Intermediate subgroups of braid groups are not bi-orderable

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Abstract: Let M be the disk or a compact, connected surface without boundary different from the sphere S^2 and the real projective plane $\mathbb{R}P^2$, and let N be a compact, connected surface (possibly with boundary). It is known that the pure braid groups $P_n(M)$ of M are bi-orderable, and, for $n \geq 3$, that the full braid groups $B_n(M)$ of M are not bi-orderable. The main purpose of this article is to show that for all $n \geq 3$, any subgroup H of $B_n(N)$ that satisfies $P_n(N) \subsetneq H \subset B_n(N)$ is not bi-orderable.

Key words: Braid groups, pure braid groups, group orderings.

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1 Introduction

A group G is said to be *left-orderable* if it can be equipped with a strict total ordering $<$ which is left-invariant. More precisely, if $x < y$, then $gx < gy$ for all $x, y, g \in G$. If the ordering $<$ is also right-invariant, then $(G, <)$ is said to be *bi-orderable*. Some examples of bi-orderable groups are free groups [49], Thompson's groups [10, 41], torsion-free nilpotent groups [43, Proposition 3.16], diagram groups [30, Theorem 6.1], and the fundamental group of some 3-manifolds [48]. Left-orderable groups that cannot be bi-ordered include the groups $\mathrm{Homeo}_+(\mathbb{R})$ and $\mathrm{Homeo}_+(\mathbb{Q} \times \mathbb{Q})$ of order-preserving homeomorphisms [40], the universal covering group of the group $SL_2(\mathbb{R})$ of 2×2 real matrices with determinant equal to one [6], and the fundamental group of the Klein bottle [48].

Let M be a compact, connected surface, and let $n \geq 1$. The set:

$$\{(z_1, \dots, z_n);\ z_i \in M \text{ and } z_i \neq z_j,\ \text{for } i \neq j\}$$

is called the *nth (ordered) configuration space of* M and is denoted by $F_n(M)$. The map $(\sigma, (z_1, \dots, z_n)) \mapsto (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ defines a free action of S_n, the symmetric group on n letters, on $F_n(M)$. The corresponding quotient, which is the orbit space $F_n(M)/S_n$, will be denoted by $C_n(M)$, and is called the *nth unordered configuration space of* M. The *pure braid group of* M *on* n *strands*, denoted by $P_n(M)$, is defined to be the fundamental group of $F_n(M)$, and the *(full) braid group of* M on n *strands*, denoted by $B_n(M)$, is defined to be the fundamental group of $C_n(M)$ [18, 21, 51].

The braid groups $B_n(D)$ of the disk D, also known as *Artin braid groups*, are residually finite [1, 4, 13, Proposition 5], Hopfian [37] and linear [7, 35]. They admit solvable word problem [2] and conjugacy problem [23], and they are not locally indicable for $n \geq 5$ [47]. Moreover, if M is a compact, connected surface without boundary, the braid groups $B_n(M)$ are torsion-free with the exception of the sphere S^2 and the real projective plane $\mathbb{R}P^2$ [18, 19, 50]. If $M \neq D, S^2$ is such that its fundamental group has trivial center, then the center of $B_n(M)$ is trivial [28, 44]. More properties of surface braid groups may be found in [8, 27, 29, 31, 44].

As proved by Dehornoy [15], braid groups of the disk are left-orderable. However, it is known that they cannot be bi-ordered, because they do not have the unique root property (see Equation 7). Since $B_n(D) \subset B_n(M)$, where D is a topological disk in M and M is a compact, connected surface different from S^2 and $\mathbb{R}P^2$ [44], $B_n(M)$ is not bi-orderable for $n \geq 3$. Garside [23] defined a partial order on the semigroup $B_n(D)^+$ of the so-called *positive braids*, i.e. non-trivial braids of the disk that can be written in terms of positive powers of the Artin generators (see Theorem 3). Later, Elrifai and Morton [16] gave a partial ordering of $B_n(D)$.

The pure braid groups of the disk and of compact, connected orientable surfaces with the exception of the sphere S^2 are bi-orderable [26, 33]. However, if N is a non-orientable surface and $n \geq 2$, González-Meneses [26] exhibited generalized torsion elements in $P_n(N)$, and therefore $P_n(N)$ is not bi-orderable. Moreover, if M is S^2 or $\mathbb{R}P^2$, then $P_n(M)$ contains torsion elements [19, 50], so it is not left-orderable, and consequently, the braid groups $B_n(S^2)$ and $B_n(\mathbb{R}P^2)$ are not left-orderable.

In this article, we will study the bi-orderability of subgroups H of $B_n(M)$ satisfying $P_n(M) \subsetneq H \subset B_n(M)$, where either $M = D$ or M is a compact, connected surface different from S^2 and $\mathbb{R}P^2$. We shall refer to such subgroups as *intermediate subgroups of* $B_n(M)$. We first discuss the case of the disk, and we then use a restriction of the embedding $B_n(D) \hookrightarrow B_n(M)$ [25, 44], induced by the inclusion of D in M, to extend our results to compact, connected surfaces. More precisely, our main results are as follows.

