ADDING COFINAL COUNTABLE SEQUENCES THROUGH MULTIPLE REGULAR CARDINALS BY SSP FORCING

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ABSTRACT. We present a direct construction of stationary set preserving forcings that make ω-cofinal all the members of some arbitrary set $\mathcal{K}$ of regular cardinals $\kappa > \omega_1$. In addition, it is made possible to ensure that no other uncountable regular cardinals from the ground model acquire countable cofinality in the forcing extension. Our method is elementary, being based on a combinatorial argument from [4] together with generalizations of typical side-condition arguments and needs no assumptions beyond ZFC.

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1. PRELIMINARIES

We make use of the following notation.

Notation 1.1. Let $\mathscr{H}$ be a transitive model of ZFC^- and $M \prec \mathscr{H}$. If $x \in \bigcup M$, we define

$$\mathrm{Hull}(M, x) = \{f(x) : f \in M, x \in \mathrm{dom}(f)\}$$

and for $x \subseteq \bigcup M$, we define

$$\mathrm{Hull}^*(M, x) = \{f(y) : f \in M, y \in x \cap \mathrm{dom}(f)\}.$$

We define the expression

$$x \parallel_{\omega_1} M,$$

to mean that

$$\mathrm{Hull}(M, x) \cap \omega_1^{\mathscr{H}} = M \cap \omega_1^{\mathscr{H}}.$$

For a regular cardinal θ and a set x, we write $x \ll \theta$ to indicate that $\mathscr{P}(\mathscr{P}(x)) \in H_\theta$.

The concept of strongly ssp forcing was defined in [1].

Definition 1.2. Let $\mathbb{P}$ be a forcing notion and $\theta \gg \mathbb{P}$ a regular cardinal. Let $\mathbb{P} \in M \prec H_\theta$. The $\mathbb{P}$-condition p^* is called $(M,\mathbb{P})$-*semigeneric* if for every $q \leq p^*$ and every predense set $D \subseteq \mathbb{P}$ with $D \in M$, there exists some $r \in D$ such that

$$r \parallel q \text{ and } r \parallel_{\omega_1} M.$$

The condition p^* is called *strongly* $(M,\mathbb{P})$-*semigeneric* if for every $q \leq p^*$ there exists $\operatorname{tr}(q|M) \in \mathbb{P}$ such that $\operatorname{tr}(q|M) \parallel_{\omega_1} M$ and for every $r \in \operatorname{Hull}(M, \operatorname{tr}(q|M)) \cap \mathbb{P}$,

$$r \leq \operatorname{tr}(q|M) \Rightarrow r \parallel q.$$

$\mathbb{P}$ is said to be *(strongly) semiproper* with respect to $M \prec H_\theta$ if for every $p \in \mathbb{P} \cap M$, there exists $p^* \leq p$ such that p^* is (strongly) $(M,\mathbb{P})$-semigeneric.

A set $\mathcal{X} \subseteq [H_\theta]^\omega$ is *projective stationary* if for every stationary subset $S \subseteq \omega_1$ the set $\{M \in \mathcal{X} : M \cap \omega_1 \in S\}$ is stationary in $[H_\theta]^\omega$.

Definition 1.3. A forcing $\mathbb{P}$ is *strongly ssp* if for every regular cardinal θ which is sufficiently large ($\theta \gg \mathbb{P}$), there exist projective stationary many countable $M \prec H_\theta$ such that $\mathbb{P}$ is strongly semiproper with respect to M.

If $\mathbb{P}$ is strongly semiproper with respect to $M \prec H_\theta$, then $\mathbb{P}$ is semiproper with respect to M (see [1, Lemma 3.3]). Furthermore, the forcing $\mathbb{P}$ is ssp if and only if for every regular cardinal θ which is sufficiently large ($\theta \gg \mathbb{P}$), there exist projective stationary many countable $M \prec H_\theta$ such that $\mathbb{P}$ is semiproper with respect to M ([3]). It follows that, as expected, strongly ssp forcings are ssp.

Throughout the construction of our forcings, we will keep the following notation fixed.

Notation 1.4.

(1) Let's fix a set $\mathcal{K}$ of uncountable regular cardinals with $\omega_1 \notin \mathcal{K}$.
(2) Define the cardinal
$$\lambda = \sup(\mathcal{K}).$$
(3) Let $\mathcal{R}_\lambda$ be the set of all uncountable regular cardinals $\kappa < \lambda$.

2. Games

The forcings $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$, $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ (to be defined below) have as side conditions certain countable models $M \prec H_\mu$ (for μ regular and sufficiently large), which are characterized by means of two games $\mathbf{G}_0^{\mathbf{Nm}}(M,\mathcal{K})$ and $\mathbf{G}_1^{\mathbf{Nm}}(M,\mathcal{K})$. In order to prove that there are projective stationary many such models, we make use of a combinatorial argument from [4], where closely related games have been analyzed. We recall the relevant notions and results from [4].

Notation 2.1. If $\mathbf{T}$ is a subtree of $\lambda^{<\omega}$ and $\sigma \in \mathbf{T}$, we write $\mathbf{T} \cap \sigma\!\uparrow$ for the set

$$\{\tau \in \mathbf{T} : \sigma \subseteq \tau\}.$$

2.1. **F-fully labelled trees.**
We will make use of certain trees that were considered in Section 3 of [4] and first introduce terminology related to those trees.

Definition 2.2. Let $F : \lambda^{<\omega} \to [\mathcal{R}_\lambda]^\omega$. An *$F$-fully labelled tree* is a pair $(\widetilde{\mathbf{T}}, \ell)$ with $\widetilde{\mathbf{T}}$ a subtree of $(\lambda^{<\omega}, \subseteq)$ and ℓ a function from $\widetilde{\mathbf{T}}$ to $\mathcal{R}_\lambda$ such that the following two conditions are satisfied:

(a) for all $\sigma \in \widetilde{\mathbf{T}}$, the set $\{\eta \in \lambda : \sigma \frown \eta \in \widetilde{\mathbf{T}}\}$ is a cofinal subset of $\ell(\sigma)$,
(b) for all $\sigma \in \widetilde{\mathbf{T}}$ and all $\kappa \in F(\sigma)$, there exist arbitrarily large $n < \omega$ such that $(\forall \tau \in \widetilde{\mathbf{T}} \cap \sigma{\uparrow} \cap \lambda^n)\ \ell(\tau) = \kappa$.

Note that for every function $F : \lambda^{<\omega} \to [\mathcal{R}_\lambda]^\omega$ there exist F-fully labelled trees (this is simply a matter of bookkeeping).

Definition 2.3. Let $(\widetilde{\mathbf{T}}, \ell)$ be an F-fully labelled tree and let $\mathbf{T}$ be a subtree of $\widetilde{\mathbf{T}}$. For $\kappa \in \mathcal{R}_\lambda$, we say that

- κ is a *branching cardinality* for $\mathbf{T}$ if $(\forall \sigma \in \mathbf{T})$
$$\ell(\sigma) = \kappa \Rightarrow |\{\eta \in \kappa : \sigma \frown \eta \in \mathbf{T}\}| = \ell(\sigma),$$
- κ is a *fixing cardinality* for $\mathbf{T}$ if $(\forall \sigma \in \mathbf{T})$
$$\ell(\sigma) = \kappa \Rightarrow |\{\eta \in \kappa : \sigma \frown \eta \in \mathbf{T}\}| = 1.$$

$\mathbf{T} \subseteq \widetilde{\mathbf{T}}$ is called an *acceptable subtree* of $\widetilde{\mathbf{T}}$ if for every node $\sigma \in \mathbf{T}$, the cardinal $\ell(\sigma)$ is either a branching cardinality for $\mathbf{T}$ or a fixing cardinality for $\mathbf{T}$.

