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The Weight of Spin(32)/$\mathbb{Z}_2$ Little Strings: T-duality and Hasse Diagrams

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Abstract

We study the worldvolume theories of stacks of $Spin(32)/\mathbb{Z}_2$ heterotic NS5-branes probing a transverse singularity $\mathfrak{g}$. We revisit and extend the original classification by Blum and Intriligator, and show that the resulting $6d$ Little String Theories (LSTs) are naturally labeled by affine dominant coweights of the singularity $\mathfrak{g}$. This in turn enables us to efficiently arrange these theories into groupings satisfying all known necessary conditions to be T-dual. Using this formulation, we then study the partial order of those coweights, and extend a recently proposed slice-subtraction algorithm to construct Hasse diagrams for LSTs directly from their six-dimensional generalized quivers, allowing us to probe certain properties of their Higgs branch. Along the way, we exploit these techniques to show that the number of duality orbits at maximal flavor rank is determined by the center of the transverse singularity $\mathfrak{g}$, and provide a simplified proof of a monotonicity theorem for this class of theories. Finally, we show how some of our techniques can be extended to other classes of 6d theories, such as Type-II LSTs and SCFTs.

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1 Introduction

Six-dimensional supersymmetric Quantum Field Theories (SQFTs) occupy a special place in the landscape of field theories due to a combination of a high degree of supersymmetry and a large number of spacetime dimensions, which severely restricts the consistent matter spectra and allowed symmetries. This leads to a very constrained set of theories that provides an attractive playground to study a host of field-theoretic features, ranging from non-perturbative effects to generalized symmetries [1].

Moreover, the presence of antisymmetric tensor fields and the BPS non-critical strings to which they couple is intimately related to the possible ultraviolet (UV) completions of six-dimensional SQFTs in the absence of gravity: when BPS strings become tensionless, new massless, non-perturbative, degrees of freedom arise [2, 3]. If all BPS strings can be made tensionless at the same time, all other scales—in particular inverse gauge coupling constants—are also eliminated by supersymmetry. One obtains a local, strongly-coupled, superconformal field theory (SCFT). On the other hand, there are cases where not all BPS strings can be made tensionless simultaneously. The remnant scale acts as an ultraviolet cutoff above which the theory develops a Hagedorn density of states [4]. The resulting non-local theories, known as Little String Theories (LSTs), share several hallmark features of ten-dimensional critical strings—–most notably T-duality—–despite being non-gravitational. Below this scale, LSTs reduce to effective Quantum Field Theories [3, 5] whose dynamics can be analyzed using conventional field-theoretic techniques.

A tool of choice to study SQFTs in six dimensions has long been geometric engineering through string theory, in particular F-theory. Indeed, one of the arguably greatest successes of F-theory has been an enumeration of all possible geometries leading to SCFTs [6, 7] and LSTs [8–10] in spacetime dimensions $d = 6$ via compactification, see reference [11] for a review.

Little String Theories have recently experienced a renewed interest, as they exhibit generalized symmetries [1]—see recent reviews [12–18] and reference therein—which can be understood with the geometric techniques of F/M-theory [19–21]. They are in particular all endowed with a unique continuous two-group symmetry [22–25], which is absent for SCFTs or in supergravity. The presence of a non-dynamical tensor multiplet $B_{\mu\nu}^{\text{LST}}$ associated with the little string induces a $\mathfrak{u}(1)^{(1)}$ one-form symmetry that mixes non-trivially with ordinary zero-form symmetries, namely the $\mathfrak{su}(2)_R$ R-symmetry, spacetime Poincaré invariance $\mathfrak{p}$, and a possible flavor symmetry $\mathfrak{f}$, to form a two-group:

$$(\mathfrak{su}(2)_R \oplus \mathfrak{p} \oplus \mathfrak{f})^{(0)} \times_{\kappa_R,\kappa_P,\kappa_F} \mathfrak{u}(1)^{(1)} \,. \tag{1.1}$$

The coefficients $\kappa_R, \kappa_P, \kappa_F$ are the structure constants of the two-group, and encode the non-trivial mixing between $B_{\mu\nu}^{\text{LST}}$ and the other background fields under a symmetry transformation.

In addition, like their critical ten-dimensional cousins, LSTs manifest a form of T-duality: upon reduction on a circle, two different LSTs may lead to the same five-dimensional theory. This feature was already recognized three decades ago by Aspinwall and Morrison, who explicitly realized the duality between $\mathfrak{so}_{32}$ and $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic strings by identifying inequivalent elliptic fibrations of the same birational Calabi–Yau threefold [26]. Geometric engineering has since been the preferred approach to verify T-duality, and has been recently applied to a plethora of models [27–34]. An important question is therefore what structure T-dual models may exhibit, and how different T-dual pairs are related. Indeed, there can also be more than two LSTs that are dual to one another, and there are many explicit example of so-called "T-n-ality" [24, 27, 28, 31–35].

A list of quantities that are invariant under T-duality has been put forward, greatly simplifying the search for dual pairs [27, 28, 31, 35]:

$$\kappa_P\,, \quad \kappa_R\,, \quad \dim(\text{CB})\,, \quad \text{rk}(\mathfrak{f})\,, \tag{1.2}$$

with dim(CB) the dimension of the resulting five-dimensional theory, given by the sum of dynamical tensor multiplets and the rank of the total gauge symmetry of the six-dimensional theory, and rk($\mathfrak{f}$) the rank of the flavor symmetry. Notably, the two-group constants above are part of the T-duality invariants and are therefore very desirable to understand.[1] It has furthermore been observed that the two-group structure constant associated with the Poincaré symmetry only take two values: $\kappa_P = 0, 2$. This has been identified in the M-theory picture as counting the number of M9-branes [24]. In string constructions, $\kappa_P = 2$ then labels those LSTs than can be engineered from $\mathfrak{so}_{32}$ or $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic strings, while $\kappa_P = 0$ are those with an origin in Type-IIB string theory.[2] The list given in Equation (1.2) is by no means exhaustive, and it is not known what the complete list of invariantss is. For instance, the defect group and one-form symmetries lead to one-form symmetries in five dimensions, and the former are exchanged under T-duality [33]. It therefore remains an open question whether the matching of the invariants given in Equation (1.2) is enough to differentiate T-dual theories, or form only a necessary condition. However, we are not aware of any case where two theories share the same invariants, but can be shown to not be T-dual.

[1] The third two-group constant κ_F is also expected to match across duality after embedding the flavor symmetry in an appropriate larger symmetry [31, 35].

[2] Possible spectra of five-dimensional theories which could potentially descend from the twisted compactification of LSTs resulting in $\kappa_P = 1$ have been identified in reference [34], but whether they admit a geometric realization is as yet unknown.

In this work, we focus on charting the network of Little String Theories with an $\mathfrak{so}_{32}$ heterotic origin. At a generic point of their tensor branch—the space spanned by expectation values of the scalar fields that are part of tensor multiplets—these theories admit a very simple field-theoretic realization: their matter spectrum, in addition to tensor and vector multiplets, involves only hypermultiplets transforming in the bifundamental representation of classical algebras $\mathfrak{su}_k\,,\mathfrak{so}_k\,,\mathfrak{sp}_k$. It was moreover shown by Blum and Intriligator [36–38] that they follow an ADE classification, and that given a choice of a simple algebra,

$$\mathfrak{g} \in \{\mathfrak{su}_n\,,\mathfrak{so}_{2n}\,,\mathfrak{e}_6\,,\mathfrak{e}_7\,,\mathfrak{e}_8\}\,, \tag{1.3}$$

all possible consistent $Spin(32)/\mathbb{Z}_2$ LSTs can be labeled by an integer Q and a choice of embedding of the finite group $\Gamma\subset SU(2)$ McKay dual to $\mathfrak{g}$ into $Spin(32)/\mathbb{Z}_2$. We will therefore denote these LSTs as:

$$\widetilde{\mathcal{K}}_Q(\mu^\vee;\mathfrak{g})\,,\qquad \mu^\vee\in \mathrm{Hom}(\Gamma, Spin(32)/\mathbb{Z}_2)\,. \tag{1.4}$$

In the heterotic string constructions, Q is related to the number of instantons probing a $\mathbb{C}^2/\Gamma$ singularity, and $\mu^\vee$ corresponds a choice of holonomy at infinity that breaks the original $\mathfrak{so}_{32}$ flavor symmetry to one of its subalgebra.

Our goal is to use this classification to find structures in the network of T-duality that can occur in heterotic LSTs. To do so, we translate the embedding of the singularity into the more modern language of dominant coweights. The embedding $\mu^\vee$ is then interpreted as a particular dominant coweight of $\mathfrak{g}$. We will therefore exploit the versatility of Lie theory to easily separate putative T-dual theories into "orbits" leading to the same duality invariants. It turns out that this procedure only depends on the coweight labeling the LST and does not rely explicitly on its geometric engineering. While the invariants only form a set of necessary rather than sufficient conditions, it converts an *a priori* complicated geometric question into a simpler linear-optimization problem easier to solve.

We will in particular show that theories with the same T-duality invariants partition the set of dominant weights of $\mathfrak{g}$ into convex integer cones, and will give a simple prescription to find their generators. This comes with certain advantages, as some properties of the T-duality invariants follows from properties of finite and affine root systems. For instance, κ_R has been conjectured to follow a monotonicity "κ-theorem" [27], and decreases monotonically along Renormalization Group (RG) flows. In our setting, this property follows directly from the partial order of dominant coweights.

In the context of 3d $\mathcal{N}=4$ quiver gauge theories, the language of coweights and related techniques using the technology of magnetic quivers have proven to be a powerful tool to explore the Higgs branch of six-dimensional theories [32, 35, 39–53]. These techniques have recently led to an algorithm called "slice subtraction" [47], used to study a large class of 6d SCFTs, and giving a remarkably simple and elegant way to recursively obtain all possible SCFTs that can be reached through an RG flow, and which operator participates in the Higgs mechanism. For these SCFTs, the algorithm uses the partial order of dominant coweights. As LSTs are similarly labeled, we propose a generalization of slice subtraction that applies to LSTs, working also for all types of $\mathfrak{so}_{32}$ LSTs.

This work is organized as follows. In Section 2 we give a review of $Spin(32)/\mathbb{Z}_2$ LSTs. Although inspired by the string theory construction, we mostly take a field-theoretic point of view of certain known results. We in particular compute the full anomaly polynomial of theories with a perturbative description Section 2.1, generalizing certain known results to the complete set of continuous flavor symmetries. In Section 2.2 we review the classification of $Spin(32)/\mathbb{Z}_2$ LSTs, and describe it in the language of coweights.

In Section 3 we show how coweights can be used to find putative T-dual orbits easily. In Section 4 we explore the partial order of coweights and its relation to Higgs-branch RG flows, leading us to a generalized slice-subtraction algorithm. In Section 5 we explore possible extensions of our results to more general classes of SCFTs and LSTs. We give our conclusions in Section 6, giving a summary of our results.

In addition, we collate all heterotic LSTs corresponding to a trivial embedding in various tables in Appendix A, give additional details of the derivation of the anomaly polynomial in Appendix B, and a short review of finite and affine root systems in Appendix C, setting our conventions.

2 $Spin(32)/\mathbb{Z}_2$ Heterotic Little String Theories

The simplest realization of Little String Theories corresponds to a small instanton in $\mathfrak{so}_{32}$ heterotic string theory that shrinks to zero size, first shown by Witten to be associated with an $\mathfrak{su}_2$ gauge theory describing the worldvolume theory of an NS5-brane [54]. A stack of Q such branes then leads to an enhanced $\mathfrak{sp}_Q$ gauge theory. To cancel gauge anomalies, 32 half hypermultiplets must be present and are rotated by an $\mathfrak{so}_{32}$ symmetry, thereby giving rise to the familiar $Spin(32)/\mathbb{Z}_2$ group of the heterotic string. This construction admits several dual descriptions, most notably Type-I string theory—which offers a perturbative regime—but also through Type-IIA and IIB (orientifold) vacua.

A further generalization is obtained by considering orbifold singularities of the form $\mathbb{C}^2/\Gamma$, with Γ a finite subgroup of $SU(2)$, in the directions transverse to the branes. This gives rise to new physical degrees of freedom corresponding to additional hyper-, tensor, and vector multiplets [55, 56]. The $\mathfrak{so}_{32}$ flavor symmetry can then be broken to obtain new LSTs. In the heterotic picture, this is done by a choice of a flat connection at infinity, encoded in the fundamental group $\pi_1(S^3/\Gamma) \simeq \Gamma$, corresponding to a choice of embedding of the finite group Γ into $Spin(32)/\mathbb{Z}_2$.

Although *a priori* not perturbative, there exists an elegant Type-IIB description of heterotic LSTs through F-theory, where many of the relevant field-theoretic features can be obtained from the data of an elliptically-fibered Calabi–Yau threefold:

$$\begin{array}{ccc} T^2 & \hookrightarrow & X_3 \\ & & \downarrow \\ & & B_2 \end{array} \tag{2.1}$$

The base B_2 must be non-compact in order to decouple gravity. Furthermore, to obtain a Little String Theory in six dimensions, we must engineer a BPS string that can never be tensionless at any point of the tensor-branch moduli space: the little string itself. In F-theory, BPS strings arise from D3-branes wrapping two-real-dimensional curves in the base, their volume setting the tension of the strings. The LST criterion is then that B_2 must have a curve that cannot be shrinked to zero volume anywhere in the Kähler moduli space of B_2, and in addition that there is a point of that moduli space where all other curves go to zero volume simultaneously. This forces the base to be in one of two classes, with the heterotic choice required to be birational to $\mathbb{P}^1 \times \mathbb{C}^1$.[3] Geometrically, this means that the base B_2 admits a compact genus-zero curve Σ_{LST} of self-intersection $\Sigma^2_{\text{LST}} = 0$, further implying that X_3 comes with a nested elliptic K3 fibration over $\mathbb{C}^1$, as expected for theories with an heterotic dual. We will not delve into a detailed explanation of the geometric construction here, and defer to the review [29] and references therein.

Rather, the key point is that the tensor-branch geometry of the theory is encoded in the curve structure of the base B_2. Let us choose a basis of curves $\Sigma^I \in H^2(B_2)$ with $I = 0 \ldots h^{1,1}(B_2)$ of genus zero. By Grauert's criterion, to be able to send the volume of all other curves than Σ_{LST} to zero simultaneously, they must have negative self-intersection. Consistency of the elliptic fibration then requires that the self-intersection of the curves $\Sigma^I \cdot \Sigma^I = -n_I$ lie within the range $0 \leq n_I \leq 12$, and have a negative semi-definite intersection form:[4]

$$\Sigma^I \cdot \Sigma^J = -\eta^{IJ} \preceq 0\,. \tag{2.2}$$

[3]On the other hand, for Type-II LSTs the base B_2 is required to be birational to $T^2 \times \mathbb{C}$ [9, 33].

[4]We use a convention where η^{IJ} is a *positive* semi-definite matrix, while the intersection pairing itself is *negative* semi-definite. We do so because η is then interpreted as the Dirac pairing of the associated BPS strings.

Without loss of generality, we do not consider product theories in this work, so that η^{IJ} is irreducible, and has a unique null vector up to rescaling. In the absence of gravity, product theories are decoupled and can therefore be treated independently. As there is one more curve than the independent generators of $H^2(B_2)$, there is one linear relation among the curves:

$$\Sigma_{\mathrm{LST}} = \ell_I \Sigma^I \,, \qquad \eta^{IJ}\,\ell_J = 0\,, \tag{2.3}$$

with $\ell_I \in \mathbb{N}$ and $\gcd(\ell_I) = 1$. It is a vector in the string charge lattice, interpreted as the charges of the little string under the Dirac pairing.

The base therefore defines the spectrum of tensor multiplets, the associated BPS strings, and their intersection pattern. On the other hand, the gauge sector is encoded in possibly-singular elliptic fibers over each curve. These singularities follow the Kodaira–Tate classification, and can be associated with an algebra that defines the gauge symmetry on that curve. Furthermore, global aspects of gauge and flavor group are encoded in the finite part of the Mordell–Weil group of X_3 [57–60]. Indeed, the dual F-theory model of the $Spin(32)/\mathbb{Z}_2$ heterotic string admits an order-two Mordell–Weil group. The matter spectrum is then associated with enhanced singularities at the intersection of two curves, and can also be easily obtained from the classification, see e.g. reference [61].

Geometric engineering of LSTs through F-theory is therefore very practical, and reduces to consider a collection of curves whose elliptic fibers are singular, and from which the spectrum can easily be obtained. Furthermore, as we have reviewed above, $\mathfrak{so}_{32}$ heterotic LSTs are very attractive due to their perturbative nature, and it follows that their F-theory realization is particularly simple: the base only involves curves of self-intersection $n_I \in \{0, 1, 2, 4\}$ that have at most a simple intersection $\eta_{IJ} = -1$, and the algebras associated with the singular fibers are only of classical types $\mathfrak{sp}_k\,, \mathfrak{so}_k\,, \mathfrak{su}_k$. This is in stark contrast with Type-II LSTs, where more complicated intersection patterns occur, or those of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$-heterotic type where any simple algebra can appear that lead to the presence of non-perturbative states.

As the transverse singularities satisfy an ADE classification, given a choice of algebra $\mathfrak{g} = \mathfrak{su}_k\,, \mathfrak{so}_{2k}\,, \mathfrak{e}_k$, the base is related to the topology of its affine Dynkin diagram. There is however a caveat: for reasons that will become clear when we review the classification of the allowed flavor and gauge symmetries, when the algebra $\mathfrak{g}$ associated with the transverse singularity of the LST admits complex representations, the topology of the base is associated with the *folded* affine version of $\mathfrak{g}$, which we denote by $\widetilde{\mathfrak{b}}$. For instance, with $\mathfrak{g} = \mathfrak{e}_6$ the base is $\widetilde{\mathfrak{b}} = F_4^{(1)}$. The correspondence is given in Table 1. Note that we have not included the case $\mathfrak{g} = \mathfrak{su}_{2k+1}$ by design, as it has additional subtelties which will be treated separately in Section 5.2.

$\mathfrak{g}$	$\mathfrak{su}_{2k}$	$\mathfrak{so}_{4k}$	$\mathfrak{so}_{4k+2}$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$\widetilde{\mathfrak{g}}$	$A_{2k-1}^{(1)}$	$D_{2k}^{(1)}$	$D_{2k+1}^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$
$\widetilde{\mathfrak{b}}$	$\boldsymbol{C_k^{(1)}}$	$D_{2k}^{(1)}$	$\boldsymbol{B_{2k}^{(1)}}$	$\boldsymbol{F_4^{(1)}}$	$E_7^{(1)}$	$E_8^{(1)}$

Table 1: Algebras arising in $Spin(32)/\mathbb{Z}_2$ heterotic LSTs and the associated bases $\widetilde{\mathfrak{b}}$, possibly folded from the affine algebra $\widetilde{\mathfrak{g}}$. The bases that are different from $\widetilde{\mathfrak{g}}$ are in bold. The special case of $\mathfrak{g} = \mathfrak{su}_{2k+1}$ is treated separately in Section 5.2.

Notation and conventions: we will often have to jump between quantities related to a simple Lie algebra $\mathfrak{g}$, its affine version $\widetilde{\mathfrak{g}}$, as well as the base algebra $\widetilde{\mathfrak{b}}$ and its non-affine counterpart $\mathfrak{b}$.

For a simple algebra $\mathfrak{g}$ we use the usual fraktur font, e.g. $\mathfrak{e}_6$, while we use Kac's notation [62] $X_r^{(n)}$ for its affine version, e.g. $E_6^{(1)}$. This makes the distinction easier, and will also enable us to treat the case $\mathfrak{g} = \mathfrak{su}_{2k+1}$, as well as other types of $\mathfrak{so}_{32}$ LSTs whose bases correspond to twisted affine algebras.

Given an affine algebra $\widetilde{\mathfrak{b}}$ representing the base, we label the nodes of its Dynkin diagram—or equivalently the curves in the base—with uppercase letters from the the middle of the Latin alphabet $I, J, \dots \in \{0, 1, \dots r_{\mathfrak{b}}\}$. If we need to exclude the affine node $I = 0$, we use lowercase letters $i, j, \dots \in \{1, \dots r_{\mathfrak{b}}\}$.

For quantities related to arbitrary algebras $\mathfrak{g}$, including the ADE classification of LSTs—but not their base—we instead use letters from the beginning of the Latin alphabet, with the same distinction between affine and simple nodes: $A, B, \dots \in \{0, 1, \dots r_{\mathfrak{g}}\}$ and $a, b, \dots \in \{1, \dots r_{\mathfrak{g}}\}$ for $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}$, respectively. The affine node of a Dynkin diagram is always labeled by $I = 0$ or $A = 0$.

In addition, if an index-free quantity of a finite algebra has an natural extension to its affine version we denote it with a tilde. For instance the highest root $\theta = \theta_a\, \alpha^a$ of $\mathfrak{g}$ is naturally extended to the null root $\widetilde{\theta} = \theta_A \alpha^A$ of $\widetilde{\mathfrak{g}}$. When indices are present we do not make the distinction, e.g. C^{AB} is an affine Cartan matrix while C^{ab} is that of the associated simple algebra.

Finally, we will also extensively use the properties of finite and affine root systems. A short review of the relevant concepts can be found in Appendix C.

Pictorial representation of LSTs: we will use the now-standard notation encoding the geometry of the elliptic fibration in terms of "generalized quivers" [7]. Consider a curve $\Sigma \subset B_2$ with self intersection $\Sigma^2 = -n$, hosting a singular fiber corresponding to an algebra $\mathfrak{g}$. We also assume that there is a flavor symmetry $\mathfrak{f}$ rotating the possible matter fields associated with that curve. We can pictorially represent this configuration as follows:

$$\underset{[\mathfrak{f}]}{\overset{\mathfrak{g}}{n}}\,. \tag{2.4}$$

If the fiber is not associated with an algebra, $\mathfrak{g} = \varnothing$, or there are no flavor symmetry associated with the configuration, we omit them. If there are more than one curve, they are arranged in a pattern reproducing the adjacency of the pairing matrix η^{IJ}: if two neighboring curves Σ^I and Σ^J intersect with $\eta^{IJ} = -1$, they are depicted side by side.

The possible combinations of n and $\mathfrak{g}$ is furthermore restricted by anomaly cancellation, or in geometric terms, the consistency of the elliptic fibration X_3. For the heterotic LSTs of type $\mathfrak{so}_{32}$ we will consider, we only have algebras of classical types, and there are only four possibilities:[5]

$$\underset{[\mathfrak{so}_m]}{\overset{\mathfrak{sp}_k}{1}}\,, \qquad \underset{[\mathfrak{sp}_m]}{\overset{\mathfrak{so}_k}{4}}\,, \qquad \underset{[\mathfrak{su}_m]}{\overset{\mathfrak{su}_k}{2}}\,, \qquad \underset{[\mathfrak{su}_m]}{\overset{\mathfrak{su}_k}{1}}\,, \tag{2.5}$$

which can be traced back to the fact that those LSTs admit a perturbative Type-I or Type-IIA dual descriptions which only have mutually-local 5-branes and O-planes.

In each case, there are a number of hypermultiplets transforming in the bifundamental representations of $\mathfrak{g} \oplus \mathfrak{f}$, where the precise rank of $\mathfrak{f}$ is set by anomaly-cancellation condition, as we will review below. The last case is special, and hosts an additional hypermultiplet transforming in the antisymmetric representation of $\mathfrak{su}_k$, and arise for LST with $\mathfrak{g} = \mathfrak{su}_{2k+1}$. The presence of this additional representation explains why we will delay discussing those theories to Section 5.2, and the majority of this work will focus on the first three cases.

Furthermore, if two compact curves $\overset{\mathfrak{g}}{m}$ and $\overset{\mathfrak{h}}{n}$ intersect, part of the hypermultiplets will transform in the bifundamental representation of $\mathfrak{g} \oplus \mathfrak{h}$, and the flavor symmetries on each curve will be reduced as there are fewer matter fields to rotate. In the F-theory picture, the precise matter content can be obtained by studying the enhanced singularity at the intersection, but the remnant flavor symmetries associated with each gauge algebras can equivalently be obtained from anomaly-cancellation conditions, as reviewed below.

As an example, let us consider $\mathfrak{so}_{32}$ heterotic LSTs with a base labeled by $\mathfrak{g} = \mathfrak{e}_6$ and instanton number Q. We have argued above—and will give additional details in the next section—that the base is associated with $\widetilde{\mathfrak{b}} = F_4^{(1)}$. With generic flavor symmetries, i.e. a generic embedding

[5]We use the usual convention $\mathfrak{sp}_1 = \mathfrak{usp}_2 \simeq \mathfrak{su}_2$ for symplectic algebras. Therefore the fundamental representation of $\mathfrak{sp}_k$ has dimension $2k$.

$\mu^\vee$, the quiver has the following shape:

$$\widetilde{\mathcal{K}}_Q(\mu^\vee;\mathfrak{e}_6): \qquad \begin{array}{ccccc} \mathfrak{sp}_{Q+p_0} & \mathfrak{so}_{2Q+p_1} & \mathfrak{sp}_{3Q+p_2} & \mathfrak{su}_{4Q+p_3} & \mathfrak{su}_{2Q+p_4} \\ 1 & 4 & 1 & 2 & 2 \\ [\mathfrak{so}_{m^0}] & [\mathfrak{sp}_{m^1}] & [\mathfrak{so}_{m^2}] & [\mathfrak{su}_{m^3}] & [\mathfrak{su}_{m^4}] \end{array}, \qquad \eta^{IJ} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{2.6}$$

We therefore have four hypermultiplets transforming in the bifundamental representation of $\mathfrak{g}_I \oplus \mathfrak{g}_{I+1}$, and five transforming in the bifundamental of $\mathfrak{g}_I \oplus \mathfrak{f}_I$. We stress that the Cartan matrix of $F_4^{(1)}$ and η^{IJ} are different. While $\eta^{IJ} = 0, -1$ off diagonal, this is not the case for the Cartan matrix, see Equation (2.10). The relation between the two, as well as the relation between the coefficients m^I and p_I and the choice of embedding $\mu^\vee$, will be clearer when we review anomaly-cancellation conditions.

We can also have trivalent intersections, for instance in the case of LSTs with $\mathfrak{g} = \mathfrak{e}_7$. With a trivial choice of embedding of the associated finite group Γ, the LST with an $\mathfrak{so}_{32}$ flavor symmetry arising from Q NS5-branes probing the corresponding ALE space is given by:

$$\widetilde{\mathcal{K}}_Q(\varnothing;\mathfrak{e}_7): \qquad \begin{array}{ccccccc} & & & \mathfrak{sp}_{2Q-20} & & & \\ & & & 1 & & & \\ \mathfrak{sp}_Q & \mathfrak{so}_{4Q-16} & \mathfrak{sp}_{3Q-24} & \mathfrak{so}_{8Q-64} & \mathfrak{sp}_{3Q-28} & \mathfrak{so}_{4Q-32} & \mathfrak{sp}_{Q-12} \\ 1 & 4 & 1 & 4 & 1 & 4 & 1 \\ [\mathfrak{so}_{32}] & & & & & & \end{array}. \tag{2.7}$$

The other quivers for LSTs with any ADE algebra $\mathfrak{g}$ and a trivial embedding can be found in Appendix A, along with their T-dual invariants.

2.1 Anomalies of $Spin(32)/\mathbb{Z}_2$ LSTs

Heterotic LSTs of type $\mathfrak{so}_{32}$ have the attractive property that all algebras involved are of classical type. As mentioned above several times, their ranks are constrained by requiring the absence of gauge anomalies. A careful analysis of these constraints was performed by Blum and Intriligator [36–38], see also references [55, 63], who achieved a classification of all consistent heterotic $Spin(32)/\mathbb{Z}_2$ LSTs in terms of embeddings of the discrete group $\Gamma \subset SU(2)$ into $Spin(32)/\mathbb{Z}_2$, giving a prescription to enumerate them in each case.

We now analyze the anomaly polynomial of theories with general classical algebras in detail. Beyond pedagogy, our purpose is twofold: first, this enables us to review the original classification, and generalize it to the complete set of continuous (flavor) anomalies, the underlying two-group structure constants, and highlight the difference with respect to other LSTs and their SCFT cousins. Moreover, this will make the relationship between $\mathfrak{so}_{32}$ heterotic LSTs and Lie theory manifest. We will furthermore be able to find several closed-form formulas for various quantities of interests, paving the way to reinterpret the classification in the modern language of coweights in the next section.

To compute the anomaly polynomial of an LST, we can use the fact that scalar fields falling in tensor multiplets are singlets under all global symmetries. These symmetries are therefore conserved under tensor-branch deformations. This enables us to compute the anomaly polynomial at a point of the tensor branch where the theory is weakly coupled, and by 't Hooft anomaly matching the obtained expression is then valid at the strongly-coupled point corresponding to the LST. This procedure is straightforward and algorithmic. Here, we only briefly recall the salient points, and defer to [11, 64–66] for detailed explanations in the case of six-dimensional SQFTs. We follow the same conventions as in references [64–66].

In six dimensions, the anomaly polynomial is made out of two different contributions:

$$I_8 = I_8^{\text{1-loop}} + I_8^{\text{GS}}\,. \tag{2.8}$$

The former is a "one-loop" contribution obtained by summing the contribution of each multiplet in the weakly-coupled regime, while the latter corresponds to a Green–Schwarz–West–Sagnotti mechanism [67, 68]—which we will refer to as a GS term for brevity—necessary to completely cancel gauge anomalies.

We will compute the anomaly polynomial for a six-dimensional quiver theory with the following spectrum:

1. a collection of $r_{\mathfrak{b}}$ dynamical tensor multiplets;
2. $r_{\mathfrak{b}}+1$ vector multiplets transforming in the adjoint representation of a *classical* gauge algebra $\mathfrak{g}_I \in \{\mathfrak{sp}_{k_I}, \mathfrak{so}_{k_I}, \mathfrak{su}_{k_I}\}$, $I = 0, 1, \ldots, r_{\mathfrak{b}}$;
3. hypermultiplets transforming in the bifundamental $(\boldsymbol{d}_I, \overline{\boldsymbol{d}}_J)$ of $\mathfrak{g}_I \oplus \mathfrak{g}_J$. Their symmetric adjacency matrix is given by $\frac{1}{2}A^{IJ}$, with $A^{II}=0$, and differentiates between half- and full hypermultiplets (see below);
4. hypermultiplets transforming in the bifundamental $(\boldsymbol{d}_I, \overline{\boldsymbol{f}}^J)$ of $\mathfrak{g}_I \oplus \mathfrak{f}_J$ with a *diagonal* adjacency matrix $\frac{1}{2}D^{IJ}$, again taking into account the difference between and half- and full hypermultiplets.