Theorem 1. *Let $n \geq 3$. If H is a bi-orderable subgroup of $B_n(D)$ such that $P_n(D) \subset H \subset B_n(D)$, then $H = P_n(D)$, i.e. $P_n(D)$ is a maximal bi-orderable subgroup of $B_n(D)$.*

Theorem 2. *Let $n \geq 3$, and let $M \neq S^2, \mathbb{R}P^2$ be a compact, connected surface. Let H be an intermediate subgroup of $B_n(M)$. Then, H is not bi-orderable.*

Besides the Introduction, this paper consists of two sections. In Section 2, we give preliminary definitions and we state some well-known results on surface braid groups and orderable groups. The goal of Section 3, which consists of four subsections, is to investigate the bi-orderability of a certain family of intermediate subgroups of the Artin braid groups, and to prove Theorems 1 and 2. We finish the paper with some consequences of Theorem 1, such as the fact that no intermediate subgroups of the braid group $B_\infty(D)$ of the disk on infinitely many strands can be bi-ordered.

2 Preliminaries

In this section, we recall some results on the braid groups of the disk, such as the classical presentation due to Artin, the description of the center and the structure of the pure braid groups. We also summarize some properties of orderable groups, and we discuss the relation between the unique root property and the existence of generalized torsion elements in a group.

2.1 Artin braid groups

Let M be a compact, connected surface, and let $n \geq 1$. The canonical projection $F_n(M) \to C_n(M)$ is a regular $n!$-fold covering map, which leads to the following short exact sequence:

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \xrightarrow{\pi} S_n \longrightarrow 1, \tag{1}$$

where $\pi\colon B_n(M) \to S_n$ associates the permutation corresponding to the endpoints of the strands of a braid. The group homomorphism π is called the *permutation homomorphism*.

When M is the disk D, Artin [1, 3] studied geometric and algebraic properties of its braid groups, and provided presentations for both $B_n(D)$ and $P_n(D)$.

Theorem 3 (Artin, [1, 3])**.** *For $n \geq 1$, $B_n(D)$ has the following presentation:*

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \text{ for } i = 1, 2, \dots, n-2 \end{array} \right\rangle. \tag{2}$$

The generators σ_i are called *Artin generators*. They may be interpreted geometrically as a clockwise half-twist of the i-th and $(i+1)$-th strands, as shown in Figure 1, where the i-th strand is colored in red and the $(i+1)$-th strand is in green.

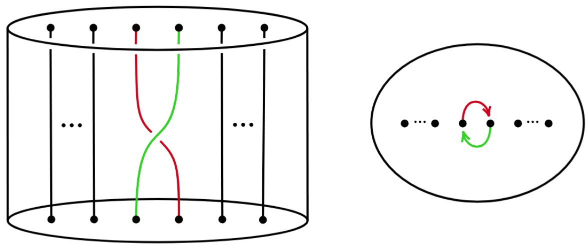

Figure 1: The braid $\sigma_i \in B_n(D)$ illustrated geometrically and as an element of $\pi_1(C_n(D))$.

Remark 4. *Let $n \geq 1$. The braid group $B_i(D)$ may be viewed as a subgroup of $B_n(D)$ for all $1 \leq i \leq n$. Indeed, it follows directly from the fact that the homomorphism $\iota \colon B_i(D) \to B_{i+1}(D)$ defined by $\iota(\sigma_k) = \sigma_k$ for all $1 \leq k \leq n-1$ is injective for all $i \geq 1$ [32, Corollary 1.14].*

We recall that the center of $B_n(D)$ is infinite cyclic [11].

Proposition 5 (Chow, [11])**.** *For $n \geq 3$, the center of $B_n(D)$ is the cyclic subgroup $\langle \Delta_n^2 \rangle$. The braid Δ_n^2 is called the full-twist, and it satisfies $\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n$.*

Remark 6. *Let $n \geq 3$ and $2 \leq k \leq n$. The braid $\Delta_k = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{k-1}\cdots\sigma_1)$ of $B_k(D)$ is called the half-twist on k strands, and its square is equal to the full-twist. We recall that:*

$$\pi(\Delta_k) = \begin{cases} (1,k)(2,k-1)\cdots(\frac{k}{2}, \frac{k}{2}+1), & \text{if } k \text{ is even;} \\ (1,k)(2,k-1)\cdots(\frac{k-1}{2}, \frac{k+3}{2}), & \text{if } k \text{ is odd.} \end{cases}$$

That is:

$$\pi(\Delta_k) = \prod_{l=1}^{\lfloor \frac{k}{2} \rfloor} (l, (k+1)-l), \tag{3}$$

a product of disjoint transpositions, where $\lfloor \frac{k}{2} \rfloor$ denotes the greatest integer less than or equal to $\frac{k}{2}$.

We now give a preliminary result, which will be used in the calculations of Subsection 3.2.

Lemma 7 (Garside, [23])**.** *Let $1 \leq i \leq k-1$, and let $2 \leq k \leq n$. Then, $\sigma_i\Delta_k = \Delta_k\sigma_{k-i}$.*

By (1), the pure braid group of the disk is a finite-index subgroup of $B_n(D)$. Moreover, $P_n(D)$ is residually torsion-free nilpotent [20], and its center coincides with the center of $B_n(D)$ for all $n \geq 3$ [8, 11]. Also, $P_n(D)$ is isomorphic to a semi-direct product of free groups [2].

Theorem 8 (Artin, [1, 3])**.** *Let $n \geq 1$. The subgroup $P_n(D)$ is finitely presented, and it is generated by the set $\{A_{i,j};\ 1 \leq i < j \leq n\}$, where $A_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1})$.*

Let $n \geq 3$, and $1 \leq i \leq n$. Let

$$\begin{array}{rccc} \varphi_i\colon & F_n(D) & \longrightarrow & F_{n-1}(D) \\ & (x_1, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \end{array}$$

be the projection that omits the i-th coordinate of an element $(x_1, \ldots, x_n) \in F_n(D)$. Then, the map φ is a locally-trivial fibration [18], and so it induces an epimorphism $(\varphi_i)_\#\colon P_n(D) \to P_{n-1}(D)$, which may be interpreted geometrically as the retraction that forgets the i-th strand of a given pure braid.

