solely from positivity geometry and is saturated by Gaussian states. It establishes a minimal entanglement content compatible with a given level of correlation as quantified by χ . The next section applies the same geometric principle to “information-type” measures, deriving an upper bound (ceiling) for the quantum Fisher information that remains finite even in all-to-all connected systems.

V. METROLOGICAL CEILINGS: THE QUANTUM FISHER INFORMATION

The dual of the entanglement floor is the *metrological ceiling*. Information-capacity measures such as the quantum Fisher information (QFI) quantify the ultimate sensitivity of a quantum state to infinitesimal parameter displacements. Because these quantities are concave with respect to convex mixtures and decrease as correlations grow beyond the positivity interior, they belong to the “decreasing” class of measures described by Eq. (13). They therefore admit *upper bounds*—metrological ceilings—that follow directly from the universal χ -inequality.

A. The quantum Fisher information

For a pure probe state $|\psi\rangle$ undergoing a unitary encoding $U(\theta) = e^{-iH\theta}$ generated by a Hermitian operator H , the quantum Fisher information is

$$F_Q[|\psi\rangle, H] = 4 \text{Var}_\psi(H) = 4(\langle H^2 \rangle - \langle H \rangle^2). \quad (23)$$

For mixed states ρ , F_Q generalizes to the symmetric logarithmic derivative form, but its geometric meaning

remains unchanged: it measures the local curvature of the quantum-state manifold in the Bures metric. The larger the variance of the generator, the greater the sensitivity to θ , and hence the greater the information content of the state.

Because the covariance of any set of observables is itself a positive matrix, positivity of $K = \langle\langle H_i, H_j \rangle\rangle - \langle H_i \rangle \langle H_j \rangle$ imposes an upper bound on the attainable variances. The same determinant-ratio invariant that bounded the entanglement from below now bounds the Fisher information from above.

B. Derivation of the metrological ceiling

Consider a collection of collective spin operators $\mathbf{J} = (J_x, J_y, J_z)$ forming an irreducible representation of $SU(2)$ with total spin J . For any state of fixed length $J^2 = J(J+1)$, the variances satisfy

$$\text{Var}(J_x) + \text{Var}(J_y) + \text{Var}(J_z) = J(J+1) - \|\langle \mathbf{J} \rangle\|^2. \quad (24)$$

The right-hand side is constrained by positivity of the Gram matrix $\langle J_i J_j + J_j J_i \rangle$. Applying the determinant inequality (3) to this covariance matrix and projecting along an arbitrary direction \hat{n} of sensitivity yields the bound

$$F_Q(\hat{n}) \leq 4[J^2(1 - s^2) + J], \quad s = \frac{\|\langle \mathbf{J} \rangle\|}{J}. \quad (25)$$

Equation (25) is the *SU(2) metrological ceiling*. It is saturated by states lying on the equator of the Bloch sphere ($s = 0$), such as the optimal Greenberger-Horne-Zeilinger (GHZ) states, and reduced by polarization ($s > 0$). This ceiling follows solely from positivity geometry: it is the χ bound applied to the covariance of generators rather than to the density matrix itself.

C. All-to-all connected systems

One might expect that increasing connectivity or interaction range could evade such limits. Indeed, an all-to-all connected network of N spins possesses $O(N^2)$ interaction links and, naively, $O(N^2)$ potential scaling of information. Positivity geometry shows otherwise. Even in the fully connected case, the collective covariance matrix is rank-1 in the symmetric subspace, so its determinant-ratio invariant χ remains finite and bounded by 1. Substituting this into Eq. (11) with decreasing monotonicity yields a finite ceiling identical in form to Eq. (25). Equivalently, in the $SU(2)$ language the total spin $J = N/2$ defines an effective curvature of state space whose metric cannot be sharpened beyond the positivity boundary. Thus even maximal connectivity cannot increase F_Q beyond the positivity-imposed light cone in state space.

This result is conceptually striking: it reveals that the Heisenberg scaling $F_Q \propto N^2$ is not merely a dynamical accident but the algebraic saturation of a positivity constraint. The “light cone” here is a hypersurface of constant χ in the space of density operators—beyond which information curvature would violate the positivity of the quantum metric tensor.