The algebras $\mathfrak{f}_I$ correspond to flavor symmetries of the theory. Moreover, $r_{\mathfrak{b}}$ will correspond to the rank of the folded base $\widetilde{\mathfrak{b}}$, see Table 1. We have also anticipated that the tensor corresponding to the LST can be thought of as being a non-dynamical background field [9]. If one consider an SCFT instead, there is an additional dynamical multiplet however this will not influence the constraints between the gauge and flavor symmetries.

We stress that we must differentiate between full and half hypermultiplets. Generically, a hypermultiplet transforms in the representation $\mathcal{R} \oplus \overline{\mathcal{R}}$ of the associated symmetry. However, if the representation is (pseudo)-real, $\mathcal{R} = \overline{\mathcal{R}}$, we are overcounting and only have a half hypermultiplet. In practice, it means that $\frac{1}{2}D^{II} = 1$ (respectively $\frac{1}{2}A^{IJ} = 1$) if $\mathfrak{g}_I \oplus \mathfrak{f}_I$ (respectively $\mathfrak{g}_I \oplus \mathfrak{g}_J$) contains an $\mathfrak{su}_k$ factor, or $\frac{1}{2}$ otherwise.

The matter interactions taking this property into account are then encoded in the matrix:

$$G^{IJ} = 2D^{IJ} - A^{IJ}\,, \qquad G^{IJ} = D^I_K\, C^{KJ}\,. \tag{2.9}$$

Note that for the cases we consider, G^{IJ} must be a symmetric matrix with $G^{II} \in \{2, 4\}$ and have off-diagonal entries $-2, -1, 0$. This is precisely equivalent to the classification of the *symmetrized* Cartan matrices of both finite and affine Lie algebras [62] excluding those with -3 off diagonal, such as $\mathfrak{g}_2$. The matrix C defined above is the associated Cartan matrix.

Indeed, recall that given the (generalized) Cartan matrix C of a Kac–Moody algebra, there exist a positive-integer-valued diagonal matrix D such that $G = DC$ is symmetric with integer values. The determinant of G differentiates between an SCFT and an LST, as for the latter it implies that is a combination of the tensor multiplets that is not dynamical.

Classification 1 (base topology of 6d SQFTs)**.** *If the irreducible quiver of a six-dimensional SQFT with a finite number of nodes involves only hypermultiplets in the fundamental representation of classical gauge symmetries, the hypermultiplet adjacency matrix G must be the symmetrized Cartan matrix of:*

- *a finite algebra $\mathfrak{b}$ for those describing the tensor branch of an SCFT;*
- *an affine (possibly-twisted) algebra $\widetilde{\mathfrak{b}}$ for those describing the tensor branch of an LST.*

Cartan Matrices with off-diagonal elements $C^{IJ} < -2$ are not realized as 6d *SQFTs with only bifundamental matter. The precise type of algebras $\mathfrak{b}$, $\widetilde{\mathfrak{b}}$ is then read off the diagonal matrix D. If it includes both half and full hypermultiplets, it is not simply-laced.*

Although found through combinatoric arguments, this is of course in agreement with the known classifications [6–10] when restricted to classical gauge algebras $\mathfrak{g}_I$ and $\mathfrak{f}_I$. Note that not all finite and affine algebras are realized. Since the base must be supplemented with a choice of algebras, it is not always possible to find a consistent choice. Furthermore, we cannot have bases for which $\det(G) < 0$ since they lead to negative defined Dirac pairing. The only exception are supergravity theories which have $\det(\eta) = -1$, see e.g. reference [8].

For the LST of type $\mathfrak{g} = \mathfrak{e}_6$ discussed around Equation (2.6), we have a base $\widetilde{\mathfrak{b}} = F_4^{(1)}$ and it is straightforward to see that we indeed have

$$G^{IJ} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -2 & 4 & -2 \\ 0 & 0 & 0 & -2 & 4 \end{pmatrix}\,, \qquad C^{IJ} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}\,, \qquad D^{IJ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}\,. \tag{2.10}$$

This explains why the algebra $\widetilde{\mathfrak{b}} = F_4^{(1)}$ arises naturally in this picture, rather than e.g. $\widetilde{\mathfrak{b}} = A_4^{(1)}$ or any other linear Dynkin diagram. A complete list of the bases that can occur can be found in Table 12 of Section 6.

Furthermore, in the definition of the spectrum we have denoted the dimension of the fundamental representation of a gauge algebra $\mathfrak{g}_I$ by d_I, while that of the flavor symmetry $\mathfrak{f}_I$ is denoted by f^I. For the rest of this work, d_I and f^I will always be associated with the fundamental representation, and all closed-form expressions we will derive will depend on them explicitly. However, for the sake of clarity, it will sometimes be more convenient to use $\mathfrak{sp}_{k_I}$ rather than $\mathfrak{sp}_{d_I/2}$. We can conveniently pass from one to the other through

$$d_I = V_I^J k_J \,, \qquad V_I^J = \begin{cases} V_I^I = 1\,, & \text{if} \quad \mathfrak{g}_I \in \{\mathfrak{su}_k, \mathfrak{so}_k\}\,; \\ V_I^I = 2\,, & \text{if} \quad \mathfrak{g}_I = \mathfrak{sp}_{k_I}\,; \\ V_I^J = 0\,, & \text{if} \quad I \neq J\,. \end{cases} \tag{2.11}$$

Of course, using either d_I or k_I is equivalent, but the dimension of the fundamental representation is the natural object to consider in the language of dominant coweights, as we will see below.

The matrix V is also useful to relate the dual Coxeter numbers $h^I = h^\vee_{\mathfrak{g}_I}$ and the quartic Casimir x_I of the gauge algebra $\mathfrak{g}_I$ in terms of the dimension of the fundamental representation of the associated flavor symmetry:

$$d_I = V_J^I h^J - 2S^I \,, \qquad x^I = D^{IJ} d_J + 8S^I \,, \qquad S^I = \begin{cases} -1\,, & \text{if} \quad \mathfrak{f}_I = \mathfrak{sp}_k\,; \\ \ \ 1\,, & \text{if} \quad \mathfrak{f}_I = \mathfrak{so}_k\,; \\ \ \ 0\,, & \text{if} \quad \mathfrak{f}_I = \mathfrak{su}_k\,. \end{cases} \tag{2.12}$$

These are easily shown by inspection. For $Spin(32)/\mathbb{Z}_2$ LSTs, the quantity S^I can be interpreted as the Frobenius–Schur indicator of the associated finite representation of Γ, as we will review in detail in the Section 2.2. These relations allows us to greatly simplify the computation of the anomaly polynomial, and will lead to simple closed-form expressions for many quantities relevant to this work. We will furthermore assume that d_I is always large enough so that the gauge algebras have a non-vanishing quartic Casimir.

"One-loop" contribution: with the above spectrum, we can easily compute the first term in Equation (2.8), obtained by summing the contribution of each supermultiplet:[6]

$$I_8^{\text{1-loop}} = r_{\mathfrak{b}} I_8^{\text{tensor}} + \sum_I I_8^{\text{vec}}(\mathfrak{g}_I) + \frac{1}{4} A^{IJ} I_8^{\text{hyp}}(\mathfrak{g}_I, \mathfrak{g}_J) + \frac{1}{2} D^{IJ} I_8^{\text{hyp}}(\mathfrak{g}_I, \mathfrak{f}_J)\,. \tag{2.13}$$

where the extra factor $\frac{1}{2}$ for gauge hypermultiplets takes into account double counting, as A^{IJ} is symmetric. The contribution of a given supermultiplet is universal, and can be written in terms of characteristic classes associated with the 0-form symmetries, such as the Chern character $\mathrm{ch}_{\mathcal{R}}$ in a representation $\mathcal{R}$ for the background flavor, gauge, and R-symmetry, denoted F_I, $\widetilde{F}_I$, and R, respectively [69]. Terms encoding gravitational background terms depend on the A-roof genus $\widehat{A}(T)$, and the Hirzebruch genus $L(T)$ of the spacetime tangent bundle T. As the spectra solely involve classical algebras and their adjoint and fundamental representations, we can use various group-theoretical relations to simplify certain expressions, see e.g. Equation (2.12). The steps necessary to obtain the one-loop contribution are given in Appendix B along with further details on characteristic classes.

We find that the one-loop contribution to the anomaly polynomial can be rewritten in a relatively compact form:[7]

$$\begin{aligned} I_8^{\text{1-loop}} = {} & \frac{r_{\mathfrak{g}}}{2}\widehat{A}(T)\mathrm{ch}_{\mathbf{2}}(R) - \frac{r_{\mathfrak{b}}}{8}L(T) + \frac{d_I}{4}\Big(2D^{IJ}\mathrm{ch}_{\boldsymbol{F}}(F_J) + A^{IJ}d_J - (S^I + D^{IJ}d_J)\mathrm{ch}_{\mathbf{2}}(R)\Big)\widehat{A}(T)\Big|_8 \\ & - \frac{1}{2}c_2(\widetilde{F}_I)(\eta^{IJ}c_2(\widetilde{F}_J) - 2J^I) + \frac{1}{48}\left(f^I - C^{IJ}d_J + 16S^I\right)\Big(D_I^K \mathrm{Tr}\widetilde{F}_K^4 + V_I^K c_2(F_K)p_1(T)\Big) \end{aligned} \tag{2.14}$$

The first line of Equation (2.14) is given in terms of the characteristic classes and the dimension of the fundamental representation d_I of each gauge symmetry, and is understood as taking the eight-form component of the resulting expression. The second line depends on the second Chern class of the background flavor symmetry, denoted $c_2(F_I)$, while $p_2(T)$ denotes the second Pontryagin classes of the spacetime background. Furthermore, we have written the first terms of the second line with the Dirac pairing η^{IJ}, see Equation (2.2), using the group-theoretic relations of classical groups. We have also defined the following quantity, which will be related to the GS term:

$$J^I = \delta^{Ia}c_2(F_a) + \frac{1}{4}a^I p_1(T) - h^I c_2(R)\,, \qquad a^I = 2 - \eta^{II}\,, \qquad h^I = h^\vee_{\mathfrak{g}_I}\,. \tag{2.15}$$

As is, the one-loop contribution to the anomaly polynomial contain gauge anomalies, which

[6]For brevity, we will ignore Abelian factors, and only reintroduce them in relevant examples.

[7]As noted above, in the case of an SCFT with classical algebras, this result holds with the exception that the number of tensors is different. Furthermore, there is no relation between $\mathfrak{g}$ and $\mathfrak{b}$ in that case, and the first two terms have slightly different coefficients.

can be separated into two classes. The first corresponds to quadratic gauge anomalies proportional to $c_2(\widetilde{F}_I) = \frac{1}{4}\mathrm{Tr}(\widetilde{F}_I^2)$. Due to the presence of dynamical tensor multiplets and their dual BPS strings, such anomalies can be canceled through a GS term [67, 68]. The other are quartic terms of the form $\mathrm{Tr}(\widetilde{F}_I^4)$ which cannot be canceled through such a mechanism, and therefore impose constraints on the matter spectrum.

Cancellation of quartic gauge anomalies: as the anomaly polynomial cannot contain terms proportional $\mathrm{Tr}\widetilde{F}_I^4$, the last term in Equation (2.14) must vanish identically, constraining the allowed values of the dimensions of the fundamental representation of the flavor symmetry f^I:

$$C^{IJ} d_J = f^I - 16 S^I \,. \tag{2.16}$$

The matrix C^{IJ} has been defined in Equation (2.9) and—through the argument given there—must correspond to the Cartan matrix of a simple (for SCFTs) or affine (for LSTs) Lie algebra. The vector S^I is defined in Equation (2.12), and will be also be interpreted in terms of group theory in Section 2.2.

Equation (2.16) is well known, and reproduces the correct curve self-intersection in the geometric approach. In fact, it is one of the central result of the classification of $Spin(32)/\mathbb{Z}_2$ heterotic Little String Theories [36], and most of the results we will obtain can have their origin traced back to this constraint.

Note that in the language of quiver gauge theory, Equation (2.16) is equivalent to demanding that the theory is "balanced". This means that all 6d SQFTs must be "good" in the sense of Gaiotto and Witten [70], and can never be "bad" nor "ugly" due to anomaly cancellation.

For LSTs, the Cartan matrix is of affine type and cannot be inverted. However, following reference [36], we can express d_I in terms of a one-parameter family of solutions:

$$d_I = 2Q\, K_I + X_{IJ}(f^J - 16 S^J)\,, \tag{2.17}$$

The integer Q is then interpreted as the instanton number in the heterotic construction. Furthermore, the matrix X_{IJ} is expressed in terms of the inverse Cartan matrix of the associated *finite* Lie algebra:

$$X_{0I} = 0 = X_{I0}\,, \qquad X_{ij} = C_{ij}\,, \qquad C_{ij}\, C^{jk} = \delta_i^k\,, \tag{2.18}$$

while the coefficients K_I correspond to the comarks of $\widetilde{\mathfrak{b}}$. For future reference, we recall that they can be obtained from the Cartan matrix, along with the marks θ_I:[8]

$$C^{IJ}K_J = 0 = \theta_I C^{IJ}\,, \qquad \theta_I = D_I^J K_J\,. \tag{2.19}$$

They are normalized such that the greatest common divisor of their elements is one. As a cross check, we can see from Table 13 that on the affine node, $Spin(32)/\mathbb{Z}_2$ LSTs have an $\mathfrak{sp}_Q$ gauge algebra, with a fundamental representation of dimension $d_0 = 2Q$.

Depending on the base algebra, this equation can *a priori* lead to non-integer fundamental representations d_I or odd-valued dimensions for $\mathfrak{sp}_k$ algebras, in which case these solutions to the quartic constraint must be discarded.

We also note that for LSTs, acting with the marks θ_A on the left in Equation (2.17), we obtain a weaker constraint, albeit giving a relation between the flavor symmetries:

$$\theta_I f^I = 16\,\theta_I S^I\,, \tag{2.20}$$

We can therefore always write the flavor symmetry on the affine node of in terms of those associated with nodes of the simple algebra.

Green–Schwarz term and two-group invariants: while cancellation of quartic anomalies impose strong constraints on the spectrum, the remaining gauge anomalies do not further constrain the quiver. They can be canceled through a Green–Schwarz–West–Sagnotti mechanism [67, 68], which generically takes the form:

$$I_{\text{GS}} = \frac{1}{2} I^i \eta_{ij} I^j\,, \qquad I^i = \eta^{iI} c_2(F_I) - J^i\,. \tag{2.21}$$

with $\eta_{ij}\,\eta^{jk} = \delta_i^k$. If we consider LSTs, we again run into a zero eigenvalue making the adjacency Dirac η^{IJ} non-invertible. In physical terms, it means that one of the tensor multiplets is non-dynamical and therefore cannot participate to a GS mechanism. Instead, we can omit the tensor associated with affine node $I = 0$. We denote the remaining ones with Latin indices $i, j = 1, \ldots, \text{rk}(\mathfrak{b})$ running over nodes of the finite algebra $\mathfrak{b}$, see the paragraph on conventions above. The second line of Equation (2.14) can then be written as:

$$-\frac{1}{2} c_2(F_I)(\eta^{IJ} c_2(F_J) - 2J^I) = -I_{\text{GS}} + \frac{1}{2} J^i \eta_{ij} J^i - c_2(F_0)(J^0 - \eta^{0i}\eta_{ij} J^j) \tag{2.22}$$

[8] The comarks are often (confusingly) given different names, such as Kac labels or Dynkin multiplicities. We follow Kac's nomenclature [62] and use the terms marks and comarks to dispel any doubts, as both will arise in computations; we denote them by θ_I and K_I, respectively.

The absence of a term of the form $c_2(F_0)^2$ which cannot be canceled by the GS term constrains the consistent adjacency matrix, and is straightforwardly shown to depend on the LST charge vector ℓ_I, see Equation (2.3).

We stress that from the field-theoretic perspective, the Dirac pairing η^{IJ}—and by extension the curve intersections $\Sigma^I \cdot \Sigma^J$—and the little string charge arises naturally from group-theoretical relations and imposing anomaly cancellation. While this reproduces the geometric expectation, it does so without any string theoretic input. The only assumption beyond the usual field-theoretic ones is completeness of the spectrum, so that each tensor couples to a BPS string. In particular, it is easy to show that both the Dirac pairing and the LST charge satisfies:

$$\ell_I = \frac{1}{2} V_I^J \, \theta_J \, , \qquad \eta^{IJ} = 2((VD)^{-1} \, G \, (VD)^{-1})^{IJ} \, , \tag{2.23}$$

Summing the two contributions in Equation (2.8), we arrive at the final formula for the anomaly polynomial of any LST containing only classical algebras:

$$\begin{aligned} I_8 = & \left(\frac{1}{4} d_I \big(2D^{IJ} \mathrm{ch}_{\mathrm{F}}(F_J) + A^{IJ} d_J - (S^I + D^{IJ} d_J) \mathrm{ch}_{\mathbf{2}}(R) \big) \widehat{A}(T) \right) \Big|_{\text{8-form}} \\ & + \left(\frac{r_{\mathfrak{g}}}{2} \widehat{A}(T) \mathrm{ch}_{\mathbf{2}}(R) - \frac{r_{\mathfrak{b}}}{8} L(T) \right) \Big|_{\text{8-form}} + \frac{1}{2} J^i \eta_{ij} J^i - c_2(F_0) \, \ell_I J^I \, . \end{aligned} \tag{2.24}$$

The last term has been shown to correspond to the give the two-group structure constants up to a possible prefactor [23, 24]. In deed, we find

$$\kappa_R = \ell_I \, h^I \, , \qquad \kappa_P = \ell_I \, a^I \, , \qquad \kappa_F(\mathfrak{f}_I) = \ell_I \, , \tag{2.25}$$

reproducing the usual formulas. Using the relations above, it is then easy to show that for $\mathfrak{so}_{32}$ heterotic LSTs:

$$\kappa_P = 2 \, , \qquad \kappa_R = \ell_I h^I = \frac{1}{2} \theta_I \, (\delta^{IJ} d_J + 2S^I) \, . \tag{2.26}$$

The former is expected, as we have already mentioned in the introduction that in the M-theory picture, κ_P counts the number of M9-branes—in this case two, consistent with an anomaly inflow argument on the little string [33]. On the other hand, κ_R changes from one LST to the other. Using the constraint between the gauge and flavor symmetries, it can be rewritten solely in terms of the flavor symmetry. As we will show below, it admits a very simple Lie-theoretic interpretation as a pairing in the root system of the associated affine algebra.

Higgs-branch dimension: another important property of a six-dimensional SQFT is the dimension of its Higgs branch. It was shown in reference [71] that it is also encoded in the anomaly polynomial through the coefficient δ of the second Pontryagin class $p_2(T)$. For $\mathfrak{so}_{32}$ heterotic LSTs,

$$I_8 \supset \frac{\delta}{24} p_2(T)\,, \qquad \delta = -\frac{1}{60}\left(30Q + r_{\mathfrak{g}} + \frac{1}{2} f^i C_{ij} (f^i - 2S^i)\right)\,. \tag{2.27}$$

where we have used the various relations above to simplify the expression; in particular C_{ij} is the inverse Cartan matrix of the simple algebra $\mathfrak{b}$ associated with the base, and the sum does not run over the affine node.[9]

Given a Higgs-branch RG flow between two theories $\mathcal{T}_{\text{UV}}$ and $\mathcal{T}_{\text{IR}}$, it can be shown that the change in the dimension of the Higgs branch is then found to be proportional to the difference between the two coefficients δ:

$$\Delta\text{HB} = \dim(\text{HB}_{\text{UV}}) - \dim(\text{HB}_{\text{IR}}) = -60(\delta_{\text{UV}} - \delta_{\text{IR}})\,. \tag{2.28}$$

This formula will be useful when discussing the Higgs branch of LSTs in Section 4.

Extra abelian flavor symmetries: we have so far neglected possible $\mathfrak{u}_1$ factors in our analysis of the anomaly polynomial. There is formally an additional $\mathfrak{u}_1$ flavor symmetry arising for each hypermultiplet transforming in a complex representation, with unit charge under that symmetry. However not all those would-be Abelian symmetries are genuine flavor of the theory. Indeed, the anomaly polynomial generically contains a term

$$I_8 \supset -\frac{1}{6} \text{Tr} \widetilde{F}_I^3\, \mathcal{Q}^{IJ}\, P_J\,, \tag{2.29}$$

where P_I is thebackground field strength for the flavor $\mathfrak{u}_1$, and the so-called charge matrix $\mathcal{Q}^{IJ}$ is made out of combinations of d_I, f^I. The presence of terms $\text{Tr}\widetilde{F}_J^3$ is related to non-trivial cubic Casimir operators for $\mathfrak{su}_k$ gauge algebras with $k > 2$. Such a gauge term in the anomaly polynomial does not make the theory inconsistent, but rather lead to an ABJ-type anomaly and the $\mathfrak{u}_1$ flavor symmetries are broken at the quantum level [72,73]. Note that this distinguishes them from flavor symmetries of type $\mathfrak{so}_2 \simeq \mathfrak{u}_1$ symmetries, which arise naturally from the quiver and cannot be broken by quantum effects since $\mathfrak{sp}_k$ does not have a cubic Casimir.

If $\mathcal{Q}^{IJ}$ has a non-trivial kernel however, there might be combinations that survive. A careful analysis of those remnant $\mathfrak{u}_1$ factors was performed for heterotic LSTs in reference [31] to show that $\text{rk}(\mathfrak{f})$ was a T-duality invariant after these subtle contributions were taken into account.

[9]For SCFTs, the constant term must be modified slightly to take into account the additional dynamical tensor multiplet, but the term depending on the flavor symmetry is the same.

Hence, we will only focus on the dimension of the kernel of $\mathcal{Q}^{IJ}$, as the number of surviving $\mathfrak{u}_1$ factors is what is relevant for T-duality, rather the minutiae of the specific charges. We however note that those charges might be "delocalized" over the whole quiver rather than only a single hypermultiplet being charged under the remnant symmetry.

It practice however, for $\mathfrak{so}_{32}$ LSTs with a generic instanton number Q, one can show that there is a remnant $\mathfrak{u}_1$ for each non-trivial $\mathfrak{su}_{f^I}$ flavor symmetry.

2.2 Classification of $Spin(32)/\mathbb{Z}_2$ LSTs and Coweights

We have seen that the allowed flavor symmetries are constrained by cancellation of quartic anomalies, see Equation (2.16). We now review how this can be used to classify all possible consistent spectra based on the McKay correspondence. We mostly follow the work of references [36–38], albeit reformulated in a more modern language that will make certain closed-form formulas simpler, and make a systematic study of the T-duality structure of these theories possible.

The McKay correspondence: consider a simply-laced simple Lie algebra $\mathfrak{g}$ of rank $r_{\mathfrak{g}}$. Each finite group $\Gamma \subset SU(2)$ can be described by the Dynkin diagram of the affine algebra $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$. Let us shortly summarize how the correspondence arises.

In the standard basis of simple roots, the (fundamental) null root $\widetilde{\theta}$ is defined as:[10]

$$\widetilde{\theta} = \sum_{A=0}^{r_{\mathfrak{g}}} \theta_A \, \alpha^A = \alpha^0 + \theta \,, \tag{2.30}$$

where the marks θ_A have already been encountered in Equation (2.19) and form the null vector of the transpose of the Cartan matrix (in the proper normalization). Note that for simply-laced algebras, the Cartan matrix is symmetric and marks and comarks are the same: $K_A = \theta_A$.

The highest root of a simple algebra $\mathfrak{g}$ is given by $\theta = \theta^a \alpha_a$ with $a = 1, \ldots, r_{\mathfrak{g}}$. The null root $\widetilde{\theta}$ therefore defines the affine simple root α^0 from the data of the finite algebra $\mathfrak{g}$. A short review of root systems can be found in Appendix C.

McKay realized that the finite subgroups $\Gamma \subset SU(2)$ can be associated with a simply-laced affine algebra $\widetilde{\mathfrak{g}}$ [74]: to each node of its affine Dynkin diagram, labeled by $A = 0, 1, \ldots, \mathrm{rk}_{\mathfrak{g}}$, one can associate one of its irreducible representation $\mathcal{R}_A$ of Γ, and vice versa. Their dimension

[10] The fundamental null root $\widetilde{\theta}$ is traditionally written as δ [62]. We however reserve the symbol δ to denote one of the coefficients of the anomaly polynomial, see Equation (2.27), as it is the standard convention in the literature of six-dimensional SQFTs.

node	S^A	type	flavor
$\color{red}{\bullet}$	$+1$	$\mathcal{R}$	$\mathfrak{so}_{f^A}$
$\color{blue}{\bullet}$	-1	$\mathcal{P}$	$\mathfrak{sp}_{f^A/2}$
$\circ$	0	$\mathcal{C}$	$\mathfrak{su}_{f^A}$
$\bullet$	0	$\overline{\mathcal{C}}$	$\mathfrak{su}_{f^A}$

Table 2: Relations between the representations of ADE finite subgroups $\Gamma \subset SU(2)$ and quiver data. S^A denotes the Frobenius–Schur indicator of the representation $\mathcal{R}_A$.

is then given by the value of the corresponding (co)mark of $\widetilde{\mathfrak{g}}$:

$$\dim(\mathcal{R}_A) = \theta_A \,. \tag{2.31}$$

The McKay correspondence therefore allows us to relates the dimensions of the irreducible representations $\mathcal{R}_A$ to quantities related to $\mathfrak{g}$. For instance, it is well known—and can easily be checked by inspection—that ADE algebras satisfy the following properties:

$$\text{ADE algebras:} \qquad \sum_{A=0}^{r_{\mathfrak{g}}} \theta_A = h^\vee \,, \qquad \sum_{A=0}^{r_{\mathfrak{g}}} (\theta_A)^2 = \Gamma \,, \tag{2.32}$$

where by abuse of notation Γ denotes both the finite subgroup of $SU(2)$ and its order.

In the context of $Spin(32)/\mathbb{Z}_2$ Little String Theories, the irreducible representations of Γ have an impact on the relevant branching rules of $Spin(32)/\mathbb{Z}_2$ classifying them [36]. We will therefore sometimes decompose the indices $A = 0, 1, \ldots, r_{\mathfrak{g}}$ depending on the type of the representation. For all real representations $\mathcal{R}_A$, we collectively denote the set of indices labelling the corresponding nodes by $\mathcal{R}$; for pseudo-real representations, they are denoted by $\mathcal{P}$. We also separate complex representations into sets $\mathcal{C}$ and $\overline{\mathcal{C}}$ so that we may distinguish conjugate representations: if $A \in \mathcal{C}$ and $\overline{\mathcal{R}}_A = \mathcal{R}_B$, then $B \in \overline{\mathcal{C}}$. This is summarized in Table 2.

We can then arrange the group-theoretical data of Γ that will be relevant for the study of LSTs into a colored Dynkin diagram, with the color differentiating between the type of representations. They are summarized, along with other useful group-theoretical quantities, in Table 3.

The Frobenius–Schur indicator: when discussing anomalies, we have defined the quantity S^I, which is a constant depending on the associated flavour symmetry of the quiver. In the context of the McKay correspondence, it simply corresponds to the Frobenius–Schur finite representation $\mathcal{R}_A$, whose values depends on whether it is (pseudo)-real or complex:

$$S^A = \begin{cases} 1\,, & \text{if} \quad A \in \mathcal{R}\,; \\ -1\,, & \text{if} \quad A \in \mathcal{P}\,; \\ 0\,, & \text{if} \quad A \in \mathcal{C} \cup \overline{\mathcal{C}}\,. \end{cases} \tag{2.33}$$

This reproduces the definition given in Equation (2.12). It is further possible to show that for any ADE algebra excluding $\mathfrak{su}_{2k+1}$, it satisfies the relation

$$\theta_A\, S^A = 2\,, \qquad \mathfrak{g} \in \{\mathfrak{su}_{2k}, \mathfrak{so}_k, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}\,. \tag{2.34}$$

These algebras are precisely those appearing in the $Spin(32)/\mathbb{Z}_2$ LSTs we are focusing on, and will give us a universal way of rewriting Equation (2.20). For $\mathfrak{g} = \mathfrak{su}_{2k+1}$ we instead have $\theta_A\, S^A = 1$, once again illustrating the special case of LSTs with that type of singularity.

Embeddings of the finite group: with the conventions we have introduced above, the classification of the possible embeddings of Γ into $Spin(32)/\mathbb{Z}_2$ then correspond to a decomposition of the fundamental representation of $Spin(32)/\mathbb{Z}_2$ into representations of Γ [36]:

$$\mathcal{R} = \bigoplus_A \mu^A\, \mathcal{R}_A\,, \qquad \mu^A\, \theta_A = 32\,. \tag{2.35}$$

Taking into account the reality condition of $Spin(32)/\mathbb{Z}_2$, we must have $\mu^I = \mu^J$ if $\overline{\mathcal{R}}_I = \mathcal{R}_J$. In addition, pseudo-real representation must be even, and are rotated by a symplectic transformation, forcing the decomposition of $Spin(32)/\mathbb{Z}_2$ to be given by:

$$\begin{aligned} Spin(32)/\mathbb{Z}_2 \longrightarrow \quad & \Gamma \times \prod_{A\in\mathcal{R}} SO(\mu^A) \times \prod_{A\in\mathcal{P}} SP(\frac{\mu^A}{2}) \times \prod_{A\in\mathcal{C}} U(\mu^A)\,. \\ \mathbf{32} \longrightarrow \quad & \bigoplus_A \theta_A\, \mu^A\,. \end{aligned} \tag{2.36}$$

The McKay correspondence therefore gives an elegant way to find the decomposition of $Spin(32)/\mathbb{Z}_2$ into subgroups that are relevant to the study of $Spin(32)/\mathbb{Z}_2$ LSTs.

Recovering the quiver: the integers μ^I can therefore be understood as forming a weighted Dynkin diagram of $\widetilde{\mathfrak{g}}$. The mapping between the coefficients μ^A and a $Spin(32)/\mathbb{Z}_2$ LSTs is then straightforward: through Equation (2.36), we see that they define the dimension of the fundamental representation of a flavor symmetry $\mathfrak{f}_I$ on each node. This flavor symmetry rotates hypermultiplets transforming in the fundamental representation of a gauge symmetry $\mathfrak{g}_I$.