Definition 2.4. If $\theta \geq \lambda$ is regular and $M \prec H_\theta$ is a countable elementary submodel of H_θ, then define $F_M : \lambda^{<\omega} \to [\mathcal{R}_\lambda]^\omega$ by
$$F_M(\sigma) = \mathcal{R}_\lambda \cap \mathrm{Hull}(M, \sigma).$$
We shall abbreviate F_M-fully labelled tree to *M-fully labelled tree.*

2.2. **The Foreman-Magidor game.**
We now recall the definition of a game defined in [4].

Definition 2.5 ([4, Proof of Lemma 9])**.** Let $\mathbf{T}$ be an acceptable subtree of an F-fully labelled tree $(\widetilde{\mathbf{T}}, \ell)$, let $\kappa \in \mathcal{R}_\lambda$ be a branching cardinality for $\mathbf{T}$, let $\delta < \kappa$ and let $B \subseteq [\mathbf{T}]$ be a set of branches of $\mathbf{T}$. The corresponding *Foreman-Magidor Game* $\mathbf{G}^{\mathbf{FM}}(\mathbf{T}, \kappa, \delta, B)$ is the following game played by players PI and PII.

Both PI and PII play ordinals in λ. After round n has been concluded, a node $\sigma_{n+1} \in \mathbf{T}$ will be defined and, to get started, we set $\sigma_0 = \emptyset$.

In round n, first PI has to play an ordinal $\rho_n < \ell(\sigma_n)$ and then PII has to play an ordinal η_n which is such that $\sigma_n \frown \eta_n =: \sigma_{n+1} \in \mathbf{T}$, where both players are additionally respecting the following rules:

- if $\kappa \leq \ell(\sigma_n)$ and $\ell(\sigma_n)$ is a branching cardinality for $\mathbf{T}$ then PII has to play such that $\eta_n > \rho_n$. Moreover, if $\ell(\sigma_n) = \kappa$, then PI has to respect $\rho_n < \delta$,
- else (that is if $\ell(\sigma_n)$ is either a fixing cardinality for $\mathbf{T}$ or $\ell(\sigma_n) < \kappa$), then PI has to play ρ_n such that $\sigma_n \frown \rho_n \in \mathbf{T}$ and PII has to play $\eta_n = \rho_n$.

PII wins if and only if the branch b through $\mathbf{T}$ determined by the game (that is $b = \bigcup_n \sigma_n$) belongs to B.

Definition 2.6. Let $\mathbf{T}$ be an acceptable subtree of an F-fully labelled tree $(\widetilde{\mathbf{T}}, \ell)$, let $\kappa \in \mathcal{R}_\lambda$ be a branching cardinality for $\mathbf{T}$, let $\delta < \kappa$, and let $B \subseteq [\mathbf{T}]$ be a set of branches of $\mathbf{T}$. Suppose that Σ is a winning strategy for PII in the corresponding Foreman-Magidor game $\mathbf{G^{FM}}(\mathbf{T}, \kappa, \delta, B)$. We say that a node $\sigma \in \mathbf{T}$ is Σ-*compatible* if there exists a run of the game in which PII follows the strategy Σ and in which the branch determined by the run contains σ.

Lemma 2.7 (Winning lemma, [4, Claim in proof of Lemma 9])**.** *Fix a regular cardinal $\theta \gg \lambda$ together with a countable elementary $N_0 \prec H_\theta$ with $\mathcal{K} \in N_0$. Let $\mathbf{T}$ be an acceptable subtree of an N_0-fully labelled tree $(\widetilde{\mathbf{T}}, \ell)$, let $\kappa \in \mathcal{R}_\lambda$ be a branching cardinality for $\mathbf{T}$. For every $\delta < \kappa$, let B^κ_δ be the set of those branches b through $\mathbf{T}$ that satisfy*

$$\sup(\mathrm{Hull}^*(N_0, b) \cap \kappa) \leq \delta.$$

Then there are club many $\delta < \kappa$ such that PII *has a winning strategy in the game* $\mathbf{G}_\delta := \mathbf{G^{FM}}(\mathbf{T}, \kappa, \delta, B^\kappa_\delta)$.

Proof. We show that there is no sequence $(\Sigma_\alpha : \alpha \in S)$ such that:

- $S \subseteq \kappa$ is stationary,
- for every $\alpha \in S$, Σ_α is a winning strategy for PI in the game $\mathbf{G}_\alpha$.

Because every game $\mathbf{G}_\alpha$ is determined, this implies the lemma.
Suppose, aiming for contradiction, that $(\Sigma_\alpha : \alpha \in S)$ were such a sequence. Pick an elementary submodel N of $(H_\theta, \in)$ such that $|N| < \kappa$, $N \cap \kappa =: \delta \in S$ and moreover, $N_0, (\Sigma_\alpha : \alpha \in S), \mathbf{T}, \ell \in N$.

We describe a run of the game $\mathbf{G}_\delta$ which is won by PII but in which PI plays as dictated by Σ_δ. Note that it suffices to ensure that PII only plays ordinals which are in N. Suppose this has been accomplished up to position $\sigma \in \mathbf{T}$, we indicate how to proceed.
<u>Case 1</u>: if there exists a unique η such that $\sigma \frown \eta \in \mathbf{T}$, then $\eta \in N$ (since inductively $\sigma \in N$).
<u>Case 2</u>: if $\ell(\sigma) < \kappa$, then (since $\ell(\sigma) \in N$) both PI and PII play an ordinal $< \delta$, hence both ordinals are in N.
<u>Case 3</u>: if $\ell(\sigma) > \kappa$, we need to check that $\ell(\sigma) \cap N \setminus \Sigma_\delta(\sigma)$ is non-empty. This suffices because N knows that for every $\eta < \ell(\sigma)$, there exists $\eta' \in \ell(\sigma) \setminus \eta$ such that $\sigma \frown \eta' \in \mathbf{T}$ and therefore PII can continue playing ordinals in N. However it is clear that $\ell(\sigma) \cap N \setminus \Sigma_\delta(\sigma)$ is indeed non-empty because it follows from regularity of $\ell(\sigma)$ that $\sup(\Sigma_\alpha(\sigma) : \alpha \in S) \in \ell(\sigma) \cap N$.
<u>Case 4</u>: if $\ell(\sigma) = \kappa$, note that, since $\Sigma_\delta(\sigma)$ is winning for PI in $\mathbf{G}_\delta$, we have $\Sigma_\delta(\sigma) < \delta$. Then $\kappa \cap N \setminus \Sigma_\delta(\sigma) \neq \emptyset$ and it is possible to find $\eta \in \kappa \cap N \setminus \Sigma_\delta(\sigma)$ such that $\sigma \frown \eta \in \mathbf{T}$. □

