Coreset for Robust Geometric Median: Eliminating Size Dependency on Outliers

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Abstract

We study the robust geometric median problem in Euclidean space $\mathbb{R}^d$, with a focus on coreset construction. A coreset is a compact summary of a dataset P of size n that approximates the robust cost for all centers c within a multiplicative error ε. Given an outlier count m, we construct a coreset of size $\tilde{O}(\varepsilon^{-2} \cdot \min\{\varepsilon^{-2}, d\})$ when $n \geq 4m$, eliminating the $O(m)$ dependency present in prior work [39, 40]. For the special case of $d = 1$, we achieve an optimal coreset size of $\tilde{\Theta}(\varepsilon^{-1/2} + \frac{m}{n}\varepsilon^{-1})$, revealing a clear separation from the vanilla case studied in [40, 1]. Our results further extend to robust (k, z)-clustering in various metric spaces, eliminating the m-dependence under mild data assumptions. The key technical contribution is a novel non-component-wise error analysis, enabling substantial reduction of outlier influence, unlike prior methods that retain them. Empirically, our algorithms consistently outperform existing baselines in terms of size-accuracy tradeoffs and runtime, even when data assumptions are violated across a wide range of datasets.

Contents

1 Introduction

Geometric median, also known as the Fermat-Weber problem, is a foundational problem in computational geometry, whose objective is to identify a center $c \in \mathbb{R}^d$ for a given dataset $P \subset \mathbb{R}^d$ of size n that minimizes the sum of Euclidean distances from each data point to c. The given objective function, while simple and user-friendly, suffers from significant robustness problems when exposed to noisy or adversarial data [13, 15, 14, 32, 24]. For example, an adversary could add a few distant noisy outliers to the main cluster. These points could deceive the geometric median algorithm into incorrectly positioning the center closer to these outliers to minimize the cost function. Such susceptibility to outliers has been a considerable obstacle in data science and machine learning, prompting a substantial amount of algorithmic research on the subject [15, 32, 29, 53, 49].

Robust geometric median. We consider *robust* versions of the geometric median problem, specifically a widely-used variant that introduces outliers [13]. Formally, given an integer $m \geq 0$, the goal of robust geometric median is to find a center $c \in \mathbb{R}^d$ that minimizes the objective function:

$$\mathrm{cost}^{(m)}(P, c) := \min_{L \subset P: |L| = m} \sum_{p \in P \setminus L} \mathrm{dist}(p, c), \tag{1}$$

where L represents the set of m outliers w.r.t. c and $\mathrm{dist}(p, c) = \|p - c\|_2$ denotes the Euclidean distance from p to center c. Intuitively, outliers capture the points that are furthest away and these are typically considered to be noise. When the number of outliers $m = 0$, the robust geometric median problem simplifies to the vanilla geometric median. This problem (together with its generalization: robust (k, z)-clustering) has been well studied in the literature [15, 7, 29, 2, 23, 53]. However, the presence of outliers introduces significant computational challenges, particularly in the context of large-scale datasets [52, 50, 51, 53]. For example, the approximation algorithm for robust geometric median proposed by [52] requires $\tilde{O}(n^{d+1}(n - m)d)$[1]; and [2] presents a fixed-parameter tractable (FPT) algorithm with $f(\varepsilon, m) \cdot n^{O(1)}$ time. These challenges have driven extensive research on data reduction algorithms designed for limited hardware and time constraints.

Coreset. To tackle the computational challenge, we study *coresets*, which are (weighted) subsets $S \subseteq P$ such that the clustering cost $\mathrm{cost}^{(m)}(S, c)$ approximates $\mathrm{cost}^{(m)}(P, c)$ within a factor of $(1 \pm \varepsilon)$ for all center sets $c \in \mathbb{R}^d$, where $\varepsilon > 0$ is a given error parameter. A coreset preserves key geometric information while substantially reducing the dataset size, thus serving as a compact proxy for the original dataset P. Consequently, applying existing approximation algorithms to the coreset significantly improves computational efficiency. Moreover, coresets can be reused in future analyses of P, reducing redundant computation and saving storage resources. Finally, the size of the coreset reflects the intrinsic complexity of the problem, making the study of optimal coreset size a research topic of independent interest.

Coreset for vanilla geometric median ($m = 0$) has been extensively studied across different dimensions d [28, 16, 18, 19, 1, 22] (see Section 1.3 for details). When $d = 1$, [37] proposed a coreset of size $\tilde{\Theta}(\varepsilon^{-1/2})$. Subsequently, [1] extended this result by constructing an optimal coreset of size $\tilde{\Theta}(\varepsilon^{-d/(d+1)})$ for dimension $d = O(1)$. Moreover, for high dimensions $d > \varepsilon^{-2}$, the optimal coreset size is $\tilde{\Theta}(\varepsilon^{-2})$ [18, 21]. In contrast, the study of coreset for robust geometric median is far from optimal, even in the simplest case of $d = 1$. Early work either required an exponentially large coreset size [27] or involved relaxing the outlier constraint [39]. A notable recent advancement by [39] overcame these limitations, proposing a coreset of size $O(m) + \tilde{O}(\varepsilon^{-3} \cdot \min\{\varepsilon^{-2}, d\})$, using a hierarchical sampling framework based on [9]. Further improvements reduced the coreset size to

[1]Here, $\tilde{O}(n)$ denotes $O(n \cdot \mathrm{polylog}(n))$, hiding logarithmic factors.

$O(m)+\tilde{O}(\varepsilon^{-2}\cdot\min\{\varepsilon^{-2},d\})$ [40]. A recent paper [42] presented a coreset size $O(m\varepsilon^{-1}+\text{Vanilla size})$ via a novel reduction from the robust case to the vanilla case.

All previous coreset sizes for robust geometric median contain the factor m. However, the outlier number m could approach $\Omega(n)$ in real-world scenarios [25, 12, 30, 10]. For instance, the PageBlocks dataset has $n = 5473$ points with $m \approx 0.1n$ outliers [10]. Such m results in an inefficient coreset size, raising a natural question:

Can we eliminate the $O(m)$ term from the coreset size?

At first glance, one might assume the answer is negative, as [39] establishes a coreset lower bound of $\Omega(m)$. However, their worst-case instance critically relies on an extremely large number of outliers, specifically with $m = n - 1$. This requirement can be relaxed to $n - m = o(n)$ (see Theorem 1.1). Motivated by this, we investigate the possibility of eliminating the $O(m)$ dependency when $n - m = \Omega(n)$, and show that this condition is both necessary and sufficient. Under this setting, we obtain a coreset of size $\tilde{O}(\varepsilon^{-2}\cdot\min\{\varepsilon^{-2},d\})$ (see Theorem 1.3).

Nevertheless, this size is substantially larger than that of the vanilla case across all dimensions. For instance, when $d = 1$, our coreset size is $\tilde{O}(\varepsilon^{-2})$, whereas the vanilla case achieves a size of only $\tilde{\Theta}(\varepsilon^{-1/2})$ [37]; thus, our bound is likely not optimal. On the other hand, our lower bound in Theorem 1.1 suggests that the optimal coreset size should indeed depend on m. This raises a natural question:

What is the optimal coreset size for robust geometric median, and how does it vary with m?

Answering this question sheds light on when the complexity of robust geometric median fundamentally diverges from that of the vanilla case.

1.1 Our contributions

In this paper, we study (optimal) coreset sizes for the robust geometric median problem. See Table 1 for a summary. We begin with a lower bound result (proof in Section B).

Theorem 1.1 (Coreset lower bound for robust geometric median)**.** *Let $0 < \varepsilon < 0.5$ and $n > m \geq 1$. There exists a dataset $P \subset \mathbb{R}$ of size n such that any ε-coreset of P for the robust geometric median problem must have size $\Omega(\frac{m}{n-m})$.*

This theorem indicates that when $n - m = o(n)$, which implies $m = \Theta(n)$, the coreset size is at least $\Omega\left(\frac{m}{n-m}\right) = \frac{m}{o(m)}$, which depends on m. This extends the previous result in [39] to general m. Thus, $n - m = \Omega(n)$ is a necessary condition for eliminating the $O(m)$ term from the coreset size.

Accordingly, we assume $n \geq 4m$ for the algorithmic results, where the constant 4 ensures the inlier number $n - m$ is sufficiently larger than the outlier number m. We first state our result when $d = 1$.