D. Noncompact analog: SU(1,1) squeezing networks

For SU(1, 1) interferometers and multimode squeezing networks [26][14–16], the relevant generators (K_x, K_y, K_z) obey the noncompact algebra $[K_x, K_y] = -iK_z$. Repeating the variance analysis with the SU(1, 1) Casimir $K_z^2 - K_x^2 - K_y^2 = K(K - 1)$ yields

$$F_Q(\hat{n}) \leq 4[K^2(1 + s^2) - K], \quad (26)$$

where $s = \|\langle \mathbf{K} \rangle\|/K$. The sign reversal relative to Eq. (25) reflects the hyperbolic geometry of the SU(1, 1) manifold. Equation (26) represents the corresponding positivity ceiling in noncompact phase space. It too originates from the same χ invariant and becomes exact for minimum-uncertainty (squeezed-vacuum) states, where squeezing acts as an irreducible resource [27].

E. Physical meaning and operational consequences

The metrological ceiling defines a universal upper limit on information curvature—the rate at which a quantum state can change distinguishably under infinitesimal parameter shifts. Because this limit arises purely from positivity geometry, it applies to all estimation protocols and to all control Hamiltonians. In practice it sets a geometric bound on achievable precision: no measurement, regardless of entanglement or connectivity, can surpass the curvature allowed by the positive cone of quantum states. In optical-interferometric settings this ceiling reproduces and generalizes known quantum-metrology limits [28].

The coexistence of the entanglement floor and the metrological ceiling illustrates the dual role of χ . As χ increases, the state becomes more correlated and therefore more entangled, but simultaneously the accessible information curvature shrinks. The positivity boundary therefore marks the ultimate tradeoff surface between entanglement and precision.

F. Summary

The quantum Fisher information provides the archetype of the “information-type” class of measures governed by the universal χ -inequality. Its upper bound (25) and its noncompact analog (26) are direct geometric consequences

of positivity and are unaffected by microscopic connectivity. Even all-to-all coupled systems cannot escape this constraint. Positivity geometry thus imposes a universal light cone in state space, within which all physically realizable metrological processes must lie. This light-cone structure of state space echoes analogous features of real-space dynamics: Lieb–Robinson bounds impose causal cones for operator propagation in spin systems, and recent analyses have shown that those cones exhibit fold-type caustics analogous to classical catastrophes [6]. The same geometric mechanism reappears here in the information–geometric domain, defining the causal surface that delimits attainable precision. The following section shows that the same geometry also determines the *universality classes of bound saturation*, revealing that every approach to these limits is governed by the catastrophe–theoretic exponents of Thom and Arnold.

VI. UNIVERSALITY OF BOUND SATURATION

This universal structure echoes the generic singularity hierarchies of catastrophe theory [41–44].

The universal χ -inequality establishes quantitative limits for all quantum correlations, but the *manner* in which these limits are approached is itself universal. Whenever a physical control parameter tunes a system toward the positivity boundary of state space, the observable measures of correlation (E_N, F_Q , etc.) exhibit characteristic singular behavior. This section shows that the local geometry of those singularities is completely determined by the structure of the positivity cone and can be classified by the Thom–Arnold hierarchy of catastrophes. Bound saturation therefore possesses the same universality that governs critical phenomena in condensed matter and singular optics.

A. Bound surfaces and control parameters

Consider a family of physical states $\rho(\boldsymbol{\lambda})$ depending smoothly on control parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$ such that $\rho(\boldsymbol{\lambda}) \succ 0$ for all admissible parameters. Each correlation measure $M(\boldsymbol{\lambda})$ obeys the universal inequality $M(\boldsymbol{\lambda}) \bowtie f_M[\chi(\boldsymbol{\lambda})]$. The hypersurface on which equality is attained,

$$\mathcal{S}_M = \{\boldsymbol{\lambda} : M(\boldsymbol{\lambda}) = f_M[\chi(\boldsymbol{\lambda})]\}, \quad (27)$$

defines the *bound surface* for M in control–parameter space. Approaching this surface corresponds to saturating the corresponding floor or ceiling. The geometry of \mathcal{S}_M near points of extremality determines the critical scaling of M with respect to $\boldsymbol{\lambda}$.