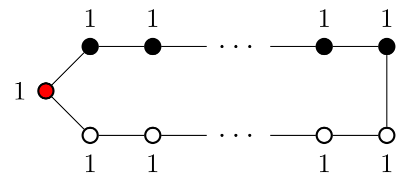
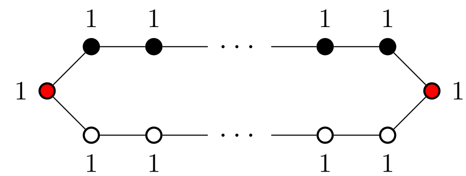
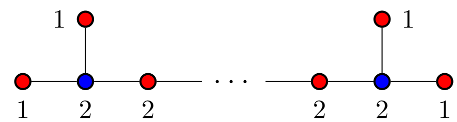
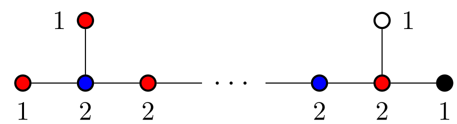
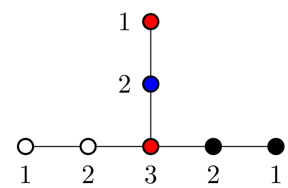
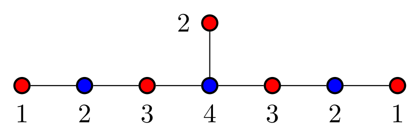
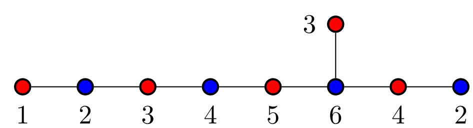
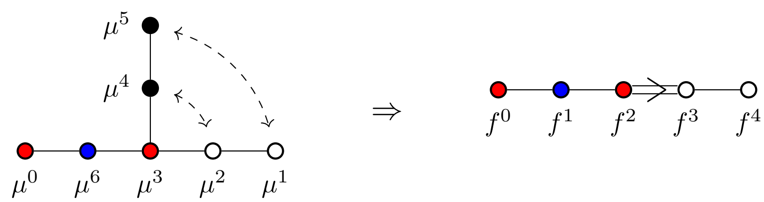
$\mathfrak{g}$	$\mathrm{rk}_{\mathfrak{g}}$	$\dim(\mathfrak{g})$	$h^\vee_{\mathfrak{g}}$	Γ	$Z(\mathfrak{g})$	Affine Dynkin diagram
$\mathfrak{su}_{K=2k+1}$	$K-1$	K^2-1	K	K	$\mathbb{Z}_K$	
$\mathfrak{su}_{K=2k}$	$K-1$	K^2-1	K	K	$\mathbb{Z}_K$	
$\mathfrak{so}_{2K=4k}$	$2k$	$\frac{K^2-K}{2}$	$2K-2$	$4K-8$	$\mathbb{Z}_2\times\mathbb{Z}_2$	
$\mathfrak{so}_{2K=4k+2}$	$2k+1$	$\frac{K^2-K}{2}$	$2K-2$	$4K-8$	$\mathbb{Z}_4$	
$\mathfrak{e}_6$	6	78	12	24	$\mathbb{Z}_3$	
$\mathfrak{e}_7$	7	133	18	48	$\mathbb{Z}_2$	
$\mathfrak{e}_8$	8	248	30	120	$\varnothing$	

Table 3: Relevant quantities of simply-laced Lie algebras. We abuse the notation and use Γ to denote the order of the finite subgroup of $SU(2)$ McKay dual to $\mathfrak{g}$. The labels of the affine Dynkin diagram refers to marks θ_A. The color of a node denotes the type of the representation of the discrete group: ● $\leftrightarrow \mathcal{R}$; ● $\leftrightarrow \mathcal{P}$; ○ $\leftrightarrow \mathcal{C}$; ● $\leftrightarrow \overline{\mathcal{C}}$.

The type of gauge symmetry can be read off the Dynkin diagram as well: if a flavor symmetry is real, i.e. of type $\mathfrak{so}_f$, it rotates gauge hypermultiplets in the fundamental representation of $\mathfrak{sp}_k$, and vice versa.

A subtlety however arises when the discrete group Γ has complex representations: since $\mathfrak{so}_{32}$ only has real representations, we must identify complex representation with their conjugate [37]. We then obtain a full hypermultiplet rotated by a $\mathfrak{su}_k$ symmetry. At the level of the affine Dynkin diagram $\widetilde{\mathfrak{g}}$, this corresponds to a *folding* of the diagram to obtain that of the base $\widetilde{\mathfrak{b}}$.

This explains why, while in the string theory construction we have five-branes probing a $\mathbb{C}^2/\Gamma$ singularity, the quiver does not always have the topology of the Dynkin diagram of $\widetilde{\mathfrak{g}}$. For instance, in the case of $\widetilde{\mathfrak{g}} = E_6^{(1)}$, we obtain the affine algebra $\widetilde{\mathfrak{b}} = F_4^{(1)}$:



(2.37)

Note that we have arranged the nodes of the $E_6^{(1)}$ Dynkin diagram differently than in Table 3 to highlight their relation to $F_4^{(1)}$. This reproduces correctly the adjacency of hypermultiplets for the LST discussed around Equation (2.10). Indeed, taking into account this folding the coefficients μ^A in Equation (2.36) precisely give the flavor fundamental f^I of the $Spin(32)/\mathbb{Z}_2$ LST, with $I = 0, 1 \ldots, r_{\mathfrak{b}}$. We see that we recover the correct spectrum from the coefficients μ^I, and the symmetrized Cartan matrix G^{IJ} of $F_4^{(1)}$ correctly reproduce the complete adjacency of the hypermultiplets spectrum: four $\mathfrak{g}_I \oplus \mathfrak{g}_J$ arranged linearly, and five $\mathfrak{g}_I \oplus \mathfrak{f}_J$. This means that given μ^A, we can directly write down the generalized quiver in the curve notation described at the beginning of this section:

$$\begin{array}{ccccc} \mathfrak{sp}_{k_1} & \mathfrak{so}_{k_2} & \mathfrak{sp}_{k_3} & \mathfrak{su}_{k_4} & \mathfrak{su}_{k_5} \\ 1 & 4 & 1 & 2 & 2 \\ [\mathfrak{so}_{f1}] & [\mathfrak{sp}_{\frac{f2}{2}}] & [\mathfrak{so}_{f3}] & [\mathfrak{su}_{f4}] & [\mathfrak{su}_{f5}] \end{array}, \tag{2.38}$$

the gauge algebras are then fixed up to an integer Q via anomaly cancellation, see Equation (2.17). Furthermore, we see that the bifundamental $\mathfrak{sp}_{k_3} \oplus \mathfrak{su}_{k_4}$ hypermultiplet corresponds to the double link of $F_4^{(1)}$, with the arrow pointing in the direction of the node hosting the complex gauge algebra $\mathfrak{su}_{k_4}$.

More generally, if we have a folding $\mathcal{F}$ mapping the indices associated with $\widetilde{\mathfrak{g}}$ to that of $\widetilde{\mathfrak{b}}$, we can define the flavor and a folded version of the Frobenius–Schur indicator as:

$$f^I = \mu^{\mathcal{F}(A)}\,, \qquad S^I = S^{\mathcal{F}(A)}\,. \tag{2.39}$$

Under that identification the relation involving θ_A and the group-theoretical quantities in Equation (2.32) holds, as the comarks of nodes associated with complex representations are doubled after folding: If $A \in \mathcal{C}$, then $\theta_{\mathcal{F}(A)} = 2\,\theta_A$. Moreover, as $S_A = 0$, the relation $\theta_I S^I = 2$ remains correct for the folded algebra. Similarly, if $\widetilde{\mathfrak{b}}$ is the algebra coming from folding $\widetilde{\mathfrak{g}}$, one can show that the relations in Equation (2.32) holds

$$\widetilde{\mathfrak{b}} = \mathcal{F}(\widetilde{\mathfrak{g}}) : \qquad \sum_{I=0}^{r_{\mathfrak{b}}} \theta_I^{\mathfrak{b}} = h_{\mathfrak{g}}^\vee\,, \qquad \sum_{I=0}^{r_{\mathfrak{b}}} (\theta_I^{\mathfrak{b}})^2 = \Gamma_{\mathfrak{g}}\,. \tag{2.40}$$

which is easily shown by inspection.

Spin(32)/$\mathbb{Z}_2$ LSTs and weight systems: the various formulas above enable us to interpret the flavor symmetries in a more Lie-algebraic context. While this is certainly interesting at an aesthetic level, it will more importantly also allow us to use results that are standard in that context in order to easily prove certain properties of LSTs.

Let us first recall that the root system of a *finite* Lie algebra $\mathfrak{g}$, has a root lattice $\mathcal{Q}$ generated by the simple roots α^a, and the coweight lattice $\mathcal{P}^\vee$ by the fundamental coweights $\omega_a^\vee$. These two lattices are dual, and we therefore have:

$$\langle \alpha^a, \omega_b^\vee \rangle = \delta_b^a = \langle \widetilde{\omega}_a, \alpha^{b\,\vee} \rangle\,, \tag{2.41}$$

where $\langle \cdot, \cdot \rangle$ the natural pairing of the root system. A coweight $\mu^\vee = \mu^a \omega_a^\vee$ is called dominant, collectively denoted $\mathcal{P}_+^\vee$, if all its coefficients μ^a are non-negative integers.

Given a finite algebra $\mathfrak{g}$, one can extend its root system to that of the affine algebra $\widetilde{\mathfrak{g}}$. We then define a set of affine roots α^A and coweights $\widetilde{\omega}_A$ satisfying

$$\langle \alpha^A, \widetilde{\omega}_B^\vee \rangle = \delta_B^A = \langle \widetilde{\omega}_A, \alpha^{B\,\vee} \rangle\,, \tag{2.42}$$

where the pairing is extended from that of the finite algebra. We have also defined the coroot $\alpha^{A\,\vee}$ and weights $\widetilde{\omega}_A$ for future reference. We review the construction of simple and affine root systems in Appendix C. For our present purpose, it will be sufficient to simply recall that a dominant affine coweight $\widetilde{\mu}^A \in \widetilde{\mathcal{P}}_+^\vee$ is then defined as being of the form

$$\widetilde{\mu}^\vee = \mu^A\,\widetilde{\omega}_A^\vee + n\,K \,\in\, \widetilde{\mathcal{P}}_+^\vee\,, \qquad \mu^A, n \geq 0\,, \tag{2.43}$$

where $K = K_A \alpha^{A\vee}$ is the so-called central element, whose coefficients K_A are the comarks of $\widetilde{\mathfrak{g}}$. The level of an affine coweight is defined as $\mathrm{lv}(\widetilde{\mu}^\vee) = \theta_A \mu^A$.

It can often be convenient to think of an *affine* coweight $\widetilde{\mu}^\vee$ of $\widetilde{\mathfrak{g}}$ as a *finite* coweight $\mu^\vee = \mu^a \omega_a^\vee$ of $\mathfrak{g}$ supplemented by a level and an integer n, encoded in the triplet:

$$\widetilde{\mu}^\vee = (\mu^\vee, \mathrm{lv}(\widetilde{\mu}^\vee), n)\,, \qquad \mathrm{lv}(\widetilde{\mu}^\vee) = \theta_A \mu^A\,. \tag{2.44}$$

In this language, all the constraints we have reviewed take a Lie-algebraic meaning. Indeed, we have already seen that the type of flavor and gauge structure depends on the choice of $\mathfrak{g}$ through the Frobenius–Schur indicator, and that the dimension of their fundamental representation is constrained by gauge-anomaly cancellation, see Equation (2.16), from which we have seen that $\theta_I f^I = 16\,\theta_I S^I = 32$. This is precisely the relation that defines the embedding of Γ into $Spin(32)/\mathbb{Z}_2$, see Equations (2.20) and (2.36), and the flavor symmetry is therefore encoded in an affine coweight of level 32.

Other properties can be rephrased in those terms. For instance, the matrix used in Equation (2.17) to solve the gauge-anomaly cancellation condition is given as the pairing $\langle \widetilde{\omega}_I, \widetilde{\omega}_J^\vee \rangle = X_{IJ}$ in the weight system of the base algebra, or equivalently for $\widetilde{\mathfrak{g}}$. In turn, we see that the dimension of the fundamental representation on the affine node $d_0 = 2Q$ is nothing but the coefficient n in the equation above.

In summary, a choice of an ADE algebra and a dominant coweight completely fixes the data defining a $Spin(32)/\mathbb{Z}_2$ LST.[11] This is precisely the classification by Blum and Intriligator.

Classification 2 ($Spin(32)/\mathbb{Z}_2$ Little String Theories [36])**.** *Given a simply-laced algebra $\mathfrak{g} \neq \mathfrak{su}_{2n+1}$, every LST $\mathcal{K}_Q(\mu^\vee; \mathfrak{g})$ is realized by a generalized quiver where the adjacency of the hypermultiplets is given by the symmetrized Cartan matrix G^{IJ} of the associated folded affine algebra $\widetilde{\mathfrak{b}}$, see Table 1. The type of gauge and flavor symmetries are obtained from the Frobenius–Schur indicator S_A, see Tables 2 and 3.*

The rest of the spectrum is then determined by a dominant coweight $\widetilde{\mu}^\vee \in \widetilde{\mathcal{P}}_+^\vee$ of $\widetilde{\mathfrak{g}}$ at level 32:

$$\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q) = \mu^A\,\widetilde{\omega}_A^\vee + 2Q\,K\,, \qquad \theta_A \mu^A = 32\,. \tag{2.45}$$

Its coefficients μ^A correspond to the dimension of the fundamental representations of the flavor symmetries, $f^I = \mathcal{F}(\mu^A)$.

[11] While this data completely fixes the data at a generic point of the tensor branch of the theory, in a few cases the spectrum at the singular point might require a choice of θ-angle for the $\mathfrak{sp}_k$ gauge algebras in order to distinguish between two different types of spectra. We will not consider this subtlety here, see reference [75].

This subset of dominant coweights, denoted $\widetilde{\mathcal{P}}^\vee_{LST}$, must furthermore satisfy the following constraints:

1. *Defining $\widetilde{S}^\vee = S^A \widetilde{\omega}^\vee_A$ from the Frobenius–Schur indicator the weight $\widetilde{\mu}^\vee - 16\widetilde{S}^\vee$ lies is a positive coroot, with components:*

$$d_A = \langle \widetilde{\omega}_A, \widetilde{\mu}^\vee - 16\widetilde{S}^\vee \rangle = 2Q\, K_A + X_{AB}(\mu^B - 16S^B) \in \mathbb{N}\,, \tag{2.46}$$

   and correspond to the dimension of the fundamental representation of the gauge algebras in the folded base algebra $d_I = \mathcal{F}(d_A)$.

2. *So that algebras of type $\mathfrak{sp}_k$ have an even fundamental representation, if $S^A = -1$ then μ^A is even, and if $S^A = 1$, then d_A is even.*
3. *Complex representations of the McKay-dual finite group Γ and their conjugates have the same coefficients: if $\mathcal{R}_A = \overline{\mathcal{R}}_B$, then $\mu^A = \mu^B$.*

One can equivalently work directly in the affine root system of the base $\widetilde{\mathfrak{b}}$, rendering the last constraint trivial, and with the Frobenius–Schur indicator extended through Equation (2.39). It is in practice more convenient, and it has several advantages when studying LSTs computationally—in particularly, generating all dominant coweights associated with an LST can be considerable faster in that case. We therefore see that a coweight of $\widetilde{\mathfrak{b}}$ completely defines an LST without needing any additional external data.

Similarly, the triplet $\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q)$ also has advantages, as one can work with the simpler finite coweight $\mu^\vee$. The flavor on the affine node is then simply $\mu^0 = 32 - \theta_a\mu^a$.

T-duality invariants and coweights: In the language above, we can rewrite the two-group constant κ_R, see Equation (2.26), in a particularly simple form:

$$\kappa_R = \frac{1}{2}\langle \widetilde{T}, \widetilde{\mu}^\vee - 16S \rangle = \Gamma_{\mathfrak{g}}(Q - r_{\mathfrak{g}} - 1) + 2 + \frac{1}{2}\mu^a C_{ab}\theta^a\,, \tag{2.47}$$

where $\widetilde{T} = \sum_A \theta_A \widetilde{\omega}_A$ is the affine weight whose coefficients are the marks—not to be confused with the null root $\widetilde{\theta} = \theta_A \alpha^A$. The pairing is taken equivalently over either $\widetilde{\mathfrak{g}}$ or $\widetilde{\mathfrak{b}}$, and we recall that lowercase indices a, b are taken over only the finite part of the algebra—where the Cartan matrix is invertible, with C_{ab} its inverse. The properties of the involved (co)weights and (co)roots are collected in Appendix C.

The Coulomb branch dimension of the five-dimensional theory can be similarly expressed in closed form. Due to the fact that the rank of $\mathfrak{so}_d$ is $\lfloor \frac{d}{2} \rfloor$, the resulting expression is a bit more involved:

$$
\begin{aligned}
\dim(\mathrm{CB}) &= 2r_{\mathfrak{b}} - r_{\mathfrak{g}} + \frac{1}{2}\langle\widetilde{\rho},\widetilde{\mu}^{\vee}\rangle + \sum_{A\in\mathcal{P}} \mathrm{frac}(\frac{d_I}{2}) \\
&= Qh^{\vee}_{\mathfrak{g}} - \dim(\mathfrak{g}) + \frac{1}{2}\rho^a C_{ab}\mu^a + \sum_{a\in\mathcal{P}} \mathrm{frac}(\frac{C_{ab}\mu^a}{2})\,,
\end{aligned}
\tag{2.48}
$$

where $\widetilde{\rho}$ is the weight with component $\rho^I = D^{II}$, and we have used that the number of $\mathfrak{su}_n$ gauge algebra on the six-dimensional quiver is always $r_{\mathfrak{g}} - r_{\mathfrak{b}}$.[12]

Finally, the total flavor rank also admit a simple closed-form. There, however, the instanton number Q of course cannot appear explicitly, and we need to restrict the coweight in Equation (2.45) to omit the central element K. For brevity, we only give the expression in components:[13]

$$
\mathrm{rk}(f) = \frac{1}{2}\rho_A\,\mu^A + \sum_{A\in\mathcal{R}} \mathrm{frac}(\frac{\mu^A}{2})\,. \tag{2.49}
$$

where we have used—as indicated around Equation (2.13)—that a $\mathfrak{su}_f$ flavor symmetry is always accompanied by a unbroken Abelian factor $\mathfrak{u}_1$. These closed-form expressions are very important as they make the search for T-dual theories much simpler. In addition, we can now use standard results in the theory of affine root systems to greatly simplify certain computations.

The κ-theorem: in two- and four-dimensional Quantum Field Theory, the central charges a and c are known to decrease monotonically along RG flows [76, 77], and are therefore interpreted as a non-perturbative counting of physical degrees of freedom. For six-dimensional QFTs, a general a-theorem remains elusive [78–83]. However, through numerical scans and due to the deep relationship between group theory and these SQFTs, it has been established for large classes of theories using properties of the underlying root systems [43, 66, 84].

On the other hand, in the case of LSTs the absence of a local energy-momentum tensor at the singular point of the tensor branch do not allow for such theorems. However, it has been conjectured that κ_R should give rise to a similar version of at least the weak a-theorem [27]: given an RG flow between two LSTs $\mathcal{K}^{\mathrm{UV}}$ and $\mathcal{K}^{\mathrm{IR}}$, we must have

$$
\kappa_R(\mathcal{K}^{\mathrm{UV}}) > \kappa_R(\mathcal{K}^{\mathrm{IR}})\,. \tag{2.50}
$$

[12] For ADE algebras, the finite part of $\widetilde{\rho}$ is the Weyl vector, with components $\rho^a = 1$. When working at the level of the folded algebras on the other hand, we stress that $\rho^I = D^{II}$. This distinction is particularly important when using the closed-form formulas, as we must insure that folded nodes are taken into account correctly.

[13] Type-A LSTs have an additional "baryonic" $\mathfrak{u}_1$ symmetry rotating all hypermultiplets simultaneously, which we did not include in Equation (2.49). However, every statement we will make about matching of the flavor symmetry rank will be correct when it is included, see e.g. references [31, 35].

This "κ-theorem" was proven in reference [35] for any LST admitting a description in terms of a generalized quiver. Due to their relation to coweights we can give an equivalent, but simpler, proof for any $\mathfrak{so}_{32}$ LST. Indeed, anticipating a more complete in the next section, dominant coweights admit a partial order. One says that $\widetilde{\nu}^\vee < \widetilde{\mu}^\vee$ if $\widetilde{\mu}^\vee - \widetilde{\nu}^\vee$ lies in the cone of positive coroots. If $\mathcal{K}^{\mathrm{UV}}$ and $\mathcal{K}^{\mathrm{IR}}$ are associated with $\widetilde{\mu}^\vee$ and $\widetilde{\nu}^\vee$, respectively, a necessary condition for an RG flow to exist is then that $\widetilde{\nu}^\vee < \widetilde{\mu}^\vee$.

While the original proof describes how κ_R decreases after performing a Higgs mechanism, the theory of coweights shows this result immediately for $\mathfrak{so}_{32}$ heterotic LSTs: it is a standard result that, given any dominant weight w, the pairing $\langle w, \cdot\rangle$ is a monotonic function with respect with the partial order [85], and we indeed recover

$$\kappa_R(\widetilde{\mu}^\vee) - \kappa_R(\widetilde{\nu}^\vee) = \frac{1}{2}\langle \widetilde{T}, \widetilde{\mu}^\vee - \widetilde{\nu}^\vee\rangle > 0\,. \tag{2.51}$$

While this proof of the κ-theorem is of course completely equivalent to that of reference [35], it does not explicitly require prior knowledge of the quiver of F-theory description. This illustrates how the language of dominant coweights can greatly simplify certain computations, as we will now see for T-dual theories.

3 T-duality of $Spin(32)/\mathbb{Z}_2$ Little String Theories

Little String Theories have the important property that they can exhibit T-duality. To avoid confusion, we now spell out carefully what we mean by such a feature, which follows from the what is commonly thought of as T-duality in ten dimensions.

Definition 1 (T-duality of LSTs)**.** *Two Little String Theories are T-dual if, upon a circle reduction, there exists a common subspace in their five-dimensional extended Coulomb-branch moduli space along which the two theories become identical.*

Note that two T-dual theories are generically not identical upon a simple circle reduction. However, due to symmetry-valued Wilson lines along the circle, the lower-dimensional moduli space is much larger than its higher-dimensional analogue. For specific values of Wilson lines—and possibly the tensor-branch moduli giving rise to part of the five-dimensional Coulomb-branch moduli space—there exists a subspace where both theories are exactly identical. A direct consequence of this requirement is that the Coulomb-branch dimension in five dimensions—corresponding to the sum of rank of the full gauge symmetry with the number of dynamical tensor multiplets in six dimension—must be the same for the two LSTs.

For LSTs of heterotic type, we will differentiate between three different classes of theories exhibiting T-duality. Namely, we will refer to a pair of T-dual LSTs as:

1. **internal (T-)dual**, when they are both heterotic LSTs of type $\mathfrak{so}_{32}$, *or* both of type $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ for a given ADE algebra $\mathfrak{g}$;
2. **fiber-base (T-)dual**, when one is of type $\mathfrak{so}_{32}$ while the other is of type $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ for a given ADE algebra $\mathfrak{g}$;
3. **exotic (T)-dual** if they relate LSTs associated with different ADE algebras $\mathfrak{g}$.

Fiber-base duality is the standard T-duality between the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ and $\mathfrak{so}_{32}$ heterotic string theories [26], its name deriving in the geometric picture from the exchange of the algebra associated with the base of the elliptic fibration with that of the elliptic fibers. The term also applies to those of Type II. There, LSTs are uniquely defined by a choice of base algebra $\mathfrak{g}_B$ and a fiber algebra $\mathfrak{g}_F$, and fiber-base duality simply exchanges them: $(\mathfrak{g}_B, \mathfrak{g}_F) \leftrightarrow (\mathfrak{g}_F, \mathfrak{g}_B)$ [24,33].

Furthermore, in the presence of discrete symmetries, a twist along the circle can be performed, opening up the possibility of new T-dual pairs [30, 34]. While we will not discuss twisted compactification in this work, these would fall into the class of internal duals.

A contrario, pairs of exotic type often arise when an LST can be thought of as originating through a Higgs-branch flow from theories associated with different algebras. This therefore especially occurs when the number of curves is low, see e.g. references [28, 31, 34]. In this work, we will consider the instanton number to be large so that there are no base-changing deformations, and will generically not consider those cases. In Section 5, we will however discuss infinite families which exhibit exotic duality [28,31] which can partly be explained by a group-theoretic property relating certain invariants of LSTs with different quiver topology.

Of course, more than two LSTs can be dual to one another, and there are many examples of such "T-n-ality" [9, 24, 27, 28, 30, 30–35]. We will refer to such groupings as **(T-)duality orbits**. These can contain a quite large number of theories. For instance, in reference [28] it was shown through a specific string-theoretic realization that a certain geometry led to sixteen different, but T-dual, Little String Theories.

3.1 Structure of Duality Orbits

Identifying LSTs falling in the same duality orbit and proving T-duality can be an arduous task, and generally requires either a brane construction or geometric engineering in the F-theory picture. Certain quantities that are invariant under T-duality have however been identified and can be used to facilitate the search for T-dual pairs [27, 28, 31, 35]:

$$\kappa_P \,, \quad \kappa_R \,, \quad \dim(\mathrm{CB}) \,, \quad \dim(\mathfrak{f}) \,. \tag{3.1}$$

When convenient, and for ease of notation, we will often denote dim(CB) simply as CB.

In reference [33], it was further shown that higher-form symmetries are preserved under T-duality, in the sense that six-dimensional one-form symmetries and the defect group give rise to one-form symmetries in five dimension, and must therefore match. However, in the case of heterotic theories the defect group is always trivial, and this will not be relevant for our purpose.

On the $\mathfrak{so}_{32}$ side, the T-dual invariants are of course not independent, as they are ultimately related to the coweight defining the theory through the closed-form formulas derived in the previous section. On the other hand, the invariants in Equation (3.1) are extremely powerful to carve out the space of heterotic LSTs into candidate duality orbits.

Consider two LSTs associated with affine coweights $\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q)$ and $\widetilde{\nu}^\vee = (\nu^\vee, 32, 2Q')$, in the notation of Equation (2.45), with $\mu^\vee, \nu^\vee \in \mathcal{P}^\vee_{\text{LST}}(\mathfrak{g})$ two dominant finite coweights of the same algebra $\mathfrak{g}$. From the closed-form expressions given in equations (2.47) and (2.48), we see that the two invariants take the form:

$$\kappa_R(\widetilde{\mu}^\vee) = \Gamma\left(Q - r_{\mathfrak{g}} - 1\right) + 2 + \delta\kappa_R(\mu^\vee)\,, \qquad \text{CB}(\widetilde{\mu}^\vee) = h^\vee Q - \dim(\mathfrak{g}) + \delta\text{CB}(\mu^\vee)\,, \quad (3.2)$$

where $\delta\kappa_R$ and δCB depend only on the finite part $\mu^\vee$of the affine coweight $\widetilde{\mu}^\vee$. A necessary condition for the two associated LSTs to be T-dual is therefore that:[14]

$$\delta\kappa_R(\mu^\vee) - \delta\kappa_R(\nu^\vee) = \Gamma(Q - Q')\,, \qquad \delta\text{CB}(\mu^\vee) - \delta\text{CB}(\nu^\vee) = h^\vee_{\mathfrak{g}}(Q - Q')\,, \quad (3.3)$$

The two $\mathfrak{so}_{32}$ theories are therefore putatively in the same T-duality orbit if their part of the invariants controlled by the finite part of the coweight coincide up to a particular multiple of $Q - Q'$.

In this section, we will therefore restrict ourselves to dominant coweights of the simple algebra $\mathfrak{g}$ and find those who lead to LSTs in the same orbit up to this multiple, rather than affine coweights. We recall that $\mathcal{P}^\vee_+$ is the set of dominant finite coweight, while $\mathcal{P}^\vee_{\text{LST}} \subset \mathcal{P}^\vee_+$ denotes those associated with an LST, as defined at the end of Section 2.2.

We can significantly simplify our work by using additional properties of the root system. In particular, it was shown by Stembridge that dominant (co)weights admit a natural partial order [85]. Indeed, two dominant coweights $\mu^\vee, \nu^\vee \in \mathcal{P}^\vee_+$ are partially ordered if their difference is a positive sum of coroots:

$$\nu^\vee \leq \mu^\vee\,, \quad \Leftrightarrow \quad \mu^\vee - \nu^\vee = m_a \alpha^{a\,\vee}\,, \quad m_a \in \mathbb{N}\,, \quad (3.4)$$

[14]We will not consider $\kappa_P = 2$, as it always matches for any heterotic LST and assuming that there are no additional discrete twists along the Kaluza–Klein circle.

with the equality satisfied only if $\mu^\vee = \nu^\vee$. This definition extends to affine coweights $\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q)$ and $\widetilde{\nu}^\vee = (\nu^\vee, 32, 2Q')$ if in addition $Q' \leq Q$.

Physically, we have seen in Section 2.2 that $\widetilde{\mu}^\vee - 16\widetilde{S}^\vee$ must be a positive coroot for the gauge algebras to be well defined. The partial order in equation (3.4) therefore simply means that an LST is "smaller than" an other if the change in the associated different dimensions d_I of the fundamental representation of a gauge symmetry is a positive integers. As we will see in the next section, this is related to the fact that there can be a Higgs-branch Renormalization Group flow between the two.

The set of dominant coweights endowed with the partial order above is seen as the poset $(\mathcal{P}_+^\vee, <)$, and has certain attractive properties [85] with important physical consequences. First, it is disconnected, with the number of components given by the order of the center $Z(\mathfrak{g}) = \mathcal{Q}^\vee/\mathcal{P}^\vee$.[15] However, an LST with $\mu^\vee \in \mathcal{P}^\vee_{\text{LST}}$ must then be in the component of the poset connected to the trivial coweight: for any LST we must have $0 < \mu^\vee$, as all d_I are positive integers.

Lets now us assume that we have found all possible coweights in a putative orbit, defined as the set of coweights leading to the same T-dual invariants up to a shift in Q, i.e. satisfying the condition in equation (3.3), and the same flavor rank. The smallest coweight $\mu^\vee$ of the putative orbit under the partial order defined in equation (3.4) defines a natural representative of that orbit, as all the other coweights are by definition obtained by adding a coroot mimicking the shift in the instanton number matching of the T-dual invariants. We denote this orbit by $\mathcal{O}_{\mu^\vee}$, and say that it is *seeded* by $\mu^\vee$.

Therefore, if $\nu^\vee \in \mathcal{O}_{\mu^\vee}$, then $\lambda^\vee = \mu^\vee - \nu^\vee$ must be a positive coroot:

$$\nu^\vee \in \mathcal{O}_{\mu^\vee}\,, \qquad \nu^\vee = \mu^\vee + \lambda^\vee\,, \quad \lambda^\vee \in \mathcal{Q}_+^\vee\,. \tag{3.5}$$

This shows that the T-duality orbits are not mere subsets of $\mathcal{P}^\vee_{\text{LST}}$, but they have an interesting structure: the set of admissible $\lambda^\vee$ is constrained by the matching of the T-dual invariants, see Equation (3.3).