Remark 9. *For $n = 3$, we have $(\varphi_1)_\#(A_{1,3}) = (\varphi_1)_\#(A_{2,3}) = A_{1,2} \neq 1 = (\varphi_1)_\#(A_{1,2})$, and $(\varphi_2)_\#(A_{1,3}) = A_{1,2} \neq 1 = (\varphi_1)_\#(A_{2,3})$.*

We now recall the fact that the inclusion of a subsurface of a surface into the surface gives rise to an embedding between their braid groups [44]. We shall use this result in the case where the subsurface is a disk.

Proposition 10 (Paris and Rolfsen, Proposition 2.2 [44])**.** *Let M be a connected surface different from S^2 and $\mathbb{R}P^2$, and let N be a connected subsurface of M such that none of the connected components of the closure $\overline{M \setminus N}$ is a disk. Let $m, n \geq 1$, and let $\psi\colon B_n(N) \to B_m(M)$ be the homomorphism induced by the natural embedding $N \hookrightarrow M$. Then ψ is injective.*

Remark 11. *Taking $m = n$ and N to be a topological disk D in Proposition 10, we obtain $B_n(D) \cong \psi(B_n(D)) \subset B_n(M)$ for all $n \geq 1$ and $M \neq S^2, \mathbb{R}P^2$ a connected surface. Hence, $B_n(D)$ may be viewed as a subgroup of $B_n(M)$. Moreover, by the proof of Proposition 10 [44, Proposition 2.2, p. 424], we have $P_n(D) \cong \psi(P_n(D)) \subset P_n(M)$, which was observed earlier by Goldberg [25, Theorem 1].*

2.2 Ordered groups

In the literature, one may find authors who study left-orderings (eg. [9, 12, 40]) and others who study right-orderings (eg. [34, 45, 47]). In this paper, we choose the convention of studying left-orderable groups. Recall that a left-orderable group $(G, <)$ is clearly right-orderable with the ordering $\prec$ defined by $x \prec y \Leftrightarrow y^{-1} < x^{-1}$.

A well-known characterization of ordered groups, first established by Levi [36] in the context of bi-invariant orderings, is given in terms of the existence of a *positive cone*.

Proposition 12 (Levi, [36])**.** *A group G is left-orderable if and only if there exists a set $P \subset G$, known as the positive cone of the ordering, such that $PP \subset P$ and $G = P \dot{\bigcup} \{1\} \dot{\bigcup} P^{-1}$, where the union is disjoint. Moreover, G is bi-orderable if and only if P is a normal set, namely $P^g = gPg^{-1} \subset P$ for all $g \in G$.*

Left-orderable groups are torsion-free. Furthermore, bi-orderable groups have the *unique root property* [24, Lemma 2.1.4], and they do not have *generalized torsion elements* [38, Lemma 2.3], as we recall in Proposition 13 (i) and (ii) respectively. We also recall that a non-trivial group element is said to be a *generalized torsion element* if there exists a non-empty finite product of its conjugates that is equal to the identity.

Proposition 13. *Let G be a bi-orderable group. Then, the following hold:*

(i) Let $g, h \in G$. If there exists $n \geq 1$ such that $g^n = h^n$, then $g = h$.

(ii) If $g \in G \setminus \{1\}$, then the product $(x_1 g x_1^{-1})(x_2 g x_2^{-1}) \cdots (x_k g x_k^{-1})$ is non-trivial for all $x_1, \ldots, x_k \in G$ and $k \geq 1$.

By the following proposition, a group with non-unique roots also has generalized torsion elements [42], and so (i) implies (ii) in Proposition 13.

Proposition 14 (Naylor and Rolfsen, [42])**.** *Let $p, q \in \mathbb{Z}$ be such that $|p|, |q| > 1$. The group*

$$G_{p,q} = \langle x, y;\ x^p = y^q \rangle$$

contains a generalized torsion element, namely the commutator $[x, y] = xyx^{-1}y^{-1}$.

By Proposition 14, if there exist $x, y \in G$ such that $x^p = y^q$, for $|p|, |q| > 1$, and x and y do not commute, then $[x, y] \in G_{p,q} \subset G$ is a generalized torsion element of G, and therefore G is not bi-orderable. We will use this result in Remark 28 to exhibit generalized torsion elements of intermediate subgroups of the braid groups of the disk.

Example 15. *With the notation of Proposition 14, let $p = q = 2$, and let us write $x^y = yxy^{-1}$ for all $x, y \in G$. To obtain a generalized torsion relation satisfied by the element $[x, y]$ of the group $G_{2,2} = \langle x, y;\ x^2 = y^2 \rangle$, observe that:*

$$\left[x^2, y^2\right] = \left[x, y^2\right]^x \left[x, y^2\right] = \left([x, y]\,[x, y]^y\right)^x [x, y]\,[x, y]^y = [x, y]^x\,[x, y]^{xy}\,[x, y]\,[x, y]^y\,.$$

Then, since $x^2 = y^2$, we have:

$$[x, y]^x\,[x, y]^{xy}\,[x, y]\,[x, y]^y = 1. \tag{4}$$

More details and properties of ordered groups may be found in [12, 39, 45]. We end this subsection with some comments on *partial orderings*, for which we refer the reader to Fuchs's [22] and Glass's [24] books. A group G is said to be *partially left-orderable* if there exists a partial strict ordering $<$ on its elements that is left-invariant. If such an ordering is also right-invariant, then $(G, <)$ is said to be *partially bi-orderable*. As in Proposition 12, a group G is partially left-orderable (resp. partially bi-orderable) if and only if there exists a semigroup Q such that $Q \cap Q^{-1} = \emptyset$ (resp. and Q is a normal set). In this case, the semigroup Q is called a *partial positive cone* of the group G. An example of a partially left-orderable group that cannot be left-ordered is the multiplicative group of all non-zero principal ideals in a commutative integral domain with unity, and an example of a partially bi-ordered group that is not bi-orderable is the additive group of all polynomial functions with real coefficients whose domain is the closed interval $[0, 1]$ [22, Section II.3].