Theorem 1.2 (Optimal coreset for robust 1D geometric median)**.** *Let $n, m \geq 1$ be integers and $\varepsilon \in (0, 1)$. Assume $n \geq 4m$. There is a linear algorithm that given dataset $P \subset \mathbb{R}$ of size n, outputs an ε-coreset for robust 1D geometric median of size $\tilde{O}(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ in $O(n)$ time. Moreover, there exists a dataset $X \subset \mathbb{R}$ of size n such that any ε-coreset must have size $\Omega(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$.*

This theorem provides the optimal coreset size for the robust 1D geometric median problem. Since $m \le n$, the coreset size is at most $\tilde{O}(\varepsilon^{-1})$, which is independent of m. Thus, we eliminate the $O(m)$ dependency in the coreset size compared to previous bounds [40, 42]. In particular, relative to the bound $O(m) + \tilde{O}(\varepsilon^{-2})$ in [40], our result also improves the ε-dependence from ε^{-2} to at most ε^{-1}.

The theorem also shows how the coreset size increases as m grows. When $m \le \sqrt{\varepsilon}n$, our coreset size is dominated by $\tilde{O}(\varepsilon^{-1/2})$ since $\frac{m}{n}\varepsilon^{-1} \le \varepsilon^{-1/2}$. This size matches the coreset size $\tilde{O}(\varepsilon^{-1/2})$ for vanilla 1D geometric median as given in [40]. As m increases from $\sqrt{\varepsilon}n$ to $\frac{n}{4}$, our coreset size is dominated by the term $\tilde{O}(\frac{m}{n}\varepsilon^{-1})$, which is a linear function of m growing from $\tilde{O}(\varepsilon^{-1/2})$ to $\tilde{O}(\varepsilon^{-1})$. This size suggests that the complexity of robust 1D geometric median is higher than vanilla 1D geometric median in this range of outlier number m.

We remark that the size lower bound $\Omega(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ holds for any dimension $d \ge 1$. In contrast, for $d = O(1)$, the tight coreset size for the vanilla geometric median is $\tilde{\Theta}(\varepsilon^{-d/(d+1)})$ [1]. This implies that when $m \ge \Omega(n \cdot \varepsilon^{d/(d+1)})$, our coreset size lower bound exceeds $\Omega(\varepsilon^{-d/(d+1)})$, resulting in a gap between the coreset sizes for the robust and vanilla versions of geometric median.

Theorem 1.3 (Coreset for robust geometric median in $\mathbb{R}^d$)**.** *Let $n, m, d \ge 1$ be integers and $\varepsilon \in (0, 1)$. Assume $n \ge 4m$. There exists a randomized algorithm that given a dataset $P \subset \mathbb{R}^d$ of size n, outputs an ε-coreset for the robust geometric median problem of size $\tilde{O}(\varepsilon^{-2} \min\{\varepsilon^{-2}, d\})$ in $O(nd)$ time.*

Compared to previous bounds $O(m) + \tilde{O}(\varepsilon^{-2} \min\{\varepsilon^{-2}, d\})$ [40] and $O(m\varepsilon^{-1}) + \tilde{O}(\varepsilon^{-2})$ [42], this theorem eliminates the $O(m)$ term in the coreset size when $n \ge 4m$. This result can be extended to various metric spaces (Section 4.2).

Finally, we extend this theorem to handle the robust (k, z)-clustering problem (Definition 5.1), which encompasses robust k-median ($z = 1$) and robust k-means ($z = 2$). To capture the additional complexity introduced by k, we propose the following geometric assumption.

Assumption 1.4 (Assumptions for robust (k, z)-clustering)**.** *Given a dataset $P \subset \mathbb{R}^d$ of n points and an ε-approximate center set $C^\star \subset \mathbb{R}^d$ of P for robust (k, z)-clustering. Define $P_I^\star := \arg\min_{P_I \subset P, |P_I| = n-m} \sum_{p \in P_I} \mathrm{dist}(p, C^\star)$ to be the inlier points w.r.t. $C^\star$, where $\mathrm{dist}(p, C^\star) := \min_{c \in C^\star} \mathrm{dist}(p, c)$. Let $\{P_1^\star, \ldots, P_l^\star\} \subset P_I^\star$ denote the k inlier clusters induced by $C^\star$, where each $P_i^\star$ contains points in $P_I^\star$ whose closest center is $c_i^\star$. We assume the followings: 1) $\min_{i \in [k]} |P_i^\star| \ge 4m$; 2) $\max_{p \in P_I^\star} \mathrm{dist}(p, C^\star)^z \le 4k \sum_{p \in P_I^\star} \mathrm{dist}(p, C^\star)^z / |P_I^\star|$.*

The first condition directly generalizes the assumption $n \ge 4m$, which requires that the size of each inlier cluster is more than the outlier number m. The second excludes "remote inlier points" to $C^\star$, which could play a similar role as outliers. This condition holds for several real-world datasets (see Table 2), demonstrating its practicality. Now we propose the following result.

Theorem 1.5 (Coreset for robust (k, z)-clustering)**.** *Let $n, k, d \ge 1$ be integers and $\varepsilon \in (0, 1)$. There exists a randomized algorithm that given a dataset $P \subset \mathbb{R}^d$ of size n satisfying Assumption 1.4, outputs an ε-coreset for robust (k, z)-clustering of size $\tilde{O}(k^2 \varepsilon^{-2z} \min\{\varepsilon^{-2}, d\})$ in $O(nkd)$ time.*

It improves upon the previous result $O(m) + \tilde{O}(k^2 \varepsilon^{-2z} \min\{\varepsilon^{-2}, d\})$ in [39, 40] by eliminating the $O(m)$ term. Furthermore, the result can also be extended to various metric spaces (Section 5.3).

Empirically, in Section 6, we evaluate the performance of our coreset algorithms on six real-world datasets. We compare the size-error tradeoff against baselines [39, 40], and across all tested sizes, our algorithms consistently achieve lower empirical error. For instance, for robust geometric median on the **Census1990** dataset, our method produces a coreset of size 1000 with an empirical error of 0.012, while the baseline produces a coreset of size 2300 with an empirical error slightly higher than 0.013 (Figure 3). Moreover, our algorithms provide a speedup compared to baselines for achieving the same level of empirical error (see e.g., Table 8). Additionally, we show that our algorithms remain practically effective, regardless of which data assumption in the theoretical results is violated, further demonstrating the practical utility of our algorithms (Section 6).

1.2 Technical overview

We outline the technical ideas and novelty for Theorems 1.2 and 1.3. Our approach introduces a novel non-component-wise error analysis for coreset construction, enabling a substantial reduction in the number of outlier points rather than preserving them all. In contrast, previous algorithms divide the dataset P into multiple components, and their approach requires aligning the number of outliers between P and S in every component. This idea, as we will show, inherently introduces an $\Omega(m)$-sized coreset. Furthermore, for the 1D case, our new algorithm offers a more refined partitioning of inlier points, enabling an adaptation of the vanilla coreset construction and leading to the optimal coreset size.

1.2.1 Overview for Theorem 1.2 ($d = 1$)

Revisiting coreset construction for vanilla 1D geometric median. Recall that [37] first partitions dataset $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}$ into $\tilde{O}(\varepsilon^{-1/2})$ buckets, where each bucket B_i is a consecutive subsequence $\{p_l, p_{l+1} \ldots, p_r\}$ (see Definition 3.1). They select a mean point $\mu(B)$, assign a weight $|B|$ for each bucket B, and output their union as a coreset such that $\forall c \in \mathbb{R}$,

$$\textstyle\sum_{i\in[T]} |\mathrm{cost}(B_i, c) - |B_i| \cdot \mathrm{dist}(\mu(B_i), c)| \leq \varepsilon \cdot \mathrm{cost}(P, c). \tag{2}$$

This follows that only the bucket containing c contributes a non-zero error at most $\varepsilon \cdot \mathrm{cost}(P, c)$ (see Lemma 3.1). To handle the additional outliers for the robust case, a natural idea is to extend Inequality (2) in the following manner: for each center $c \in \mathbb{R}$ and each tuple $(m_1, \ldots, m_T) \in \mathbb{Z}^T_{\geq 0}$ of outlier numbers per bucket (with $\sum_{i\in[T]} m_i = m$),

$$\textstyle\sum_{i\in[T]} |\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i) \cdot \mathrm{dist}(\mu(B_i), c)| \leq \varepsilon \cdot \mathrm{cost}^{(m)}(P, c). \tag{3}$$

This bounds the induced error $|\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i) \cdot \mathrm{dist}(\mu(B_i), c)|$ of each bucket B_i when aligning the outlier numbers within this bucket for dataset P and coreset S.