Let us first consider κ_R; from Equation (2.26), we must have

$$\kappa_R: \qquad \frac{1}{2}\langle T, \lambda^\vee\rangle = \frac{1}{2}\theta^a\, C_{ab}\, \lambda^b \equiv 0 \mod \Gamma\,, \tag{3.6}$$

where we recall that $T = \sum_a \theta^a \omega_a$, and C_{ab} is the inverse Cartan matrix of $\mathfrak{g}$. This congruence equation defines an integer cone $\mathcal{C}_\Gamma \subset \mathcal{Q}_+^\vee$ to which $\lambda^\vee$ must belong. Its generators are easily

[15] The center Z is that of the simply-connected Lie group associated with $\mathfrak{g}$. As we work with an ADE algebra, we write $Z(\mathfrak{g}) = \mathcal{P}^\vee/\mathcal{Q}^\vee$ to emphasize that it can be obtained via the root system of $\mathfrak{g}$.

obtained by e.g. computing the Hilbert basis through standard linear-optimization techniques for any class of LSTs. Since $\langle\theta, P^\vee_{\rm LST}\rangle \leq 32$, see Classification 2, the elements of $\mathcal{O}_{\mu^\vee}$ form a truncated cone pointed at $\mu^\vee$.

We note that the ordering of dominant coweight is not total: there are elements $\mu^\vee, \nu^\vee \in \mathcal{P}^\vee_+$ satisfying neither $\nu^\vee < \mu^\vee$ nor $\mu^\vee < \nu^\vee$. This means that given a putative T-duality orbit, there can be multiple coweights $\mu_i^\vee$ for which there are no $\nu^\vee < \mu_i^\vee$. In other words, the orbit is seeded by more than one coweight, and given by $\cup_i \mathcal{O}_{\mu_i^\vee}$. This phenomenon is fairly common among all possible $\mathfrak{g}$; however, we have checked explicitly by construction that the majority of putative T-dual orbits are seeded by a single coweight $\mu^\vee$. As an example, since $0 < \nu^\vee \in \mathcal{P}^\vee_{\rm LST}$, the orbit containing the theory with an $\mathfrak{so}_{32}$ flavor symmetry is only seeded by the trivial coweight.

The constraint on $\lambda^\vee$ due to κ_R must be supplemented by also matching of CB and rk($\mathfrak{f}$). As we need to consider the rank $\lfloor k/2 \rfloor$ of $\mathfrak{so}_k$ we need to separate cases with odd and even of such flavor and gauge symmetries, which for the Coulomb branch is given in components by:

$$\text{CB}: \qquad \frac{1}{2}\rho^a C_{ab}(\nu^a - \mu^a) + \sum_{\alpha\in\mathcal{P}} \left(\text{frac}(\frac{C_{\alpha a}\nu^a}{2}) - \text{frac}(\frac{C_{\alpha a}\mu^a}{2}) \right) \equiv 0 \mod h^\vee \,, \tag{3.7}$$

and similarly for the flavor rank. The appearance of floor functions therefore adds a layer of computational subtlety, but does not substantially changes the implementation of those constraints: instead of the matching of the flavour rank and CB cutting out a unique integer subcone of $\mathcal{C}_\Gamma$, it may rather result into a union of such subcones. Generically, we therefore find the set of dominant coweights $\mathcal{P}^\vee_{\rm LST}$ associated with a $Spin(32)/\mathbb{Z}_2$ LST has the following properties:

Classification 3 (Structure of putative T-duality orbits)**.** *Given a simple Lie algebra $\mathfrak{g}$ and a fixed instanton number Q, the set of dominant finite coweights $\mathcal{P}^\vee_{LST}$ associated with a $Spin(32)/\mathbb{Z}_2$ LST is partitioned into disjoint unions of sets of coweights distinguished by their T-dual invariants κ_R, CB, rk($\mathfrak{f}$). Each orbit is seeded by (possibly multiple) minimal coweight(s) $\mu_i^\vee$, and we have*

$$\mathcal{P}^\vee_{LST}(\mathfrak{g}) = \bigcup_{\mu_i^\vee} \mathcal{O}_{\mu_i^\vee} \,. \tag{3.8}$$

Up to a shift in the instanton number Q, elements of $\mathcal{O}_{\mu_i^\vee}$ satisfy all the known necessary conditions to be internal T-duals. The orbits themselves are unions of truncated integer cones translated to $\mu_i^\vee$ which are obtained by solving Equations (3.6) *and* (3.7)*, and matching the flavor rank.*

We stress that the matching of the T-dual invariants is only a necessary condition rather than a *sufficient* condition, and the properties of the coweight lattice cannot, as is, ensure that putative pairs are in fact T-dual without an explicit geometric construction. Heterotic LSTs are particularly amenable to toric constructions and large classes of models have been constructed [24,27,28,31,34]. Although these methods cannot accommodate a description of all LSTs at any instanton number, we are not aware of a single example where two different putative T-dual LSTs—that is two theories with the same invariants—and a geometric realization can be shown to be part of two inequivalent geometries, thereby ruling T-duality. In all known constructions, matching of the invariants will result in T-duality. This also extends to cases where one allows twists by discrete symmetries along the compactification circle; putative duals with the same invariants continue to admit a description as inequivalent fibrations of $6d$ F-theory vacua [34].

T-duality and low-rank algebras $\mathfrak{g}$: using the closed-form expressions in terms of the coweights, we observe that for $\mathfrak{g} = \mathfrak{so}_8, \mathfrak{so}_{10}, \mathfrak{e}_6$, only two T-duality invariants are sufficient to define a duality orbit. This can be shown explicitly by finding a linear relation between the invariants, which follows from the numerology of the related algebras. We have furthermore checked that one indeed only need two invariants by explicitly scanning over the set of dominant coweights. The same can easily be shown for $\mathfrak{g} = \mathfrak{su}_K$, since $\Gamma = K = h^\vee$. Furthermore, the orbifold $\mathbb{C}^2/\mathbb{Z}_K$ being obtained from an Abelian discrete group, the flavor cannot be broken and all theories have maximal flavor rank.

While this property does not extend to other ADE algebras $\mathfrak{g}$, it has interesting consequences for the geometry of F-theory. The T-duality invariants in Equation (1.2) encode certain geometric properties of the non-compact Calabi–Yau threefold on which F-theory is compactified to get the LSTs. For example, the flavor rank is reflected the number of non-compact divisors in the geometry, whereas the dim(CB) encodes compact divisors. The two-group constant κ_R encodes both information about the singularity structure as well as the intersection data of the base curves. Our observation—at least for the algebras above—suggests a deeper non-trivial relation between these quantities at the geometric level. It would be interesting to study this relation in future work, as well as explaining why it seems to fails for other singularities.

3.2 Orbits at Maximal Rank

While the complexity of the orbits grows as the total flavor rank decreases from the matching of rk($\mathfrak{f}$) and CB—particularly due to floor functions—the orbits at maximal rank are particularly simple, which we now use as an opportunity to exemplify the procedure laid out above.

In the sequel, we will work solely at the level of the algebra $\mathfrak{g}$ rather than that of the base. Similar arguments may be given in terms of the Lie theory of the folded algebras, but using simply-laced case is more straightforward as it involves a relation between the order of the

ADE discrete group Γ and the center of $Z(\mathfrak{g})$.

Consider an affine coweight $\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q) \in \widetilde{\mathcal{P}}^\vee_{\text{LST}}$. Using that its level gives the constraint $\theta_A \mu^A = 32$, from the closed-form expressions we have derived for the flavor rank in Equation (2.49), it is easy to show that the flavor symmetry of any $Spin(32)/\mathbb{Z}_2$ LST must satisfy

$$\mu^A(\theta_A - \rho_A) \leq 32 - 2\,\text{rk}(\mathfrak{f})\,. \tag{3.9}$$

This expression can also be used to simplify the scans of valid dominant coweights. The inequality is saturated only at maximal flavor rank, where we obtain

$$\text{rk}(\mathfrak{f}) = 16: \qquad \mu^A(\theta_A - \rho_A) = 0\,. \tag{3.10}$$

with $A = 0, 1, \ldots, \text{rk}(\mathfrak{g})$. Since $\rho_A = 1 \leq \theta_A$ for simply-laced algebras, this equation is extremely constraining: we can only have non-zero values of μ^A when $\theta_A = 1$. In terms of the McKay correspondence, it means we can only decompose the fundamental of $Spin(32)/\mathbb{Z}_2$ in terms of representation of Γ of dimension one.

By inspection, see Table 3, we see that such representations have two important consequences. First, the Abelianization Γ_{ab} of Γ—whose order is the number of one-dimensional representation—is given by the center: $\Gamma_{\text{ab}} = Z(\mathfrak{g}) = \mathcal{P}^\vee/\mathcal{Q}^\vee$. Moreover the associated flavor symmetries are either $\mathfrak{so}_f$ or $\mathfrak{su}_f$, which means that the corresponding μ_A are not *a priori* restricted to be even.

To solve the flavor constraints, we therefore only have to consider the indices α such that $\theta_\alpha = 1$. From Equation (2.47), in order to find the possible values of κ_R we must consider

$$\frac{1}{2}\langle T, \mu^\vee \rangle = \frac{1}{2}\theta^b\, C_{b\alpha}\, \mu^\alpha = \frac{1}{2|Z|}(\theta\, \text{Ad}(C))_\alpha\, \mu^\alpha\,, \tag{3.11}$$

where $\text{Ad}(C) = \det(C) \cdot C^{-1}$ is the adjugate of the Cartan matrix of $\mathfrak{g}$, and $\det C = |Z|$ is the order of its center $Z(\mathfrak{g})$. The matrix $\text{Ad}(C)$ is by definition integer valued, and it is straightforward to further show that $\frac{1}{2}(\theta \text{Ad}(C))_\alpha \in \mathbb{N}$.

The number of *a priori* distinct duality orbits is found by solving Equation (3.3). For the matching of κ_R, we therefore have the congruence equation:

$$\frac{1}{2}\langle T, \mu^\vee \rangle \equiv c \mod |\Gamma|\,, \tag{3.12}$$

for some integer c. Using the integer properties of the adjugate matrix above, one can see that to solve this equation, there must also exist an integer b related to c for which $\frac{1}{2}(\theta\, \text{Ad}(C))_\alpha \mu^\alpha \equiv b \mod |Z|$. Since $|Z|$ divides $|\Gamma|$, this means that there are *at most* $|Z|$ distinguished solutions to equation (3.12). The *actual* number of solutions is obtained by relating the integers b to c. A case-by-case analysis shows that the number of putative orbits at maximal flavor rank is indeed given by the order of the center, except for $\mathfrak{g} = \mathfrak{so}_{4k}$, where there are only two orbits, rather than four.

A more Lie-algebraic explanation for this fact is as follows. By considering only μ^α for which $\theta_\alpha = 1$, we only take into account the generators of the center Z, and the congruence equation (3.12) can be thought as a map $Z(\mathfrak{g}) \to \mathbb{Z}_{|\Gamma|}$. Since $|Z|$ always divides $|\Gamma|$, the map is straightforward when the center is cyclic. This is the case for every algebra except for $Z(\mathfrak{so}_{4n}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ where the embedding is made only through its diagonal subgroup $\mathbb{Z}_2$.

The same reasoning applies for the matching of the Coulomb branch, where the condition is the same after exchanging Γ with $h^\vee$. It is then straightforward to see that anomaly cancellation in that case only leads to even fundamental representations of the gauge symmetry, so that there are no fractional parts involved, and we are led to the same conclusion as above, namely:

Classification 4 (Number of putative orbits at maximal flavor rank)**.** *For any $Spin(32)/\mathbb{Z}_2$ LSTs of type $\mathfrak{g}$, the number of orbits at maximal flavor rank $rk(\mathfrak{f})$ is given by the order of the center $Z = \mathcal{P}^\vee/\mathcal{Q}^\vee$ related to $\mathfrak{g}$, except in the case $\mathfrak{g} = \mathfrak{so}_{4k}$, where it is given by the order of the diagonal subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_2 \times \mathbb{Z}_2$ instead.*

Example $\mathfrak{g} = \mathfrak{e}_6$ at flavor rank 16: let us illustrate this result for $\mathfrak{g} = \mathfrak{e}_6$. From Table 3, we see we have only three nodes of the affine Dynkin diagram of $\widetilde{\mathfrak{g}} = E_6^{(1)}$ with $\theta_A = 1$. Two of the one-dimensional representations of Γ are conjugate, and we must set $\mu^5 = \mu^1$. The finite dominant coweight is therefore constrained to be of the form $\mu^\vee = \mu^1(\omega_1^\vee + \omega_5^\vee) \in \mathcal{P}^\vee_{\text{LST}}$, and the congruence equation is given by

$$\mathfrak{g} = \mathfrak{e}_6 : \qquad \frac{1}{2}\langle T, \mu^\vee \rangle = 16\,\mu^1 \equiv c \mod 24\,. \tag{3.13}$$

The representatives of the orbits are defined by $\mu^1 = 0, 1, 2$, which matches the order of $Z = \mathbb{Z}_3$. We can further see that the integer cones are simply generated by $\mathcal{C}_\Gamma = \text{span}(3\,\omega_1^\vee)$. After folding, the flavor is located on the last node with $f^4 = \mu^1 + 3n$, and the affine node is constrained to be $f^0 = 32 - 2(\mu^1 + 3n)$ so that the level of the affine coweight satisfies $\theta_A \mu^A = 32$; this giving an upper bound on n.

The ranks of the gauge symmetry are then fixed by anomaly cancellation, enabling us to indeed find three orbits. From smallest to largest with respect to the partial order, the quivers are given by:

$$\begin{array}{lccccccccccc}
\mathcal{O}_{0\cdot(\omega_1^\vee+\omega_5^\vee)}: & \underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}} & \overset{\mathfrak{so}_{4Q-16}}{4} & \overset{\mathfrak{sp}_{3Q-24}}{1} & \overset{\mathfrak{su}_{4Q-32}}{2} & \overset{\mathfrak{su}_{2Q-16}}{2} & \ldots & \underset{[\mathfrak{so}_{2}]}{\overset{\mathfrak{sp}_Q}{1}} & \overset{\mathfrak{so}_{4Q+14}}{4} & \overset{\mathfrak{sp}_{3Q+6}}{1} & \overset{\mathfrak{su}_{4Q+13}}{2} & \underset{[\mathfrak{su}_{15}]}{\overset{\mathfrak{su}_{2Q+14}}{2}} \\
\mathcal{O}_{1\cdot(\omega_1^\vee+\omega_5^\vee)}: & \underset{[\mathfrak{so}_{30}]}{\overset{\mathfrak{sp}_Q}{1}} & \overset{\mathfrak{so}_{4Q-14}}{4} & \overset{\mathfrak{sp}_{3Q-22}}{1} & \overset{\mathfrak{su}_{4Q-29}}{2} & \underset{[\mathfrak{su}_{1}]}{\overset{\mathfrak{su}_{2Q-14}}{2}} & \ldots & \overset{\mathfrak{sp}_Q}{1} & \overset{\mathfrak{so}_{4Q+16}}{4} & \overset{\mathfrak{sp}_{3Q+8}}{1} & \overset{\mathfrak{su}_{4Q+16}}{2} & \underset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_{2Q+16}}{2}} \\
\mathcal{O}_{2\cdot(\omega_1^\vee+\omega_5^\vee)}: & \underset{[\mathfrak{so}_{28}]}{\overset{\mathfrak{sp}_Q}{1}} & \overset{\mathfrak{so}_{4Q-12}}{4} & \overset{\mathfrak{sp}_{3Q-20}}{1} & \overset{\mathfrak{su}_{4Q-26}}{2} & \underset{[\mathfrak{su}_{2}]}{\overset{\mathfrak{su}_{2Q-12}}{2}} & \ldots & \underset{[\mathfrak{so}_{4}]}{\overset{\mathfrak{sp}_Q}{1}} & \overset{\mathfrak{so}_{4Q+12}}{4} & \overset{\mathfrak{sp}_{3Q+4}}{1} & \overset{\mathfrak{su}_{4Q+10}}{2} & \underset{[\mathfrak{su}_{14}]}{\overset{\mathfrak{su}_{2Q+12}}{2}}
\end{array} \tag{3.14}$$

Recall from the discussion around Equation (2.29) that for each non-trivial $\mathfrak{su}_f$ flavor symmetry, there is an additional $\mathfrak{u}_1$ factor possibly delocalized over all complex hypermultiplets, and all LSTs have $\mathrm{rk}(\mathfrak{f}) = 16$. Furthermore, with the bound on n above, one finds that each orbits contains $\{|\mathcal{O}_{i\cdot(\omega_1^\vee+\omega_5^\vee)}|, i = 0, 1, 2\} = \{6, 6, 5\}$ LSTs that are putatively T-dual.

Lower-rank orbits: as we have already argued above, solving the congruence constraints for $\mathrm{rk}(\mathfrak{f}) < 16$ is more involved, both because the flavor relation in Equation (3.9) admits more solutions, and because we need to treat dim(CB) and κ_R separately due to floor functions.

In particular, we have mainly relied on the fact that $C^{-1} = |Z|^{-1}\mathrm{Ad}(C)$. One could therefore naively expect that at lower flavor rank, the number of orbits can be predicted by the possible values of $\kappa_R \mod \Gamma$ and are again of the order of the center, compounded by the different values that the Coulomb branch might take. However, the correct number orbits is difficult to predict. First, as it might be that that instead of being counted by the cyclic group $\mathbb{Z}_{|Z|}$, the numerology forces it to be a smaller subgroup. Furthermore, a particular value of κ_R might correspond to two different values of the Coulomb-branch dimension. The constraints are however easily implemented on a computer to efficiently obtain the number of orbits for a specific algebra $\mathfrak{g}$ at any flavor rank.

We have performed such a scan for all type-DE LSTs up to $r_{\mathfrak{g}} = 8$. At maximal flavor rank, one can see from Table 4 that we indeed have a number of orbits corresponding to the order of the center, except for for $\mathfrak{g} = \mathfrak{so}_8$, which only has two for the reason showed above. Note that as the dominant affine coweights realizing an LST must have level 32, the minimal flavor rank is constrained, and depends on the largest value the mark θ_A can take. In particular, no $Spin(32)/\mathbb{Z}_2$ LST can be flavorless. This is in contradistinction with LSTs of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic type, where there are theories with low-rank flavor symmetry, and even no flavor [86]. In those cases, we expect the fiber-base dual theories to be associated with twisted compactifications.

Furthermore, due to their low rank, LSTs of type $\mathfrak{g} = \mathfrak{so}_8\,, \mathfrak{so}_{10}$ have an interesting structure as the number of orbits is the same at all flavor rank with the exception of very low flavor where almost no models are available. In general, we observe that the number of orbits

rk($\mathfrak{f}$)	$\mathfrak{so}_8$	$\mathfrak{so}_{10}$	$\mathfrak{so}_{12}$	$\mathfrak{so}_{14}$	$\mathfrak{so}_{16}$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
16	2	4	2	4	2	3	2	1
15	2	4	4	8	6	3	4	2
14	2	4	6	12	12	6	8	5
13	2	4	8	16	20	6	14	10
12	2	4	10	20	28	9	24	22
11	2	4	12	24	36	9	36	40
10	2	4	14	28	44	12	50	73
9	2	4	16	32	52	12	66	121
8	1	4	17	36	59	12	83	190
7	1	4	15	40	62	12	97	286
6	—	—	—	—	20	11	96	388
5	—	—	—	—	—	2	75	440
4	—	—	—	—	—	—	27	396
3	—	—	—	—	—	—	2	191
2	—	—	—	—	—	—	—	22
1	—	—	—	—	—	—	—	—
0	—	—	—	—	—	—	—	—

Table 4: Number of putative duality orbits per flavor rank for type-DE $\mathfrak{so}_{32}$ LSTs of rank up to eight.

monotonically grows as the flavor rank decreases, possibly plateaus, and then monotonically decreases. While this pattern continues for $r_{\mathfrak{so}_{2k}} > 8$, we however do not have an explanation, either Lie theoretic or geometric, for this behavior. We leave a more detailed study for future works.

3.3 Example: Duality Orbits of Heterotic LSTs on $\mathfrak{g} = \mathfrak{so}_8$.

In the following we give a complete classification of the T-duality orbits of the heterotic string appearing in Table 4 for $\mathfrak{g} = \mathfrak{so}_8$. These theories have the convenient property that the number of duality orbits is at most two. Moreover we will show that this property is the same for all other heterotic LSTs and that all such theories fall into the same two possible duality orbits at each flavor rank.

The $Spin(32)/\mathbb{Z}_2$ theory: for theories with $\mathfrak{g} = \mathfrak{so}_8$, the base is $\widetilde{\mathfrak{b}} = D_4^{(1)}$. It is then straightforward to show that the anomaly constraints in Equation (2.16) forces the quiver to take the simple form

$$
\begin{array}{ccc}
\overset{[\mathfrak{so}_{a+4m_0}]}{\overset{\mathfrak{sp}_Q}{1}} & & \overset{[\mathfrak{so}_{a+4m_3}]}{\overset{\mathfrak{sp}_{Q-m_0+m_3}}{1}} \\
& \underset{[\mathfrak{sp}_{8-a-N}]}{\overset{\mathfrak{so}_{4Q+16-a-4m_0}}{4}} & \\
\underset{[\mathfrak{so}_{a+4m_1}]}{\overset{\mathfrak{sp}_{Q-m_0+m_1}}{1}} & & \underset{[\mathfrak{so}_{a+4m_4}]}{\overset{\mathfrak{sp}_{Q-m_0+m_4}}{1}}
\end{array}
\qquad
\begin{aligned}
\mathrm{rk}(\mathfrak{f}) &= 8+4\lfloor\frac{a}{2}\rfloor+N-a\,; \\
\dim(\mathrm{CB}) &= 6\,Q+4+N-6\,m_0-\lfloor\frac{a}{2}\rfloor\,; \\
\kappa_R &= 8\,Q+18+N-8\,m_0-a\,.
\end{aligned}
\tag{3.15}
$$

with integers $a \in 0\ldots3$ and m_i such that $N = \sum_i m_i$, and must satisfy $a + N \leq 8$. The coefficients of the affine coweight $\mu^A \in \{a + m_0, \ldots, a + 4m_4]$ can therefore be obtained from a 4-partition of N. Note that the quiver admits an S_4 symmetry permuting the (-1)-curves and one must take care of not double-counting equivalent quivers. As the T-duality invariants only depend on N and the value m_0 on the affine node, this does not affect the number of putative T-duality orbits.

As we can be very explicit in that case, we find at most two duality orbits at each rank, labeled only by $\tilde{a} = \lfloor\frac{a}{2}\rfloor = 0, 1$ and that flavor rank $\mathrm{rk}(\mathfrak{f})$. This shows that with $\mathfrak{g} = \mathfrak{so}_8$, an orbit is only characterized by two T-duality invariants. For ease of reading we will denote an orbit via this data rather than the coweight $\mu^\vee$ seeding it: $\mathcal{O}_{\mu^\vee} = \mathcal{O}_{\tilde{a},\mathrm{rk}(\mathfrak{f})}$. The other two invariants are then given by:

$$
\mathcal{O}_{\tilde{a},\mathrm{rk}(\mathfrak{f})}: \qquad \begin{pmatrix} \kappa_R \\ \dim(\mathrm{CB}) \end{pmatrix} = \mathrm{rk}(\mathfrak{f}) + \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \tilde{a} \begin{pmatrix} 4 \\ -3 \end{pmatrix} \mod \begin{pmatrix} 8 \\ 6 \end{pmatrix}, \tag{3.16}
$$

where the congruence is in terms of $(\Gamma, h^\vee) = (8, 6)$ takes into account the need for possible shifts in the number of five-branes Q so as to reproduce the constraints in Equation (3.3). We therefore clearly see that there can only be at most two putative T-duality orbits at each flavor rank labeled by $\tilde{a} = 0, 1$.

Anticipating a discussion in Section 5, these $Spin(32)/\mathbb{Z}_2$ can be dual to another type of $\mathfrak{so}_{32}$ heterotic theory corresponding to an example of internal duality, as well as fiber-base duality to LSTs arising from compactifications of the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic string.

The $(\mathbf{SU(16)} \times \mathbf{U(1)})/\mathbb{Z}_2$ theory: the first case corresponds to LSTs whose base is described by the twisted affine algebra $D_3^{(2)}$, see Section 5.1. After taking into account cancellations of quartic anomalies, it is constrained to take the form:

$$
[\mathfrak{u}_{4n+\tilde{a}}]\ \overset{\mathfrak{su}_{2Q}}{2}\ \underset{[\mathfrak{so}_{2b}]}{\overset{\mathfrak{sp}_{2Q-2n-\tilde{a}}}{1}}\ \overset{\mathfrak{su}_{2Q-4n-2\tilde{a}-b+8}}{2}\ [\mathfrak{u}_{16-(4n+a)-2b}] \qquad \mathrm{rk}(\mathfrak{f}) = 16-b\,. \tag{3.17}
$$

The parameter $\tilde{a} = 0, 1$ can again take on exactly two values, and decide the T-duality orbits $\mathcal{O}_{\tilde{a},\mathrm{rk}(\mathfrak{f})}$ the theory belong. It is easy to see that the flavor rank is bounded by $\mathrm{rk}(\mathfrak{f}) \geq 8$. For this class of theory, we therefore find the same formulas as in Equation (3.16) and precisely the same structure for putative T-duality orbits as for $Spin(32)/\mathbb{Z}_2$ theories.

The $\mathbf{E_8 \times E_8}$ theory: the last class of T-duality that can occur at arbitrary number of instantons Q is "fiber-base" duality between the $\mathfrak{so}_8$ $Spin(32)/\mathbb{Z}_2$ LSTs and those of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ type. As briefly reviewed in Section 5.3, these theories are classified by a pair of embeddings of $\Gamma_{\mathfrak{so}_8}$ into the group E_8: $\lambda_L, \lambda_R \in \mathrm{Hom}(\Gamma_{\mathfrak{so}_8}, E_8)$. Individually, they can be understood as independent breaking of the $\mathfrak{e}_8$ factors of two SCFTs called orbi-instantons, which have $\mathfrak{e}_8 \oplus \mathfrak{so}_8$ flavor symmetry [87]. The list of quivers for these SCFTs and all possible choices $\lambda \in \mathrm{Hom}(\Gamma_{\mathfrak{so}_8}, E_8)$ can be found in reference [86].

Interestingly, the contribution $\delta\kappa_R(\lambda), \delta\mathrm{CB}(\lambda)$ from each SCFT to the T-duality invariants of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ LSTs can be computed independently. Furthermore, even though the SCFTs do not exhibit a two-group symmetry, $\delta\kappa_R(\lambda)$ can still be obtained directly at the level of the orbi-instanton since it descends from the anomaly polynomial. The key observation is that, as for the other two classes of LSTs with $\mathfrak{g} = \mathfrak{so}_8$, there are only two types for a given SCFT, labeled by an integer $\tilde{a} = 0, 1$ and the rank of the flavor symmetry $\mathfrak{f}$ associated with a choice of embedding $\lambda \in \mathrm{Hom}(\Gamma_{\mathfrak{so}_8}, E_8)$:

$$\begin{pmatrix} \delta\kappa_R(\lambda) \\ \delta\mathrm{CB}(\lambda) \end{pmatrix} = \mathrm{rk}(\mathfrak{f}) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \tilde{a} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \mod \begin{pmatrix} 8 \\ 6 \end{pmatrix}, \tag{3.18}$$

In Table 5, we have collated all nineteen orbi-instanton theories and the value of $\tilde{a}$ they give rise to. We see that the minimal flavor rank of those SCFTs is four, which in turn means that at the level of the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ LST, the minimal flavor rank is eight. Interestingly, as opposed to what occured on the $\mathfrak{so}_{32}$ side of the duality, there cannot be any LST with flavor rank seven if there are no twist along the compactification circle.

Starting with two orbi-instanton SCFTs of type $\mathfrak{so}_8$ whose quivers are fixed by choices $\lambda_L, \lambda_R \in \mathrm{Hom}(\Gamma_{\mathfrak{so}_8}, E_8)$, it is straightforward to show that, taking into account the contribution of the vector multiplet used to gauge the common $\mathfrak{so}_8$ flavor symmetry to produce an LST, we have

$$\mathcal{O}_{\tilde{a}_{\mathrm{tot}},\mathrm{rk}(\mathfrak{f}_L \oplus \mathfrak{f}_R)} : \begin{pmatrix} \kappa_R(\lambda_L, \lambda_R) \\ \mathrm{CB}(\lambda_L, \lambda_R) \end{pmatrix} = \begin{pmatrix} \delta\kappa_R(\lambda_L) \\ \delta\mathrm{CB}(\lambda_L) \end{pmatrix} + \begin{pmatrix} \delta\kappa_R(\lambda_R) \\ \delta\mathrm{CB}(\lambda_R) \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix} \mod \begin{pmatrix} 8 \\ 6 \end{pmatrix}, \tag{3.19}$$

with $\tilde{a}_{\mathrm{tot}} = a_L + a_R \mod 2$. Therefore, while we have *a priori* more freedom in $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LSTs, as we may choose two independent embeddings rather than one. In the end, we once again find only at most two duality orbits.

rk($\mathfrak{f}$)	flavors with $\tilde{a}=0$	flavors with $\tilde{a}=1$
8	$\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{16}, \mathfrak{so}_8^2, \mathfrak{so}_{12} \oplus \mathfrak{so}_4$	$\mathfrak{su}_8 \oplus \mathfrak{u}_1, \mathfrak{e}_6 \oplus \mathfrak{u}_1^2$
7	$\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_8 \oplus \mathfrak{su}_2^3$	$\mathfrak{su}_6 \oplus \mathfrak{u}_1^2$
6	$\mathfrak{sp}_2^3, \mathfrak{so}_9 \oplus \mathfrak{sp}_2, \mathfrak{so}_{13}$	$\mathfrak{so}_7 \oplus \mathfrak{su}_2^3$
5	$\mathfrak{f}_4 \oplus \mathfrak{su}_2, \mathfrak{sp}_4 \oplus \mathfrak{su}_2$	$\mathfrak{so}_8 \oplus \mathfrak{su}_2$
4	$\mathfrak{sp}_4$	—

Table 5: All nineteen $\mathfrak{g} = \mathfrak{so}_8$ orbi-instanton theories [86] identified by their flavor ranks $\mathfrak{f}$ and the coefficient $\tilde{a} = 0, 1$ that defines the duality orbit of the associated $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LST.