3 Bi-orderability of intermediate subgroups

This section consists of four subsections. Although the braid groups of the disk are not bi-orderable, in Subsection 3.1, we show that they are partially bi-ordered. In Subsection 3.2, we investigate some intermediate subgroups of $B_n(D)$. In Subsection 3.3, by studying the permutation of non-pure braids, we prove Theorem 1. Finally, in Subsection 3.4, we state some consequences of Theorem 1 and give the proof of Theorem 2.

3.1 Partial bi-ordering

In this section, we prove that intermediate subgroups of $B_n(D)$ can be partially bi-ordered.

Proposition 16. *Let G be a group, and let $f\colon G \to N$ be a group homomorphism. If $\operatorname{Im}(f)$ is non-trivial and N is partially bi-orderable, then G is partially bi-orderable.*

Proof. Let $Q(N)$ be the partial positive cone of N. Then the set $Q = f^{-1}(Q(N))$ is a partial positive cone of G. $\square$

Corollary 17. *Intermediate subgroups of the braid groups of the disk are partially bi-orderable. In particular, $B_n(D)$ is partially bi-orderable.*

Proof. Let H be an intermediate subgroup of $B_n(D)$. Consider the Abelianization homomorphism $\exp\colon B_n(D) \to \mathbb{Z}$ given by $\exp(\sigma_i) = 1$ for all $1 \leq i \leq n-1$. Since $2\mathbb{Z} = \exp(P_n(D)) \subset \exp(H)$, the image $\operatorname{Im}(\exp|_H)$ of the restriction of exp to H is non-trivial, and so the result follows from Proposition 16 and the fact that $\mathbb{Z}$ is clearly partially bi-orderable with the usual ordering. $\square$

3.2 Non-bi-orderability of certain intermediate subgroups

In this subsection, we show that certain intermediate subgroups of braid groups of the disk are not bi-orderable.

Definition 18. *Let $n \geq 2$. Given $\beta \in B_n(D) \setminus P_n(D)$, let $H_\beta = \langle P_n(D), \beta \rangle$, where $\langle P_n(D), \beta \rangle$ is the subgroup generated by $P_n(D)$ and β.*

We first study the family of intermediate subgroups described in Definition 18. Note that $B_1(D)$ is trivial, and $B_2(D)$ is isomorphic to $\mathbb{Z}$. So from now on, we suppose that $n \geq 3$. Our goal is to show that the subgroup H_β is not bi-orderable for every braid $\beta \in B_n(D) \setminus P_n(D)$, which will imply that no intermediate subgroup of the braid groups of the disk can be bi-ordered.

Lemma 19. *Let $f\colon G \to H$ be a group homomorphism. Then $\langle \operatorname{Ker} f, \beta \rangle = f^{-1}(\langle f(\beta) \rangle)$ for all $\beta \in G$.*

Proof. First, note that $f(\langle \operatorname{Ker} f, \beta \rangle) = \langle f(\beta) \rangle$, hence $\langle \operatorname{Ker}(f), \beta \rangle \subset f^{-1}(\langle f(\beta) \rangle)$. Conversely, let $y \in f^{-1}(\langle f(\beta) \rangle)$. Thus, $f(y) = f(\beta)^\epsilon$ for some $\epsilon \in \mathbb{Z}$. Then, we have $f(y\beta^{-\epsilon}) = 1$, and so there exists $\alpha \in \operatorname{Ker} f$ such that $y\beta^{-\epsilon} = \alpha$. Hence $y = \alpha\beta^\epsilon \in \langle \operatorname{Ker} f, \beta \rangle$, and we conclude that $\langle \operatorname{Ker} f, \beta \rangle = f^{-1}(\langle f(\beta) \rangle)$. $\square$

Consider the short exact sequence in (1). Given $\beta \in B_n(D)$, by Lemma 19, we have $\pi^{-1}(\langle \pi(\beta) \rangle) = \langle \operatorname{Ker} \pi, \beta \rangle = H_\beta$. Thus if $\gamma \in B_n(D)$ is such that $\pi(\gamma) = \pi(\beta)$, then $H_\beta = H_\gamma$, and so to decide whether H_β is bi-orderable, it suffices to study H_γ for some fixed braid γ. Moreover, we shall prove that the bi-orderability of H_β depends only on the cycle type of the permutation of the braid β.