Prior work [39, 40] utilized this component-wise error analysis for robust coreset construction. They show that at most three buckets B_i could induce a non-zero error, including a bucket that contains c and two "partially intersected" buckets with $0 < m_i < |B_i|$ that contain both inlier and outlier points relative to c. Thus, to prove Inequality (3), it suffices to ensure that the maximum induced error $|\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i) \cdot \mathrm{dist}(\mu(B_i), c)|$ of these three buckets is at most $\varepsilon \cdot \mathrm{cost}^{(m)}(P, c)/3$. However, it is possible that $\mathrm{cost}^{(m)}(P, c) \ll \mathrm{cost}(P, c)$, which presents a significant challenge for this guarantee compared to the vanilla case. To overcome this obstacle, prior work [39, 40] includes the "outmost" m points of P into the coreset to ensure zero induced error from them, resulting in an $O(m)$ size dependency.

First attempt to eliminate the $O(m)$ dependency. We first observe that a subset $P_M = \{p_{m+1}, \ldots, p_{n-m}\} \subset P$ of size $n - 2m$ acts as inliers w.r.t. any center $c \in \mathbb{R}$. The condition $n \geq 4m$ guarantees the existence of this P_M with $|P_M| = n - 2m \geq 2m$. Since $\mathrm{cost}^{(m)}(P, c) \geq \mathrm{cost}(P_M, c)$ by the construction of P_M, $\mathrm{cost}^{(m)}(P, c)$ is intuitively not "too small", which is useful for eliminating the $O(m)$ dependency in analysis. A natural idea is to apply the vanilla coreset construction method given by [37] to P_M and also include $P - P_M$ in the coreset S. This attempt yields a coreset of size

$2m + \tilde{O}(\varepsilon^{-1/2})$, which already improves the previous bound of $O(m) + \tilde{O}(\varepsilon^{-2})$ by [40]. To eliminate the $O(m)$ term, it is essential to significantly reduce points in $P - P_M$.

Our first attempt is to further decompose $P - P_M$ into buckets and utilize the component-wise error analysis in Inequality (3). As discussed earlier, the critical step is to ensure $|\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i) \cdot \mathrm{dist}(\mu(B_i), c)| \le \frac{\varepsilon}{3} \cdot \mathrm{cost}^{(m)}(P, c)$ for two partially intersecting buckets induced by c with $0 < m_i < |B_i|$. To achieve this, we study the relative location of such buckets with respect to c (Lemma 3.3), enabling us to partition the buckets based on the scale of $\mathrm{cost}^{(m)}(P, c)$, similar to the vanilla method (see Lines 1-5 of Algorithm 1). However, this partitioning approach is applicable only for $c \in P_M$, where the scale of $\mathrm{cost}^{(m)}(P, c)$ is well-controlled (see Lemma 3.5). Consequently, the challenge remains to control the induced error of these buckets for $c \notin P_M$.

Obstacle for $c \notin P_M$. Unfortunately, to ensure Inequality (3) holds for $c \notin P_M$, the number of constructed buckets in $P - P_M$ must be at least $\Omega(m)$. To see this, we provide an illustrative example in Section A, where there is a collection P_1 of $(n - m)$ points condensed into a small interval, and a collection P_2 of m points that are exponentially far from each other. We show that for any bucket containing two points $p, q \in P_2$ with $p < q$, the induced error of this bucket could be much larger than $\mathrm{cost}^{(m)}(P, c)$. Thus, to ensure Inequality (3) holds, all points in P_2 must be included in the coreset, leading to a size of $\Omega(m)$. Therefore, a new error analysis beyond Inequality (3) is required.

Key approach. W.L.O.G., we assume $c > p_{n-m}$, i.e., c is to the right of P_M. The key idea is to develop a novel non-component-wise error analysis beyond Inequality (3): we regard $|\mathrm{cost}^{(m)}(P, c) - \mathrm{cost}^{(m)}(S, c)|$ as a single entity and study how it changes as c shifts to the right from p_{n-m}. Let $f'_P(c)$ and $f'_S(c)$ denote the derivative of $\mathrm{cost}^{(m)}(P, c)$ and $\mathrm{cost}^{(m)}(S, c)$ respectively. Then the induced error can be rewritten as $|\mathrm{cost}^{(m)}(P, c) - \mathrm{cost}^{(m)}(S, c)| \le |\mathrm{cost}^{(m)}(P, p_{n-m}) - \mathrm{cost}^{(m)}(S, p_{n-m})| + \int_{p_{n-m}}^{c} |f'_P(x) - f'_S(x)| dx$. We can ensure $\mathrm{cost}^{(m)}(P, p_{n-m}) = \mathrm{cost}^{(m)}(S, p_{n-m})$ by an extra bucket-partitioning step (see Line 8 of Algorithm 1). Thus, it suffices to ensure $\int_{p_{n-m}}^{c} |f'_P(x) - f'_S(x)| dx \le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$.

A key geometric observation is that $f'_P(c)$ equals the difference between the number of inliers in P located to the left of c and those to the right of c; a similar relationship holds for $f'_S(c)$. For c, let m_i and m'_i denote the number of outliers in bucket B_i relative to dataset P and coreset S, respectively. By the geometric observation, we conclude that $|f'_P(c) - f'_S(c)| \le \sum_i |m_i - m'_i| + 2|B_c|$, where B_c denotes the bucket containing c. Therefore, $\int_{p_{n-m}}^{c} |f'_P(x) - f'_S(x)| dx \le (\sum_i |m_i - m'_i| + 2|B_c|) \cdot \mathrm{dist}(p_{n-m}, c)$ (see Lemma 3.7). Thus, it suffices to ensure $(\sum_i |m_i - m'_i| + 2|B_c|) \cdot \mathrm{dist}(p_{n-m}, c) \le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$. This desired property can be achieved by limiting the size of each bucket in $P - P_M$ to be at most $O(\varepsilon n)$, which ensures $\sum_i |m_i - m'_i| \le O(\varepsilon n)$ for all $c \in P_M$ (see Lemma 3.6).

We remark that, due to the misalignment of outlier counts, a single bucket may induce an arbitrarily large error in our analysis. However, these bucket-level errors can cancel out, and the overall error remains well controlled. To the best of our knowledge, this represents the first non-component-wise error analysis in the coreset literature, which may be of independent research interest. To demonstrate the power of our new error analysis, we apply it to the aforementioned obstacle instance—a case that previous component-wise analyses cannot solve (Appendix A).

Coreset size analysis. Note that the coreset size equals the number of buckets. We partition P_M into $\tilde{O}(\varepsilon^{-1/2})$ buckets using the vanilla coreset construction method [37]. The analysis for $c \in P_M$ and $c \notin P_M$ results in at most $\tilde{O}(\varepsilon^{-1/2})$ buckets for the former and $O\left(\frac{m}{n}\varepsilon^{-1}\right)$ buckets for the latter. Therefore, the total coreset size is $\tilde{O}(\varepsilon^{-1/2} + \frac{m}{n}\varepsilon^{-1})$ (see Lemma 3.4).

Table 1: Comparison of the state-of-the-art coreset size and our results for robust geometric median and robust k-median in $\mathbb{R}^d$. Robust k-median is a generalization of robust geometric median; see Definition 5.1 when $z = 1$.

Parameters d,k	Prior Results	Our Results (assuming $n \geq 4m$)
$d = 1$ $k = 1$	$O(m) + \tilde{O}(\varepsilon^{-2})$[40] $\tilde{O}(m\varepsilon^{-1})$ + Vanilla size[42] $\Omega(m)$ [39]	$\tilde{O}(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ (Theorem 1.2) $\Omega(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ (Theorem 1.2)
$d > 1$ $k = 1$	$O(m) + \tilde{O}(\varepsilon^{-2} \cdot \min\{\varepsilon^{-2}, d\})$[40] $\tilde{O}(m\varepsilon^{-1})$ + Vanilla size[42] $\Omega(m)$ [39]	$\tilde{O}(\varepsilon^{-2} \cdot \min\{\varepsilon^{-2}, d\})$ (Theorem 1.3) $\Omega(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ (Theorem 1.2)
$d > 1$ $k > 1$	$O(m) + \tilde{O}(k^2\varepsilon^{-2} \cdot \min\{\varepsilon^{-2}, d\})$[40] $\tilde{O}(\min\{km\varepsilon^{-1}, m\varepsilon^{-2}\})$ + Vanilla size[42] $\Omega(m)$[39]	$\tilde{O}(k^2\varepsilon^{-2} \cdot \min\{\varepsilon^{-2}, d\})$ (Theorem 1.5, under Assumption 1.4)

This coreset size is shown to be tight. To see this, we construct a worst-case example in Section 3.5, where the m outliers are partitioned into $\frac{m}{n}\varepsilon^{-1}$ intervals, each containing εn points, with the interval scales increasing exponentially. If the coreset omits all points from any such interval, each point within it contributes an error of $\frac{2 \cdot \mathrm{cost}^{(m)}(P,c)}{n}$, resulting in a total error of $2\varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$—which is unacceptably large. Thus, to control the error, the coreset must include at least one point from each interval, yielding a lower bound of $\frac{m}{n}\varepsilon^{-1}$ on the coreset size.