B. Local expansion and reduced potential

Let λ_{\perp} denote the coordinate normal to the bound surface and λ_{\parallel} the remaining tangential coordinates. Expanding the correlation measure near the saturation point gives

$$\Delta M = M - f_M(\chi) \simeq a_2 \lambda_{\perp}^2 + a_3 \lambda_{\perp}^3 + a_4 \lambda_{\perp}^4 + \dots, \quad (28)$$

where the coefficients a_k depend smoothly on λ_{\parallel} . The exponent governing the leading term in Eq. (28) determines the rate at which the inequality tightens. When the quadratic term dominates ($a_2 \neq 0$), the approach is generic and follows a *fold* singularity with critical exponent

$$\Delta M \propto (\lambda_{\perp} - \lambda_{\perp}^*)^{\alpha_2}, \quad \alpha_2 = 2. \quad (29)$$

When the quadratic term vanishes by symmetry ($a_2 = 0$) but the cubic term is present, the local structure becomes a *cusp* catastrophe with exponent

$$\Delta M \propto (\lambda_{\perp} - \lambda_{\perp}^*)^{\alpha_3}, \quad \alpha_3 = \frac{4}{3}. \quad (30)$$

Higher degeneracies give rise to swallowtail (A_4) and more complex catastrophes. In all cases the exponents $(k+1)/k$ coincide with those of the standard A_k sequence.

C. Examples: folds and cusps in compact and noncompact systems

In $SU(2)$ spin ensembles, positivity is realized on a compact manifold (the Bloch sphere). Approaching the metrological ceiling of Eq. (25) corresponds to moving tangentially along this sphere toward its equatorial boundary. Because the positivity constraint is quadratic in small deflections of the Bloch vector, $\Delta F_Q \propto (\delta s)^2$, the scaling is of fold type ($\alpha_2 = 2$). This ‘‘Heisenberg fold’’ describes how precision increases and then stagnates as states become maximally entangled. Fold singularities of this kind are familiar from wave and optical caustics and have recently been observed in the Lieb–Robinson light cones of quenched spin chains [6], reinforcing the geometric unity of these phenomena.

In contrast, $SU(1, 1)$ squeezing networks live on a noncompact hyperbolic manifold where the positivity constraint becomes cubic in the normal coordinate to the bound surface. Approaching the entanglement floor of Eq. (19) in such systems therefore exhibits cusp-type scaling with exponent $\alpha_3 = 4/3$. This ‘‘squeezing cusp’’ provides the canonical example of a noncompact positivity catastrophe.

D. General classification and universality

The fold and cusp examples illustrate the general rule: each independent constraint or symmetry that forces ad-

ditional degeneracy raises the catastrophe order by one. If r independent control parameters tune the system to the positivity boundary, the generic singularity class is A_{r+1} . Because the local structure depends only on the algebraic order of the positivity constraint and not on microscopic details, the resulting critical exponents are *universal*. Every correlation measure—entanglement, coherence, Fisher information, or otherwise—shares the same local catastrophe structure near its bound.

E. Conceptual implications

The universality of bound saturation endows positivity geometry with the same predictive power that universality has in statistical mechanics. The χ -level sets act as isoenergetic surfaces in a generalized potential landscape, and the catastrophe classification provides the normal forms of their singularities. Bound saturation thus defines a new type of critical phenomenon in state space: as a control parameter pushes a quantum system toward the positivity boundary, all correlation measures exhibit algebraically universal scaling governed by the Thom–Arnold hierarchy. These exponents serve as the *critical indices of quantum correlation*.

This universal structure echoes the generic singularity hierarchies of catastrophe theory [41–44].

F. Summary

Positivity geometry therefore not only defines the absolute bounds of quantum correlation but also prescribes the universal behavior of all systems that approach those bounds. Compact manifolds yield fold-type ceilings; noncompact manifolds yield cusp-type floors; higher degeneracies produce higher catastrophes. Together they establish a complete taxonomy of quantum correlation saturation rooted in the geometry of the positive cone.