Lemma 20. *Let H and N be subgroups of a group G. Suppose that there exists $g \in G$ such that $H^g = N$. Then, H is bi-orderable if and only if N is bi-orderable.*

Proof. Suppose that H is bi-orderable, and let $P(H)$ denote its positive cone. The semigroup $P = P(H)^g = \{ghg^{-1};\ h \in P(H)\}$ is a positive cone for N and it is invariant under conjugation by elements of N. Therefore, N is bi-orderable. Moreover, $H^g = N$ if and only if $H = N^{g^{-1}}$, and this concludes the proof. $\square$

Proposition 21. *Let $\beta, \gamma \in B_n(D) \setminus P_n(D)$ be such that the permutations $\pi(\beta)$ and $\pi(\gamma)$ have the same cycle type. Then, the subgroups H_β and H_γ are conjugate. Moreover, H_β is bi-orderable if and only if H_γ is bi-orderable.*

Proof. Since $\pi(\beta)$ and $\pi(\gamma)$ have the same cycle type in S_n, there exists $\tau \in S_n$ such that $\tau\pi(\beta)\tau^{-1} = \pi(\gamma)$. The map π is an epimorphism, so there exists $\alpha \in B_n(D)$ such that $\pi(\alpha) = \tau$. Thus, $\pi(\alpha\beta\alpha^{-1}) = \pi(\gamma)$, and we obtain:

$$H_\gamma = \langle P_n(D), \gamma\rangle = \langle P_n(D), \alpha\beta\alpha^{-1}\rangle = \langle P_n(D), \beta\rangle^\alpha = H_\beta^\alpha,$$

because $P_n(D)$ is a normal subgroup of $B_n(D)$. Since H_β and H_γ are conjugate, by Lemma 20, H_β is bi-orderable if and only if H_γ is bi-orderable. □

By Proposition 21, to decide whether the subgroup H_β is bi-orderable, where $\beta \in B_n(D)$, it suffices to analyze the bi-orderability of H_γ for a braid $\gamma \in B_n(D)$ whose permutation has the same cycle type as $\pi(\beta)$. The rest of this section is devoted to studying H_β, where $\pi(\beta)$ is a transposition, a product of disjoint transpositions or a cycle of length greater than 2. We start with the case where $\pi(\beta)$ is a transposition.

Proposition 22. *Let $n \geq 3$, and let $\beta \in B_n(D)$ be such that $\pi(\beta)$ is a transposition. The subgroup H_β of $B_n(D)$ is not bi-orderable.*

Proof. By Proposition 21, it suffices to show that H_{σ_1} is not bi-orderable. Using the Artin relations (2), we have:

$$(\sigma_1\sigma_2^2)^2 = \sigma_1\sigma_2(\sigma_2\sigma_1\sigma_2)\sigma_2 = (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_2) = \sigma_2(\sigma_1\sigma_2\sigma_1)\sigma_2\sigma_1 = (\sigma_2^2\sigma_1)^2. \quad (5)$$

Note that $\sigma_1\sigma_2^2$ and $\sigma_2^2\sigma_1$ belong to H_{σ_1}. If H_{σ_1} is bi-orderable, by Proposition 13, we have:

$$\sigma_1 A_{2,3} = \sigma_1\sigma_2^2 = \sigma_2^2\sigma_1 = A_{2,3}\sigma_1.$$

However, $A_{2,3} \neq \sigma_1^{-1}A_{2,3}\sigma_1 = A_{1,3}$, by Remark 9. Hence, H_{σ_1} does not have the unique root property, and so, by Proposition 13, it is not bi-orderable. □

We now give a preliminary result, which we use to show that H_β is not bi-orderable in the case that $\pi(\beta)$ is a product of disjoint transpositions.

Lemma 23. *Let $3 \leq i \leq n$. In $B_i(D) \subset B_n(D)$, the braids $A_{1,2}$ and Δ_i do not commute.*

Proof. Suppose that there exists $k \geq 3$ such that $A_{1,2}$ commutes with Δ_k. By Lemma 7, we have $\sigma_1\Delta_k = \Delta_k\sigma_{k-1}$. Then:

$$A_{1,2} = \sigma_1^2 = \Delta_k^{-1}\sigma_1^2\Delta_k = (\Delta_k^{-1}\sigma_1\Delta_k)^2 = \sigma_{k-1}^2 = A_{k-1,k},$$

which holds if and only if $k = 2$. Therefore, $A_{1,2}$ and Δ_i do not commute if $3 \leq i \leq n-1$. □

Proposition 24. *Let $3 \leq i \leq n$. Then, the subgroup $H_{\Delta_i} = \langle P_n(D), \Delta_i\rangle$ is not bi-orderable.*

Proof. Let $i \in \{3, \ldots, n\}$. Since the center of $B_i(D)$ is $\langle\Delta_i^2\rangle$ by Proposition 5, we have:

$$(A_{1,2}\Delta_i A_{1,2}^{-1})^2 = A_{1,2}\Delta_i^2 A_{1,2}^{-1} = \Delta_i^2 = (\Delta_i)^2. \quad (6)$$

So, if H_{Δ_i} is bi-orderable, then $A_{1,2}\Delta_i A_{1,2}^{-1} = \Delta_i$, which contradicts Lemma 23. □

Finally, we study the intermediate subgroup H_β for a braid $\beta \in B_n(D)$ whose permutation is a cycle of length strictly greater than 2.