1.2.2 Overview for Theorem 1.3 (general d)

The obstacle of the traditional component-wise analysis remains in the high-dimensional case, motivating us to adopt the non-component-wise analysis framework. However, due to the increased complexity of candidate center distribution, a straightforward extension of the 1D analysis would involve using an ε-net to partition the center space, as in [35], but this introduces an $O(\varepsilon^{-d})$ factor in the coreset size. To overcome this, we leverage the concept of the ball range space, as explored in previous works [9, 39], which allows us to effectively describe high-dimensional spaces.

Our Algorithm 2 takes a uniform sample S_O of size $\tilde{O}(\varepsilon^{-2}\min\{\varepsilon^{-2}, d\})$ from the "outmost" m points of P to include in the coreset, in contrast to including them all as in [39, 38]. To analyze the error induced by S_O, we examine errors caused by "outlier-misaligned" points—those that act as outliers (or inliers) in P but as inliers (or outliers) in S with respect to a fixed c. The induced error for each such point is bounded by $\frac{\mathrm{cost}^{(m)}(P,c)}{m}$ when $n \geq 4m$ (see Lemmas 4.4, 4.5, and 4.6). To ensure the total error remains within $O(\varepsilon) \cdot \mathrm{cost}^{(m)}(P, c)$, it suffices for the number of such outlier-misaligned points to be $O(\varepsilon m)$. The key geometric insight is that this condition holds when S_O serves as an ε-approximation for the ball range space on $L^\star$ (see Lemma 4.7). This approximation is guaranteed by ensuring $|S_O| = \tilde{O}(\varepsilon^{-2}\min\{\varepsilon^{-2}, d\})$, as established in Lemma 4.3.

To our knowledge, this analysis is the first to apply the range space argument to outlier points. The range space argument leverages the fact that the VC dimension of $\mathbb{R}^d$ is at most $O(d)$ and can be further reduced to $\tilde{O}(\varepsilon^{-2})$ by dimension reduction. This enables us to generalize results to various metric spaces via the notion of VC dimension (or doubling dimension), as well as to robust (k, z)-clustering; see Section 4.2 and 5.

1.3 Other related work

Coreset for robust clustering. A natural extension of robust geometric median is called robust (k, z)-Clustering, attracting considerable interest for its coreset construction techniques in the literature [27, 38, 39, 41, 53]. In early work, [27] proposed a coreset construction method for the robust k-median problem, which requires an exponentially large size $(k+m)^{O(k+m)}(\varepsilon^{-1} d \log n)^2$. Recently, [39] improved the coreset size to $O(m) + \tilde{O}(k^3 \varepsilon^{-3z-2})$ via a hierarchical sampling framework proposed by [9]. Following this, [40] further improved the size to $O(m) + \tilde{O}(k^2 \varepsilon^{-2z-2})$. More recently, [42] proposed a new coreset of size $O(\min\{km\varepsilon^{-1}, m\varepsilon^{-2z}\})$+Vanilla size. We give Table 1 to compare our theoretical results with prior work.

Coreset for other clustering problems. Coreset construction for other variants of (k, z)-Clustering problems has also been extensively studied, including vanilla clustering [35, 16, 26, 9, 18, 19, 37, 41], capitalized clustering [9, 20], fair clustering [17, 5] and fault-tolerant clustering [44, 33]. Specifically, for vanilla (k, z)-clustering recent advancements by [19, 18, 41], produced a coreset of size $\tilde{O}(\min\left\{k\varepsilon^{-z-2}, k^{\frac{2z+2}{z+2}}\varepsilon^{-2}\right\})$. When $\varepsilon \geq k^{-\frac{1}{z+2}}$, the coreset upper bound $k\varepsilon^{-z-2}$ is shown to be optimal by a recent breakthrough [41]. Furthermore, a recent study [37] has investigated the coreset bounds when d is small.

2 Preliminaries

In this section, we define the coreset for robust geometric median. For preparation, we first generalize the cost function $\mathrm{cost}^{(m)}$ to handle weighted datasets.

Definition 2.1 (Cost function for weighted dataset)**.** *Let m be an integer. Let $S \subseteq \mathbb{R}^d$ be a weighted dataset with weights $w(p)$ for each point $p \in S$. Let $w(S) := \sum_{p \in S} w(p)$. Define a collection of weight functions $\mathcal{W} := \{w' : S \to \mathbb{R}^+ \mid \sum_{p \in S} w'(p) = w(S) - m \wedge \forall p \in S, w'(p) \leq w(p)\}$. Moreover, we define the following cost function on S: $\forall c \in \mathbb{R}^d$, $\mathrm{cost}^{(m)}(S, c) := \min_{w' \in \mathcal{W}} \sum_{p \in S} w'(p) \cdot \mathrm{dist}(p, c)$.*

Intuitively, to compute $\mathrm{cost}^{(m)}(S, c)$, we find a weighted subset S' of S with total weight $w(S) - m$ that minimizes its vanilla cost to c. Thus, this $\mathrm{cost}^{(m)}(S, c)$ serves as the cost function for robust geometric median on a weighted set. Note that for the unweighted case where $w(p) = 1$ for all $p \in S$, this cost function reduces to that in Equation (1).

Next, we define the notion of coreset for robust geometric median.

Definition 2.2 (Coreset for robust geometric median)**.** *Given a point set $P \subset \mathbb{R}^d$ of size $n \geq 1$, integer $m \geq 1$ and $\varepsilon \in (0, 1)$, we say a weighted subset $S \subseteq P$ together with a weight function $w : S \to \mathbb{R}^+$ is an ε-coreset of P for robust geometric median if $w(S) = n$ and for any center $c \in \mathbb{R}^d$, $\mathrm{cost}^{(m)}(S, c) \in (1 \pm \varepsilon) \cdot \mathrm{cost}^{(m)}(P, c)$.*

This formulation ensures that the weighted coreset S provides an accurate approximation of the original dataset P's cost for all centers c, within a tolerance specified by ε.

3 Optimal coreset size when $d = 1$

Let $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}$ with $p_1 < p_2 < \ldots < p_n$. Denote $L^{(c)} := \arg\min_{L \subseteq P, |L| = m} \mathrm{cost}(P - L, c)$ to be the set of outliers of P w.r.t. a center c, and $P_I^{(c)} = P - L^{(c)}$ to be the set of inliers. Let $c^\star := \arg\min_{c \in \mathbb{R}} \mathrm{cost}^{(m)}(P, c)$ be an optimal center. Let $L^\star = L^{(c^\star)}$ and $P_I^\star = P_I^{(c^\star)}$.

Buckets and cumulative error. We introduce the definition of bucket proposed by [35] associated with related notions, which is useful for coreset construction when $d = 1$ [34, 37].

Definition 3.1 (Bucket and associated statistics)**.** *A bucket B is a continuous subset $\{p_l, p_{l+1}, \ldots, p_r\}$ of P for some $1 \le l \le r \le n$. Let $N(B) := r - l + 1$ represents the number of points within B, $\mu(B) := \frac{\sum_{p\in B} p}{N(B)}$ represents the mean point of B, and $\delta(B) := \sum_{p\in B} |p - \mu(B)|$ represents the cumulative error of B.*

A basic idea for vanilla coreset construction in 1D case is to partition P into multiple buckets B and then retain a point $\mu(B)$ with weight $N(B)$ as the representative point of B in coreset. This idea works since each bucket induces an error of at most $\delta(B)$; see Lemma 3.1.