Lemma 2.8 (Thinning lemma, [4, Lemma 9])**.** *Fix a regular cardinal $\theta \gg \lambda$ together with a countable elementary $N_0 \prec H_\theta$ with $\mathcal{K} \in N_0$. Let $\mathbf{T}$ be an acceptable subtree of an N_0-fully labelled tree $(\widetilde{\mathbf{T}}, \ell)$. Suppose that $\kappa \in N_0$ is a branching cardinality for $\mathbf{T}$. Suppose in addition that $S \subseteq \kappa$ is stationary and contains only ordinals of countable cofinality. Then there exists an acceptable subtree $\mathbf{T}'$ of $\mathbf{T}$ such that the following conditions (1), (2), (3) hold:*

(1) $\{\ell(\sigma) : \sigma \in \mathbf{T}', \ell(\sigma)$ is branching for $\mathbf{T}\} \setminus \{\kappa\}$ is the set of branching cardinalities for $\mathbf{T}'$,

(2) $\{\ell(\sigma) : \sigma \in \mathbf{T}', \ell(\sigma)$ *is fixing for* $\mathbf{T}\} \cup \{\kappa\}$ *is the set of fixing cardinalities for* $\mathbf{T}'$*,*

(3) there exists $\delta \in S$ *such that*

$$\sup(\mathrm{Hull}^*(N_0, b) \cap \kappa) = \delta,$$

for every branch b *through* $\mathbf{T}'$*.*

Proof. Using the Winning lemma, pick $\delta \in S$ and a winning strategy Σ for PII in the game $\mathbf{G}_\delta = \mathbf{G}^{\mathbf{FM}}(\mathbf{T}, \kappa, \delta, B^\kappa_\delta)$. Fix a sequence $(\delta_m : m < \omega)$ which is increasing and cofinal in δ. We define $\mathbf{T}' \cap \lambda^n$ recursively for $n < \omega$.

Let $\sigma \frown \eta \in \mathbf{T} \cap \lambda^{n+1}$. If $\sigma \notin \mathbf{T}' \cap \lambda^n$ or σ is not Σ-compatible, then we decide that $\sigma \frown \eta \notin \mathbf{T}' \cap \lambda^{n+1}$. Else, we separate different cases:

- if $\ell(\sigma)$ is a fixing cardinality for $\mathbf{T}$, then $\{\eta' \in \lambda : \sigma \frown \eta' \in \mathbf{T}\} = \{\eta\}$ and we let $\sigma \frown \eta \in \mathbf{T}'$,

now suppose $\ell(\sigma)$ is a branching cardinality for $\mathbf{T}$:

- if $\ell(\sigma) < \kappa$, then we let $\sigma \frown \eta \in \mathbf{T}'$,
- if $\ell(\sigma) = \kappa$, let $\sigma \frown \eta \in \mathbf{T}' \cap \lambda^{n+1}$ if and only if η is the response of Σ to PI playing δ_n in position σ in the game $\mathbf{G}_\delta$,
- if $\ell(\sigma) > \kappa$, let $\sigma \frown \eta \in \mathbf{T}' \cap \lambda^{n+1}$ if and only if there exists $\rho < \ell(\sigma)$ such that η is the response of Σ to PI playing ρ in position σ.

We now check that $\mathbf{T}'$ satisfies 2.8(1), 2.8(2) and 2.8(3).

2.8(1): if $\sigma \in \mathbf{T}'$ and $\ell(\sigma)$ is a branching cardinality for $\mathbf{T}$, not equal to κ, then either:

- $\ell(\sigma) < \kappa$ and $\{\eta < \lambda : \sigma \frown \eta \in \mathbf{T}\} = \{\eta < \lambda : \sigma \frown \eta \in \mathbf{T}'\}$, so $\ell(\sigma)$ is also a branching cardinality for $\mathbf{T}'$,
- $\ell(\sigma) > \kappa$ and then $\{\eta < \lambda : \sigma \frown \eta \in \mathbf{T}'\}$ is cofinal in $\ell(\sigma)$ since Σ is winning for PII.

2.8(2): if $\sigma \in \mathbf{T}'$ and $\ell(\sigma)$ is a fixing cardinality for $\mathbf{T}$, then it is also a fixing cardinality for $\mathbf{T}'$, because there is a unique $\eta \in \lambda$ such that $\sigma \frown \eta \in \mathbf{T}$ and this $\sigma \frown \eta$ is also an element of $\mathbf{T}'$.

κ is also a fixing cardinality for $\mathbf{T}'$ because if $\sigma \in \mathbf{T}'$ and $\ell(\sigma) = \kappa$, then there exists a unique η such that $\sigma \frown \eta \in \mathbf{T}'$.

2.8(3): let b be a branch through $\mathbf{T}'$, then there exists a run z of the game $\mathbf{G}_\delta$ in which PII follows Σ and such that z determines b.

Then, because Σ is winning, $\mathrm{Hull}^*(N_0, b) \cap \kappa \subseteq \delta$, so it suffices to show that for every $n < \omega$, there exists $k < \omega$ such that $b(k) \in \kappa \setminus \delta_n$.

Because $\mathbf{T}'$ is an acceptable subtree of an N_0-fully labelled tree, there exists $k \geq n$ such that for every $\sigma \in \mathbf{T}' \cap \lambda^k$, $\ell(\sigma) = \kappa$. Then $b(k)$ is the response of Σ to PI playing δ_k in position $b \restriction k$ in the game $\mathbf{G}_\delta$, hence $b(k) \in \kappa \setminus \delta_k$. □

Definition 2.9. Given a regular cardinal $\theta \gg \lambda$ and $\lambda \in M \prec H_\theta$, the multiple Namba game $\mathbf{G}^{\mathbf{Nm}}_0(M, \mathcal{K})$ for $(M, \mathcal{K})$ proceeds as follows. Let $M_0 = M$. In round $n < \omega$, PI plays a pair (κ_n, ν_n), with $\kappa_n \in \mathcal{K} \cap M_n$ and $\nu_n \in \kappa_n$. PII has to answer by playing an ordinal $\xi_n \in \kappa_n \setminus \nu_n$ such that

$$M_{n+1} \cap \omega_1 = M_n \cap \omega_1,$$

with M_{n+1} defined in the following way:

$$M_{n+1} = \mathrm{Hull}(M_n, \xi_n).$$

PII loses in round n of the game if PII fails to play such ξ_n, else the game continues. PII wins the game if PII does not lose in any round $n < \omega$ of the game.

Notation 2.10. When $\mathtt{r}$ is a finite run of the game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$ consisting of n rounds, let PII($\mathtt{r}$) denote the sequence $(\xi_i : i < n)$ of moves played by PII in $\mathtt{r}$.

Notation 2.11. If $\mathtt{r}$ is a finite run of the game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$, consisting of n rounds, then $\mathtt{r}$ defines a model M_n, namely the n-th model constructed during the game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$ as described in the above Definition 2.9. We then denote by $M \ltimes \mathtt{r}$ this model $M_n = \mathrm{Hull}(M, \mathrm{PII}(\mathtt{r}))$.

We are now ready to prove the following Proposition 2.12.

Proposition 2.12. *For every set of uncountable regular cardinals $\mathcal{K} \in H_\theta$ with $\omega_1 \notin \mathcal{K}$, the set of countable elementary submodels $M \prec H_\theta$ for which* PII *has a winning strategy in the corresponding generalised Namba game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$ is a projective stationary subset of $[H_\theta]^\omega$.*

Proof. Let $S \subseteq \omega_1$ be stationary with $\mathrm{Hull}^*(N_0, \delta) = \delta$ for all $\delta \in S$. It suffices to show that for any countable $N_0 \prec H_\theta$ with $\mathcal{K} \in N_0$, there exists $\delta \in S$ such that PII has a winning strategy in the generalised Namba game $\mathbf{G}_0^{\mathbf{Nm}}(\mathrm{Hull}^*(N_0, \delta), \mathcal{K})$.