Finally we remark that the above considerations can also be extended to twisted compactifications following reference [34]. Although beyond the scope of this work, it can be shown that all $\mathfrak{so}_8$ twisted heterotic theories fall into at most two duality orbits as in untwisted cases. This hints at a much deeper structure that the duality orbits may have even when including possible discrete symmetries.

4 Higgs Branches and the Partial Order of Coweights

With a concise way of encoding many properties of $Spin(32)/\mathbb{Z}_2$ LSTs in terms of dominant affine coweights, one may ask what can be learned about how they are related using the partial order defined in equation (3.4). As anticipated above, it can be interpreted as a consistent change of the gauge symmetries between two LSTs and is related to Higgs-branch Renormalization Group (RG) flows.

Concretely, let us consider two LSTs with affine coweights $\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q)$ and $\widetilde{\nu}^\vee = (\nu^\vee, 32, 2Q)$ of a fixed base algebra $\widetilde{\mathfrak{b}}$, in the notation of Equation (2.44). We assume for now that Q is set to a fixed value for simplicity. If the two LSTs are related by a consistent RG flow, then the dimension $d_I = \langle \widetilde{\omega}_I, \widetilde{\mu}^\vee - 16\widetilde{S}^\vee \rangle$ of the fundamental representation of each gauge symmetry can only change by an integer. As we have seen around Classification 2, in terms of the affine root system it means that the difference $\mu^\vee - \nu^\vee$ is a positive coroot.

For any Higgs-branch RG flow to take place, the two coweights must therefore be ordered:

$$\widetilde{\mathcal{K}}_Q(\mu^\vee; \mathfrak{g}) \quad \xrightarrow{\text{RG}} \quad \widetilde{\mathcal{K}}_Q(\nu^\vee; \mathfrak{g}) \qquad \Rightarrow \qquad \nu^\vee < \mu^\vee \,. \tag{4.1}$$

While it is true that the partial order of dominant coweights is related to Higgs-branch flows, there are important caveats. First, for low values of Q, symmetry enhancements may occur.

For instance, an undecorated (-1)-curve is well known to support an $\mathfrak{e}_8$ flavor symmetry rather than a classical algebra, which is not captured by Classification 2—at least not in a direct way. Worse, deep inside the Higgs branch, one ultimately encounters LSTs described by zero curves $\overset{\mathfrak{g}}{0}$, which have been shown to have a very rich Higgs-branch structure [32]. There are also Higgs-branch deformations that can change the curve configuration of the base—or even lead to product theories [35, 44, 47, 65, 88]—which also cannot be captured by the partial order above. While certain formulas can sometimes be "analytically continued" to cover such cases [71, 87], they are beyond the scope of this work.

To avoid such subtleties, we will restrict ourselves to theories where those cases do not arise. In particular, we will always assume Q to be "large enough"—that is arbitrary, but large. The precise numerology depends on the type of theory, but in practice one usually requires $Q \gtrsim h^\vee_{\mathfrak{g}}$. In the sequel, we will therefore only consider flows between LSTs whose quivers are described by affine coweights and whose flavor and gauge symmetry follows directly from them.

The Higgs branch of theories with at least eight supercharges—which include $6d$ $\mathcal{N} = (1, 0)$ SQFTs—can sometimes be studied via a three-dimensional auxiliary quiver gauge theory. The strategy is to find a so-called magnetic quivers: a $3d\,\mathcal{N} = 4$ theory whose Coulomb branch is the same as the Higgs branch of the higher-dimensional theory. This has led to advances in our understanding of the structure of the Higgs branch of six-dimensional theories, and has recently been applied to a host of cases, such as LSTs on a zero-curve [32], SCFTs with classical symmetries [48, 49, 89–91], as well as certain classes of SCFTs and heterotic LSTs with $\mathfrak{g} = \mathfrak{su}_K$ that can be built by "fusing" these SCFTs together [35, 39–46].

For those SCFTs that can be constructed via $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic string theory with $\mathfrak{g} = \mathfrak{su}_K$, it has been shown that they are classified by dominant affine coweight of $E_8^{(1)}$ at level K [39, 43]. The partial order of coweights then reproduces the network of Higgs-branch RG flows of the SCFTs, which is encoded in the geometry of a group-theoretic object referred to as the (double) affine Grassmannian [43, 92].

Although we will not delve into the mathematical minutiae of affine Grassmannians, we will argue that the partial order of coweights arising for $Spin(32)/\mathbb{Z}_2$ LSTs encodes at least part of the Higgs branch. Our goal in this section is to obtain an algorithm to, starting with any $Spin(32)/\mathbb{Z}_2$ LST, obtain all other possible LSTs that can in principle be reached via a Higgs-branch RG flow.

Studying the partial order of these theories is in some sense an easier task that was done for SCFTs arising from $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic string theory: going from an affine coweight of $E_8^{(1)}$ to the six-dimensional generalized quiver is not as direct as in our case [39, 43]. One the other hand, for $Spin(32)/\mathbb{Z}_2$ LSTs, all types of classical algebras may arise as gauge or flavor symmetries, while affine-Grassmannian techniques are typically studied only for unitary quivers [39,43,44]. We will nonetheless find that, despite needing to deal with all classical algebras, there are only a reasonable number of inequivalent minimal transitions that can occur between dominant coweights. The number of LSTs for a given algebra being finite, these are straightforward to identify.

To make our task easier, the closed-form formulas we have found in the previous section reveal to be useful. In particular, they enable us to easily find how the T-duality invariant change between two theories. Indeed, for Q kept fixed—which we will assume—we have

$$\begin{aligned}
\Delta\kappa_R &= \langle T, \mu^\vee - \nu^\vee \rangle \\
\Delta\mathrm{CB} &= \langle \rho, \mu^\vee - \nu^\vee \rangle + \sum_{\alpha\in\mathcal{P}} \left(\mathrm{frac}(\frac{C_{\alpha a}\mu^a}{2}) - \mathrm{frac}(\frac{C_{\alpha a}\nu^a}{2}) \right) \\
\Delta\mathrm{HB} &= \frac{1}{2}(\mu^\vee, \mu^\vee) - \frac{1}{2}(\nu^\vee, \nu^\vee) - (\mu^\vee - \nu^\vee, S)\,.
\end{aligned} \tag{4.2}$$

All pairing are taken over the finite algebra $\mathfrak{g}$, and we defined $(\mu^\vee, \nu^\vee) = \mu^a C_{ab} \nu^b$ the natural scalar product on the coweight lattice, with C_{ab} the inverse Cartan matrix, see equation (2.9).

4.1 Slice Subtraction

An algorithm referred to as *slice subtraction* has recently been devised for a large of six-dimensional SCFTs [47]. It enables one to reconstruct the complete Hasse diagram encoding the network of Higgs-branch flows of an SCFT. Even though its validity was proved using similar quiver-subtraction algorithms for 3d $\mathcal{N} = 4$ theories [48, 93, 94] using their magnetic quiver, the algorithm is performed directly at the level of the six-dimensional generalized quiver. It therefore offers a very elegant way of obtaining all possible theories that can be obtained by Higgs-branch RG flows from a given SCFT.

Given an SCFT $\mathcal{K}$, one can reach a new theory $\mathcal{K}'$ though a subtraction operation. One finds that there is a single *minimal transverse slice* (or slice, for brevity) $\mathcal{S}$ such that $\mathcal{K}' = \mathcal{K} - \mathcal{S}$. This procedure does not require to know the magnetic quiver, and encodes the field theory data that are involved in the RG flow, such as which operator is given a vacuum expectation value, or which moduli are turned on when a brane picture is available. Once all allowed minimal slices $\mathcal{S}$ have been found, the complete Hasse diagram can be recreated by successive subtractions. In three dimensions the slices have been shown to correspond to singular symplectic varieties, and have been used to study the Higgs mechanism for these theories. In this work, we will not explore their geometric properties, but rather simply use them as a tool encoding how the generalized quiver of a given LST must change to obtain another consistent theory. We instead refer to reference [95] for a mathematical introduction to the topic.

Although not quite phrased in those terms, the authors of reference [47] relied on the properties of the partial order of dominant coweights discussed above to determine the slice-subtraction algorithm. It is therefore very tempting to try to generalize it to LSTs admitting a perturbative description, as we have a straightforward way to construct the set of all dominant coweights $\widetilde{\mathcal{P}}^\vee_{\text{LST}}$ corresponding to consistent LSTs.

In the following, we will work for convenience with a fixed value of Q and directly at the level of the finite part of the algebra of the base $\mathfrak{b}$ rather than the algebra $\mathfrak{g}$ associated with the orbifold singularity. Before presenting the general algorithm, let us first consider a specific example to illustrate the procedure: consider the two dominant coweights $\mu^\vee = [0,0,0,2]$ and $\mu^\vee = [2,0,2,0]$ of $\mathfrak{b} = \mathfrak{so}_9$. It is possible to show that there are no other LST with dominant coweight $\lambda^\vee$ such that $\mu^\vee < \lambda^\vee < \nu^\vee$, and the transition between the two must therefore be minimal. In terms of generalized quivers, we have:

$$\begin{array}{cccc} & [\mathfrak{so}_{26}] & & \\ & \overset{\mathfrak{sp}_Q}{1} & & \\ \overset{\color{blue}\mathfrak{sp}_{Q-6}}{1} & \overset{\color{red}\mathfrak{so}_{4Q-10}}{4} & \overset{\color{blue}\mathfrak{sp}_{2Q-12}}{1} & \overset{\mathfrak{su}_{2Q-12}}{2} \\ {\color{red}[\mathfrak{so}_2]} & & {\color{red}[\mathfrak{so}_2]} & \end{array} \quad \xrightarrow{\begin{array}{ccccc} {\color{blue}2} & - & {\color{red}2} & - & {\color{blue}2} \\ | & & & & | \\ {\color{red}[2]} & & & & {\color{red}[2]} \end{array}} \quad \begin{array}{cccc} & [\mathfrak{so}_{28}] & & \\ & \overset{\mathfrak{sp}_Q}{1} & & \\ \overset{\color{blue}\mathfrak{sp}_{Q-7}}{1} & \overset{\color{red}\mathfrak{so}_{4Q-12}}{4} & \overset{\color{blue}\mathfrak{sp}_{2Q-13}}{1} & \overset{\mathfrak{su}_{2Q-12}}{2} \\ & & & [\mathfrak{su}_2] \end{array} \tag{4.3}$$

The curve corresponding to the affine node is the one with $\overset{\mathfrak{sp}_Q}{1}$. For ease of reading, we have have colored the algebras that change, with ${\color{blue}\mathfrak{sp}_k}$ algebras in blue and ${\color{red}\mathfrak{so}_k}$ in red, following the notation of table 2 as well as the standard conventions of $3d\,\mathcal{N}=4$ quiver gauge theories. Over the arrow, we have indicated by how much the dimension of a fundamental representation d_I and f^I of either gauge and flavor symmetries changes, omitting those who remain the same. To make it resemble a $3d\,\mathcal{N}=4$ quiver and following the notation introduced in reference [47], we only show d_I and f^I. The quiver encoding the change of spectrum between the two LSTs is the minimal slice $\mathcal{S}$ discussed above:

$$\mathcal{S}: \qquad \begin{array}{ccccc} {\color{blue}2} & - & {\color{red}2} & - & {\color{blue}2} \\ | & & & & | \\ {\color{red}[2]} & & & & {\color{red}[2]} \end{array} . \tag{4.4}$$

In the following we will only denote the dimension of the fundamental representations. We therefore see that as advertised, starting from the LSTs with coweight $\mu^\vee$, the subtraction operation $\mathcal{K} - \mathcal{S}$ is simply given by changing the dimensions d_I and f^I by an amount given by those of $\mathcal{S}$. Note that if we simply perform the subtraction, we obtain a quiver with flavor symmetries that leads to a gauge-anomalous spectrum. This is a standard feature of subtraction algorithms [48, 93, 94]: after subtracting $\mathcal{S}$ to the original quiver, the flavor symmetry must be "rebalanced" according to Equation (2.16).

Interpreting $\mathcal{S}$ as a $3d$ $\mathcal{N}=4$ quiver, it can be shown that this RG flow corresponds to a so-called Kraft–Procesi transition [95, 96], with a change to the dimensions of the Coulomb and Higgs branches of -3 and -1, respectively. One can easily see that this is precisely what is obtained from the closed-form expressions derived in Section 2.1.

For general LSTs, as we have given a way to construct all $\mu^\vee \in P^\vee_{\text{LST}}(\mathfrak{b})$, we can easily find all the possible minimal transverse slices $\mathcal{S}$ in the same fashion as the example above. We say that given two coweights, $\mu^\vee$ *covers* $\nu^\vee$, denoted $\nu^\vee \lessdot \mu^\vee$, if there is no third coweight $\lambda^\vee$ such that $\nu^\vee < \lambda^\vee < \mu^\vee$. This definition also applies to affine dominant coweights. In each case, we can always uniquely associate a slice $\mathcal{S}$. Since, given an algebra $\mathfrak{b}$ associated with the base of an LST, the number of possible coweights is finite, it is straightforward to obtain an exhaustive list of all transverse minimal slices $\mathcal{S}$.

The example above shows how, knowing the quiver of a particular slice $\mathcal{S}$, one is able to find the next theory. For more general cases, the following algorithm holds:

Algorithm 1 (Slice Subtraction)**.** *Consider an LST $\mathcal{K} = \widetilde{\mathcal{K}}_Q(\mu^\vee; \mathfrak{g})$ associated with an affine dominant coweight $\widetilde{\mu}^\vee \in \widetilde{P}^\vee_{LST}(\widetilde{\mathfrak{b}})$. A second LSTs $\mathcal{K}'$ can be reached through the following procedure:*

1. *The quiver $\mathcal{S}$ of a transverse slice appearing in Tables 6–10 can be subtracted from the generalized quiver of $\mathcal{K}$ if $\mathcal{S}$ appears as one of its subquiver, with matching types of flavor and gauge symmetries.*
2. *The quiver corresponding to a subtraction operation $\mathcal{K}' = \mathcal{K} - \mathcal{S}$ has gauge symmetries obtained by subtracting the dimensions of the fundamental representation d_I, f^I of $\mathcal{K}$ with those of $\mathcal{S}$. The gauge algebras that do not align with those of $\mathcal{S}$ remain unchanged.*
3. *After performing the subtraction, the dimension of fundamental representations of the flavor symmetries f^I of $\mathcal{K}'$ must be "rebalanced" to ensure cancellation of quartic gauge anomalies:*
$$f^I = C^{IJ} d_J + 16 S^I\,, \tag{4.5}$$
with C^{IJ} the Cartan matrix of $\widetilde{\mathfrak{b}}$, and S^I the associated Frobenius–Schur indicator.

The resulting LST $\mathcal{K}'$ is associated with an affine coweight $\widetilde{\nu}^\vee \in \widetilde{P}^\vee_{LST}(\widetilde{\mathfrak{b}})$ such that $\widetilde{\nu}^\vee \lessdot \widetilde{\mu}^\vee$.

Algorithm 1 therefore provides us with an easy way to generate a large part of the set $\mathcal{P}^\vee_{\text{LST}}$ from a single LST. For low values of Q, one might encounter the subtleties mentioned at the beginning of this section, and the algorithm can break down.

We must now find all possible minimal slices $\mathcal{S}$ appearing in $Spin(32)/\mathbb{Z}_2$ LSTs. Note that our task is slightly different than the original classification of coverings of coweights by Stembridge [85]. In our case, given two dominant coweights $\mu^\vee, \nu^\vee \in \mathcal{P}_+^\vee$ satisfying $\nu^\vee \lessdot \mu^\vee$ might not correspond to LSTs. We are therefore only interested in coverings within $\mathcal{P}_{\mathrm{LST}}^\vee \subset \mathcal{P}_+^\vee$. Given the finite part $\mathfrak{b}$ of a base algebra, we can then find all minimal slices $\mathcal{S}$ that can occur by enumerating all the possible coverings in $\mathcal{P}_{\mathrm{LST}}^\vee$. As the level of the dominant coweights is set to 32, this can be done systematically for low-rank algebras $\mathfrak{b}$. For infinite classical series, it is easy to convince oneself that no new types of slices arise at large rank.

Our task is now to find all possible coverings for an algebra $\mathfrak{b}$ that are related to $Spin(32)/\mathbb{Z}_2$ LST. For simplicity, we focus here only on *finite* coweights, meaning that we are only considering LSTs with a fixed rank Q. However we find that the algorithm extends to general coweights, although there might be additional minimal slices. In fact, the slices we find for finite coweights do often appear between coweights with different values of Q.

Classical Kraft–Procesi transitions: The simplest cases occur when a single curve supporting a gauge symmetry $\mathfrak{su}_n, \mathfrak{so}_n, \mathfrak{sp}_n$ has n reduced by one. These correspond to the closure of the minimal orbit of the associated flavor $\mathfrak{f}_I$, usually denoted by a_k, b_k, c_k, d_k, depending on the type of the algebra of the flavor symmetry. For instance, for b_k, we have:

$$\cdots \underset{[\mathfrak{so}_{2k+1}]}{\overset{\mathfrak{sp}_n}{1}} \cdots \quad \xrightarrow{\;b_k \,=\, \overset{2}{\underset{[2k+1]}{|}}\;} \quad \cdots \underset{[\mathfrak{so}_{2k-3}]}{\overset{\mathfrak{sp}_{n-1}}{2}} \cdots \tag{4.6}$$

The other types of minimal slices corresponding to the closure of minimal nilpotent orbits of classical algebras are collated in Table 6.

These minimal slices are the simplest transitions appearing in the Higgs branch of SQFTs with eight supercharges, and correspond to giving a particular vacuum expectation value (vev) to the moment map operator, the gauge-invariant composite scalar operator in the same supermultiplet as the flavor current. As it transforms in the adjoint representation of the flavor algebra, one may choose the vev to be the minimal nilpotent orbit, corresponding to the simplest decomposition $\mathfrak{f} \to \mathfrak{su}_2 \oplus \mathfrak{f}_{\mathrm{IR}}$. The algebra $\mathfrak{f}_{\mathrm{IR}}$ is then the same as in Equation (4.6). It is then well known that the change in the Coulomb-branch dimension always decreases by one, as can see from above, and that of the Higgs branch decreases by $h_{\mathfrak{f}}^\vee - 1$, which is the dimension of the minimal orbit of $\mathfrak{f}$.

These RG flows are part of a larger class of transitions referred to as classical Kraft–Proecesi transitions, which—loosely speaking—arise in physics when studying brane systems that can be labelled by integer partitions. We defer to a more in-depth discussion of Kraft–Procesi

Quiver:	1 $\vert$ $[k]$	2 $\vert$ $[2k+1]$	1 $\vert$ $[k]$	2 $\vert$ $[2k]$
slice:	a_k	b_k	c_k	d_k

Table 6: Slices corresponding to the closure of a minimal nilpotent orbits. The integer at each node of the quiver denotes the dimension of the fundamental representation, with $\mathfrak{so}_k$ in red, $\mathfrak{sp}_k$ in blue, and $\mathfrak{su}_k$ in black.

transitions to references [95,96], which also contain a mathematical introduction to the topic.

The other types of Kraft–Procesi transitions are associated with Kleinian singularities $\mathbb{C}/\Lambda$ where $\Lambda \subset SU(2)$, and are denoted by the associated ADE algebra. For instance a transition associated with the Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_n$ is denoted A_{n-1}. Physically, the RG flow is triggered by a baryon-like operator getting a vev. We stress that the singularity relates to the geometry of the Higgs branch of the theory, and *not* the type of LST we consider. These are engineered in heterotic string theory on an orbifold singularity $\mathbb{C}^2/\Gamma$. The slices $\mathcal{S}$ appearing in $Spin(32)/\mathbb{Z}_2$ LSTs and the associated quivers are shown in Table 7.

For Kraft–Procesi transitions, whether associated with the closure of a nilpotent orbit or a Kleinian singularity, there is always an RG flow associated with those types of coverings in six dimensions, which has been checked in a plethora of cases via magnetic quivers [32,35,39, 44,47–53]. The changes in the dimension of the Coulomb and Higgs branches are exchanged with respect to that of nilpotent orbits: the former changes by $h^\vee - 1$ and the latter by one. There is an exception in cases involving $\overset{\mathfrak{so}_k}{4}$ configurations: in most cases, the change in k is even and there are no issues, but when it is odd, the change in the Coulomb branch might be different that the predicted $h^\vee - 1$, as the rank of $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$ is the same. This difference in the predicted and actual change in the Coulomb-branch dimension is a known occurrence, is related to the so-called X-collapse of the partition labeling the branes [96].

Exotic transitions: in addition to the usual Kraft–Procesi transitions, we also find coverings where we can associate a quiver $\mathcal{S}$ for which we do not know the corresponding type of transition. We separate them into two classes. The first appears in theories whose base is non-simply laced, while the second corresponds to orthosymplectic quivers whose gauge nodes are shaped like an exceptional algebra.

The slices related to non-simply-laced bases are collected in Table 8. One can see that in each case, the quiver corresponding to the slice arranges like the Dynkin diagram of a (possibly-twisted) affine algebra, with the flavor symmetry on the affine node. Interestingly, the change in the dimension of the fundamental representation at each node of the slice is equal to the

Slice type	Quiver	ΔCB	ΔHB
A_k	$[1]-\underbrace{1-\cdots-1}_{k}-[1]$	$-k$	-1
A_1	$[3]-2-1$	-2 ($\overset{\mathfrak{so}_k}{4}$ even) -1 ($\overset{\mathfrak{so}_k}{4}$ odd)	-1
A_{2k-1}	$[2]-\underbrace{2-2-\cdots-\overset{\overset{[1]}{\vert}}{2}-1}_{2k\ \text{gauge}}$	$-2k$ ($\overset{\mathfrak{so}_k}{4}$ even) $-(2k-1)$ ($\overset{\mathfrak{so}_k}{4}$ odd)	-1
$A_{2k-1}\cup A_{2k-1}$	$[2]-\underbrace{2-\cdots-2}_{2k-1}-[2]$	$-(2k-1)$	-1
$A_3\simeq D_3$	$1-\underset{\underset{[2]}{\vert}}{2}-1$	-1 ($\overset{\mathfrak{so}_k}{4}$ even) -3 ($\overset{\mathfrak{so}_k}{4}$ odd)	-1
D_k	$1-\underbrace{\overset{\overset{[1]}{\vert}}{2}-2-\cdots-\overset{\overset{[1]}{\vert}}{2}}_{2k-1\ \text{gauge}}-1$	$-(2k-3)$ ($\overset{\mathfrak{so}_k}{4}$ even) $-(2k-5)$ ($\overset{\mathfrak{so}_k}{4}$ odd)	-1

Table 7: Quivers appearing in the slice-subtraction algorithm that, interpreted as a $3d$ $\mathcal{N}=3$ theory, corresponds to Kraft–Procesi transitions associated with Kleinian singularities, taken from [96, Tables 6, 7]. The integer at each node of the quiver denotes the dimension of the fundamental representation, with $\mathfrak{so}_k$ in red, $\mathfrak{sp}_k$ in blue, and $\mathfrak{su}_k$ in black. In the presence of $\mathfrak{so}_k$ algebras, if the quiver of the slice leads to an odd change in the dimension of the fundamental representation, the change in the Coulomb-branch dimension depends on the UV theory, and whether the associated $\mathfrak{so}_k$ symmetry is even or odd.

comarks of the algebra. Note that, similarly to Kraft–Procesi transitions of the Kleinian type, when the quiver of the slice has only a few gauge nodes, the flavor symmetry of the slice can be larger what could be expected from the comarks.

For the second class, collated in Tables 9 and 10, we distinguish two subclasses: the first corresponds to quivers shaped like an $\mathfrak{e}_6$ Dynkin diagram. Such quiver slices occur solely in the case of $\mathfrak{g}=\mathfrak{so}_{2k},\mathfrak{e}_k$ due to their trivalent node. The remaining quivers, collated in Table 8 occur only in the exceptional cases $\mathfrak{g}=\mathfrak{e}_7,\mathfrak{e}_8$. Contrary to the other types of exceptional minimal slices, they induce change in the dimension of the fundamental that is greater than the comarks of the corresponding Dynkin diagrams. To the best of our knowledge, when thought of as $3d$ $\mathcal{N}=4$ quivers, these types of slices have not appeared in the literature.

Quiver	ΔCB	ΔHB
$\begin{array}{c} 1 - 2 \nRightarrow 1 \\ \vert \\ [1] \end{array}$	-2	-1
$\begin{array}{c} 1 - 2 \nRightarrow 1 - [1] \\ \vert \\ [1] \end{array}$	-2	-0
$\begin{array}{c} 2 \\ \vert \\ [2k+1] \end{array}$	$-k$ ($\overset{\mathfrak{so}_k}{4}$ even) $-(k-1)$ ($\overset{\mathfrak{so}_k}{4}$ odd)	0
$\begin{array}{c} 2 \nRightarrow 2 - 1 \\ \vert \\ [1] \end{array}$	-4	-0
$[2] - 2 \nRightarrow 1$	-2	0
$[3] - 2 \nRightarrow 1$	-2	-1
$[2] - 2 \nRightarrow 1 - [1]$	-2	-1
$[3] - 2 \nRightarrow 1 - [1]$	-2	-2
$\begin{array}{c} 1 \\ \vert \\ [k] \end{array}$	$-k$	0
$2 - 4 \nRightarrow 3 - 2 - [1]$	-8	0

Table 8: Quivers appearing in the slice-subtraction algorithm for LSTs with bases $\widetilde{\mathfrak{b}}$ that are non-simply laced. The integer at each node of the quiver denotes the dimension of the fundamental representation, with $\mathfrak{so}_k$ in red, $\mathfrak{sp}_k$ in blue and $\mathfrak{su}_k$ in black. In the presence of $\mathfrak{so}_k$ gauge algebras, if the quiver associated with the slice leads to an odd change in the dimension of the fundamental representation, the change in the Coulomb-branch dimension depends on the UV theory, and whether the fundamental representation of the corresponding $\mathfrak{so}_k$ gauge symmetry is even or odd.

Quiver	ΔCB	ΔHB	Appears in $\mathfrak{g}$
$\begin{array}{ccccccccc} &&&&[1]&&&& \\ &&&&\vert&&&& \\ &&&&2&&&& \\ &&&&\vert&&&& \\ 1&-&2&-&3&-&2&-&1 \end{array}$	-4	0	$\mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccc} &&&&[2]&&&& \\ &&&&\vert&&&& \\ &&&&2&&&& \\ &&&&\vert&&&& \\ 1&-&2&-&3&-&2&-&1 \end{array}$	-7	-1	$\mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccc} &&&&[1]&&&& \\ &&&&\vert&&&& \\ &&&&2&&&& \\ &&&&\vert&&&& \\ 1&-&2&-&3&-&2&-&[1] \end{array}$	-4	0	$\mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccc} &&&&[2]&&&& \\ &&&&\vert&&&& \\ &&&&2&&&& \\ &&&&\vert&&&& \\ 1&-&2&-&3&-&2&-&[1] \end{array}$	-6 ($\overset{\mathfrak{so}_k}{4}$ even) -4 ($\overset{\mathfrak{so}_k}{4}$ odd)	-1	$\mathfrak{so}_{2k}, \mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccc} &&&&[1]&&&& \\ &&&&\vert&&&& \\ &&&&2&&&& \\ &&&&\vert&&&& \\ [1]&-&2&-&3&-&2&-&[1] \end{array}$	-4	0	$\mathfrak{so}_{2k}, \mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccc} &&&&[2]&&&& \\ &&&&\vert&&&& \\ &&&&2&&&& \\ &&&&\vert&&&& \\ [1]&-&2&-&3&-&2&-&[1] \end{array}$	-5 ($\overset{\mathfrak{so}_k}{4}$ even) -4 ($\overset{\mathfrak{so}_k}{4}$ odd)	-1	$\mathfrak{so}_{2k}, \mathfrak{e}_7, \mathfrak{e}_8$

Table 9: Slices appearing in the partial order of coweights related to D-type algebras $\mathfrak{so}_{2k}$, and exceptional algebras $\mathfrak{e}_7$ and $\mathfrak{e}_8$. The integer at each node of the quiver denotes the dimension of the fundamental representation, with $\mathfrak{so}_k$ in red, and $\mathfrak{sp}_k$ in blue. For the change in Coulomb-branch dimensions, the different possibilities refer to whether the fundamental representation of the $\mathfrak{so}_k$ gauge algebra of UV theory on the trivalent node is even or odd.

Quiver	CB	HB	Appears in $\mathfrak{g}$
$\begin{array}{ccccccccccc} &&&&&&[1]&&2&& \\ &&&&&&\vert&&\vert&& \\ [1]&-&2&-&3&-&4&-&4&-&2\end{array}$	8	0	$\mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccccccc} &&&&&&[1]&&2&&&& \\ &&&&&&\vert&&\vert&&&& \\ [1]&-&2&-&3&-&4&-&4&-&2&-&[1]\end{array}$	-8 ($\frac{\mathfrak{so}_k}{4}$ even) -9 ($\frac{\mathfrak{so}_k}{4}$ odd)	-1	$\mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccccccc} &&&&&&[1]&&2&&&& \\ &&&&&&\vert&&\vert&&&& \\ 1&-&2&-&3&-&4&-&4&-&2&-&[1]\end{array}$	-8 ($\frac{\mathfrak{so}_k}{4}$ even) -10 ($\frac{\mathfrak{so}_k}{4}$ odd)	-1	$\mathfrak{e}_8$
$\begin{array}{ccccccccccc} &&&&&&[1]&&2&& \\ &&&&&&\vert&&\vert&& \\ 1&-&2&-&3&-&4&-&4&-&2\end{array}$	-8	0	$\mathfrak{e}_8$
$\begin{array}{ccccccccccccc} &&&&&&&&2&&[1]&& \\ &&&&&&&&\vert&&\vert&& \\ [1]&-&2&-&3&-&4&-&5&-&4&-&2\end{array}$	-12	-1	$\mathfrak{e}_8$
$\begin{array}{ccccccccccccc} &&&&&&&&2&&[1]&& \\ &&&&&&&&\vert&&\vert&& \\ 1&-&2&-&3&-&4&-&5&-&4&-&2\end{array}$	-13	0	$\mathfrak{e}_8$
$\begin{array}{ccccccccccc} &&&&&&4&&[1]&& \\ &&&&&&\vert&&\vert&& \\ 2&-&4&-&6&-&8&-&6&-&3\end{array}$	-16	0	$\mathfrak{e}_7, \mathfrak{e}_8$
$\begin{array}{ccccccccccccc} &&[1]&&&&&&6&&&& \\ &&\vert&&&&&&\vert&&&& \\ 3&-&6&-&8&-&10&-&12&-&8&-&4\end{array}$	-28	0	$\mathfrak{e}_8$

Table 10: Slices only appearing in the partial order of coweights related to exceptional algebras $\mathfrak{e}_7$ and $\mathfrak{e}_8$. The integer at each node of the quiver denotes the dimension of the fundamental representation, with $\mathfrak{so}_k$ in red, and $\mathfrak{sp}_k$ in blue. For the change in Coulomb-branch dimensions, the different possibilities refer to the whether the fundamental representation of the $\mathfrak{so}_k$ gauge algebra of UV theory on the trivalent node is even or odd.