Lemma 3.1 (Error analysis for buckets [34])**.** *Let $B = \{p_l, \ldots, p_r\} \subseteq P$ for $1 \le l \le r \le n$ be a bucket and $c \in \mathbb{R}$ be a center. We have*

1. *if $c \in (p_l, p_r)$, $|\mathrm{cost}(B, c) - N(B) \cdot \mathrm{dist}(\mu(B), c)| \le \delta(B)$;*
2. *if $c \notin (p_l, p_r)$, $|\mathrm{cost}(B, c) - N(B) \cdot \mathrm{dist}(\mu(B), c)| = 0$.*

Thus, we have the following theorem that provides the optimal coreset for vanilla 1D geometric median [37]

Theorem 3.2 (Coreset for vanilla 1D geometric median [37])**.** *There exists algorithm $\mathcal{A}$, that given an input data set $P \subset \mathbb{R}$ and $\varepsilon \in (0, 1)$, $\mathcal{A}(P, \varepsilon)$ outputs an ε-coreset of P for vanilla 1D geometric median with size $\tilde{O}(\varepsilon^{-\frac{1}{2}})$ in $O(|P|)$ time.*

We will apply $\mathcal{A}$ to the "inlier subset" of P, say the set of middle $n-2m$ points $P_M = \{p_{m+1}, \ldots, p_{n-m}\}$. Let $P_L = \{p_1, \ldots, p_m\}$ and $P_R = \{p_{n-m+1}, \ldots, p_n\}$. Any continuous subsequence of length $n - m$ of P must contain P_M, implying that all points in P_M must be inliers w.r.t. any center $c \in \mathbb{R}$. This motivates us to apply the vanilla method $\mathcal{A}$ to P_M.

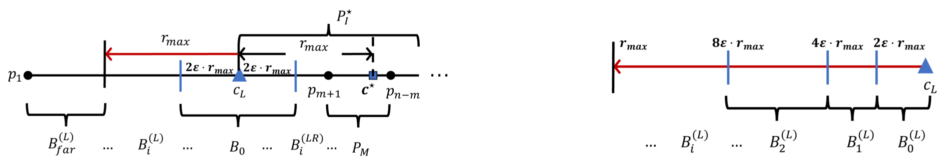
As discussed in Section 1.2, we then partition P_L an P_R into buckets, which requires an understanding of the relative position between the inlier subset $P_I^{(c)}$ and c. Define $r_{\max} := \max_{p\in P_I^\star} |p - c^\star|$, $c_L := c^\star - r_{\max}$ and $c_R := c^\star + r_{\max}$. We present the following lemma.

Lemma 3.3 (Location of $P_I^{(c)}$)**.** *Let $c \in P_M$ be a center with $c < c^\star$ and $P_I^{(c)} \ne P_I^\star$. Let p_l be the leftmost point of $P_I^{(c)}$. Then $\mathrm{dist}(p_l, c_L) \le 2 \cdot \mathrm{dist}(c, c^\star)$.*

Proof. We prove the lemma by contradiction. Suppose there exists a center $c \in P_M$ satisfying that $c < c^\star$ and $P_I^{(c)} \ne P_I$, and the leftmost point p_l of $P_I^{(c)}$ satisfies $\mathrm{dist}(p_l, c_L) > 2\mathrm{dist}(c, c^\star)$. Then by the assumption of c, we have $p_l < c_L$, thus $p_l \notin P_I^\star$. For any points p in $P_I^\star$, we have

$$\begin{aligned}
\mathrm{dist}(p, c) &\le \mathrm{dist}(p, c^\star) + \mathrm{dist}(c, c^\star) && \text{(Triangle Inequality)} \\
&\le r_{\max} + \mathrm{dist}(c, c^\star) && \text{(Definition of } r_{\max}) \\
&< r_{\max} + \mathrm{dist}(p_l, c_L) - \mathrm{dist}(c, c^\star) && (\mathrm{dist}(p_l, c_L) > 2\mathrm{dist}(c, c^\star)) \\
&= \mathrm{dist}(p_l, c)
\end{aligned}$$

This implies that any point in $P_I^\star$ must be in $P_I^{(c)}$. However, since $|P_I^\star| = |P_I^{(c)}| = n - m$, $P_I^{(c)}$ cannot contain any other point not in $P_I^\star$. This contradicts $p_l \notin P_I^\star$. Thus, $\mathrm{dist}(p_l, c_L) \le 2\mathrm{dist}(c, c^\star)$ holds. ☐



(a) (b)

Figure 1: Illustration of the block partition. The blue square marks the optimal solution $c^\star$, and the blue triangle c_L denotes the left boundary of the inlier set $P_I^\star$, with distance $r_{\max} = \mathrm{dist}(c_L, c^\star)$. Figure 1(a) partitions the one-dimensional space left of P_M into disjoint blocks based on each point's position relative to c_L: points farther than $r_{\max}$ form B_{far}, and those within $2\varepsilon r_{\max}$ form B_0. Figure 1(b) shows the logarithmic subdivision of inner blocks $B_i^{(L)}$ within distance $r_{\max}$ from c_L.

A symmetric observation can be made for $c > c^\star$. Note that $\mathrm{cost}^{(m)}(P, c) > \mathrm{cost}(P_M, c) \geq O(n) \cdot \mathrm{dist}(c, c^\star) \geq O(n) \cdot \mathrm{dist}(p_l, c_L)$. This lower bound for $\mathrm{cost}^{(m)}(P, c)$ motivates us to select the cumulative error bound for the bucket containing point p_l as $\varepsilon n \cdot \mathrm{dist}(p_l, c_L)$. Thus, we partition $P_L \cup P_R$ into disjoint *blocks* according to points' distance to c_L and c_R, a concept inspired by [37]. Concretely, we partition the four collections $P_L \cap (-\infty, c_L)$, $P_L \cap [c_L, \infty)$, $P_R \cap (-\infty, c_R]$, $P_R \cap (c_R, \infty)$ into blocks, respectively. Due to symmetry, we visualize only the partition of $P_L \cap (-\infty, c_L)$ in Figure 1.

Outer blocks (B_{far}): Define $B_{far}^{(L)}$ and $B_{far}^{(R)}$ as the set of points that are far from c_L and c_R, where

$$
\begin{aligned}
B_{far}^{(L)} &:= \{\, p \in P_L \cap (-\infty, c_L) \mid \mathrm{dist}(p, c_L) \geq r_{\max} \,\}, \\
B_{far}^{(R)} &:= \{\, p \in P_R \cap (c_R, \infty) \mid \mathrm{dist}(p, c_R) \geq r_{\max} \,\}.
\end{aligned} \tag{4}
$$

Inner blocks (B_i): Define $B_0^{(L)}$, $B_0^{(LR)}$, $B_0^{(R)}$, and $B_0^{(RL)}$ as the set of points that are close to c_L or c_R, where

$$
\begin{aligned}
B_0^{(L)} &:= \{\, p \in P_L \cap (-\infty, c_L) \mid \mathrm{dist}(p, c_L) < 2\varepsilon r_{\max} \,\}, \\
B_0^{(LR)} &:= \{\, p \in P_L \cap [c_L, \infty) \mid \mathrm{dist}(p, c_L) < 2\varepsilon r_{\max} \,\}, \\
B_0^{(R)} &:= \{\, p \in P_R \cap (c_R, \infty) \mid \mathrm{dist}(p, c_R) < 2\varepsilon r_{\max} \,\}, \\
B_0^{(RL)} &:= \{\, p \in P_R \cap (-\infty, c_R] \mid \mathrm{dist}(p, c_R) < 2\varepsilon r_{\max} \,\}.
\end{aligned} \tag{5}
$$

For the remaining points, partition them into blocks $B_i^{(L)}$, $B_i^{(LR)}$, $B_i^{(R)}$ and $B_i^{(RL)}$ based on exponentially increasing distance ranges for $i = 1, \ldots, \lceil \log_2(\varepsilon^{-1}) \rceil$, where

$$
\begin{aligned}
B_i^{(L)} &:= \{\, p \in P_L \cap (-\infty, c_L) \mid 2^i \varepsilon r_{\max} \leq \mathrm{dist}(p, c_L) < 2^{i+1} \varepsilon r_{\max} \,\}, \\
B_i^{(LR)} &:= \{\, p \in P_L \cap [c_L, \infty) \mid 2^i \varepsilon r_{\max} \leq \mathrm{dist}(p, c_L) < 2^{i+1} \varepsilon r_{\max} \,\}, \\
B_i^{(R)} &:= \{\, p \in P_R \cap (c_R, \infty) \mid 2^i \varepsilon r_{\max} \leq \mathrm{dist}(p, c_R) < 2^{i+1} \varepsilon r_{\max} \,\}, \\
B_i^{(RL)} &:= \{\, p \in P_R \cap (-\infty, c_R] \mid 2^i \varepsilon r_{\max} \leq \mathrm{dist}(p, c_R) < 2^{i+1} \varepsilon r_{\max} \,\}.
\end{aligned} \tag{6}
$$

Algorithm 1 Coreset construction for 1D

Input: A dataset $P = \{p_1, \ldots, p_n\} \subset \mathbb{R}$ with $p_1 < \ldots < p_n$, and $\varepsilon \in (0, 1)$.