VII. DISCUSSION AND OUTLOOK

The results presented here reveal that the ultimate limits of quantum correlation and information stem from a single geometric principle: the positivity of quantum state space. From this principle arises the determinant–ratio invariant χ , a scalar that captures the intrinsic correlation geometry of any bipartition and serves as a universal generator of bounds. Every monotone measure of nonclassical correlation—entanglement, coherence, or metrological sensitivity—obeys a χ -dependent inequality that defines either a floor or a ceiling. These bounds are exact for Gaussian normal forms and remain valid for all states sharing the same second-moment data. Comprehensive reviews of entanglement and quantum correlation

measures may be found in [46, 47]. For Gaussian-state realizations and operational measures, see [36–40]. The resulting hierarchy of universal quantum bounds mirrors Zeeman’s classification of physical catastrophes [45].

A. Unified picture

The invariant χ provides a common language for all known quantitative limits of quantum mechanics. It recovers the Cramér–Rao inequality and the Heisenberg and Lieb–Robinson bounds as special cases, extends them to arbitrary architectures, and predicts new constraints such as the exact entanglement floor of Eq. (19) and the metrological ceiling of Eq. (25), which remains finite even in fully connected spin networks. Positivity geometry therefore unifies the entire hierarchy of quantum limits into a single framework based on the same algebraic structure.

B. Positivity light cone in state space

The coexistence of entanglement floors and metrological ceilings reveals a new geometric object in quantum theory—a *light cone in state space*. Just as the Minkowski light cone limits the propagation of signals in spacetime, positivity defines a causal domain in the manifold of quantum states. In dynamical settings, operator spreading and quantum information flow generate analogous causal structures: Lieb–Robinson bounds constrain local Hamiltonian evolution, while studies of out-of-time-ordered correlators and quantum scrambling have mapped the shapes of these “information light cones” in chaotic and long-range systems [29–32]. The positivity light cone plays the same role in the static geometry of state space, delineating the frontier of physically allowed correlations. Within this domain, correlations and information can vary freely; outside it, the Gram matrix would cease to be positive. Approaching the positivity light cone corresponds to pushing a quantum system to the brink of physical realizability: entanglement diverges, Fisher information saturates, and all correlation measures exhibit universal fold- or cusp-type scaling. This causal interpretation of positivity geometry links the kinematic constraints of quantum mechanics to the dynamical constraints of information flow.

C. Experimental and theoretical implications

The invariant χ is directly computable from experimentally accessible covariance data, making the derived bounds testable across a wide range of platforms—from photonic and atomic squeezing networks to superconducting qubits and hybrid sensors. Measuring χ provides a certified lower bound on entanglement and a certified upper bound on achievable precision, without requiring full

state reconstruction. In optical-interferometric settings this ceiling reproduces and extends the known quantum-metrology limits established in optical interferometry and atomic ensembles [28]. The catastrophe-theoretic exponents predicted here can be probed by scanning control parameters (squeezing strength, interaction range, or polarization) and observing the algebraic scaling of performance as the positivity boundary is approached.

On the theoretical side, positivity geometry opens several directions. First, the construction can be extended to multipartite partitions, defining a hierarchy of invariants $\chi_{A|BC}$, $\chi_{AB|CD}$, and beyond, that generate bounds for networked and higher-order correlations. Second, dynamic versions of the framework may lead to state-space Lieb–Robinson bounds—light-cone inequalities that limit the rate of change of χ under local Hamiltonian evolution. Finally, the geometric program proposed here resonates with developments in information geometry, from the Uhlmann and Petz formulations of quantum-state metrics [33, 34] to broader physical constraints on information and privacy [35]. Together these connections suggest a reformulation of quantum mechanics itself as a theory of positive geometries and their invariants.

D. Outlook

Positivity geometry provides a unifying structure for the quantitative laws of quantum theory. It replaces a landscape of measure-specific inequalities with a single geometric invariant that governs them all. From this invariant emerge the entanglement floor, the metrological ceiling, and the universal exponents of bound saturation. Together they define a new “periodic table” of quantum limits. The discovery of this structure suggests that the familiar constants of quantum mechanics— \hbar , c , and now χ —are the natural scales of quantum geometry, setting not only how states evolve but how correlated they can become.

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