Let $(\widetilde{\mathbf{T}}, \ell)$ be an N_0-fully labelled tree and use the Thinning lemma (Lemma 2.8) to find an acceptable subtree $\mathbf{T}$ of $\widetilde{\mathbf{T}}$ such that

- ω_1 is a fixing cardinality for $\mathbf{T}$,
- every element of $\mathcal{K}$ is a branching cardinality for $\mathbf{T}$,
- there exists an ordinal $\delta \in S$ such that for every branch b through $\mathbf{T}$ it is the case that $\mathrm{Hull}^*(N_0, b) \cap \omega_1 = \delta$.

Let $M = \mathrm{Hull}^*(N_0, \delta) = \bigcup_{n<\omega} \mathrm{Hull}(N_0, \delta_n)$, where $(\delta_n : n < \omega)$ is cofinal in δ. Then $M \cap \omega_1 = \delta \in S$ and what remains is to describe a winning strategy for PII in the multiple Namba game on M.

Suppose PI starts by playing (κ_0, ν_0). Then $\kappa_0 \in M$, so there exists $\tau_0 \in \mathbf{T}$ such that $\kappa_0 \in \mathrm{Hull}(N_0, \tau_0)$.

Then, there exists $\tilde{\sigma}_0 \in \mathbf{T} \cap \tau_0{\uparrow}$ such that $\ell(\tilde{\sigma}_0) = \kappa_0$. PII answers

$$\xi_0 = \min\{\eta < \kappa_0 \setminus \nu_0 : \tilde{\sigma}_0 \frown \eta \in \mathbf{T}\}.$$

Put $\sigma_0 = \tilde{\sigma}_0 \frown \xi_0$.

Suppose at move $n > 0$, PI plays (κ_n, ν_n). Then for certain $k < \omega$,

$$\kappa_n \in \mathrm{Hull}(N_0, \{\delta_k, \xi_0, \ldots, \xi_{n-1}\}),$$

so there exists $\tau_n \in \mathbf{T} \cap \sigma_{n-1}{\uparrow}$ such that $\kappa_n \in \mathrm{Hull}(N_0, \tau_n)$.

There in turn exists $\tilde{\sigma}_n \in \mathbf{T} \cap \tau_n{\uparrow}$ such that $\ell(\tilde{\sigma}_n) = \kappa_n$. PII answers

$$\xi_n = \min\{\eta < \kappa_n \setminus \nu_n : \tilde{\sigma}_n \frown \eta \in \mathbf{T}\}.$$

Put $\sigma_n = \tilde{\sigma}_n \frown \xi_n$.

If $z = ((\nu_n : n < \omega), (\xi_n : n < \omega))$ is the resulting run of the game, then the corresponding $(\sigma_n : n < \omega)$ determines a branch b of $\mathbf{T}$ such that

$$\bigcup_{n<\omega} \mathrm{Hull}(M, \xi_n) \subseteq \mathrm{Hull}^*(N_0, b).$$

From which it follows that

$$\bigcup_{n<\omega} \mathrm{Hull}(M, \xi_n) \cap \omega_1 = \delta.$$

□

3. Multiple Namba forcing

Recall that we have fixed a set of uncountable regular cardinals $\mathcal{K}$ not containing ω_1 and $\lambda = \sup \mathcal{K}$. We now also fix some $\mu \gg \lambda$ regular. We denote by $\mathcal{W}_0$ the set of all countable models $M \prec H_\mu$ with $\mathcal{K} \in M$ for which PII has a winning strategy in the game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$. As an application of the multiple Namba game $\mathbf{G}_0^{\mathbf{Nm}}(N, \mathcal{K})$, we construct a forcing $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ that is strongly stationary set preserving and collapses $|H_\mu| = 2^{<\mu}$ to ω_1 in such a way that for every $\kappa \in \mathcal{K}$,

$$\Vdash_{\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})} \mathrm{cf}(\kappa) = \omega.$$

The forcing $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ has size $2^{<\mu}$ and therefore forces $(2^{<\mu})^+$ to become the new ω_2.

Definition 3.1. A $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-condition is a finite set p that satisfies the following:

- every element of p is a three-tuple $(M, \mathtt{r}, \Sigma)$ with $M \in \mathcal{W}_0$, Σ a winning strategy for PII in the game $\mathbf{G}_0^{\mathbf{Nm}}(N, \mathcal{K})$ and $\mathtt{r}$ a finite run of the game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$ in which PII follows Σ,
- every two elements $(M_1, \mathtt{r}_1, \Sigma_1)$ and $(M_2, \mathtt{r}_2, \Sigma_2)$ of p that satisfy $M_1 \cap \omega_1 = M_2 \cap \omega_1$ are equal,
- for every two elements $(M_1, \mathtt{r}_1, \Sigma_1)$ and $(M_2, \mathtt{r}_2, \Sigma_2)$ of p, if $M_1 \cap \omega_1 < M_2 \cap \omega_1$ then $(M_1, \mathtt{r}_1, \Sigma_1) \in M_2 \ltimes \mathtt{r}_2$.

If p, q are two $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-conditions, then $q \leq p$ if and only if:

for every $(M, \mathtt{r}, \Sigma) \in p$ there exists $(M', \mathtt{r}', \Sigma') \in q$ such that $M = M'$, $\Sigma = \Sigma'$ and the finite run $\mathtt{r}'$ of the game is an extension of the run $\mathtt{r}$.

Notation 3.2. If $G \subseteq \mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ is a V-generic filter, define C^G to be the set of all ordinals $\delta < \omega_1$ for which there exists $(M, \mathtt{r}, \Sigma) \in p$ with $p \in G$ such that $\delta = M \cap \omega_1$. Note that C^G is clearly unbounded in ω_1, so let $(\delta_\alpha^G : \alpha < \omega_1)$ denote the increasing enumeration of C^G. Furthermore, for every $\alpha < \omega_1$, define M_α^G to be the union of all models of the form $M \ltimes \mathtt{r}$ with $(M, \mathtt{r}, \Sigma) \in p \in G$ and $M \cap \omega_1 = \delta_\alpha^G$.

Proposition 3.3. *$\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ forces that for every $\kappa \in \mathcal{K}$, for all but countably many $\alpha < \omega_1$, the ω-sequence*

$$\bigcup \{\mathrm{PII}(\mathtt{r}) : (M, \mathtt{r}, \Sigma) \in p \in G, M \cap \omega_1 = \delta_\alpha^G\}$$

is cofinal in κ.