Output: An ε-coreset S

1: Set $P_M \leftarrow \{p_{m+1}, \ldots, p_{n-m}\}$, $P_L \leftarrow \{p_1, \ldots, p_m\}$, $P_R \leftarrow \{p_{n-m+1}, \ldots, p_n\}$.
2: Construct $S_M \leftarrow \mathcal{A}(P_M, \frac{\varepsilon}{3})$ by Theorem 3.2.
3: Compute an optimal center $c^\star$ of P for robust 1D geometric median. Let $P_I^\star$ be the set of inliers w.r.t. $c^\star$, and $r_{\max} := \max_{p \in P_I^\star} \mathrm{dist}(p, c^\star)$, $c_L \leftarrow c^\star - r_{\max}$, $c_R \leftarrow c^\star + r_{\max}$.
4: Given c_L and c_R, divide P_L and P_R into outer blocks $B_{far}^{(L)}$ and $B_{far}^{(R)}$ by Equation (4), and inner blocks $B_i^{(L)}$, $B_i^{(LR)}$, $B_i^{(RL)}$ and $B_i^{(R)}$ $(0 \le i \le \lceil \log_2(\varepsilon^{-1}) \rceil)$ by Equations (5)-(6).
5: For each non-empty inner block B_i, divide B_i into disjoint buckets $\{B_{i,j}\}_{j \ge 0}$ in a greedy way: each bucket $B_{i,j}$ is a maximal set with $\delta(B_{i,j}) \le \frac{2^i \cdot \varepsilon^2 n r_{\max}}{288}$ and $N(B_{i,j}) \le \frac{\varepsilon n}{16}$.
6: If $B_{far}^{(L)}$ is non-empty, divide B_{far} into disjoint buckets $\{B_{far,j}^{(L)}\}_{j \ge 0}$ in a greedy way: each bucket $B_{far,j}^{(L)}$ is a maximal set with $N(B_{far,j}^{(L)}) \le \frac{\varepsilon n}{16}$. The same for $B_{far}^{(R)}$.
7: Compute the inlier set $P_I^{(p_{m+1})}$ with respect to center $c = p_{m+1}$ and the inlier set $P_I^{(p_{n-m})}$ with respect to center $c = p_{n-m}$ respectively.
8: If there exists some bucket B such that both $B \cap P_I^{(p_{m+1})}$ and $B \setminus P_I^{(p_{m+1})}$ are non-empty, divide B into two buckets $B \cap P_I^{(p_{m+1})}$ and $B \setminus P_I^{(p_{m+1})}$. Do the same thing for $P_I^{(p_{n-m})}$.
9: For every B, add $\mu(B)$ with weight $N(B)$ into S_O.
10: Return $S \leftarrow S_O \cup S_M$.

The algorithm. Our algorithm (Algorithm 1) consists of three stages. In Stage 1, we construct a coreset S_M for P_M using Algorithm $\mathcal{A}$ for vanilla 1D geometric median (Theorem 3.2), ensuring that $\mathrm{cost}(S_M, c) \in (1 \pm \varepsilon) \cdot \mathrm{cost}(P_M, c)$ for any center $c \in \mathbb{R}$. In Stage 2, we divide sets P_L and P_R into outer and inner blocks by Equations (4)-(6), and greedily partition these blocks into disjoint buckets B with bounded $\delta(B)$ and $N(B)$ in Lines 3-6. In Stage 3, we ensure that $\mathrm{cost}^{(m)}(P, p_{n-m}) = \mathrm{cost}^{(m)}(S, p_{n-m})$ and $\mathrm{cost}^{(m)}(P, p_{m+1}) = \mathrm{cost}^{(m)}(S, p_{m+1})$ to control induced error when $c \notin P_M$. Finally, we add the mean point $\mu(B)$ of each bucket B with weight $N(B)$ into S_O, and return $S_O \cup S_M$ as the coreset of P.

By construction, the coreset size $|S|$ is exactly the number of buckets. Therefore, we have the following lemma that proves the coreset size in Theorem 1.2.

Lemma 3.4 (Number of buckets)**.** *Algorithm 1 constructs at most $\tilde{O}(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ buckets.*

Proof. Note that in Line 2, the number of buckets is $\tilde{O}(\varepsilon^{-\frac{1}{2}})$ in S_M by Theorem 3.2. For Line 4, there are at most $O(\log(\frac{1}{\varepsilon}))$ non-empty blocks. For Line 6, the constraint on $N(B_{i,j})$ generates at most $O(\frac{m}{n}\varepsilon^{-1})$ buckets. For Lines 7-8, there are $O(1)$ new one-point buckets.

What remains is to show that each non-empty block contains $\tilde{O}(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ buckets in Line 5. Note that each block contains at most m points, thus the constraint on $N(B_{i,j})$ generates at most $O(\frac{m}{n}\varepsilon^{-1})$ buckets. Thus we only have to consider the constraint on $\delta(B_{i,j})$.

Suppose we divide an inner block B_i into t buckets $\{B_{i,j}\}_{1 \le j \le t}$ due to controlling $\delta(B_{i,j})$. Since each B_{ij} is the maximal bucket with $\delta(B_{i,j}) \le \frac{2^i \cdot \varepsilon^2 n r_{\max}}{288}$, we have $\delta(B_{i,j} \cup B_{i+1,j}) > \frac{2^i \cdot \varepsilon^2 n r_{\max}}{288}$. Denote $B_{i,2j-1} \cup B_{i,2j}$ by C_j for $j \in \{1, \ldots, \lfloor \frac{t}{2} \rfloor\}$. Let $len(B) := \max_{p \in B} p - \min_{p \in B} p$ be the length of B. Note that $\delta(B) \le N(B) \cdot len(B)$ holds for every bucket B. Thus we have:

$$
\begin{aligned}
m2^i\varepsilon r_{\max} &\ge N(B_i) \cdot len(B_i) && (N(B_i) < m, len(B_i) < 2^i\varepsilon r_{\max}) \\
&\ge \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} N(C_j) \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} |C_j| \\
&\ge (\sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} N(C_j)^{\frac{1}{2}} |C_j|^{\frac{1}{2}})^2 && \text{(Cauchy-Schwarz Inequality)} \\
&\ge (\sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} \delta(C_j)^{\frac{1}{2}})^2 \\
&> (\lfloor \frac{t}{2} \rfloor)^2 \cdot \frac{2^i \cdot \varepsilon^2 n r_{\max}}{288}.
\end{aligned}
\tag{7}
$$

So we have $(\lfloor \frac{t}{2} \rfloor)^2 \cdot \frac{2^i \cdot \varepsilon^2 n r_{\max}}{288} < m2^i\varepsilon r_{\max}$, which implies $t \le O(\varepsilon^{-\frac{1}{2}})$. The proof above is similar to Lemma 2.8 in [40]. Similarly, it is trivial to prove that B_0 also satisfies the above inequality. Since there are $O(\log(\varepsilon^{-1}))$ non-empty blocks, the constraint on $\delta(B_{i,j})$ generates at most $\tilde{O}(\varepsilon^{-\frac{1}{2}})$ buckets. Thus Lemma 3.4 holds. □

3.1 Proof of the upper bound in Theorem 1.2

Now we show that our coreset S obtained by Algorithm 1 is an ε-coreset of P.

In the following discussion, we define $S_I^{(c)}$ and $S_I^\star$ as follows. Given a weighted set S of total weight n and a center $c \in \mathbb{R}$, let $S_I^{(c)} := \arg\min_{S_I^{(c)} \subseteq S, \sum_{s \in S_I^{(c)}} w(s) = n-m} \mathrm{cost}(S_I^{(c)}, c)$ be the set of inliers of S, where $w(s)$ represents the weight of s in $S_I^{(c)}$ and is at most the weight of s in S. Moreover, let $S_I^\star$ represents $S_I^{(c^\star)}$, which exactly contains every $\mu(B)$ of each bucket B in $P_I^\star$ with weight $|B|$.