An example with $\widetilde{\mathfrak{b}} = B_4^{(1)}$: having made an inventory of the slices $\mathcal{S}$ that appear in the algorithm, in Figure 4.1 we illustrate how Algorithm 1 correctly reproduces the Hasse diagram of dominant coweights for fixed Q for $Spin(32)/\mathbb{Z}_2$ LSTs of type $\mathfrak{g} = \mathfrak{so}_{10}$. The finite part of the base algebra is therefore $\mathfrak{b} = \mathfrak{so}_9$. In the diagram we also indicate the coweights who seed a putative duality orbit $\mathcal{O}_{\mu^\vee}$ in a blue box, as explained in the previous section.

Dominant coweight ordering and RG flows: we have referred to the slice-subtraction algorithm as a way, given an LST, to reconstruct all LSTs with smaller coweights. As we have seen at the beginning of this section, for two LSTs to be related, it is a necessary condition that the LST at the end of the flow is smaller with respect to the partial order of dominant coweights than that of the ultraviolet theory. On the other hand, this does not *a priori* provide a sufficient condition for the partial order of coweights to be the same as the network of Higgs-branch RG flow.

However, to our knowledge, partial orders that have been explored in the recent literature have been shown to be equivalent to Higgs-branch RG flows [43,47,92]. In those case, up to possible subtleties at very low gauge-algebra ranks, there is a one-to-one correspondence between each dominant coweight and the magnetic quiver reproducing the partial order through quiver subtraction. However LSTs differ from those works in two ways. First, we are dealing with ortho-symplectic "electric" theories, while previous work have focused on unitary quivers. Since, for instance, the dimension of the fundamental representation of an $\mathfrak{sp}_k$ flavor symmetry must be even, not all dominant coweight are associated with an LST. This means that the cone of dominant coweight is not fully populated. Furthermore, anomaly cancellation forces coweights to have a fixed level of 32, which is not the case in SCFTs.

Despite these differences, it seems natural to expect RG flows to be related to that partial order. In particular, it has been shown that the closure of nilpotent orbits and other Kraft–Procesi transitions discussed above indeed correspond to RG flows in six-dimensional SCFTs [35, 43, 48, 49]. However, some of the exotic slices we have found have a peculiar property: they do not induce a change in the dimension of the Higgs branch. As an example, consider again the $\mathfrak{g} = \mathfrak{so}_{10}$ case, and let us focus on the last transition in Figure 4.1, involving LSTs associated to the next-to-trivial and trivial finite coweights:

$$\begin{array}{cccc} & [\mathfrak{so}_{30}] & & \\ & \mathfrak{sp}_Q & & \\ & 1 & & \\ \mathfrak{sp}_{Q-7} & \mathfrak{so}_{4Q-14} & \mathfrak{sp}_{2Q-15} & \mathfrak{su}_{2Q-15} \\ 1 & 4 & 1 & 2 \\ [\mathfrak{so}_2] & & & \end{array} \xrightarrow{\begin{array}{c} 2-2-2-2 \Rrightarrow 1 \\ | \\ [2] \end{array}} \begin{array}{cccc} & [\mathfrak{so}_{32}] & & \\ & \mathfrak{sp}_Q & & \\ & 1 & & \\ \mathfrak{sp}_{Q-8} & \mathfrak{so}_{4Q-16} & \mathfrak{sp}_{2Q-16} & \mathfrak{su}_{2Q-16} \\ 1 & 4 & 1 & 2 \end{array} \qquad (4.7)$$

One can verify that there is indeed no change in the Higgs-branch dimension in agreement with Table 8. If there is a Higgs-branch deformation between the two theories, one would—

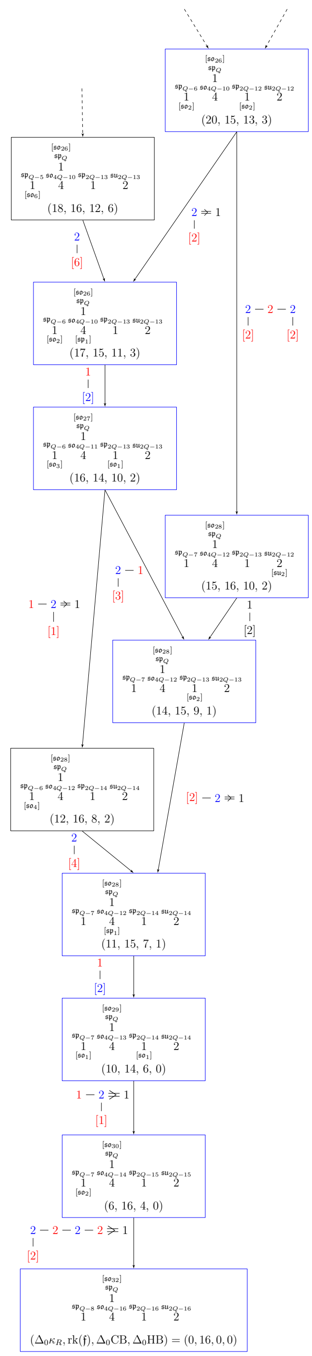


Figure 4.1: Part of the Hasse diagram of $Spin(32)/\mathbb{Z}_2$ LSTs with $\mathfrak{g} = \mathfrak{so}_{10}$ at fixed value of Q following the partial order of *finite* coweights. Theories in a blue box indicate are the representative of a duality orbit. Each arrow indicates the subtracted quiver corresponding to the slice $\mathcal{S}$. The integers associated with each LST indicate the total flavor rank or the change from the theory with trivial coweight, e.g. $\Delta_0\kappa_R(\mu^\vee) = \kappa_R(\mu^\vee) - \kappa_R(\varnothing)$.

albeit maybe too naively—expect that as the R-symmetry is broken along the RG flow, there must be at least one hypermultiplet acting as its Nambu–Goldstone mode that decouple from the UV theory. We therefore expect the change in the Higgs-branch dimension to always be strictly negative.

In Tables 8–10 one can find several slices with that feature. One may therefore be tempted to say that these slices cannot be physical. From a geometric point of view however, it does not—again, at least naively—appear to be problematic: breaking a gauge algebra to one of its subalgebra e.g. $\mathfrak{su}_{2Q-14}$ to $\mathfrak{su}_{2Q-16}$ in Equation (4.7) corresponds to smoothing the fiber singularity from a Kodaira singularity I_{2Q-14} to I_{2Q-16}. Individually, this is always possible, see e.g. reference [97].

If the deformation cannot be done geometrically, a problem must occur due to a conspiracy between singularities involving multiple curves. Ruling out these transitions would require to explicitly find the Weierstrass model associated with both LSTs and check whether a complex-structure deformation between the two is possible. While this has been done for LSTs of rank zero to explore the Higgs-branch Hasse diagram and match with magnetic-quiver techniques [32], in practice this is however notoriously difficult to achieve for a large number of curves, and we are not aware of a reference that has attempted to explicitly construct the deformation for a multi-curve Weierstrass model. Furthermore, the LSTs where these curious slices appear do not have, to our knowledge, a known description in terms of a magnetic quiver. Whether this puzzle can be reconciled in either pictures appears to be an interesting avenue for future work.

5 Extension to Other Types of LSTs and SCFTs

We have so far solely focused on $Spin(32)/\mathbb{Z}_2$ heterotic LSTs. In this section, we briefly explore possible extensions of our results to other theories. Indeed, other types of six-dimensional quivers and the associated LSTs or SCFTs involving only classical algebras can also be rephrased in the language of coweights, and similar analyses can be performed.

The main reason we have focused on $Spin(32)/\mathbb{Z}_2$ heterotic LSTs stems from the fact that they involve only bifundamental hypermultiplets, and the type of flavor and gauge symmetries are uniquely set by the McKay correspondence, as reviewed in Section 2.2. However, as we will now see, it also applies straightforwardly to another class of $\mathfrak{so}_{32}$ heterotic LSTs, with their data encoding the coweight lattice of *twisted* affine algebras.[16] It may be used to also incorporate cases involving two-anti-symmetric representations of $\mathfrak{su}_k$, arising from a curve

[16] In this section, we always mean twisted in the Lie-algebraic sense, rather than to refer to twisted compactification. The latter will not be discussed here, see however e.g. reference [34].

configuration $\overset{\mathfrak{su}_k}{1}$ under a very mild modification. We will further argue that certain properties relating to dominant coweights also apply to SCFTs, offering a generalization of the slice-subtraction algorithm also in that case, so that our result cover the overwhelming majority of known consistent six-dimensional theories with only classical symmetries.

5.1 $(SU(16) \times U(1))/\mathbb{Z}_2$ Little String Theories

The $\mathfrak{so}_{32}$ heterotic Little String Theories we have discussed above were labeled by (folded) ADE algebras through the McKay correspondence, and fall in the classification by Blum and Intriligator [36], see Classification 2. In references [28, 34], other types of $\mathfrak{so}_{32}$ LSTs not captured by the original classification were discovered via explicit geometric construction.

Instead of the $Spin(32)/\mathbb{Z}_2$ global symmetry that descends from the heterotic construction, they are characterised by one of its maximal rank-preserving subgroups:

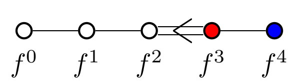
$$Spin(32)/\mathbb{Z}_2 \quad \longrightarrow \quad (SU(16) \times U(1))/\mathbb{Z}_2\,. \tag{5.1}$$

While such a breaking could naively be thought of as being obtained from a choice of non-trivial affine coweight of the ADE algebra $\widetilde{\mathfrak{g}}$ that encodes the decomposition $\mathfrak{so}_{32} \to \mathfrak{su}_{16} \oplus \mathfrak{u}_1$, it turns out that the gauge and intersection pattern in the F-theory description are different. For instance, one may obtain the following quiver:

$$\underset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_Q}{2}}\ \overset{\mathfrak{su}_{2Q-16}}{2}\ \overset{\mathfrak{su}_{3Q-32}}{2}\ \overset{\mathfrak{sp}_{2Q-24}}{1}\ \overset{\mathfrak{so}_{Q-16}}{4}\ , \tag{5.2}$$

which has the advertised flavor symmetry, the $\mathfrak{u}_1$ symmetry arising through the arguments discussed around Equation (2.29). Its linear shape is reminiscent of the $Spin(32)/\mathbb{Z}_2$ LST with $\mathfrak{g} = \mathfrak{e}_6$, see Equation (2.38). However the gauge symmetries—and consequently the curve self-intersections—are indeed different. The hypermultiplet adjacency matrix G discussed in Classification 1 is therefore distinct from that of $F_4^{(1)}$, and corresponds instead to the symmetrized Cartan matrix of the twisted algebra $\widetilde{\mathfrak{b}} = E_6^{(2)}$. For a generic quiver of the type above, we can therefore make the following identification:

$$\underset{[\mathfrak{su}_{f^0}]}{\overset{\mathfrak{su}_{d_0}}{2}}\ \underset{[\mathfrak{su}_{f^1}]}{\overset{\mathfrak{su}_{d_1}}{2}}\ \underset{[\mathfrak{su}_{f^2}]}{\overset{\mathfrak{su}_{d_2}}{2}}\ \underset{[\mathfrak{so}_{f^3}]}{\overset{\mathfrak{sp}_{d_3/2}}{1}}\ \underset{[\mathfrak{sp}_{f^4/2}]}{\overset{\mathfrak{so}_{d_4}}{4}} \qquad \longleftrightarrow \tag{5.3}$$



As before, we have labeled the Dynkin diagrams with the dimension of the fundamental representation of the associated flavor symmetries, coloring them in the same fashion as for $Spin(32)/\mathbb{Z}_2$ LSTs, see Table 2. There are only three classes of LSTs of that type, all with a

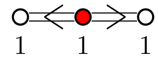
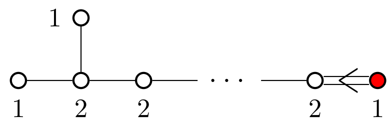
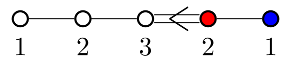
$\widetilde{\mathfrak{b}}$	$h^\vee_{\mathfrak{b}}$	Twisted Dynkin Diagram	$(SU(16)\times U(1))/\mathbb{Z}_2$ LST
$D_3^{(2)}$	4		$\underset{[\mathfrak{su}_{f0}]}{\overset{\mathfrak{su}_{d_0}}{2}}\ \underset{[\mathfrak{so}_{f1}]}{\overset{\mathfrak{sp}_{d_1/2}}{1}}\ \underset{[\mathfrak{su}_{f2}]}{\overset{\mathfrak{su}_{d_2}}{2}}$
$A_{2K-1}^{(2)}$	$2K$		$\underset{[\mathfrak{su}_{f1}]}{\overset{\mathfrak{su}_{d_1}}{2}}\ \underset{[\mathfrak{su}_{d_1}]}{\overset{\overset{[\mathfrak{su}_{f0}]}{\overset{\mathfrak{su}_{d_0}}{2}}}{\overset{\mathfrak{su}_{d_2}}{2}}}\ \underset{[\mathfrak{su}_{f_3}]}{\overset{\mathfrak{su}_{d_3}}{2}}\ \cdots\ \underset{[\mathfrak{su}_{f_{K-1}}]}{\overset{\mathfrak{su}_{d_{K-1}}}{2}}\ \underset{[\mathfrak{so}_{f_K}]}{\overset{\mathfrak{sp}_{d_K/2}}{1}}$
$E_6^{(2)}$	12		$\underset{[\mathfrak{su}_{f0}]}{\overset{\mathfrak{su}_{d_0}}{2}}\ \underset{[\mathfrak{su}_{f1}]}{\overset{\mathfrak{su}_{d_1}}{2}}\ \underset{[\mathfrak{su}_{f2}]}{\overset{\mathfrak{su}_{d_2}}{2}}\ \underset{[\mathfrak{so}_{f3}]}{\overset{\mathfrak{sp}_{d_3/2}}{1}}\ \underset{[\mathfrak{sp}_{f4/2}]}{\overset{\mathfrak{so}_{d_4}}{4}}$

Table 11: LSTs of type $(SU(16)\times U(1))/\mathbb{Z}_2$ and group-theoretic quantities associated with the twisted Lie algebra of its base $\widetilde{\mathfrak{b}}$. The labels of the Dynkin diagram corresponds to the marks θ_I of $\widetilde{\mathfrak{b}}$.

base corresponding to a twisted algebra:[17]

$$(SU(16)\times U(1))/\mathbb{Z}_2 \quad \text{LST}: \qquad \widetilde{\mathfrak{b}} \in \{A_{2k-1}^{(2)}, D_3^{(2)}, E_6^{(2)}\} \tag{5.4}$$

The corresponding generalized quivers are given in Table 11, along with useful quantities.

In the same fashion as in previous cases, one can show that the quartic anomaly-cancellation condition takes precisely the expected form:

$$C^{IJ} d_J = f^I - 16 S^I\,, \tag{5.5}$$

where C^{IJ} is now the Cartan matrix of the twisted algebra $\widetilde{\mathfrak{b}}$. S^I is the generalization of the Frobenius–Schur indicators to twisted algebras. While the McKay correspondence applies only to ADE algebras, we can proceed similarly as we did for folding in Equation (2.39): if $\mathcal{T}$ is the map transforming the root system of $\widetilde{\mathfrak{g}}$ to that of its twisted version $\widetilde{\mathfrak{b}}$, we simply define $S^I = \mathcal{T}(S^A)$ with S^A the Frobenius–Schur indicator of the A-th node of the Dynkin diagram of $\widetilde{\mathfrak{g}}$. For instance, for $\widetilde{\mathfrak{b}} = E_6^{(2)}$, we have $S^I = (0,0,0,-1,1)$. As before, the values of S_I are a Lie-algebraic proprety of $\widetilde{\mathfrak{b}}$, which can be obtained directly from the colored Dynkin diagram, where we have used the same conventions as in Table 11.

There is however a significant difference between the $Spin(32)/\mathbb{Z}_2$ and $(SU(16)\times U(1))/\mathbb{Z}_2$

[17] In the first two cases, there are also two analogous classes involving a $\overset{\mathfrak{su}_k}{1}$ configuration. These are discussed in Section 5.2.

LSTs: it is easily shown that for the latter, Equation (5.5) leads to the constraints

$$(SU(16)\times U(1))/\mathbb{Z}_2: \qquad f^I\theta_I = 16\,, \qquad \theta_I S^I = 1\,, \qquad d_I = Q\,K_I + X_{IJ}(f^J - 16S^J)\,. \tag{5.6}$$

We recall that the coefficients θ_I and K_I are the marks and comarks of $\widetilde{\mathfrak{b}}$, and X_{IJ} inverts the Cartan matrix up to an integer factor Q of its nullspace, see Equation (2.19).

These constraints are similar to those we obtained for the folded algebras, see equations (2.17) and (2.36). In terms of coweights, they imply that for these theories, the dimensions f^I of the fundamental representation of flavor symmetries define an affine coweight $\widetilde{\mu}^\vee$ of $\widetilde{\mathfrak{b}}$ at level 16 rather than 32. Furthermore the value of the scaling element—the coefficient n in Equation (2.43)—is now Q rather than $2Q$. This is explained by the fact that for these twisted algebras, on the zeroth node the fundamental representation of the $\mathfrak{su}_Q$ gauge symmetry has a dimension $d_0 = Q$, rather than $d_0 = 2Q$ for $\mathfrak{sp}_Q$.

The T-duality invariants of these theories are then easily computed. For the two-group constant κ_R, one finds:

$$\kappa_R = (\theta_I K^I)(Q - 2r_{\mathfrak{b}} - 2) + 2 + \theta^i C_{ij} f^j = \langle \widetilde{T}, \widetilde{\mu}^\vee - 16\widetilde{S}^\vee \rangle\,. \tag{5.7}$$

with, as before, $\widetilde{T} = \sum_I \theta_I \widetilde{\omega}_I$ and C_{ij} the inverse Cartan matrix when omitting the zeroth node. This form is very reminiscent of the one we found in Equation (2.47). A similar expression can be found for the Coulomb-branch dimension, albeit slightly more complicated due to the floor function related to the rank of $\mathfrak{so}_k$ gauge algebras, but is of analogue form to Equation (2.48).

We therefore reach the conclusion that all $\mathfrak{so}_{32}$ heterotic LSTs involving only bifundamental hypermultiplets, either of type $Spin(32)/\mathbb{Z}_2$ or $(SU(16)\times U(1))/\mathbb{Z}_2$, are classified by affine dominant coweights of their base $\widetilde{\mathfrak{b}}$. While we will not go over a complete analysis of the T-duality orbit and the slice subtraction algorithm, one may proceed in a similar fashion.

To close this section, we note that the coefficients multiplying Q in the T-dual invariants are given in terms of the marks of $\widetilde{\mathfrak{b}}$, and satisfy an interesting numerology. As was observed in references [28, 34], for $D_3^{(2)}, A_{2K-1}^{(2)}, E_6^{(2)}$ these coefficients are *half* that of $Spin(32)/\mathbb{Z}_2$ with $\mathfrak{g} = \mathfrak{so}_8, \mathfrak{so}_{4K}, \mathfrak{e}_7$, respectively. For instance, with $\widetilde{\mathfrak{b}} = E_6^{(2)}$, we have $K^I\theta_I = \Gamma_{\mathfrak{e}_6} = 24 = \frac{1}{2}\Gamma_{\mathfrak{e}_7}$. In fact, for all three types of algebras, not only do these quantities satisfy this numerical curiosity, but when $Q = 2Q'$, the T-dual invariant exactly match those of the $Spin(32)/\mathbb{Z}_2$ theories with number of instanton Q' for appropriate choices of finite coweights. The T-duality can be verified geometrically [28, 34], and in the terminology described at the beginning of Section 3, corresponds to infinite families of exotic T-dualities. For trivial coweights, these theories can be found in Table 15 in Appendix A.

5.2 Including Anti-symmetric Representations of $\mathfrak{su}_k$

Throughout this work, we have always carefully excluded the case of $Spin(32)/\mathbb{Z}_2$ with $\mathfrak{g} = \mathfrak{su}_{2n+1}$, as one of the node of its quiver includes an additional hypermultiplet transforming in the two-anti-symmetric representation of an $\mathfrak{su}_k$ algebra, and therefore does not fall in Classification 1, where we have considered only bifundamental hypermultiplets.

Geometrically, the inclusion of a single hypermultiplet in the two-anti-symmetric representation is well known to be described by a curve configuration $\overset{\mathfrak{su}_k}{1}$. Field theoretically, this is once again explained at the level of cancellation of quartic anomalies: one hypermultiplet in the antisymmetric, in addition to those in the fundamental representation, mimics the contribution to the quartic gauge anomaly of hypermultiplets transforming in the fundamental representation of an $\mathfrak{sp}_k$ gauge symmetry.

The absence of quadratic gauge anomalies are then shown to indeed reproduce a Dirac pairing corresponding to that of a (-1)-curve. This conspiracy allows us to treat the anti-symmetric representation as if that node was associated with an $\mathfrak{sp}_k$ symmetry. Indeed, from the F-theory classification [9], the LST with $\mathfrak{g} = \mathfrak{su}_{2n+1}$ is the collection of n curves forming the following generalized quiver:

$$\underset{[\mathfrak{so}_{f^0}]}{\overset{\mathfrak{sp}_{k_0}}{1}} \underset{[\mathfrak{su}_{f^1}]}{\overset{\mathfrak{su}_{k_1}}{2}} \cdots \underset{[\mathfrak{su}_{f^{n-1}}]}{\overset{\mathfrak{su}_{k_{n-1}}}{2}} \underset{[\mathfrak{su}_{f^n}]}{\overset{\mathfrak{su}_{k_n}}{1}} . \tag{5.8}$$

Using the same techniques used in Section 2.1 to find the anomaly polynomial, it is straightforward to see that requiring cancellation of quartic gauge anomalies imposes the condition

$$C^{IJ} d_J = \mu^I - 16 S^I , \qquad \mu^J = (f^0, f^1, \dots, 2f^n) , \qquad d_I = (2k_0, k_1, \dots, k_n) . \tag{5.9}$$

with C^{IJ} is the Cartan matrix of $\widetilde{\mathfrak{b}} = C_n^{(1)}$, and $S^I = 1$ if $I = 0, n$ and zero otherwise.

We therefore see that we have a similar condition as for $\mathfrak{g} = \mathfrak{su}_{2n}$: the Cartan matrix and the Frobenius–Schur indicators are that of $C_n^{(1)}$ in both cases. We therefore expect μ^I to be associated with an affine coweight of that algebra. However the set $\widetilde{\mathcal{P}}^\vee_{\mathrm{LST}}$ leading to those two types of LSTs are different: μ^n must be even to ensure $f^n = \mu^n/2$ is a positive integer, while the coefficient d_I of the last coroot must not necessarily be even. For theories with $\mathfrak{g} = \mathfrak{su}_{2n}$, these two constraints are reversed.

We therefore conclude that even in the presence of a single anti-symmetric representation of $\mathfrak{su}_{k_n}$, all the arguments described in the previous sections go through with only minor modifications.

A similar analysis also applies to the two generalized quivers corresponding to $(SU(16) \times U(1))/\mathbb{Z}_2$ LSTs shown in Table 11, where we can again replace the $\mathfrak{sp}_k$ algebra on the (-1)-

curve with an $\mathfrak{su}_k$ symmetry to introduce an anti-symmetric representation:

$$
\underset{[\mathfrak{su}_{\mu^0}]}{\overset{\mathfrak{su}_{d_0}}{2}}\ \underset{[\mathfrak{su}_{\mu^1/2}]}{\overset{\mathfrak{su}_{d_1}}{1}}\ \underset{[\mathfrak{su}_{\mu^2}]}{\overset{\mathfrak{su}_{d_2}}{2}}\,, \qquad \qquad \underset{[\mathfrak{su}_{\mu^1}]}{\overset{\mathfrak{su}_{d_1}}{2}}\ \underset{[\mathfrak{su}_{\mu^1}]}{\overset{\overset{[\mathfrak{su}_{\mu^0}]}{\overset{\mathfrak{su}_{d_0}}{2}}}{\overset{\mathfrak{su}_{d_2}}{2}}}\ \underset{[\mathfrak{su}_{\mu^3}]}{\overset{\mathfrak{su}_{d_3}}{2}} \cdots \underset{[\mathfrak{su}_{\mu^{K-1}}]}{\overset{\mathfrak{su}_{d_{K-1}}}{2}}\ \underset{[\mathfrak{su}_{\mu^K/2}]}{\overset{\mathfrak{su}_{d_K}}{1}}\,. \tag{5.10}
$$

Together with an integer Q, the coefficients μ^I define an affine coweight at level 16 of $D_3^{(2)}$ and $A_{2k-1}^{(2)}$ respectively.

Both LSTs with an $\mathfrak{su}$ or $\mathfrak{sp}$ on a (-1)-curve are therefore described by coweights of the same algebra, and we can apply the techniques described in Section 3 to find putative T-duality orbits in a similar way, we expect the observations made in Section 4 to be different. The slice-subtraction algorithm should still hold, but since the set of coweights and gauge algebras changed, so will the involved slices. We therefore possibly expect new minimal slices to appear.

Furthermore, the presence of a hypermultiplet in the antisymmetric representation opens up new transitions between e.g. $Spin(32)/\mathbb{Z}_2$ LSTs of type $\mathfrak{g} = \mathfrak{su}_{2k+1}$ and $\mathfrak{su}_{2k}$. Indeed, giving a vacuum expectation value to this field leads to the decomposition $\mathfrak{su}_{2k} \to \mathfrak{sp}_k$. For LSTs on a 0-curve, this was shown to be associated with the exotic slice $h_{2,k}$ [32]. This type of deformation is indeed always possible [98], as expected from field theory, and we have:

$$
\cdots \underset{[\mathfrak{su}_{f^{n-1}}]}{\overset{\mathfrak{su}_{k_{n-1}}}{2}}\ \underset{[\mathfrak{su}_{f^n}]}{\overset{\mathfrak{su}_{2k_n}}{1}} \qquad \longrightarrow \qquad \cdots \underset{[\mathfrak{su}_{f^{n-1}}]}{\overset{\mathfrak{su}_{k_{n-1}}}{2}}\ \underset{[\mathfrak{so}_{\mu^n}]}{\overset{\mathfrak{sp}_{k_n}}{1}}\,. \tag{5.11}
$$

We leave a more extensive analysis of the space of dominant coweight involving anti-symmetric representations and the possible transitions between different types of LSTs for future work.

5.3 $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ Heterotic Little String Theories

Due to their perturbative nature in the low-energy regime, we have only considered $\mathfrak{so}_{32}$ heterotic LSTs in this work. Via the standard duality between the $\mathfrak{so}_{32}$ and $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic strings, we can have access to a larger landscape of theories, which can exhibit what we referred to as "fiber-base" T-dualities in Section 3. These LSTs are constructed similarly through stacks NS5-branes probing a $\mathbb{C}^2/\Gamma$ orbifold.

They are labeled by a pair of embeddings of the finite group in to the *group* E_8: $\mu_L^\vee, \mu_R^\vee \in \mathrm{Hom}(\Gamma, E_8)$ corresponding choices of connection at infinity for the "left" and "right" $\mathfrak{e}_8$ factors. In those cases, the presence of curves with self-intersections different than we have encountered in the rest of this work implies the existence of non-perturbative states. While field-theoretic techniques can be employed to study some of their properties, in particular their anomalies [43, 64, 66, 99–101], one usually resorts to geometric engineering.

When both embeddings are trivial, $\mu_L^\vee = \varnothing = \mu_R^\vee$, one obtains an $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ flavor symmetry, with generalized quivers given in Table 14 of Appendix A. One observes, starting with Q or M NS5-branes probing the same $\mathbb{C}^2/\Gamma$ orbifold in the $\mathfrak{so}_{32}$ and $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ picture, respectively, that the theories with trivial embedding can only be T-dual if

$$\kappa_R = \Gamma(M + r_\mathfrak{g} + 2) + 2\,, \qquad \dim(\mathrm{CB}) = h^\vee M - \dim(\mathfrak{g})\,, \qquad M = Q + n_\mathfrak{g} - 2(r_\mathfrak{g} + 2)\,, \tag{5.12}$$

where $n_g = 2, 4, 6, 8, 12$ for $\mathfrak{g} = \mathfrak{su}_K, \mathfrak{so}_{2K}, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$. The difference between the two instanton numbers Q and M on each side of the duality is related to the fractionalization of the five-branes, see e.g. references [27, 28, 71, 102]. In the F-theory picture, $n_\mathfrak{g}$ correspond to (minus) the smallest self-intersection appearing in the quiver, as can be seen in Table 14.

A subtlety with T-duality: the discussion above matches the T-duality invariants at a generic point of the tensor-branch moduli space. As was first observed in reference [26], on the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ side when reaching the singular point corresponding to the actual Little String Theory, geometrically, one must supplement this data with a choice of a contraction map. Given a brane system, this is equivalent to a choice of partition $M = M_L + M_R$ indicating how many five-branes are collapsed to the "left" and "right" nine-branes. This was studied in detail for $\mathfrak{g} = \mathfrak{su}_K$ in reference [35], where it was explicitly shown that different choices of M_L and M_R lead to different Higgs-branch or equivalently different magnetic quivers. The Coulomb branches of theories with different (M_L, M_R) is therefore disconnected, but related geometrically by flop transitions as those are simply contractions of the same underlying theory.

This is why we must allow to move in the full *extended* Coulomb branch moduli space to call two theories T-dual in Definition 1 which in geometry corresponds to the *extended* Kähler moduli space. Hence the different 6d F-theory models must come from the same *birational* equivalent CY-threefolds.

Non-trivial embeddings: more general theories corresponds to generic choices of flat connections. The embeddings $\mathrm{Hom}(\Gamma, E_8)$ and the possible quivers are in principle classified [7, 86]; however the precise mapping between the two is not known in full generality. For $\mathfrak{g} = \mathfrak{su}_K$, embeddings $\mathrm{Hom}(\mathbb{Z}_K, E_8)$ are known to be classified by affine coweights of $E_8^{(1)}$ at level K [62], see also reference [43] for recent work in the context of type-A SCFTs. An algorithm to pass from the coweight to the generalized quiver has been proposed in reference [39].

For other types of algebras, one also expects to associate a precise affine coweight of $E_8^{(1)}$

to an SCFT as they are the natural object to classify the embeddings. Two coweights $\mu_L^\vee$ and $\mu_R^\vee$ can then be used to "glue" two such SCFTs to reach any $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ LST [9, 87]. Using the formula for κ_R, this suggest that one simply sums the contributions from both theories independently.