Proof. Given $\kappa \in \mathcal{K}$, let $p \in \mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ such that there exists $(M, \mathtt{r}, \Sigma) \in p$ with $\kappa \in M$ (by Proposition 2.12 such p are dense in $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$). It suffices to prove that for every $\nu \in \kappa$ there exists a $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-condition $q \leq p$ for which there is $(M, \mathtt{r}_M^q, \Sigma_M^q) \in q$ with $\mathrm{ran}(\mathrm{PII}(\mathtt{r}_M^q)) \cap \kappa \setminus \nu \neq \emptyset$. Let $(N_i, \mathtt{r}_i, \Sigma_i) \in p$ for $i < k$ enumerate the elements of

$$\{(N, \mathtt{r}_N^p, \Sigma_N^p) \in p : M \cap \omega_1 \leq N \cap \omega_1\},$$

with $N_i \cap \omega_1 \leq N_{i+1} \cap \omega_1$ for each $i < k - 1$. Let p_{k-1} be the $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-condition strengthening p that is obtained by removing from p the triple $(N_{k-1}, \mathtt{r}_{k-1}, \Sigma_{k-1})$

and adding the triple $(N_{k-1}, \mathtt{r}^*_{k-1}, \Sigma_{k-1})$, where $\mathtt{r}^*_{k-1}$ is the extension of the game-run $\mathtt{r}_{k-1}$ by one round in which PI plays (κ, ν) and PII answers following the strategy Σ_{k-1}. If $k = 1$ then the condition p_{k-1} already has the required property. Else, go on to define $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-conditions $p_{k-1} \geq p_{k-2} \geq \ldots \geq p_0$ and finite game-runs $\mathtt{r}^*_{k-1}, \mathtt{r}^*_{k-2}, \ldots, \mathtt{r}^*_0$ as follows. Having defined the condition p_{i+1} and game-run $\mathtt{r}^*_{i+1}$, let p_i be the condition obtained by removing from p_{i+1} the triple $(N_i, \mathtt{r}_i, \Sigma_i)$ and adding the triple $(N_i, \mathtt{r}^*_i, \Sigma_i)$, where $\mathtt{r}^*_i$ is the extension of the game-run $\mathtt{r}_i$ by one round in which PI plays (κ, ν_i), with ν_i the maximal ordinal below κ which is played by PII in $\mathtt{r}^*_{i+1}$, and PII answers following the strategy Σ_i. The thus defined $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-condition $p_0 \leq p$ has the required property. $\square$

Corollary 3.4. *For every $\kappa \in \mathcal{K}$,*

$$\Vdash_{\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})} \ \mathrm{cf}(\kappa) = \omega.$$

Proposition 3.5. *Let $\theta \gg \mu$ be a regular cardinal. Then $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ is strongly semiproper with respect to every countable $N \prec H_\theta$ such that $N \cap H_\mu \in \mathcal{W}_0$.*

Proof. Let $N \prec H_\theta$ such that $N \cap H_\mu \in \mathcal{W}_0$ and $p \in N$ a $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-condition. Define $p^* = p \cup \{(N \cap H_\mu, \emptyset, \Sigma_N)\}$, where Σ_N is an arbitrary winning strategy for PII in the game $\mathbf{G}_0^{\mathbf{Nm}}(N \cap H_\mu, \mathcal{K})$. Given any $q \leq p$, define $\mathrm{tr}(q|N)$ to be the $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$-condition $\{(M, \mathtt{r}^q_M, \Sigma^q_M) \in q : M \cap \omega_1 \in N\}$. There now exists some element of q that has the form $\{(N \cap H_\mu, \mathtt{r}^q_N, \Sigma_N)\}$. Note that $\mathrm{Hull}(N, \mathrm{tr}(q|N))$ contains the same countable ordinals as N, because Σ_N is a winning strategy for PII in the game $\mathbf{G}_0^{\mathbf{Nm}}(N \cap H_\mu, \mathcal{K})$ and because $\mathrm{tr}(q|N) \in N \cap H_\mu \ltimes \mathtt{r}^q_N$. Furthermore, for every $r \in \mathrm{Hull}(N, \mathrm{tr}(q|N)) \cap \mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$, it is the case that $r \in (N \cap H_\mu) \ltimes \mathtt{r}^q_N$, so if also $r \leq \mathrm{tr}(q|N)$, then

$$r \cup \{(M, \mathtt{r}_M, \Sigma_M) \in q : N \cap \omega_1 \notin N\}$$

is a mutual lowerbound for r and q. $\square$

By combining Proposition 2.12 and Proposition 3.5, we find that $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ is strongly semiproper with respect to projective stationary many countable submodels of H_θ, hence:

Corollary 3.6. *The forcing $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ is strongly ssp.*

4. The cofinality of the remaining cardinals

We further modify the forcing $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ so that for every uncountable regular cardinal $\kappa \in \lambda \setminus \mathcal{K}$ the trivial condition will force that $\mathrm{cf}^{V[G]}(\kappa) = \omega_1$. In order to do this we need to adapt the Namba game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$.

Definition 4.1. Given a regular cardinal $\theta \gg \lambda$ and $\lambda \in M \prec H_\theta$, the closed game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$ proceeds as follows.

- Set $M_0 = M$ and set $\mathrm{Promise}_0 = \emptyset$.
- In round $n < \omega$,
  - PI first plays a regular cardinal $\lambda_n \in M_n$ with
  $$\omega_1 < \lambda_n < \lambda \text{ and } \lambda_n \notin \mathcal{K}.$$
  - PII has to play a pair (λ_n, β_n) with $\beta_n \in \lambda_n$ and we define
  $$\mathrm{Promise}_{n+1} = \mathrm{Promise}_n \cup \{(\lambda_n, \beta_n)\}.$$

– Then PI is again to play and plays a pair (κ_n, ν_n), with

$$\kappa_n \in \mathcal{K} \cap M_n \text{ and } \nu_n \in \kappa_n.$$

– PII has to answer by playing an ordinal $\xi_n \in \kappa_n \setminus \nu_n$ such that

$$M_{n+1} \cap \omega_1 = M_n \cap \omega_1,$$

and such that also for every $(\kappa, \beta) \in \text{Promise}_{n+1}$,

$$\sup(M_{n+1} \cap \kappa) \leq \beta,$$

where again M_{n+1} is defined in the following way:

$$M_{n+1} = \text{Hull}(M_n, \xi_n).$$

- PII loses in round n of the game if PII fails to play such ξ_n, else the game continues. PII wins the game if and only if PII does not lose in any round $n < \omega$ of the game.

Remark here that the way the models M_n are defined in the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$ differs from the previously considered game $\mathbf{G}_0^{\mathbf{Nm}}(M, \mathcal{K})$, where all moves of PII get automatically added to M_n in order to define M_{n+1}. In the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$, PII plays first a pair (λ_n, β_n) which doesn't get added to M_n and then the ordinal ξ_n that gets added to M_n. The pair (λ_n, β_n) merely accounts for a promise of PII not to add any ordinals in the interval $\lambda_n \setminus (\beta_n + 1)$ in any of the models M_k formed in future moves. This is also reflected in the way we use the notation $M \ltimes \mathtt{r}$ when $\mathtt{r}$ is a finite run of $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$.

Notation 4.2 (Compare Notation 2.11)**.** If $\mathtt{r}$ is a finite run of length n of the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$, we write

$$M \ltimes \mathtt{r} := M_n = \text{Hull}(M, (\xi_k : k < n)),$$

where M_n and ξ_k are as in Definition 4.1.

Lemma 4.3. *For every set of uncountable regular cardinals $\mathcal{K} \in H_\theta$ with $\omega_1 \notin \mathcal{K}$, the set of countable elementary submodels M of H_θ for which* PII *has a winning strategy in the corresponding game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$ is a projective stationary subset of $[H_\theta]^\omega$.*

Proof. Let $S \subseteq \omega_1$ be stationary and $N_0 \prec H_\theta$. It suffices to show that there exists $\delta \in S$ such that PII has a winning strategy in the game $\mathbf{G}_1^{\mathbf{Nm}}(\text{Hull}(N_0, \delta), \mathcal{K})$.