First, we consider the case that $c \in P_M$. Note that regardless of the position of c, at most three buckets can induce the error: the bucket containing c, and the buckets containing the endpoints of $P_I^{(c)}$ on either side. Actually, the coreset error in P_M is already controlled by the vanilla coreset construction algorithm $\mathcal{A}$. Thus, we only have to consider the cumulative error of the buckets that partially intersect with $P_I^{(c)}$ on either side. We will show that this error is controlled by the carefully selected cumulative error bound of each inner block. Moreover, we adapt the analysis for the vanilla case in [37] to the robust case, as shown in Lemma 3.5. This allows us to avoid considering the error caused by the misaligned outliers in P and S, which still suffice to ensure that the coreset error is bounded. The proof of Lemma 3.5 is deferred to Section 3.2.

Lemma 3.5 (Error analysis for $c \in P_M$)**.** *When center $c \in (p_{m+1}, p_{n-m})$, $|\mathrm{cost}^{(m)}(P, c) - \mathrm{cost}^{(m)}(S, c)| \le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$.*

Now we analyze the case that c is not in P_M. Assume P is divided into disjoint buckets $B_1, \ldots, B_q$, from left to right. Fix a center $c \in \mathbb{R}$, for each bucket B_i ($i \in [q]$), define $m_i := |B_i \setminus P_I^{(c)}|$ and $m_i' := |B_i \setminus S_I^{(c)}|$, which represent the number of outliers in B_i with respect to c. Obviously we have $\sum_i m_i = \sum_i m_i' = m$. The following lemma shows that the number of inliers in each bucket for $P_I^{(c)}$ and $S_I^{(c)}$ remains roughly consistent, which indicates that $\mathrm{cost}^{(m)}(P, c)$ and $\mathrm{cost}^{(m)}(S, c)$ increase almost equally.

Lemma 3.6 (Total misalignment)**.** *For any center $c \in \mathbb{R}$, $\sum\limits_{i\in[q]} |m_i - m_i'| \le \frac{\varepsilon n}{4}$.*

Let $\Gamma := \sup_{c\in\mathbb{R}} \sum\limits_{i\in[q]} |m_i - m_i'|$, then we have $\Gamma \le \frac{\varepsilon n}{4}$. For $c \notin P_M$, we consider the derivative of the cost value with respect to c, and gives an upper bound of the induced error in Lemma 3.7. Combined with the upper bound of Γ, we conclude that the induced error is $O(\varepsilon n \cdot \mathrm{dist}(c, c^\star))$, which is bounded by $O(\varepsilon) \cdot \mathrm{cost}^{(m)}(P, c)$ obviously. The main idea is similar to that of the proof of Theorem 2.1 in [37]. We defer the proof of Lemma 3.6 and 3.7 to Sec 3.3 and 3.4, respectively.

Lemma 3.7 (Error analysis for $c \notin P_M$)**.** *When the center $c \le p_{m+1}$ or $c \ge p_{n-m}$, $|\mathrm{cost}^{(m)}(P, c) - \mathrm{cost}^{(m)}(S, c)| \le (\Gamma + \frac{\varepsilon n}{8}) \cdot \mathrm{dist}(c, P_M) \le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$.*

Now we are ready to prove the upper bound in Theorem 1.2.

Proof of Theorem 1.2 (upper bound). Given a dataset P of size n, we apply Algorithm 1 to P and obtain the output weighted set S. By Lemma 3.4, Algorithm 1 divides P into $\tilde{O}(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$ buckets. Note that S contains only the mean point $\mu(B)$ of each bucket B. Thus we have $|S| = \tilde{O}(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$. Combined with Lemma 3.5 and 3.7, we conclude that S is an ε-coreset of P.

For the runtime, Line 3 cost $O(n)$ time to obtain the optimal center. Recall that $P_I^\star = \{p_i, \dots, p_j\}$ is a continuous subsequence of P and $c^\star = p_{\lfloor \frac{j+i}{2} \rfloor}$, since we only consider the one-dimensional case. Thus, we can compute $P_I^\star$ and $c^\star$ in $O(n)$ time, since we can sequentially replace the leftmost point in the current inliers with the next point from the outliers, and the resulting cost difference can be computed in $O(1)$ time. Lines 4-6 cost $O(n)$ time, since we can sequentially check whether the next point can be added to the current bucket, otherwise, we place it into a new bucket. Obviously Lines 7-8 also cost $O(n)$ time. This completes the proof. ◻

3.2 Proof of Lemma 3.5: Error analysis for $c \in P_M$

Proof of Lemma 3.5. Let $L_O := P_L \cap L^\star$, $R_O := P_R \cap L^\star$. Recall that $L^\star$ denote the set of outliers w.r.t. the optimal center $c^\star$. W.L.O.G, assume $c > c^\star$, thus $P_I^{(c)} \cap L_O = \emptyset$. Next, we analyze the induced error in two cases, based on the scale of $\mathrm{dist}(c, c^\star)$. When $\varepsilon r_{\max} > \mathrm{dist}(c, c^\star)$, the center c is sufficiently close to $c^\star$, so $\mathrm{cost}^{(m)}(P, c) \approx \mathrm{cost}^{(m)}(P, c^\star)$, resulting in a small error. When $\varepsilon r_{\max} \le \mathrm{dist}(c, c^\star)$, we have $\mathrm{cost}^{(m)}(P, c) > \Omega(n \cdot \mathrm{dist}(c, c^\star)) > \Omega(\varepsilon n r_{\max})$, which matches the error from any outlier-misaligned bucket B, whose error is at most $O(\varepsilon n)(\mathrm{dist}(c, c^\star) + \varepsilon r_{\max})$ by Lemma 3.3.

Case 1: $\mathrm{dist}(c, c^\star) > \frac{\varepsilon}{6} \cdot r_{\max}$**.** If $P_I^{(c)} \cap R_O = \emptyset$, then we have $P_I^{(c)} = P_I^\star$, $S_I^{(c)} = S_I^\star$. In this case, we directly have $\mathrm{cost}^{(m)}(S, c) \in (1 \pm \varepsilon)\mathrm{cost}^{(m)}(P, c)$ by Theorem 3.2.

Next we assume $P_I^{(c)} \cap R_O \ne \emptyset$. Same as the above lemma, denote the leftmost and rightmost buckets intersecting $P_I^{(c)}$ as B_L, B_R, respectively. Recall that the coreset constructed by $\mathcal{A}(P_M, \frac{\varepsilon}{3})$ is S_M. By Theorem 3.2, Algorithm $\mathcal{A}$ ensures that

$$|\mathrm{cost}(S_M, c) - \mathrm{cost}(P_M, c)| \le \frac{\varepsilon}{3} \cdot \mathrm{cost}(P_M, c) < \frac{\varepsilon}{3} \cdot \mathrm{cost}^{(m)}(P, c).$$

Next, we bound the cumulative error of B_L and B_R. Define $\gamma := \max(0, \lceil \log(\frac{\mathrm{dist}(c,c^\star)}{2\varepsilon \cdot r_{\max}}) \rceil)$. Obviously $\frac{\varepsilon}{6} r_{\max} \le \mathrm{dist}(c, c^\star) < r_{\max}$, thus $\gamma \in [0, \lceil \log(\varepsilon^{-1}) \rceil - 1]$. By the definition of γ, we have $\mathrm{dist}(c, c^\star) \le 2^{\gamma+1}\varepsilon r_{\max}$. Denote the rightmost point of B_R as p_r, then it follows that

$$\begin{aligned} \mathrm{dist}(p_r, c_R) \le&\ 2\mathrm{dist}(c, c^\star) && \text{(Lemma 3.3)} \\ \le&\ 2^{\gamma+2}\varepsilon r_{\max}. && (\mathrm{dist}(c, c^\star) \le 2^{\gamma+1}\varepsilon r_{\max}) \end{aligned}$$