Lemma 25. *Let $n \geq 4$. In $B_{i+1}(D) \subset B_n(D)$, with $2 \leq i \leq n-1$, the braids $A_{1,2}$ and $\sigma_1\sigma_2\cdots\sigma_i$ do not commute.*

Proof. Suppose on the contrary that there exists $i \in \{2, \dots, n-1\}$ such that $A_{1,2}$ commutes with $\sigma_1\sigma_2\cdots\sigma_i$. Using the relation $\sigma_j\sigma_k = \sigma_k\sigma_j$, whenever $|j-k| \geq 2$, we have:

$$\begin{aligned} 1 = [A_{1,2}, \sigma_1\sigma_2\sigma_3\cdots\sigma_i] &= \sigma_1^2\sigma_1\sigma_2(\sigma_3\cdots\sigma_i)\sigma_1^{-2}(\sigma_3\cdots\sigma_i)^{-1}\sigma_2^{-1}\sigma_1^{-1} \\ &= \sigma_1^2\sigma_1\sigma_2\sigma_1^{-2}(\sigma_3\cdots\sigma_i)(\sigma_3\cdots\sigma_i)^{-1}\sigma_2^{-1}\sigma_1^{-1} \\ &= \sigma_1 A_{1,2}A_{1,3}^{-1}\sigma_1^{-1}. \end{aligned}$$

Thus $A_{1,2} = A_{1,3}$, which is absurd by Remark 9, and this proves the lemma. □

Proposition 26. *Let $n \geq 3$, and let $2 \leq i \leq n-1$. Let $\beta \in B_n(D)$ be such that $\pi(\beta)$ is a cycle of length $i+1$. Then, H_β is not bi-orderable.*

Proof. Since $\pi(\sigma_1\cdots\sigma_i)$ is an $(i+1)$-cycle, by Proposition 21, it suffices to prove the statement for $H_{\sigma_1\cdots\sigma_i} = \langle P_n, \sigma_1\cdots\sigma_i\rangle$. First let $n=3$, so $i=2$, and consider the subgroup $H_{\sigma_1\sigma_2}$. From Lemma 19, the braid $\sigma_2\sigma_1$ belongs to $\pi^{-1}(\langle\pi(\sigma_1\sigma_2)\rangle) = H_{\sigma_1\sigma_2}$, because $\pi(\sigma_2\sigma_1) = (1,2,3) = (1,3,2)^2 = \pi(\sigma_1\sigma_2)^2$. Note that:

$$(\sigma_1\sigma_2)^3 = (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_2) = (\sigma_2\sigma_1\sigma_2)(\sigma_1\sigma_2\sigma_1) = (\sigma_2\sigma_1)^3. \tag{7}$$

If $H_{\sigma_1\sigma_2}$ is bi-orderable, Equation (7) implies that $\sigma_1\sigma_2 = \sigma_2\sigma_1$, which yields a contradiction, since $\pi(\sigma_1\sigma_2) = (1,3,2) \neq (1,2,3) = \pi(\sigma_2\sigma_1)$.

Now suppose that $n \geq 4$. In $B_{i+1}(D) \subset B_n(D)$, since $\Delta_{i+1}^2 = (\sigma_1\cdots\sigma_i)^{i+1}$ and this element is central in B_{i+1}, we have:

$$\big(A_{1,2}(\sigma_1\cdots\sigma_i)A_{1,2}^{-1}\big)^{i+1} = (\sigma_1\cdots\sigma_i)^{i+1}.$$

If $H_{\sigma_1\cdots\sigma_i}$ is bi-orderable, by Proposition 13, it follows that $A_{1,2}(\sigma_1\cdots\sigma_i)A_{1,2}^{-1} = \sigma_1\cdots\sigma_i$. So $A_{1,2}$ commutes with $\sigma_1\cdots\sigma_i$, which contradicts Lemma 25. Hence, the subgroup $H_{\sigma_1\cdots\sigma_i}$ is not bi-orderable. □

3.3 Proof of Theorem 1

We now prove Theorem 1.

Proof. Let us show that $H_\beta \subset H$ cannot be bi-ordered for all $\beta \in H \setminus P_n(D)$, which will imply that H is not bi-orderable.

First, let $n=3$. Since any non-trivial permutation of S_3 is either a transposition or a 3-cycle, by Proposition 21, the intermediate subgroup H_β of B_3 is either conjugate to H_{σ_1} or $H_{\sigma_1\sigma_2}$. From Propositions 22 and 26, we see that H_β is not bi-orderable. Now let $n \geq 4$, and assume that $P_n(D) \subsetneq H$.

Now, let $\beta \in H \setminus P_n(D)$. We shall divide the proof into two cases:

(a) $\pi(\beta)$ is a product of disjoint transpositions;

(b) $\pi(\beta)$ is a product of disjoint cycles, one of which is of length greater than or equal to 3.

For case (a), we may write $\pi(\beta) = \displaystyle\prod_{l=1}^{m}(i_l, j_l)$, a product of disjoint transpositions, where $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$ and $i_1, j_1, \dots, i_m, j_m$ are pairwise distinct. For $2 \leq i \leq m$, recall that $\Delta_i = \sigma_1(\sigma_2\sigma_1)\cdots(\sigma_{i-1}\cdots\sigma_2\sigma_1)$. Then, by Equation (3), we have:

$$\pi(\Delta_{2m}) = \prod_{k=1}^{m}(k, (i+1)-k),$$

a product of m disjoint transpositions, therefore Δ_{2m} and β have the same cycle type, and so H_β and $H_{\Delta_{2m}}$ are conjugate by Proposition 21. If $m = 1$, $H_{\Delta_2} = H_{\sigma_1}$, and Proposition 22 shows that H_{Δ_2} is not bi-orderable, and if $m \geq 2$, $H_{\Delta_{2m}}$ is not bi-orderable by Proposition 24.