Let $(\widetilde{\mathbf{T}}, \ell)$ be an N_0-fully labelled tree and use the Thinning lemma to find an acceptable subtree $\mathbf{T}_0$ of $\widetilde{\mathbf{T}}$ such that

- ω_1 is a fixing cardinality for $\mathbf{T}_0$,
- every other element of $\mathcal{R}_\lambda$ is a branching cardinality for $\mathbf{T}_0$,
- there exists an ordinal $\delta \in S$ such that for every branch b through $\mathbf{T}_0$ it is the case that $\text{Hull}^*(N_0, b) \cap \omega_1 = \delta$.

Let $M = \text{Hull}^*(N_0, \delta) = \bigcup_{n<\omega} \text{Hull}(N_0, \delta_n)$, where $(\delta_n : n < \omega)$ is strictly increasing and cofinal in δ. Then $M \cap \omega_1 = \delta \in S$ and what remains is to describe a winning strategy for PII in the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$. Suppose PI starts by playing the cardinal $\lambda_0 \in M \cap \mathcal{R}_\lambda$. Use the Thinning lemma to find an acceptable subtree $\mathbf{T}_0'$ of $\mathbf{T}_0$ such that

- for every cardinal different from λ_0, if it was a fixing cardinality for $\mathbf{T}_0$ it still is for $\mathbf{T}'_0$ and if it was a branching cardinality for $\mathbf{T}_0$, it also still is for $\mathbf{T}'_0$,
- λ_0 is turned from a branching cardinality for $\mathbf{T}_0$ into a fixing cardinality for $\mathbf{T}'_0$,
- there exists an ordinal $\beta_0 \in \lambda_0$ such that for every branch b through $\mathbf{T}'_0$ it is the case that $\sup(\mathrm{Hull}^*(N_0, b) \cap \lambda_0) = \beta_0$.

PII then plays (λ_0, β_0). Next, PI plays (κ_0, ν_0). Then $\kappa_0 \in M$, so there exists $\tau_0 \in \mathbf{T}'_0$ such that $\kappa_0 \in \mathrm{Hull}(N_0, \tau_0)$. There exists $\tilde{\sigma}_0 \in \mathbf{T}'_0 \cap \tau_0\!\uparrow$ such that $\ell(\tilde{\sigma}_0) = \kappa_0$. PII answers

$$\xi_0 = \min\{\xi < \kappa_0 \setminus \nu_0 : \tilde{\sigma}_0 \frown \xi \in \mathbf{T}'_0\}.$$

Define

$$\sigma_0 = \tilde{\sigma}_0 \frown \xi_0,$$
$$\mathbf{T}_1 = \{\tau \in \lambda^{<\omega} : \tilde{\sigma}_0 \frown \xi_0 \frown \tau \in \mathbf{T}'_0\}.$$

We now explain how PII survives round $n > 0$. Suppose that PI first plays the cardinal $\lambda_n \in M_n$. Use the Thinning lemma to find an acceptable subtree $\mathbf{T}'_n$ of $\mathbf{T}_n$ such that

- for every cardinal different from λ_n, if it was a fixing cardinality for $\mathbf{T}_n$ it still is for $\mathbf{T}'_n$ and if it was a branching cardinality for $\mathbf{T}_n$, it also still is for $\mathbf{T}'_n$,
- if λ_n was a branching cardinality for $\mathbf{T}_n$, it is turned into a fixing cardinality for $\mathbf{T}'_n$,
- there exists an ordinal $\beta_n \in \lambda_n$ such that for every branch b through $\mathbf{T}'_n$ it is the case that $\sup(\mathrm{Hull}^*(\mathrm{Hull}(N_0, \sigma_{n-1}), b) \cap \lambda_n) = \beta_n$.

PII then plays (λ_n, β_n). Suppose that PI plays next a pair (κ_n, ν_n). Then

$$\kappa_n \in \mathrm{Hull}(N_0, \sigma_{n-1} \frown \tau_n),$$

for some $\tau_n \in \mathbf{T}'_n$, so there exists $\tilde{\sigma}_n \in \mathbf{T}'_n \cap \tau_n\!\uparrow$ such that $\ell(\sigma_{n-1} \frown \tilde{\sigma}_n) = \kappa_n$. PI answers

$$\xi_n = \min\{\eta < \kappa_n \setminus \nu_n : \tilde{\sigma}_n \frown \eta \in \mathbf{T}'_n\}.$$

Define $\sigma_n = \sigma_{n-1} \frown \tilde{\sigma}_n \frown \xi_n$, $\mathbf{T}_{n+1} = \{\tau \in \lambda^{<\omega} : \sigma_n \frown \tau \in \mathbf{T}_n\}$.

If z is a resulting play of the game in which PII has played $(\xi_n : n < \omega)$ and $((\lambda_n, \beta_n) : n < \omega)$, then unioning the terms in the sequence $(\sigma_n : n < \omega)$ determines a branch b of $\mathbf{T}$ such that

$$\bigcup_{n<\omega} \mathrm{Hull}(M, \xi_n) \subseteq \mathrm{Hull}^*(N_0, b).$$

From which it follows that

$$\bigcup_{n<\omega} \mathrm{Hull}(M, \xi_n) \cap \omega_1 = \delta.$$

Similarly, we see that for every $n < \omega$,

$$\bigcup_{n<\omega} \mathrm{Hull}(M, \xi_n) \cap \lambda_n \leq \beta_n.$$

□

Now back to the problem of adding ω-cofinal sequences through the cardinals in $\mathcal{K}$ and simultaneously forcing the uncountable regular cardinals in $\lambda \setminus \mathcal{K}$ to get cofinality ω_1. Fix again $\mu \gg \lambda$ and denote $\mathcal{W}_1$ the set of all countable models $M \prec H_\mu$ with $\mathcal{K} \in M$ for which PII has a winning strategy in the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$.

Definition 4.4. A $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$-condition is a finite set p that satisfies the following conditions:

- every element of p is a 4-tuple $(M, \mathtt{r}, \Sigma, d)$ with $M \in \mathcal{W}_1$, with Σ a winning strategy for PII in the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$, with $\mathtt{r}$ a finite run of the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$ in which PII follows Σ and with $d \in H_\mu^{<\omega}$,
- every two elements $(M_1, \mathtt{r}_1, \Sigma_1, d_1)$ and $(M_2, \mathtt{r}_2, \Sigma_2, d_2)$ of p that satisfy $M_1 \cap \omega_1 = M_2 \cap \omega_1$ are equal,
- for every two elements $(M_1, \mathtt{r}_1, \Sigma_1, d_1)$ and $(M_2, \mathtt{r}_2, \Sigma_2, d_2)$ of p, if $M_1 \cap \omega_1 < M_2 \cap \omega_1$ then $(M_1, \mathtt{r}_1, \Sigma, d_1) \in M_2 \ltimes \mathtt{r}_2$.

If p, q are two $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$-conditions, then $q \leq p$ if and only if:

> for every $(M, \mathtt{r}, \Sigma, d) \in p$ there exists $(M', \mathtt{r}', \Sigma', d') \in q$ such that $M = M', \Sigma = \Sigma'$, $d \subseteq d'$, and the finite run $\mathtt{r}'$ of the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$ is an extension of the run $\mathtt{r}$.