Considering the various results in this work for $\mathfrak{so}_{32}$ LSTs, it is very tempting to conjecture that one should be able to always find two affine coweights such that κ_R is obtained as a purely Lie-theoretic quantity

$$\kappa_R \overset{?}{\propto} \langle \widetilde{T}, \widetilde{\mu}_L^\vee + \widetilde{\mu}_R^\vee \rangle \,, \tag{5.13}$$

for some affine weight $\widetilde{T}$. Note however that the precise mapping is not clear. For $\mathfrak{so}_{32}$ LSTs, the coweight is associated the algebra corresponding to the orbifold singularity rather than $\mathfrak{so}_{32}$, while type-A LSTs are associated with coweights of $E_8^{(1)}$. It would therefore be interesting to investigate whether a similar classification to that of $\mathfrak{so}_{32}$ LSTs is possible. This would have important ramifications for both LSTs and the SCFTs that they are made out of. Not only could certain quantities be efficiently computed without needing to explicitly know the generalized quiver [43, 66], but we also expect the induced partial order to hold information on the associated network of RG flows, as well as the network of internal T-duality with similar tools developed in this work.

RG flows: the duality between $\mathfrak{so}_{32}$ and $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ could in principle also shed some light on whether some of the transition discussed in Section 4 are indeed associated with RG flows although there is no change in the Higgs-branch dimension. For type-A LSTs, it has been argued that if there is a Higgs-branch flow between two LSTs, there must be a similar Higgs-branch flow on the other side of the T-duality, and vice versa [35].

As an example, consider the following potential RG flow between two $Spin(32)/\mathbb{Z}_2$ LSTs with $\mathfrak{g} = \mathfrak{e}_6$, corresponding to the covering between the next-to-minimal and the trivial coweights:

$$\underset{[\mathfrak{so}_{30}]}{\overset{\mathfrak{sp}_Q}{1}} \; \overset{\mathfrak{so}_{4Q-14}}{4} \; \overset{\mathfrak{sp}_{3Q-22}}{1} \; \overset{\mathfrak{su}_{4Q-29}}{2} \; \underset{[\mathfrak{su}_1]}{\overset{\mathfrak{su}_{2Q-14}}{2}} \quad \xrightarrow{\color{red}{2} - \color{blue}{4} \Rrightarrow 3 - 2 - [1]} \quad \underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}} \; \overset{\mathfrak{so}_{4Q-16}}{4} \; \overset{\mathfrak{sp}_{3Q-24}}{1} \; \overset{\mathfrak{su}_{4Q-32}}{2} \; \overset{\mathfrak{su}_{2Q-16}}{2} \tag{5.14}$$

The minimal slice $\mathcal{S}$ is depicted above; the change in the dimension of the fundamental representation of the gauge group arranges like the comarks of $E_6^{(2)}$. This is an example of a minimal slice where there is no change in the dimension of the Higgs branch. It is therefore natural to ask what happens on the other side of a fiber-base duality. Focusing for simplicity only on $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ duals with one of the embedding trivial, say $\widetilde{\mu}_L^\vee = \varnothing$, we find only one candidate:

$$\underset{[\mathfrak{so}_{30}]}{\overset{\mathfrak{sp}_Q}{1}}\ \overset{\mathfrak{so}_{4Q-14}}{4}\ \overset{\mathfrak{sp}_{3Q-22}}{1}\ \overset{\mathfrak{su}_{4Q-29}}{2}\ \underset{[\mathfrak{su}_1]}{\overset{\mathfrak{su}_{2Q-14}}{2}} \quad \overset{\text{T-dual}}{\longleftrightarrow} \quad \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \underbrace{\overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1 \cdots 1\ \overset{\mathfrak{su}_3}{3}\ 1}_{Q-10}\ \overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ \underset{[\mathfrak{u}_1]}{1}\ \overset{\mathfrak{so}_{10}}{4}\ \underset{[\mathfrak{so}_{14}]}{\overset{\mathfrak{sp}_2}{1}}\,, \tag{5.15}$$

with $\kappa_R = 24Q-150$ and $\dim(\text{CB}) = 12Q-70$. On the other hand for the second $Spin(32)/\mathbb{Z}_2$ theory, we find two candidates with the property above:

$$\begin{array}{ccc}
\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \overset{\mathfrak{so}_{4Q-16}}{4}\ \overset{\mathfrak{sp}_{3Q-24}}{1}\ \overset{\mathfrak{su}_{4Q-32}}{2}\ \overset{\mathfrak{su}_{2Q-16}}{2} & \overset{\text{T-dual}}{\longleftrightarrow} & \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \underbrace{\overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1 \cdots 1\ \overset{\mathfrak{su}_3}{3}\ 1}_{Q-10}\ \overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}_2}{2}\ 2\ \underset{[\mathfrak{e}_8]}{1} \\
 & & \updownarrow \text{T-dual} \\
 & & \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{f}_4}{5}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \underbrace{\overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1 \cdots 1\ \overset{\mathfrak{su}_3}{3}\ 1}_{Q-10}\ \overset{\mathfrak{e}_6}{6}\ 1\ \overset{\mathfrak{su}_3}{3}\ 1\ \overset{\overset{[\mathfrak{su}_3]}{1}}{\overset{\mathfrak{e}_6}{6}}\ 1\ \overset{\mathfrak{su}_3}{3}\ \underset{[\mathfrak{e}_6]}{1}
\end{array} \tag{5.16}$$

with $\kappa_R = 24Q - 166$ and $\dim(\text{CB}) = 12Q - 78$. We have checked that the theories in equations (5.15) and (5.16) are indeed T-dual: using the methods of reference [31] we have found explicit toric realizations from which these theories originate.

Furthermore, on the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$, no RG flow can take place between these theories. Generically, in the generalized quiver picture an RG flow corresponds to either smoothing fiber singularities, corresponding to break a gauge algebra to one of its subgroup, blowing down an undecorated (-1)-curve with an appropriate change in the self-intersection of the neighboring curves, or a combination of both, see e.g. reference [50].

One may therefore be tempted to say that the minimal slice in equation (5.14) is unphysical as there is no possible RG flow on the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic side. However, we have found examples where, while a minimal slice of $\mathfrak{so}_{32}$ heterotic LSTs induces no change in the Higgs-branch dimension, an RG flow is possible for at least one of the T-dual $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LSTs.

Indeed, let us come back to the $\mathfrak{g} = \mathfrak{so}_{10}$ example we have already encountered in Section 4.1:

$$\underset{[\mathfrak{so}_2]}{\overset{\mathfrak{sp}_{Q-7}}{1}}\ \overset{\overset{\overset{[\mathfrak{so}_{30}]}{\mathfrak{sp}_Q}}{1}}{\overset{\mathfrak{so}_{4Q-14}}{4}}\ \overset{\mathfrak{sp}_{2Q-15}}{1}\ \overset{\mathfrak{su}_{2Q-15}}{2} \quad \xrightarrow{\begin{smallmatrix} 2-2-2-2 \Rrightarrow 1 \\ | \\ [2] \end{smallmatrix}} \quad \overset{\mathfrak{sp}_{Q-8}}{1}\ \overset{\overset{\overset{[\mathfrak{so}_{32}]}{\mathfrak{sp}_Q}}{1}}{\overset{\mathfrak{so}_{4Q-16}}{4}}\ \overset{\mathfrak{sp}_{2Q-16}}{1}\ \overset{\mathfrak{su}_{2Q-16}}{2} \tag{5.17}$$

This transition also does not change the dimension of the Higgs branch. As above, for the first theory we find the following putative T-dual $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LSTs:

$$
\overset{\mathfrak{sp}_{Q-7}}{\underset{[\mathfrak{so}_2]}{1}}\ \overset{\overset{[\mathfrak{so}_{30}]}{\overset{\mathfrak{sp}_Q}{1}}}{\overset{\mathfrak{so}_{4Q-14}}{4}}\ \overset{\mathfrak{sp}_{2Q-15}}{1}\ \overset{\mathfrak{su}_{2Q-15}}{2} \quad \overset{\text{T-dual?}}{\longleftrightarrow} \quad \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}_9}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underbrace{\overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1} \cdots \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1}}_{Q-9}\ \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underset{[\mathfrak{u}_1]}{\overset{\mathfrak{so}_{10}}{3}}\ \underset{[\mathfrak{so}_{14}]}{\overset{\mathfrak{sp}_2}{1}} \tag{5.18}
$$

with $\kappa_R = 12Q - 64$ and CB $= 8Q - 41$. On the other hand, the IR theory has four putative T-dual $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LSTs:

$$
\begin{aligned}
\overset{\mathfrak{sp}_{Q-8}}{1}\ \overset{\overset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}}{\overset{\mathfrak{so}_{4Q-16}}{4}}\ \overset{\mathfrak{sp}_{2Q-16}}{1}\ \overset{\mathfrak{su}_{2Q-16}}{2} \quad \overset{\text{T-dual?}}{\longleftrightarrow} \quad & \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}_9}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underbrace{\overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1} \cdots \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1}}_{Q-9}\ \overset{\mathfrak{so}_9}{4}\ 1\ \overset{\mathfrak{g}_2}{3}\ \overset{\mathfrak{su}_2}{2}\ 2\ \underset{[\mathfrak{e}_8]}{1} \\
& \qquad \updownarrow \text{T-dual?} \\
& \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}_9}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underbrace{\overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1} \cdots \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1}}_{Q-9}\ \overset{\mathfrak{so}_9}{4}\ 1\ \underset{[\mathfrak{sp}_1]}{\overset{\mathfrak{so}_7}{3}}\ \overset{\mathfrak{su}_2}{2}\ \underset{[\mathfrak{e}_7]}{1} \\
& \qquad \updownarrow \text{T-dual?} \\
& \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}_9}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underbrace{\overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1} \cdots \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1}}_{Q-9}\ \overset{\mathfrak{so}_9}{4}\ 1\ \overset{\mathfrak{so}_7}{3}\ \underset{[\mathfrak{so}_{16}]}{\overset{\mathfrak{sp}_2}{1}} \\
& \qquad \updownarrow \text{T-dual?} \\
& \underset{[\mathfrak{e}_8]}{1}\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{g}_2}{3}\ 1\ \overset{\mathfrak{so}_9}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underbrace{\overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1} \cdots \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{su}_2}{1}}_{Q-9}\ \overset{\mathfrak{so}_{10}}{4}\ \overset{\mathfrak{sp}_1}{1}\ \underset{[\mathfrak{su}_8]}{\overset{\mathfrak{su}_5}{2}} \oplus [\mathfrak{u}_1]
\end{aligned} \tag{5.19}
$$

with $\kappa_R = 12Q - 70$ and CB $= 8Q - 45$. We unfortunately did not find a toric realization of these theories; they however have the same T-dual invariants. Contrary to the $\mathfrak{g} = \mathfrak{e}_6$ example discussed above, one can reach the last two $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LSTs through an RG flow starting from the theory in equation (5.18). On the other hand there cannot be a deformation leading to the first two, as the number of curves is larger than that of the UV theory. This is peculiar, as it contradicts the observation that was made for type-A heterotic LSTs, namely that there is an RG flow on one side of the duality, there should be one in the other.

In the absence of a complete classification of $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ in terms of coweights, we cannot perform an exhaustive analysis of how Hasse diagrams behave under (putative) T-duality. These two examples however show that the mapping is subtle. Geometrically, if two theories are related by an RG flow, there should be a complex-structure deformation from one to the other. On the other hand, as T-dual models correspond to different elliptic fibrations of the same Calabi–Yau, it is not guaranteed that the deformation will preserve the fibration structure. It would therefore be interesting to study how T-duality acts on the Higgs-branch of specific models, and whether new patterns arise.

5.4 Type-II LSTs

Another class of Little String Theories can be obtained from Type-II string theory, giving them their name [24]. In the F-theory construction, they are engineered from an elliptically-fibred Calabi–Yau over a base birational to $(T^2 \times \mathbb{C})/\Lambda$ with $\Lambda \subset SU(2)$ [9]. Both the fiber and the base are therefore described by a Kodaira singularity $(\mathfrak{g}_F,\ \mathfrak{g}_B)$. T-duality corresponds to exchanging the two singularities—and is an example of fiber-base duality in the nomenclature of Section 3—and was established through explicit geometric constructions [33]. Such theories allow for more complicated intersection patterns. However, when the singularity $\mathfrak{g}_F$ corresponds to one of the classical algebras, the adjacency pattern follows once again Classification 1, and corresponds to an affine algebra $\widetilde{\mathfrak{b}}$. The precise gauge algebras are uniquely fixed by the pair $(\mathfrak{g}_F,\ \mathfrak{g}_B)$. However, contrary to $\mathfrak{so}_{32}$ heterotic LSTs, the gauge content is not fixed by the McKay correspondence, in the sense that on a given node, the algebra does not follow directly from the type of representation of the corresponding finite subgroup of $SU(2)$.

On the other hand, they have similar properties to the $\mathfrak{so}_{32}$ LSTs. For instance, the cancellation of quartic anomalies still takes the form

$$C^{AB} d_B = f^A - 16 S^A \,, \tag{5.20}$$

where S^A follows more generally from Equation (2.12) rather than the Frobenius–Schur indicator of the algebra $\widetilde{\mathfrak{b}}$ associated to the Cartan matrix C^{AB}. Contrary to the LSTs we have discussed throughout this work, they however always satisfies $\theta_A S^A = 0$. From what we have argued several time, it means that type-II LSTs whose fiber singularity $\mathfrak{g}_F$ allows for a high rank are classified by an affine coweight of $\widetilde{\mathfrak{b}}$ at level zero:

$$\text{type-II LSTs:} \qquad \widetilde{\mu}^\vee = (0, 0, Q) \in \widetilde{\mathcal{P}}^\vee(\widetilde{\mathfrak{b}}) \,, \tag{5.21}$$

and as before $\widetilde{\mu}^\vee - 16\widetilde{S}^\vee$ must be a positive coroot to ensure that gauge algebras have integer rank.

It follows that there cannot be any continuous flavor symmetry associated with such LSTs, up to Abelian factors delocalized over the whole quiver. This gives an alternative field-theoretic proof of a more general bound: for any type-II LSTs, an analysis of the worldsheet theory of the Little String shows that the total flavor rank can be at most two [33].

In each case, the partial order of affine coweights suggests a simple RG flow lowering Q, as the allowed affine coweights can only be of the form $\widetilde{\nu}^\vee = (0, 0, Q - c) \lessdot \widetilde{\mu}^\vee = (0, 0, Q)$ for a given constant c depending on the type of algebras involved.

5.5 SCFTs and Coweights

In this work, we have only discussed Little String Theories. However, most of our analysis of the cancellation of quartic anomalies and RG flows also applies to six-dimensional SCFTs with classical gauge algebras. Indeed, in our analysis of the anomaly polynomial, we remained for the most part agnostic on whether we were considering an LST or an SCFT, and most of the discussion in Section 2.1 applies to SCFTs as well.

In fact, SCFTs are arguably simpler than LSTs: the cancellation of quartic anomalies given in Equation (2.16) can be inverted, as C^{IJ} is the Cartan matrix of a *finite* Lie algebra $\mathfrak{b}$. We therefore expect all SCFTs with classical gauge algebras of high-enough rank to be similarly labeled uniquely by a coweight $\mu^\vee$ of $\mathfrak{b}$.

For gauge algebras of type $\mathfrak{su}_k$, requiring that $\mu^\vee - 16S^\vee$ is a coroot so as to obtain well-defined gauge algebras is precisely the condition demanded in reference [92] to reconstruct the Hasse diagram of RG flows associated with unitary three-dimensional $\mathcal{N} = 4$ quiver from slices in the corresponding affine Grassmannian. This therefore explains why the original slice-subtraction algorithm [47] correctly reproduced the Hasse diagram obtained from via magnetic-quivers methods: it simply corresponds to the partial order of coweights, with (in the case considered there) only Kraft–Procesi transitions allowed.

We therefore expect our generalization of this algorithm to also extend to type-D conformal matter in the same way we found the minimal slices in Section 4, possibly with additional slices than those we have found here.

Furthermore, the overwhelming majority of SCFTs fall into infinite families labelled by a choice of finite ADE algebra $\mathfrak{b}$ describing the base, and a choice of algebra $\mathfrak{g}_F$ describing the gauge symmetry—which is associated with the fiber singularity at the singular point in the F-theory description. As the base corresponds to a finite, rather than affine, algebra, SCFTs allow for a rich network of flavor symmetries. A choice of tuple $(\mathfrak{b}, \mathfrak{g}_F)$ then defines a "parent" theory, from which all other SCFTs falling in the same class can be reached through Higgs-branch RG flows. As hinted above, these RG flows are labelled by quantities related to the group theory of $\mathfrak{g}_F$.

For type-A algebras $\mathfrak{b}$ describing the base, one describes the RG flows by breaking the flavor symmetry on either side of the quiver: for instance, in the context of conformal matter SCFTs [102], one makes a choice of a pair of nilpotent orbits of $\mathfrak{g}_F$ [103]. From the perspective of this work, it seems more natural to encode the data of those two nilpotent orbits directly in a dominant coweight of $\mathfrak{b}$ rather than treating the breaking on both sides independently. In addition to a simplification of the description, it also enables one to consider more types of breaking, and simultaneously describe all possible choices of classical algebras $\mathfrak{g}_F$ in the same framework: RG flows between theories with $\mathfrak{g}_F = \mathfrak{su}_k$ and $\mathfrak{g}_F = \mathfrak{su}_{k-n}$ are both described by a dominant coweight of $\mathfrak{b}$.[18]

Furthermore, it can be shown that by working at a certain point of the tensor branch, questions related to anomalies can be treated in a uniform manner for any given algebra, including those of exceptional type [66]. Similarly to what we have found for classical algebras, the complete anomaly polynomial only depends on the choice of the base and group-theory data of the nilpotent orbits. In particular, as in this work, most of the results do not depend on the minutia of the F-theory description but only necessitate the Lie-theoretic quantities related to $\mathfrak{g}_F$. It would therefore be interesting to see whether the language of coweights can be applied to exceptional algebras using some of the ideas developed here.

6 Conclusions and Summary of Results

Six-dimensional $\mathfrak{so}_{32}$ LSTs with minimal supersymmetry admit only classical flavor and gauge zero-form symmetries. This fact enabled us to understand several of their field-theoretical properties on the tensor branch in terms of dominant coweights of the affine algebra $\widetilde{\mathfrak{b}}$ defining the topology of their generalized quiver independent of their string/F-theory realization. For instance, in Classification 1 we have shown that the topology of the quiver of any $6d\ \mathcal{N} = (1,0)$ SQFT with hypermultiplets only in the bifundamental representations of classical algebras correspond to the Dynkin diagram of a Lie algebra.

The main constraint that we have used throughout this work is the cancellation of quartic gauge anomalies, which allows for an interpretation in terms of coweights and enabled us to use properties of root systems. The flavor and gauge algebras are then uniquely determined by properties of $\widetilde{\mathfrak{b}}$, rather than being an additional input. The remaining freedom then corresponds to the rank of these algebras, which is encoded in an affine dominant coweight of $\widetilde{\mathfrak{b}}$. We have checked this for all known $\mathfrak{so}_{32}$ heterotic LSTs, including those whose spectrum include hypermultiplets transforming in the anti-symmetric representation of an $\mathfrak{su}_k$ gauge algebra. This therefore generalizes the original classification [36] (see Classification 2), to all known cases:

[18]The authors are grateful to Craig Lawrie for discussions on this topic and for sharing an advanced copy of the manuscript of reference [104] with us, where it is shown through Class-$\mathcal{S}$ and 6d SCFTS techniques that a breaking corresponding to two nilpotent orbits can be understood in terms of a single nilpotent orbit of a larger algebra.

Classification 5 (all $\mathfrak{so}_{32}$ LSTs)**.** *All known $\mathfrak{so}_{32}$ heterotic LSTs can be labeled by a choice of affine dominant coweight of the allowed affine algebras $\widetilde{\mathfrak{b}}$ given in Table 12. We differentiate between two classes of LSTs:*

1. *$Spin(32)/\mathbb{Z}_2$ LSTs: the coweight is of the form $\widetilde{\mu}^\vee = (\mu^\vee, 32, 2Q)$*
2. *$(SU(16) \times U(1))/\mathbb{Z}_2$ LSTs: the coweight is of the form $\widetilde{\mu}^\vee = (\mu^\vee, 16, Q)$*

The components μ^I are the dimensions of the fundamental representations of the flavor symmetry on the I-th node of the Dynkin diagram, and must satisfy the following constraints.

- *Given the affine coweight $\widetilde{S}^\vee = S^I \widetilde{\omega}_I^\vee$ whose components $S^I \in \{-1, 0, 1\}$ can be read off the color of a node of the Dynkin diagram, $\widetilde{\mu}^\vee - 16\widetilde{S}^\vee$ must be a positive coroot. The components S^I define the type of gauge algebras, and $d_I = \langle \widetilde{\omega}_I, \widetilde{\mu}^\vee - 16\widetilde{S}^\vee \rangle$ sets their fundamental representations.*
- *if $S^I = 1$ then μ^I is even, and if $S^I = -1$ then d_I is even.*
- *If the LST hosts a configuration $\underset{[\mathfrak{su}_{f^I}]}{\overset{\mathfrak{su}_{d_I}}{1}}$, then $\mu^I = 2f^I$.*

This labeling is unique except in cases with a curve configuration $\overset{\mathfrak{su}_{d_I}}{1}$, as the spectrum includes an antisymmetric representation that may mimic another configuration. Passing from a particular generalized quiver describing an LST to its coweight and vice versa is explained in more detail in Sections 2.2, 5.1, and 5.2.

This Lie-algebraic description of $\mathfrak{so}_{32}$ heterotic LSTs enable us to simplify the search for T-dual pairs. Each of the known T-duality invariants can be written in terms of closed-form expressions involving only $\widetilde{\mu}^\vee$. These can then be used to efficiently partition the cone of *finite* dominant coweights, where each block of the partition is made out of coweights that lead to LSTs with the same T-dual invariants up to a shift in the instanton number Q, and is explained in Classification 3.

This therefore reduces the search for putative T-dual pairs to a simple linear-optimization problem that is amenable to numerical scans given a particular algebra $\widetilde{\mathfrak{b}}$. For instance, in Classification 4, we have shown that at maximal flavor rank, the number of putative T-duality orbits is given by the order of the center of the algebra $\mathfrak{g}$—except for $\mathfrak{g} = \mathfrak{so}_{4k}$, where there are only two orbits.

In the absence of a way to easily geometrically engineer all possible $\mathfrak{so}_{32}$ LSTs, the partitioning of the cone of dominant coweights gives a necessary condition for T-duality. It is not known whether there are additional invariants that call for a refinement of the methods developed in this work. However, we have shown that for certain low-rank algebras, they satisfy linear

$Spin(32)/\mathbb{Z}_2$ LST	$(SU(16)\times U(1))/\mathbb{Z}_2$ LST

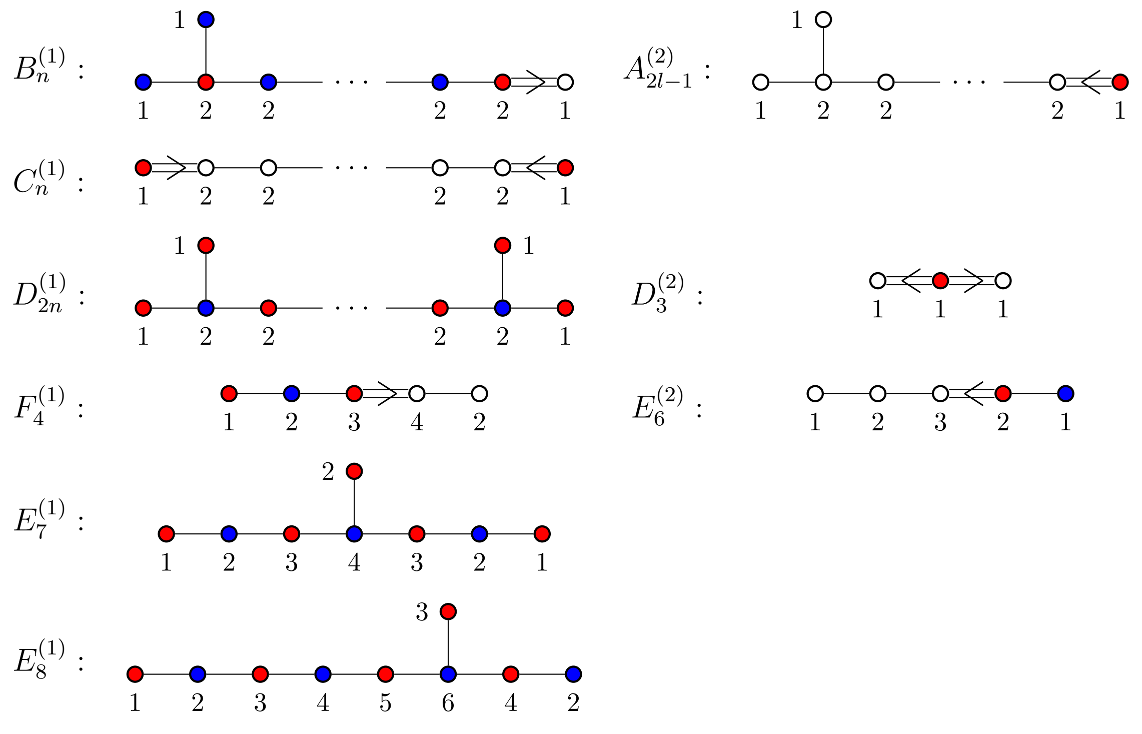


Table 12: Algebras $\widetilde{\mathfrak{b}}$ corresponding to the base of an $\mathcal{N}=(1,0)$ six-dimensional Little String Theory. The integers on each node indicate the marks θ_A of the affine algebra, and its color the associated flavor symmetry: red $\bullet \leftrightarrow \mathfrak{so}_k$; blue $\bullet \leftrightarrow \mathfrak{sp}_k$; $\circ \leftrightarrow \mathfrak{su}_k$. A choice of $\widetilde{\mathfrak{b}}$ must then be supplemented by an affine coweight representing the dimensions of the fundamental representations of the flavor symmetries. The complete data of the LST needed to study T-duality can then be recovered from the coweight. The other affine or twisted algebras do not lead to consistent LSTs.

relations, and only two out of three of the invariants we considered here are needed. As we are not aware of any geometric construction where the known invariants fail to label T-duality orbits, it is tempting to conjecture that our method lead to *sufficient* conditions to determine T-duality.

We have furthermore studied the partial order associated with dominant coweights. This has led us to put forward slice-subtraction Algorithm 1 that generalizes the one recently put forward for six-dimensional generalized unitary quivers associated with SCFTs [47]. Our generalization should apply to all unitary-orthosymplectic generalized quivers. Given an LST, the approach efficiently reconstructs the Hasse diagrams of all theories lower in the partial order. We show an example of the algorithm for $Spin(32)/\mathbb{Z}_2$ LSTs with $\mathfrak{g}=\mathfrak{so}_{10}$ in Figure

4.1.

In many cases, minimal changes from one coweight to another correspond to a Kraft–Procesi transition [95, 96], and reproduce the expected change in the dimension of the Higgs and Coulomb branch. We however, find several "exotic" slices, collated in Tables 8–10 for which we do not know the associated symplectic singularity. It would be interesting to construct the magnetic quivers for these models, and investigate if they correspond to known cases, or whether new transitions occur.

While we expect this partial order to reproduce the network of Higgs-branch Renormalization Group flows, we find certain slices that do not change the dimension of the Higgs-branch. To the best of our knowledge, such minimal slices have not appeared previously in the literature, and call for further investigation through quiver methods. Geometrically, these deformations seem naively possible, but F-theory geometries that include deformations involving configurations with multiple curves have so far not been investigated explicitly in the literature, and both techniques offer interesting avenues to settle the question of whether these correspond to *bona fide* RG flows.

We have also explored possible generalizations of our methods to $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic and Type-II LSTs, as well as SCFTs, where our results apply in certain cases. For general six-dimensional theories with arbitrary gauge algebras, the biggest hindrance to a generalization is the absence of quartic Casimir for exceptional algebras. Indeed, the main constraint we have used follows from the cancellation of quartic gauge anomalies, and therefore does not apply to exceptional algebras. However, $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic LSTs as well as conformal matter suggests similar relationships with Lie theory. Furthermore, we have not considered dualities with twisted compactifications [30, 105, 106], or those involving frozen singularities and the correspondingly heterotic compactification without vector structure [107–110]. It would be interesting to check whether similar group-theoretic considerations can also be found in those cases, as the geometry of frozen F-theory vacua is still not fully understood.

Having similar classifications in terms of explicit coweights would therefore allow for an extension of our results not only to "internal" T-duality between $\mathfrak{so}_{32}$ theories, but fiber-base dualities as well, opening up a systematic analysis of a larger part of the landscape of dualities of Little String Theories.

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A Heterotic Little String Theories with Trivial Coweights

In this Appendix, we tabulate all heterotic LSTs with a trivial coweight, that is with flavor symmetry $\mathfrak{e}_8 \oplus \mathfrak{e}_8$, $\mathfrak{so}_{32}$, or $\mathfrak{su}_{16} \oplus \mathfrak{u}_1$. We additionally give the T-duality invariants for these theories, matching the closed-form expression found in Section 2.2.