For case (b), we may write $\pi(\beta) = i_1 \cdots i_l$, a product of disjoint cycles i_k of length j_k, with $1 \leq k \leq l$, $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$, $2 \leq j_k \leq n$, $j_p \geq 3$ for some $p \in \{1, \ldots, l\}$, and $\sum_{k=1}^{l} j_k \leq n$. Since the cycles are disjoint, up to renumbering $i_1, \ldots, i_l$, we may suppose that $p = 1$, and so $j_1 \geq 3$. Now, define $\alpha \in B_n(D)$ by:

$$\alpha = \prod_{k=1}^{l} \alpha_k, \text{ with } \alpha_k = \sigma_{\sum_{m=1}^{k-1} j_m + 1} \cdots \sigma_{\sum_{m=1}^{k} j_m - 1}, \tag{8}$$

that is:

$$\alpha = (\sigma_1 \cdots \sigma_{j_1 - 1})(\sigma_{j_1+1} \cdots \sigma_{j_1+j_2-1}) \cdots (\sigma_{j_1+\cdots+j_{l-1}+1} \cdots \sigma_{j_1+\cdots+j_l-1}).$$

Note that:

$$\pi(\alpha) = \prod_{k=1}^{l} \pi(\alpha_k) = \prod_{k=1}^{l} \left(\sum_{m=1}^{k} j_m, \sum_{m=1}^{k} j_m - 1, \cdots, \sum_{m=1}^{k-1} j_m + 1 \right).$$

Then, $\pi(\alpha)$ is a product of disjoint cycles, where the cycle $\pi(\alpha_k)$ is of length j_k, and so the permutations of α and β have the same cycle type. By Proposition 21, H_β is bi-orderable if and only if H_α is. We shall show that the unique root property does not hold in H_α, and consequently, by Proposition 13, this subgroup cannot be bi-ordered. By the Artin relations (2) and Equation (8), the braids α_i and α_j commute pairwise for every $i, j \in \{1, \ldots, l\}$. Note also that $A_{1,2} = \sigma_1^2$ commutes with α_i for every $i \in \{2, \ldots, l\}$ because $j_1 \geq 3$. Since $\alpha_1^{j_1} = (\sigma_1 \cdots \sigma_{j_1-1})^{j_1} = \Delta_{j_1}^2$ is a central element of $B_{j_1}(D) \subset B_n(D)$, by Proposition 5, we have:

$$\begin{aligned}(A_{1,2}\alpha A_{1,2}^{-1})^{j_1} &= A_{1,2}\alpha^{j_1} A_{1,2}^{-1} = A_{1,2}\alpha_1^{j_1}\alpha_2^{j_1} \cdots \alpha_l^{j_1} A_{1,2}^{-1} = \alpha_1^{j_1} A_{1,2}(\alpha_2^{j_1} \cdots \alpha_l^{j_1}) A_{1,2}^{-1} \\ &= \alpha_1^{j_1}\alpha_2^{j_1} \cdots \alpha_l^{j_1} A_{1,2} A_{1,2}^{-1} = \alpha^{j_1}.\end{aligned} \tag{9}$$

If the subgroup H_α is bi-orderable, by Proposition 13, we have $A_{1,2}\alpha A_{1,2}^{-1} = \alpha$. However:

$$\begin{aligned}[A_{1,2}, \alpha] &= A_{1,2}\alpha A_{1,2}^{-1}\alpha^{-1} = \sigma_1^2 \alpha_1 (\alpha_2 \cdots \alpha_l) \sigma_1^{-2} (\alpha_2 \cdots \alpha_l)^{-1} \alpha_1^{-1} \\ &= \sigma_1^2 \alpha_1 (\alpha_2 \cdots \alpha_l)(\alpha_2 \cdots \alpha_l)^{-1} \sigma_1^{-2} \alpha_1^{-1} = \sigma_1^2 \alpha_1 \sigma_1^{-2} \alpha_1^{-1} \\ &= \sigma_1^2 \sigma_1 \sigma_2 (\sigma_3 \cdots \sigma_{j_1-1}) \sigma_1^{-2} (\sigma_3 \cdots \sigma_{j_1-1})^{-1} \sigma_2^{-1} \sigma_1^{-1} \\ &= \sigma_1^2 \sigma_1 \sigma_2 (\sigma_3 \cdots \sigma_{j_1-1})(\sigma_3 \cdots \sigma_{j_1-1})^{-1} \sigma_1^{-2} \sigma_2^{-1} \sigma_1^{-1} \\ &= \sigma_1^2 \sigma_1 (\sigma_1^{-1} \sigma_2^{-2} \sigma_1) \sigma_1^{-1} = A_{1,2} A_{2,3}^{-1},\end{aligned}$$

and so $A_{1,2} = A_{2,3}$, which contradicts Remark 9. Therefore, H_α cannot be bi-ordered, and so neither can H_β.

From cases (a) and (b), we conclude that H_β is not bi-orderable for all possible cycle types of the permutation $\pi(\beta) \in S_n$. Hence, if H is an intermediate subgroup of $B_n(D)$, then H contains H_β for some $\beta \in B_n(D) \setminus P_n(D)$, and so H is not bi-orderable. $\square$

3.4 Consequences and proof of Theorem 2

This subsection is devoted to giving some consequences of Theorem 1. We recall that the braid group $B_\infty(D)$ of the disk on an infinite number of strands is composed of braids with an infinite number of strands but finitely many crossings. More precisely, $B_\infty(D)$ is the direct limit of $B_n(D)$ under the injective homomorphism $\iota\colon B_n(D) \to B_{n+1}(D)$ given by $\iota(\sigma_i) = \sigma_i$ for all $1 \le i \le n-1$ (see [14, 17] and Remark 4). There exists an epimorphism $\pi_\infty\colon B_\infty \to S_n$ such that the restriction $\pi_\infty|_{B_{n(D)}}$ is the permutation homomorphism $\pi\colon B_n(D) \to S_n$. The kernel of the map π_∞ is defined to be $P_\infty(D)$, which is the direct limit of $P_n(D)$ under the restriction $\iota|_{P_n(D)}$. We now show that intermediate subgroups of $B_\infty(D)$ are not bi-orderable.