For generic filters $G \subseteq \mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ we use the same notations C^G, $(\delta_\alpha^G : \alpha < \omega_1)$ and $(M_\alpha^G : \alpha < \omega_1)$ as introduced in Notation 3.2. In particular, for every $\alpha < \omega_1$, we denote by M_α^G the union of all models of the form $M \ltimes \mathtt{r}$ with $(M, \mathtt{r}, \Sigma, d) \in p \in G$ and $M \cap \omega_1 = \delta_\alpha^G$. Note that for every $\alpha < \omega_1$ the set M_α^G is a countable increasing union of elementary submodels of H_μ^V and therefore itself a countable elementary submodel of H_μ^V.

Including the fourth coordinate $d \in H_\mu^{<\omega}$ in the elements $(M, \mathtt{r}, \Sigma, d)$ enables one to run a standard density argument to show that the sequence $(M_\alpha^G : \alpha < \omega_1)$ is continuous (and this is our sole intent for including these fourth coordinates).

Lemma 4.5. $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ *forces that the sequence* $(M_\alpha^G : \alpha < \omega_1)$ *is continuous.*

We next establish that in case $\lambda = \sup(\mathcal{K})$ is regular, $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ forces that $\mathrm{cf}^{V[G]}(\lambda) = \omega_1$.

Suppose therefore that the cardinal λ is regular. Because $(M_\alpha^G : \alpha < \omega_1)$ is continuous increasing and the models M_α^G together cover λ, if we can show that $M_\alpha^G \cap \lambda$ is bounded in λ for every α, it will readily follow that $\mathrm{cf}^{V[G]}(\lambda) = \omega_1$. That this is the case follows from the following simple observation.

Lemma 4.6. *Suppose $M \prec H_\mu$ and $\kappa < \lambda$ both belong to M and λ is a regular cardinal. Then*

$$\sup(\mathrm{Hull}(M, \alpha) \cap \lambda) = \sup(M \cap \lambda),$$

for all $\alpha < \kappa$.

Proof. For any $\alpha < \kappa$ and any function $f : \kappa \to \lambda$ we have that if $f \in M$, then $f(\alpha) \leq \sup(f) \in M \cap \lambda$. $\square$

Corollary 4.7. *Suppose $\lambda = \sup(\mathcal{K})$ is regular. Suppose G is V-generic for $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$. Let $p \in G$ and $(M, \mathtt{r}_M^p, \Sigma_M^p, d_M^p) \in p$, and let $\alpha < \omega_1$ such that $M_\alpha^G \cap \omega_1 = M \cap \omega_1$. Then*

$$\sup(M \cap \lambda) = \sup(M_\alpha^G \cap \lambda).$$

Lemma 4.8. *Suppose G is V-generic for $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$. If $\lambda = \sup(\mathcal{K})$ is regular, then in $V[G]$ the ordinal λ has cofinality ω_1.*

Proof. It follows from Lemma 4.5 and Corollary 4.7 that

$$(\sup(M_\alpha^G \cap \lambda) : \alpha < \omega_1)$$

is a closed unbounded subset of λ. □

We can now summarize the main properties of the forcing $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$.

Theorem 4.9. *The forcing $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ has the following properties:*

(1) if $\theta \gg \mu$ is a regular cardinal, then $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ is strongly semiproper with respect to every countable $N \prec H_\theta$ such that $N \cap H_\mu \in \mathcal{W}_1$,

(2) $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ is strongly ssp,

(3) for every $\kappa \in \mathcal{K}$, forcing with $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ changes the cofinality of κ to ω,

(4) for every regular $\kappa \in \lambda \setminus \mathcal{K} \cup \{\lambda\}$, forcing with $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ changes the cofinality of κ to ω_1,

(5) forcing with $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ makes $(2^{<\mu})^+$ into the new ω_2 and changes the cofinality of every regular cardinal between λ and $2^{<\mu}$ to ω_1.

Proof. The statements (1), (2), (3) follow from the proofs of respectively Proposition 3.5, Corollary 3.6 and Proposition 3.3, as these proofs remain, mutatis mutandis, valid for the forcing $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$.

The case $\kappa = \lambda$ of (4) is just Lemma 4.8. To conclude the proof of (4), we prove that if $\kappa \in \lambda \setminus \mathcal{K}$ is regular, then $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ forces the cofinality of κ to become ω_1. For this, we first establish that for every $\alpha < \omega_1$, the forcing $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ forces that $M_\alpha^G \cap \kappa$ is bounded in κ. Suppose otherwise. Then there would exist $p \in \mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ and $(M, r_M^p, \Sigma_M^p, d_M^p) \in p$ such that

(a) p forces that $\delta_\alpha^G = M \cap \omega_1$,

(b) p forces that $M_\alpha^G \cap \kappa$ is cofinal in κ.

This is however impossible because we can find $q \leq p$ with certain $(N, r_N^q, \Sigma_N^q, d_N^q) \in q$ such that $p, \kappa \in N$ and such that in r_N^q PII has made a promise (κ, β) for some $\beta < \kappa$. Then q forces that $M_\alpha^G \cap \kappa \subseteq \beta$, which is a contradiction. So $\sup(M_\alpha^G \cap \kappa) < \kappa$ for every α and since $(M_\alpha^G : \alpha < \omega_1)$ is continuous, it follows that $(\sup(M_\alpha^G \cap \kappa) : \alpha < \omega_1)$ is a closed unbounded subset of κ.

For statement (5), the continuous sequence $(M_\alpha^G : \alpha < \omega_1)$ covers H_μ^V, so H_μ^V acquires cardinality $\aleph_1$ in the forcing extension. Moreover, if κ is a regular cardinal with $\lambda < \kappa \leq 2^{<\mu}$, let $X \subseteq H_\mu$ of size κ and $\leq_X$ be a wellordering of X of ordertype κ. Then $(\mathrm{Hull}^*(M_\alpha^G, \lambda) \cap X : \alpha < \omega_1)$ is a continuous sequence of ground-model subsets of X of size λ that together cover X. It follows that the map mapping $\alpha < \omega_1$ to the $\leq_X$-supremum of $\mathrm{Hull}^*(M_\alpha^G, \lambda) \cap X$ is a continuous cofinal increasing map from ω_1 to $(X, \leq_X)$. □

5. An application to the countable-cofinality-constructible model

Recall that the countable-cofinality-constructible model C^* introduced in [6] is defined by

$$C^* = \bigcup C_\alpha^*, \text{ where}$$

$$C_0^* = \emptyset,\ C_\beta^* = \bigcup_{\alpha < \beta} C_\alpha^* \text{ when } \beta \text{ is limit, and}$$

$C^*_{\alpha+1}$ is the set of all sets $\{a \in C^*_\alpha \mid C^*_\alpha \models \varphi(a, \bar{b})\}$ with $\varphi(., \bar{b})$ an $\mathcal{L}(Q^{\mathrm{cf}}_\omega)$-formula with parameters in C^*_α.

In [6] and [9], Namba forcings have been found useful for controlling C^* through the coding of countable sets by means of making specific regular cardinals ω-cofinal. Using the forcings $\mathbb{P}^{\mathbf{Nm}}_1(\mathcal{K})$, also larger objects can be coded in this way.