$\mathfrak{g}$	LST	κ_R	dim(CB)
$\mathfrak{su}_{K=2k}$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \underbrace{\overset{\mathfrak{su}_{2Q-8}}{2}\ \cdots\ \overset{\mathfrak{su}_{2Q-8\cdot i}}{2}\ \cdots\ \overset{\mathfrak{su}_{2Q-8(k-1)}}{2}}_{k-1}\ \overset{\mathfrak{sp}_{Q-4k}}{1} \quad \times\ \mathfrak{u}_1$	$\Gamma(Q-r_{\mathfrak{g}}-1)+2$	$h^\vee Q-\dim(\mathfrak{g})$
$\mathfrak{su}_{K=2k+1}$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \underbrace{\overset{\mathfrak{su}_{2Q-8}}{2}\ \cdots\ \overset{\mathfrak{su}_{2Q-8\cdot i}}{2}\ \cdots\ \overset{\mathfrak{su}_{2Q-8(k-1)}}{2}}_{k-1}\ \overset{\mathfrak{su}_{2Q-8k}}{1} \quad \times\ \mathfrak{u}_1$	$\Gamma(Q-r_{\mathfrak{g}}-1)+2$	$h^\vee Q-\dim(\mathfrak{g})$
$\mathfrak{so}_8$	$\begin{array}{ccc} \overset{\substack{[\mathfrak{so}_{32}]\\ \mathfrak{sp}_Q}}{1} & & \overset{\mathfrak{sp}_{Q-8}}{1} \\ & \overset{\mathfrak{so}_{4Q-16}}{1} & \\ \overset{\mathfrak{sp}_{Q-8}}{1} & & \overset{\mathfrak{sp}_{Q-8}}{1} \end{array}$	$8Q-38$	$6Q-28$
$\mathfrak{so}_{2K=2(2k)}$	$[\mathfrak{so}_{32}]\ \overset{\mathfrak{sp}_Q}{1}\ \overset{\substack{\mathfrak{sp}_{Q-1}\\ 1}}{\overset{\mathfrak{so}_{4Q-16}}{4}}\ \underbrace{\overset{\mathfrak{sp}_{2Q-16}}{1}\ \overset{\mathfrak{so}_{4Q-32}}{4}\ \cdots\ \overset{\mathfrak{sp}_{2Q-8(k-i)}}{1}\ \overset{\mathfrak{so}_{4Q-16(k-i)}}{4}\ \cdots\ \overset{\mathfrak{sp}_{2Q-8(k-1)}}{1}\ \overset{\substack{\mathfrak{sp}_{Q-4k}\\ 1}}{\overset{\mathfrak{so}_{4Q-16(k-1)}}{4}}}_{2(k-2)\text{ curves}}\ \overset{\mathfrak{sp}_{Q-4k}}{1}$	$\Gamma(Q-r_{\mathfrak{g}}-1)+2$	$h^\vee Q-\dim(\mathfrak{g})$
$\mathfrak{so}_{2K=2(2k+1)}$	$[\mathfrak{so}_{32}]\ \overset{\mathfrak{sp}_Q}{1}\ \overset{\substack{\mathfrak{sp}_{Q-1}\\ 1}}{\overset{\mathfrak{so}_{4Q-16}}{4}}\ \underbrace{\overset{\mathfrak{sp}_{2Q-16}}{1}\ \overset{\mathfrak{so}_{4Q-32}}{4}\ \cdots\ \overset{\mathfrak{sp}_{2Q-8(k-i)}}{1}\ \overset{\mathfrak{so}_{4Q-16(k-i)}}{4}\ \cdots\ \overset{\mathfrak{sp}_{2Q-8(k-1)}}{1}\ \overset{\mathfrak{so}_{4Q-16(k-1)}}{4}}_{2(k-2)\text{ curves}}\ \overset{\mathfrak{sp}_{2Q-8k}}{1}\ \overset{\mathfrak{su}_{2Q-8k}}{2}$	$\Gamma(Q-r_{\mathfrak{g}}-1)+2$	$h^\vee Q-\dim(\mathfrak{g})$
$\mathfrak{e}_6$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{4}}\ \overset{\mathfrak{so}_{4Q-16}}{1}\ \overset{\mathfrak{sp}_{3Q-24}}{4}\ \overset{\mathfrak{su}_{4Q-32}}{2}\ \overset{\mathfrak{su}_{2Q-16}}{2}$	$24Q-166$	$12Q-78$
$\mathfrak{e}_7$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \overset{\mathfrak{so}_{4Q-16}}{4}\ \overset{\mathfrak{sp}_{3Q-24}}{1}\ \overset{\substack{\mathfrak{sp}_{2Q-20}\\ 1}}{\overset{\mathfrak{so}_{8Q-64}}{4}}\ \overset{\mathfrak{sp}_{3Q-28}}{1}\ \overset{\mathfrak{so}_{4Q-32}}{4}\ \overset{\mathfrak{sp}_{Q-12}}{1}$	$48Q-382$	$18Q-133$
$\mathfrak{e}_8$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \overset{\mathfrak{so}_{4Q-16}}{4}\ \overset{\mathfrak{sp}_{3Q-24}}{1}\ \overset{\mathfrak{so}_{8Q-64}}{4}\ \overset{\mathfrak{sp}_{5Q-48}}{1}\ \overset{\substack{\mathfrak{sp}_{3Q-32}\\ 1}}{\overset{\mathfrak{so}_{12Q-112}}{4}}\ \overset{\mathfrak{sp}_{4Q-40}}{1}\ \overset{\mathfrak{so}_{4Q-32}}{4}$	$120Q-1078$	$30Q-248$

Table 13: Heterotic Little String Theories of type $Spin(32)/\mathbb{Z}_2$ with a $\mathfrak{so}_{32}$ flavor symmetry, corresponding to a trivial choice of finite coweight of the associated algebra $\mathfrak{g}$.

$\mathfrak{g}$	LST	κ_R	dim(CB)
$\mathfrak{su}_K$	$[\mathfrak{e}_8]\, 1\, 2\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{su}_3}{2} \cdots \overset{\mathfrak{su}_{K-1}}{2}\, \underbrace{\overset{\mathfrak{su}_K}{2} \ldots}_{M} \overset{\mathfrak{su}_K}{2}\, \overset{\mathfrak{su}_{K-1}}{2} \cdots \overset{\mathfrak{su}_3}{2}\, \overset{\mathfrak{su}_2}{2}\, 2\, 1\, [\mathfrak{e}_8]$	$KM + (K^2 - K + 2)$	$KM + (K^2 + 1)$
$\mathfrak{so}_{2K}$	$[\mathfrak{e}_8]\, 1\, 2\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{g}_2}{3}\, 1\, \overset{\mathfrak{so}_9}{4}\, \overset{\mathfrak{sp}_1}{1}\, \overset{\mathfrak{so}_{11}}{4} \cdots \overset{\mathfrak{sp}_{K-4}}{1}\, \underbrace{\overset{\mathfrak{so}_{2K}}{4}\, \overset{\mathfrak{sp}_{K-4}}{1} \cdots \overset{\mathfrak{so}_{2K}}{4}}_{M}\, \overset{\mathfrak{sp}_{K-4}}{1} \cdots \overset{\mathfrak{so}_{11}}{4}\, \overset{\mathfrak{sp}_1}{1}\, \overset{\mathfrak{so}_9}{4}\, 1\, \overset{\mathfrak{g}_2}{3}\, \overset{\mathfrak{su}_2}{2}\, 2\, 1\, [\mathfrak{e}_8]$	$\Gamma M + 4(K-2)^2 + 2$	$\Gamma M + 2(K+1)^2 - 9K$
$\mathfrak{e}_6$	$[\mathfrak{e}_8]\, 1\, 2\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{g}_2}{3}\, 1\, \overset{\mathfrak{f}_4}{5}\, 1\, \overset{\mathfrak{su}_3}{3}\, 1\, \underbrace{\overset{\mathfrak{e}_6}{6}\, 1\, \overset{\mathfrak{su}_3}{3}\, 1 \cdots 1\, \overset{\mathfrak{su}_3}{3}\, 1}_{M}\, \overset{\mathfrak{e}_6}{6}\, 1\, \overset{\mathfrak{su}_3}{3}\, 1\, \overset{\mathfrak{f}_4}{5}\, 1\, \overset{\mathfrak{g}_2}{3}\, \overset{\mathfrak{su}_2}{2}\, 2\, 1\, [\mathfrak{e}_8]$	$24M + 50$	$12M + 42$
$\mathfrak{e}_7$	$[\mathfrak{e}_8]\, 1\, 2\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{g}_2}{3}\, 1\, \overset{\mathfrak{f}_4}{5}\, 1\, \overset{\mathfrak{g}_2}{3}\, \overset{\mathfrak{su}_2}{2}\, 1\, \underbrace{\overset{\mathfrak{e}_7}{8}\, 1\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{so}_7}{3}\, \overset{\mathfrak{su}_2}{2}\, 1 \cdots 1\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{so}_7}{3}\, \overset{\mathfrak{su}_2}{2}\, 1}_{M}\, \overset{\mathfrak{e}_7}{8}\, 1\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{g}_2}{3}\, 1\, \overset{\mathfrak{f}_4}{5}\, 1\, \overset{\mathfrak{g}_2}{3}\, \overset{\mathfrak{su}_2}{2}\, 2\, 1\, [\mathfrak{e}_8]$	$48M + 50$	$18M + 47$
$\mathfrak{e}_8$	$[\mathfrak{e}_8]\, 1\, 2\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{g}_2}{3}\, 1\, \overset{\mathfrak{f}_4}{5}\, 1\, \overset{\mathfrak{g}_2}{3}\, \overset{\mathfrak{su}_2}{2}\, 2\, 1\, \underbrace{\overset{\overset{1}{\mathfrak{e}_8}}{(12)} \ldots \overset{\overset{1}{\mathfrak{e}_8}}{(12)}}_{M-2}\, 1\, 2\, \overset{\mathfrak{su}_2}{2}\, \overset{\mathfrak{g}_2}{3}\, 1\, \overset{\mathfrak{f}_4}{5}\, 1\, \overset{\mathfrak{g}_2}{3}\, \overset{\mathfrak{su}_2}{2}\, 2\, 1\, [\mathfrak{e}_8]$	$120M + 29$	$30M - 8$

Table 14: Heterotic $E_8 \times E_8$ Little String Theories. M refers to insertions of conformal matter so that the curve $\overset{\mathfrak{g}}{n}_{\mathfrak{g}}$ occurs M times. In the M-theory picture this corresponds to the or equivalently to the number of $M5$ branes.

$\mathfrak{g}$	$\widetilde{\mathfrak{b}}$	$Spin(32)/\mathbb{Z}_2$ LST	$\widetilde{\mathfrak{b}}$	$(SU(16)\times U(1))/\mathbb{Z}_2$ LST
$\mathfrak{su}_{K=2N\geq 4}$	$C_N^{(1)}$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \underbrace{\overset{\mathfrak{su}_{2Q-8}}{2}\ \cdots\ \overset{\mathfrak{su}_{2Q-8(N-1)}}{2}}_{N-1}\ \overset{\mathfrak{sp}_{Q-4N}}{1}\ \times\ \mathfrak{u}_1$	—	$[\mathfrak{su}_{16}]\ \underset{[\mathfrak{su}_2]}{\overset{\mathfrak{su}_{K^2+(Q-1)K}}{0}}\ \times\ \mathfrak{u}_1$
$\mathfrak{so}_8$	$D_4^{(1)}$	$\begin{array}{ccc} \overset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}} & & \overset{\mathfrak{sp}_{Q-8}}{1} \\ & \overset{\mathfrak{so}_{4Q-16}}{1} & \\ \overset{\mathfrak{sp}_{Q-8}}{1} & & \overset{\mathfrak{sp}_{Q-8}}{1} \end{array}$	$D_3^{(2)}$	$\underset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_{2Q}}{2}}\ \overset{\mathfrak{sp}_{2Q-8}}{1}\ \overset{\mathfrak{su}_{2Q-8}}{2}\ \times\ \mathfrak{u}_1$
$\mathfrak{so}_{10}$	$D_5^{(1)}$	$\begin{array}{cccc} & \overset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}} & & \\ \overset{\mathfrak{sp}_{Q-8}}{1} & \overset{\mathfrak{so}_{4Q-16}}{4} & \overset{\mathfrak{sp}_{2Q-16}}{1} & \overset{\mathfrak{su}_{2Q-16}}{2} \end{array}$	$D_3^{(2)}$	$\underset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_{2Q}}{2}}\ \overset{\mathfrak{su}_{4Q-16}}{1}\ \overset{\mathfrak{su}_{2Q-8}}{2}\ \times\ \mathfrak{u}_1$
$\mathfrak{so}_{K=4N}$	$D_{2N}^{(1)}$	$\overset{\mathfrak{sp}_{Q-1}}{1}\ \overset{\overset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}}{\overset{\mathfrak{so}_{4Q-16}}{4}}\ \underbrace{\overset{\mathfrak{sp}_{2Q-16}}{1}\ \overset{\mathfrak{so}_{4Q-32}}{4}\ \cdots\ \overset{\mathfrak{sp}_{2Q-8(N-1)}}{1}\ \overset{\overset{\mathfrak{sp}_{Q-4N}}{1}}{\overset{\mathfrak{so}_{4Q-16(N-1)}}{4}}}_{2(N-2)\text{ curves}}\ \overset{\mathfrak{sp}_{Q-4N}}{1}$	$A_{2N-1}^{(2)}$	$\overset{\mathfrak{su}_{2Q-8}}{2}\ \overset{\overset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_{2Q}}{2}}}{\overset{\mathfrak{su}_{4Q-16}}{2}}\ \underbrace{\overset{\mathfrak{su}_{4Q-24}}{2}\ \cdots\ \overset{\mathfrak{su}_{4Q-8i}}{2}\ \cdots}_{N-3}\ \overset{\mathfrak{sp}_{2Q-4N}}{1}\ \times\ \mathfrak{u}_1$
$\mathfrak{so}_{K=4N+2}$	$B_k^{(1)}$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{1}}\ \underbrace{\overset{\mathfrak{su}_{2Q-12}}{2}\ \cdots\ \overset{\mathfrak{su}_{2Q+4-8N}}{2}}_{N-1}\ \overset{\mathfrak{su}_{Q-2-4N}}{1}$	$A_{2N-1}^{(2)}$	$\overset{\mathfrak{su}_{2Q-8}}{2}\ \overset{\overset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_{2Q}}{2}}}{\overset{\mathfrak{su}_{4Q-16}}{2}}\ \underbrace{\overset{\mathfrak{su}_{4Q-24}}{2}\ \cdots\ \overset{\mathfrak{su}_{4Q-8i}}{2}\ \cdots}_{N-3}\ \overset{\mathfrak{su}_{4Q-8N}}{1}\ \times\ \mathfrak{u}_1$
$\mathfrak{e}_7$	$E_7^{(1)}$	$\underset{[\mathfrak{so}_{32}]}{\overset{\mathfrak{sp}_Q}{4}}\ \overset{\mathfrak{so}_{4Q-16}}{1}\ \overset{\mathfrak{sp}_{3Q-24}}{4}\ \overset{\overset{\mathfrak{sp}_{2Q-20}}{1}}{\overset{\mathfrak{so}_{8Q-64}}{1}}\ \overset{\mathfrak{sp}_{3Q-28}}{4}\ \overset{\mathfrak{so}_{4Q-32}}{1}\ \overset{\mathfrak{sp}_{Q-12}}{4}$	$E_6^{(2)}$	$\underset{[\mathfrak{su}_{16}]}{\overset{\mathfrak{su}_{2Q}}{2}}\ \overset{\mathfrak{su}_{4Q-16}}{2}\ \overset{\mathfrak{su}_{6Q-32}}{2}\ \overset{\mathfrak{sp}_{4Q-24}}{1}\ \overset{\mathfrak{so}_{4Q-16}}{4}\ \times\ \mathfrak{u}_1$

Table 15: Exotic T-duality between $Spin(32)/\mathbb{Z}_2$ and $(SU(16)\times U(1))/\mathbb{Z}_2$ theories whose defining coweight correspond to different algebras. $\mathfrak{g}$ indicate the type of singularity in the F-theory realization.

B Derivation of the Anomaly Polynomial

The anomaly polynomial of a $\mathcal{N} = (1,0)$ SQFT can be written in terms of characteristic classes. We give here some of the definition needed to recover our results. To wit, the A-roof genus $\widehat{A}(T)$, The Hirzebruch genus $L(T)$, and the chern character $\mathrm{ch}(F)$ are given up to order eight as:

$$\begin{aligned}
\widehat{A}(T) &= 1 - \frac{1}{24}p_1(T) + \frac{1}{5760}\big(7p_1(T)^2 - 4p_2(T)\big) + \cdots\,, \\
L(T) &= 1 + \frac{1}{3}p_1(T) - \frac{1}{45}\big(p_1(T)^2 - 7p_2(T)\big) + \cdots\,, \\
\mathrm{ch}_{\boldsymbol{R}}(F) &= \mathrm{tr}_{\boldsymbol{R}}\, e^{iF} = \dim(\boldsymbol{R}) - \frac{1}{2}\mathrm{tr}_{\boldsymbol{R}}F^2 + \frac{1}{24}\mathrm{tr}_{\boldsymbol{R}}F^4 + \cdots\,, \\
\mathrm{ch}_{\mathbf{2}}(R) &= 2 - c_2(R) + \frac{1}{12}c_2(R)^2 + \cdots\,, \qquad c_2(R) = \frac{1}{4}\mathrm{Tr}(R^2)\,,
\end{aligned} \tag{B.1}$$

where T, R are the background fields corresponding to the spacetime, R-symmetry, respectively, and F the field strength of a gauge of flavor symmetry. The first and second Pontryagin classes are denoted $p_1(T), p_2(T)$, respectively. The "one-loop" contributions of the various multiplets to the anomaly polynomial are then

$$\begin{aligned}
I_8^{\mathrm{hyp}}(F) = \widehat{A}(T)\mathrm{ch}_{\mathcal{R}}(F)\Big|_{\text{8-form}}\,, \qquad I_8^{\mathrm{vec}}(F) = -\frac{1}{2}\,\widehat{A}(T)\mathrm{ch}_{\mathbf{2}}(R)\mathrm{ch}_{\mathbf{adj}}(F)\Big|_{\text{8-form}}\,, \\
I_8^{\mathrm{tensor}} = \left(\frac{1}{2}\mathrm{ch}_{\mathbf{2}}(R)\widehat{A}(T) - \frac{1}{8}L(T)\right)\Big|_{\text{8-form}}\,,
\end{aligned} \tag{B.2}$$

If we need to differentiate between the field strength associated with flavor and gauge symmetries, we will denote them as F_I and $\widetilde{F}_I$, respectively.

Furthermore, in the case of gauge or flavor symmetries, the traces, $\mathrm{tr}_{\boldsymbol{R}}F^n$, over the background field F depend on the representation $\boldsymbol{R}$ under which the associated fields transform. As different representations appear in the spectrum, it is important to convert them to so-called one-instanton normalized traces $\mathrm{Tr}F^n$. Given a field strength F associated with a Lie algebra, we have the conversion relations:[19]

$$\mathrm{tr}_{\boldsymbol{R}}F^2 = h_{\boldsymbol{R}}\mathrm{Tr}F^2\,, \qquad \mathrm{tr}_{\boldsymbol{R}}F^4 = x_{\boldsymbol{R}}\mathrm{Tr}F^4 + y_{\boldsymbol{R}}(\mathrm{Tr}F^2)^2\,. \tag{B.3}$$

For the fundamental representation $\boldsymbol{F}$ of a classical algebra, we have $(x_{\boldsymbol{F}}, y_{\boldsymbol{F}}) = (1,0)$ and $h_{\boldsymbol{F}} = \frac{1}{2}$ for $\mathfrak{su}_k, \mathfrak{sp}_k$ or $h_{\boldsymbol{F}} = 1$ for $\mathfrak{so}_k$. The second-order quartic Casimir of the adjoint representation is given by $y_{\boldsymbol{adj}} = \frac{3}{4}, 3, \frac{3}{2}$ for $\mathfrak{sp}_k$, $\mathfrak{so}_k$, $\mathfrak{su}_k$, respectively, while $x_{\boldsymbol{adj}}$ satisfy the relation given in equation (B.3). To encompass all nodes of the generalized quiver at the same

[19] We follow the standard conventions for six-dimensional theories [11,64,65], see Appendix A of reference [66] for a review of how trace relation are obtained for general representations.

time, we use the following notation:

$$\begin{aligned}
&\mathrm{Tr}_{\boldsymbol{F}} F_I^2 = h_I^J\, \mathrm{Tr} F_J^2\,, \qquad \mathrm{Tr}_{\boldsymbol{F}} \widetilde{F}_I^2 = \widetilde{h}_I^J\, \mathrm{Tr} F_J^2\,, \qquad \sum_I \mathrm{Tr}_{\boldsymbol{adj}} \widetilde{F}_I^2 = h^I \mathrm{Tr} \widetilde{F}_I^2\,,\\
&\mathrm{tr}_{\boldsymbol{F}} \widetilde{F}_I^4 = \mathrm{Tr} \widetilde{F}_I^4\,, \qquad \sum_I \mathrm{Tr}_{\boldsymbol{adj}} \widetilde{F}_I^4 = x^I \mathrm{Tr} \widetilde{F}_I^4 + 16 y^{IJ} c_2(\widetilde{F}_I) c_2(\widetilde{F}_J)\,.
\end{aligned} \tag{B.4}$$

with $h^I = h^\vee_{\mathfrak{g}_I}$ and $h^I_J, \widetilde{h}^I_J, y^{IJ}$ diagonal matrices whose elements are the corresponding coefficients of the trace relations in Equation (B.3).

Due to the fact that $\mathfrak{so}_{32}$ LSTs have only classical gauge and flavor symmetries, we can make use of certain group-theoretical relations to reduce the number of the relevant quantities appearing in the anomaly polynomial, such as the dimension of the fundamental representation d_J and the Frobenius–Schur indicator S^I, see Equation (2.12).

Let us compute the one-loop anomaly polynomial given in equation (2.13). The matrices A^{IJ} and D^{IJ} encode the adjacency of gauge-gauge and gauge-flavor bidunfamental hypermultiplets, differentiating between half and full hypermultiplet, see discussion around equation (2.8). Expanding the anomaly polynomial to make the gauge contributions $c_2(\widetilde{F}_I) = \frac{1}{2}\mathrm{Tr} F_I^2$ and $\mathrm{Tr}\widetilde{F}_I^4$ explicit, one finds:

$$\begin{aligned}
I_8^{\text{1-loop}} = &\left(\frac{1}{2} \widehat{A}(T) d_I D^{IJ} \mathrm{ch}_{\boldsymbol{F}}(F_J) + \frac{1}{4} A^{IJ} d_J \widehat{A}(T) - \frac{1}{2} \sum_I \dim(\mathfrak{g}_I) \widehat{A}(T) \mathrm{ch}_{\boldsymbol{2}}(R) \right)\Bigg|_{8-form}\\
&+ r_{\mathfrak{b}} I_8^{\text{tensor}} + c_2(\widetilde{F}_I) \left(((\widetilde{h} A \widetilde{h})^{IJ} - \frac{2}{3} y^{IJ}) c_2(\widetilde{F}_J) + \delta^{IJ} c_2(F_J) - h^I c_2(R) \right)\\
&+ \frac{1}{24} c_2(\widetilde{F}_I) \left((D\widetilde{h})^I_J f^J + 2(\widetilde{h}A)^{IJ} d_J - 2h^I \right) + \frac{1}{48} \left(f^I D^J_I + d_I A^{IJ} - 2x^J \right) \mathrm{Tr} \widetilde{F}_J^4
\end{aligned} \tag{B.5}$$

Using the relations between the quartic Casimir in Equation (2.12) and the definition of the Cartan matrix in Equation (2.9), we can see that the last terms is given by

$$\mathrm{Tr}\widetilde{F}_J^4 \left(D^J_I f^I + A^{JI} d_I - 2x^J \right) = \mathrm{Tr}\widetilde{F}_J^4 D^J_I (f^I - 16 S^I - C^{IK} d_K)\,, \tag{B.6}$$

which leads to the anomaly-cancellation condition used throughout this work. We have used that since $S^I = 0$ for $\mathfrak{su}_K$ algebras, then $D^I_J S^J = S^I$. The rest of the anomaly polynomial can be further simplified by noticing that since $\mathfrak{sp}_k$ gauge multiplets are rotated by $\mathfrak{so}_n$ flavor symmetries—and vice versa—we have

$$h^J_I = \frac{1}{2} V^J_J\,, \qquad \widetilde{h}^I_J = (D^{-1} V^{-1})^I_J \qquad y^{IJ} = 3 (V^{-2} D^{-1})^{IJ}\,, \tag{B.7}$$

It is then straightforward to find that the Dirac pairing can be written as

$$\frac{1}{2}\eta^{IJ} = \frac{2}{3}y^{IJ} - (\widetilde{h}A\widetilde{h})^{IJ} = (V^{-1}CD^{-1}V^{-1})^{IJ} \tag{B.8}$$

The second expression reproduces the coefficient of the term proportional to $c_2(\widetilde{F})_I \cdot c_2(\widetilde{F})_J$ in Equation (B.5). Using the other relations, we are led to the expression given in Equation (2.14).

C Of Roots and Weights

We give here a short review of the properties of root systems for finite algebras and their (possibly-twisted) affine version.

Let us consider a simple Lie algebra $\mathfrak{g}$ of rank $r_\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. The integer root lattice $\mathcal{Q} \subset \mathfrak{h}^*$ is generated by simple roots $\alpha^a\,, a = 1, \ldots, r_\mathfrak{g}$. Given the usual invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$, the Cartan matrix is obtained through the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$:

$$C^{ab} = \left\langle \alpha^a, \alpha^{b\vee} \right\rangle = \frac{(\alpha^a,\alpha^b)}{(\alpha^b,\alpha^b)}\,, \qquad , \tag{C.1}$$

with the coroots defined as $\alpha^{b\vee} = 2\alpha^b/(\alpha^b,\alpha^b)$, and span the coroot lattice $\mathcal{Q}^\vee$. Lowercase Latin indices run over the rank of $\mathfrak{g}$: $a, b = 1, \ldots, r_\mathfrak{g}$.

The simple roots are taken to be in a normalization proportional to the diagonal matrix $D^{aa} = \frac{1}{2}(\alpha^a,\alpha^a)$, i.e. the longest simple root is taken to have length two. This is the convention for D used in Section 2.1, and is furthermore used to define the *symmetric Cartan matrix*, $G = CD = DC$.

The fundamental coweights $\omega_a^\vee$ generating the coweight lattice $\mathcal{P}^\vee$ are then defined as the dual basis of the root lattice $\mathcal{Q}$:

$$\langle \alpha^a, \omega_b^\vee \rangle = \delta_b^a\,, \tag{C.2}$$

In addition, we will also need the notion of dominant coweights, which are those with non-negative integer coefficients in the basis of fundamental coweights:

$$\mathcal{P}_+^\vee = \{\mu^\vee = \mu^a\,\omega_a^\vee \in \mathcal{P}^\vee \,|\, \mu^a \in \mathbb{N}\ \forall a\}\,. \tag{C.3}$$

The weight lattice $\mathcal{P}$ and the set of dominant weights $\mathcal{P}_+$ are defined in a similar manner.

Affine root systems: to study LSTs, we need to consider the affine version $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$. The root lattice $\mathcal{Q}$ of $\mathfrak{g}$ is then augmented to its affine version $\widetilde{\mathcal{Q}}$ by adding the zeroth root α^0, such that the affine Cartan matrix is given by

$$\langle \alpha^A, \alpha^{B\vee} \rangle = C^{AB} \,, \tag{C.4}$$

Uppercase Latin indices include the zeroth component: $A, B = 0, 1, \ldots, r_{\mathfrak{g}}$. The null root $\widetilde{\theta} = \theta_A \alpha^A$ is then defined as the affine root whose coefficients, called the marks, satisfy $\theta_A C^{AB} = 0$. Similarly, the comarks K_A are defined as satisfying $C^{AB} K_B = 0$.

The weight and coweight lattices $\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{P}}^\vee$ are then constructed by supplementing the affine (co)roots with a weight Λ_0 and two coweights $\mathcal{D}, K$ referred to as the scaling weight and the central element, respectively:[20]

$$\widetilde{P} = P \quad \oplus \mathbb{Z}\widetilde{\theta} \oplus \mathbb{Z}\,\Lambda_0 \,, \qquad \widetilde{P}^\vee = P^\vee \oplus \mathbb{Z}\,\mathcal{D} \oplus \mathbb{Z}\,K \,. \tag{C.5}$$

We have already encountered the central element—whose coefficients are the comarks—above, and the other two generators satisfy the relations:

$$\langle \alpha^A, \mathcal{D} \rangle = \delta^{A0} = \langle \Lambda_0, \alpha^{A\vee} \rangle \,, \qquad K = K_A\, \alpha^{A\vee} \tag{C.6}$$

To ensure that the coweights are generated by a basis dual to the affine roots, we define the fundamental coweights as their finite version, shifted by $\mathcal{D}$:

$$\widetilde{\omega}_0^\vee = \mathcal{D} \,, \qquad \widetilde{\omega}_a^\vee = \omega_a^\vee + \theta_a\, \mathcal{D} \,, \qquad \left\langle \alpha^A, \widetilde{\omega}_B^\vee \right\rangle = \delta_B^A \,, \tag{C.7}$$

An affine coweight can then be expanded in this basis as [21]

$$\mu^\vee = \mu^A\, \widetilde{\omega}_A^\vee + n\, K \;=\; \mu^a \omega_a^\vee + (\mu^0 + \theta_a \mu^a)\mathcal{D} + n\, K \,, \tag{C.8}$$

and is dominant if $\mu^A, n \in \mathbb{N}$. Equivalently, we can see that if we choose a dominant coweight of the simple algebra $\mu^\vee \in P_+^\vee$, we can lift it to a dominant coweight of $\widetilde{\mathfrak{g}}$, $\widetilde{\mu}^\vee \in \widetilde{P}_+^\vee$ by choosing a positive integer n, and a level which uniquely fixes the component μ^0. Every dominant affine coweight is therefore given by a triple $(\mu^\vee, k, n)$ where k is the level of the coweight:

$$\mathrm{lv}(\widetilde{\mu}^\vee) = \langle \widetilde{\theta}, \mu^\vee \rangle = \theta^A\, \mu_A \,. \tag{C.9}$$

Twisted affine algebras can be obtained from their non-twisted version by quotienting by an

[20] While we mostly follow Kac's conventions [62], to avoid confusion with the dimension of the fundamental representations d_I and the coefficient δ of the anomaly polynomial, we denote the scaling element and null root by $\mathcal{D}$ and $\widetilde{\theta}$, respectively.

[21] The only exception is $A_{2k}^{(2)}$, which has $\theta_0 = 2$, and the affine cowieghts must be rescaled appropriately. All other affine Lie algebras have $\theta_0 = 1$ [62].

automorphism of its Dynkin diagram. For the purpose of this work, we will only need to know that the root system through its Cartan matrix in Equation (C.4). Finally, the affine weights $\widetilde{\omega}_I$ are defined in a similar fashoin as in equation (C.8):

$$\widetilde{\omega}_0 = \Lambda_0 \,, \qquad \widetilde{\omega}_a = \omega_a + K_a \, \Lambda_0 \,. \tag{C.10}$$

From which we find

$$\langle \alpha^I , \widetilde{\omega}_J^\vee \rangle = \delta^I_J = \langle \widetilde{\omega}_J , \alpha^{I\,\vee} \rangle \,, \qquad \langle \widetilde{\omega}_I , \widetilde{\omega}_J^\vee \rangle = X_{IJ} \tag{C.11}$$

The matrix X_{AB} defined as $X_{0B} = X_{A0} \,, X_{ab} = (C^{-1})_{ab}$ is precisely the matrix appearing when inverting the quartic-anomaly cancellation condition, see Equation (2.17).

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