Corollary 27. *Let H be an intermediate subgroup of $B_\infty(D)$, i.e. $P_\infty(D) \subsetneq H \subset B_\infty(D)$. Then, H is not bi-orderable.*

Proof. Let $\beta \in H \setminus P_\infty(D)$. Thus, there exist $m \in \mathbb{N}$ and $\beta' \in P_m(D)$ such that m is the least positive integer that satisfies $i(\beta') = \beta$, where $i\colon P_m(D) \to P_\infty(D)$ is the natural inclusion. By Theorem 1, the subgroup $\langle P_m(D), \beta\rangle$ is not bi-orderable. Identifying this subgroup with its image under i, we have $\langle P_m(D), \beta'\rangle \subset \langle P_\infty(D), \beta\rangle \subset H$, and so H is not bi-orderable. $\square$

We now prove Theorem 2, which is an extension of Theorem 1 to intermediate subgroups of compact, connected surfaces different from the sphere and the real projective plane.

Proof. Since $P_n(M) \subsetneq H$, there exists $\beta \in H \setminus P_n(M)$. Let $\alpha \in B_n(D) \subset B_n(M)$ be such that $\pi(\alpha) = \pi(\beta)$, where $\pi\colon B_n(M) \to S_n$ is the permutation homomorphism. Hence, $\alpha\beta^{-1} \in P_n(M)$. So there exists $\gamma \in P_n(M)$ such that $\alpha = \gamma\beta$ and, consequently,

$$\langle P_n(M), \alpha\rangle = \langle P_n(M), \gamma\beta\rangle = \langle P_n(M), \beta\rangle \subset H.$$

Thus, by Remark 11, we have $H \supset \langle P_n(M), \alpha\rangle \supset \langle P_n(D), \alpha\rangle$. Since $\langle P_n(D), \alpha\rangle$ is not bi-orderable, by Theorem 1, it follows that H is not bi-orderable either. $\square$

If M is a compact, connected, non-orientable surface without boundary, recall that the pure braid groups $P_n(M)$ have generalized torsion for $n \ge 2$ [26, Theorem 1.7], and therefore are not bi-orderable. This immediately implies that no intermediate subgroup of $B_n(M)$ can be bi-orderable for such a surface M.

Remark 28. *Let M be as in Theorem 2, and let $n \ge 3$. Every intermediate subgroup of $B_n(M)$ has generalized torsion in its commutator subgroup. To see this, using Proposition 14 and the fact that $\langle P_n(D), \beta\rangle \subset \langle P_n(M), \beta\rangle$ for all $\beta \in B_n(D)$ (see Remark 11), it suffices to show that H_β does not have the unique root property for certain braids $\beta \in B_n(D)$, which we now describe. For $n = 3$, by Equations (5) and (7), the commutators $[\sigma_1 A_{2,3}, A_{2,3}\sigma_1]$ and $[\sigma_1\sigma_2, \sigma_2\sigma_1]$ are generalized torsion elements of H_{σ_1} and $H_{\sigma_1\sigma_2}$ respectively. For $n \ge 4$, let us first consider the case where the cycle decomposition of $\pi(\beta)$ has a cycle of length greater than or equal to 3. With the notation defined in the proof of Theorem 1, the commutator $\left[A_{1,2}\alpha A_{1,2}^{-1}, \alpha\right]$ is a generalized torsion element of H_α, by Equation (9). Now, suppose that $\pi(\beta)$ is a product of disjoint transpositions. Then $\left[A_{1,2}\Delta_i A_{1,2}^{-1}, \Delta_i\right]$ is a generalized torsion element of H_{Δ_i}, by Equation (6), for $i \ge 3$. Moreover, the elements $[\sigma_1 A_{2,3}, A_{2,3}\sigma_1], [\sigma_1\sigma_2, \sigma_2\sigma_1]$ and $\left[A_{1,2}\Delta_i A_{1,2}^{-1}, \Delta_i\right]$ satisfy Equation (4).*

To finish the paper, we make the following remarks about the intermediate subgroups of 2-braid groups of surfaces.

Remark 29. *Let M be a compact, connected surface without boundary. If H is a subgroup of $B_2(M)$ such that $P_2(M) \subsetneq H$, then $H = B_2(M)$. Indeed, for such a subgroup H, there* *exists $\gamma \in H \setminus P_2(M)$, so $\langle P_n(M), \gamma \rangle \subset H$. Let $\beta \in B_2(M)$ be such that $\pi(\beta) = (1\ 2)$. So $\pi(\beta) = \pi(\gamma)$, and thus $\langle P_n(M), \beta \rangle = \langle P_n(M), \gamma \rangle \subset H$. Therefore, $\beta \in H$ for any $\beta \in B_2(M)$, which implies that $H = B_2(M)$.*

Remark 30. *By Theorem 3, we have $B_2(D) \cong \mathbb{Z}$. Therefore, $B_2(D)$ is bi-orderable. The groups $B_2(S^2)$ and $B_2(\mathbb{R}P^2)$ are finite [50], so they are not left-orderable. For the torus T, $B_2(T)$ is left-orderable [46, Theorem 4.2.1], whereas it is not bi-orderable [5]. However, in general, it is not known if 2-braid groups of compact, connected surfaces without boundary are left-orderable. For the Klein bottle K, $B_2(K)$ cannot be bi-ordered [26].*

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