Proposition 5.1. *For every forcing extension of the constructible universe $L \subseteq L[G]$, there exists a further extension $L[G] \subseteq L[G][H]$ such that:*

- $C^*(L[G][H]) = L[G]$,
- *and for every $S \subseteq \omega_1^{L[G]}$:*

$$L[G] \models S \textit{ is stationary} \Leftrightarrow L[G][H] \models S \textit{ is stationary.}$$

Proof. Let $\mathbb{P} \in L$ and let G be $\mathbb{P}$-generic. Furthermore, let $(\kappa_p : p \in \mathbb{P}) \in L$ be a sequence of regular (in L) cardinals $\geq \omega_2$ such that $|\mathbb{P}|^L < \kappa_p \neq \kappa_q$ for every two dinstinct $p, q \in \mathbb{P}$. In $L[G]$, consider the set $\mathcal{K}_G = \{\kappa_p : p \in G\}$ and let H be $L[G]$-generic for the forcing $\mathbb{P}^{\mathbf{Nm}}_1(\mathcal{K}_G)$.

Then indeed $C^*(L[G][H]) = L[G]$ because of the following two considerations. Firstly, clearly $G \in C^*(L[G][H])$ because $p \in G$ iff $\mathrm{cf}^{L[G][H]}(\kappa_p) = \omega$. Secondly, $C^*(L[G][H]) \subseteq L[G]$ because for every ordinal β,

$$\begin{aligned}&\{\alpha < \beta : \mathrm{cf}^{L[G][H]}(\alpha) = \omega\}\\ =&\{\alpha < \beta : (\mathrm{cf}^{L[G]}(\alpha) = \omega) \vee (\mathrm{cf}^{L[G]}(\alpha) = \kappa_p \text{ for some } p \in G)\} \in L[G],\end{aligned}$$

which by induction on α gives that $C^*_\alpha(L[G][H]) \subseteq L[G]$ for every α. □

Corollary 5.2. *Every theory T that consistently holds in a forcing extension of L, consistently holds in C^*.*

6. Semiproperness

As closing remarks, we mention the following further analysis of semiproperness of $\mathbb{P}^{\mathbf{Nm}}_0(\mathcal{K})$ and $\mathbb{P}^{\mathbf{Nm}}_1(\mathcal{K})$.

Proposition 6.1. *Suppose $\mathbb{P}$ is a forcing such that for every $\kappa \in \mathcal{K}$, $\Vdash_{\mathbb{P}} \mathrm{cf}(\kappa) = \omega$. Let $\theta \gg \mathbb{P}$ regular and $M \prec H_\theta$ countable.*

(1) If there exists $p \in \mathbb{P}$ that is $(M, \mathbb{P})$-semigeneric, then PII *has a winning strategy for the game $\mathbf{G}^{\mathbf{Nm}}_0(M, \mathcal{K})$.*
(2) If moreover also $p \Vdash_{\mathbb{P}} \mathrm{cf}(\kappa) \geq \omega_1$, for every uncountable regular cardinal $\kappa < \lambda$ with $\kappa \notin \mathcal{K}$, then PII *has a winning strategy for the game $\mathbf{G}^{\mathbf{Nm}}_1(M, \mathcal{K})$.*

Corollary 6.2.

(1) The forcing $\mathbb{P}^{\mathbf{Nm}}_0(\mathcal{K})$ is semiproper if and only if for every regular $\theta \gg \lambda$, there are club many countable $M \prec H_\theta$ such that PII *has a winning strategy for the game $\mathbf{G}^{\mathbf{Nm}}_0(M, \mathcal{K})$.*
(2) The forcing $\mathbb{P}^{\mathbf{Nm}}_1(\mathcal{K})$ is semiproper if and only if for every regular $\theta \gg \lambda$, there are club many countable $M \prec H_\theta$ such that PII *has a winning strategy for the game $\mathbf{G}^{\mathbf{Nm}}_1(M, \mathcal{K})$.*

We give the proof of part (2) of Proposition 6.1, the proof of part (1) is similar.

Proof of Proposition 6.1. Let $p_0 \in \mathbb{P}$ be $(M, \mathbb{P})$-semigeneric and suppose that $p_0 \Vdash_{\mathbb{P}} \mathrm{cf}(\kappa) \geq \omega_1$, for every uncountable regular cardinal $\kappa < \lambda$ with $\kappa \notin \mathcal{K}$. Set $M_0 := M$. For every $\kappa \in \mathcal{K}$, let $\dot{x}_\kappa$ be a $\mathbb{P}$-name such that $p_0 \Vdash \dot{x}_\kappa : \omega \to \kappa$ is cofinal, we can assume the sequence $(\dot{x}_\kappa : \kappa \in \mathcal{K})$ belongs to M. We indicate how PII wins the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$.

In round $n \geq 0$, PI plays $\lambda_n \in R_\lambda \cap M_n$ with $\lambda_n \notin \mathcal{K}$.

PII picks $p_n' \in \mathbb{P}$ with $p_n' \leq p_n$ and $\beta_n \in \lambda_n$ such that $p_n' \Vdash \sup(M[G] \cap \lambda_n) = \beta_n$.

PI then plays (λ_n, β_n).

Next, PI plays $\kappa_n \in \mathcal{K} \cap M_n$ and $\nu_n < \kappa_n$. Then PII picks $k_n < \omega$, $\xi_n < \kappa_n$ and $p_n \leq p_n'$ such that $p_n \Vdash \dot{x}_\kappa(k_n) = \xi_n \in \kappa_n \setminus \nu_n$.

PII wins this way because for every n, $p_n \Vdash M_n \subseteq M[G]$, so in particular $M_n \cap \omega_1 = M \cap \omega_1$ and $M \cap \lambda_n \leq \beta_n$ for every n. □

One natural situation in which the forcings $\mathbb{P}_0^{\mathbf{Nm}}(\mathcal{K})$ and $\mathbb{P}_1^{\mathbf{Nm}}(\mathcal{K})$ are semiproper is the case in which all elements of $\mathcal{K}$ are measurable cardinals.

Proposition 6.3. *If all elements of $\mathcal{K}$ are measurable, then for every countable $M \prec H_\theta$,* PII *has a winning strategy for the game $\mathbf{G}_1^{\mathbf{Nm}}(M, \mathcal{K})$.*

Proof. In round $n \geq 0$, PI plays $\lambda_n \in R_\lambda \cap M_n$ with $\lambda_n \notin \mathcal{K}$.

PII plays $(\lambda_n, \sup(M_n \cap \lambda_n))$.

Next, PI plays $\kappa_n \in \mathcal{K} \cap M_n$ and $\nu_n < \kappa_n$. Then PII uses measurability of κ_n to play some $\xi_n < \kappa_n$ such that $\xi_n \geq \max(\nu_n, \sup(\kappa_n \cap M_n))$, and

$$\mathrm{Hull}(M_n, \xi_n) \cap \xi_n = M_n \cap \xi_n.$$

To see that PII wins this way, let's suppose that PII loses in round n and infer a contradiction. For PII to lose in this round, there would exist some $i \leq n$ such that

$$\sup(M_{n+1} \cap \lambda_i) > \sup(M_n \cap \lambda_i).$$

If $\lambda_i < \kappa_n$, then from $\lambda_i \in M_i$ we get $\lambda_i < \xi_n$ and it would follow that

$$M_{n+1} \cap \lambda_i = M_{n+1} \cap \xi_n \cap \lambda_i = M_n \cap \xi_n \cap \lambda_i = M_n \cap \lambda_i.$$

This is impossible and therefore $\lambda_i > \kappa_n$, but this would contradict Lemma 4.6. □

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