Recall that block $B_i^{(R)} = \{ p \in P_R \mid p > c_R, 2^i \varepsilon r_{\max} \le \mathrm{dist}(p, c_R) < 2^{i+1}\varepsilon r_{\max}\}$. So the block B_i which contains bucket B_R satisfies $i \le \gamma + 1$. By Line 5 in Algorithm 1, any bucket $B_{i,j}$ in inner block B_i satisfies $\delta(B_{i,j}) \le \frac{2^i \cdot \varepsilon^2 n r_{\max}}{288}$. Thus,

$$\delta(B_R) \le \frac{2^{\gamma+1} \cdot \varepsilon^2 n r_{\max}}{288}.$$

Similarly,

$$\delta(B_L) \le \frac{2^{\gamma+1} \cdot \varepsilon^2 n r_{\max}}{s288}.$$

Note that at least $\lfloor \frac{n-m}{2} \rfloor$ points are on the left of $c^\star$, among which there are at least $(\lfloor \frac{n-m}{2} \rfloor - m)$ inliers w.r.t center c. Moreover, when $\gamma = 0$, we simply have $\mathrm{dist}(c, c^\star) > \frac{2^\gamma \varepsilon r_{\max}}{6}$ by the assumption; when $\gamma > 0$, we have $\mathrm{dist}(c, c^\star) > 2^\gamma \varepsilon r_{\max}$ by the definition of γ. Thus,

$$\begin{aligned}
\mathrm{cost}^{(m)}(P, c) &\ge (\lfloor \frac{n-m}{2} \rfloor - m) \cdot \mathrm{dist}(c, c^\star) \\
&> \frac{n}{8} \cdot \mathrm{dist}(c, c^\star) && (n \ge 4m) \\
&> \frac{2^\gamma \varepsilon n r_{\max}}{48}. && (\mathrm{dist}(c, c^\star) > \frac{2^\gamma \varepsilon r_{\max}}{6})
\end{aligned} \tag{8}$$

Now we are ready to prove our goal $|\mathrm{cost}^{(m)}(P, c) - \sum_{j \in [q]} (|B_j| - m_j) \cdot \mathrm{dist}(\mu(B_j), c)| \le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$. Recall that $m_j := |B_j \setminus P_I^{(c)}|$ for each bucket B_j. Thus for each bucket B_j that is between B_L and B_R, we have $m_j = 0$; for B_L and B_R, we have $|B_L| - m_L = |B_L \cap P_I^{(c)}|$ and $|B_R| - m_R = |B_R \cap P_I^{(c)}|$. Then we have

$$\begin{aligned}
&|\mathrm{cost}^{(m)}(P, c) - \sum_{j \in [q]} (|B_j| - m_j) \cdot \mathrm{dist}(\mu(B_j), c)| \\
&\le |\mathrm{cost}(S_M, c) - \mathrm{cost}(P_M, c)| + \sum_{p \in B_L \cap P_I^{(c)}} \mathrm{dist}(p, \mu(B_L)) + \sum_{p \in B_R \cap P_I^{(c)}} \mathrm{dist}(p, \mu(B_R)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Triangle Inequality)} \\
&\le |\mathrm{cost}(S_M, c) - \mathrm{cost}(P_M, c)| + \delta(B_L) + \delta(B_R) && \text{(Definition of } \delta(B)) \\
&< \frac{\varepsilon}{3} \cdot \mathrm{cost}^{(m)}(P, c) + \frac{2^\gamma \cdot \varepsilon^2 n r_{\max}}{72} \\
&\le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c). && \text{(Inequality (8))}
\end{aligned}$$

Thus by the definition of $S_I^{(c)}$,

$$\mathrm{cost}^{(m)}(S, c) \le \sum_{j \in [q]} (|B_j| - m_j) \cdot \mathrm{dist}(\mu(B_j), c) < (1 + \varepsilon)\mathrm{cost}^{(m)}(P, c).$$

Moreover, it is easy to prove that the rightmost block $B_{i'}$ intersecting with $S_I^{(c)}$ still satisfies $i' \le \gamma + 1$. Thus, similarly to the previous discussion, we have:

$$\mathrm{cost}^{(m)}(P, c) < (1 + \varepsilon)\mathrm{cost}^{(m)}(S, c).$$

Thus $\mathrm{cost}^{(m)}(S, c) \in (1 \pm \varepsilon)\mathrm{cost}^{(m)}(P, c)$.

Case 2: $\mathrm{dist}(c, c^\star) \le \frac{\varepsilon}{6} \cdot r_{\max}$. Let $w_l := \sum_{p \in P_I^{(c)} \setminus P_I^\star} w(p)$. Consider the points in $P_I^{(c)} \setminus P_I^\star$, we have:

$$\mathrm{cost}^{(m)}(P, c) > w_l(r_{\max} - \mathrm{dist}(c, c^\star)) \ge (1 - \frac{\varepsilon}{6}) w_l r_{\max} > \frac{5}{6} w_l r_{\max}. \tag{9}$$

Thus,

$$\begin{aligned} \mathrm{cost}^{(m)}(P, c) &= \mathrm{cost}(P_I^{(c)} \cap P_I^\star, c) + \mathrm{cost}(P_I^{(c)} \setminus P_I^\star, c) \\ &\ge \mathrm{cost}(P_I^\star, c) - 2w_l \cdot \mathrm{dist}(c, c^\star) && \text{(Definition of } w_l\text{)} \\ &\ge \mathrm{cost}(P_I^\star, c) - \frac{\varepsilon}{3} w_l r_{\max} \\ &> \mathrm{cost}(P_I^\star, c) - \frac{2\varepsilon}{5} \cdot \mathrm{cost}^{(m)}(P, c) && \text{(Inequality (9)).} \end{aligned} \tag{10}$$

By the definition of $\mathrm{cost}^{(m)}(P, c)$, we have

$$\mathrm{cost}^{(m)}(P, c) \le \mathrm{cost}(P_I^\star, c). \tag{11}$$

Similarly, we have

$$\mathrm{cost}(S_I^\star, c) - \frac{2\varepsilon}{5} \cdot \mathrm{cost}^{(m)}(S, c) \le \mathrm{cost}^{(m)}(S, c) < \mathrm{cost}(S_I^\star, c). \tag{12}$$

By Theorem 3.2, Algorithm $\mathcal{A}$ ensures that $|\mathrm{cost}(S_M, c) - \mathrm{cost}(P_M, c)| < \frac{\varepsilon}{3} \cdot \mathrm{cost}(P_M, c)$. According to the definition of the block B_0, we can obtain that there is no bucket that partially intersects with $P_I^\star$. Thus,

$$\begin{aligned} |\mathrm{cost}(S_I^\star, c) - \mathrm{cost}(P_I^\star, c)| &= |\mathrm{cost}(S_M, c) - \mathrm{cost}(P_M, c)| \\ &< \frac{\varepsilon}{3} \cdot \mathrm{cost}(P_M, c) \\ &< \frac{\varepsilon}{3} \cdot \mathrm{cost}(P_I^\star, c). \end{aligned} \tag{13}$$

Combining the above equations, we have

$$\begin{aligned} \mathrm{cost}^{(m)}(S, c) &< \mathrm{cost}(S_I^\star, c) && \text{(Inequality (12))} \\ &< (1 + \frac{\varepsilon}{3}) \mathrm{cost}(P_I^\star, c) && \text{(Inequality (13))} \\ &< (1 + \frac{\varepsilon}{3})(1 + \frac{2\varepsilon}{5}) \mathrm{cost}^{(m)}(P, c) && \text{(Inequality (10))} \\ &< (1 + \varepsilon) \mathrm{cost}^{(m)}(P, c). \end{aligned}$$

Similarly we have $\mathrm{cost}^{(m)}(P, c) < (1 + \varepsilon)\mathrm{cost}^{(m)}(S, c)$, thus $\mathrm{cost}^{(m)}(S, c) \in (1 \pm \varepsilon) \cdot \mathrm{cost}^{(m)}(P, c)$. □

3.3 Proof of Lemma 3.6: Number of misaligned outliers

Proof of Lemma 3.6. Note that for any center c, $P_I^{(c)}$ is a continuous subset of size $n - m$. Let $P_I^{(c)} = \{p_l, \ldots, p_{l+n-m-1}\}$. Assume $p_l \in B_L$ and $p_{l+n-m-1} \in B_R$. Consider a point $p_j \in P_L$ that satisfies $\mathrm{dist}(p_j, c) < \mathrm{dist}(p_{j+n-m}, c)$ and $\mathrm{dist}(\mu(B), c) > \mathrm{dist}(\mu(B'), c)$, where $p_j \in B$ and $p_{j+n-m} \in B'$. Next we show that the total number of points satisfying the above conditions is at most $O(\varepsilon n)$, which also provides an upper bound for $\sum_i |m_i - m_i'|$.

In this case $p_j \in P_I^{(c)}$. By the definition of $P_I^{(c)}$, we have $j \geq l$. If $B \neq B_L$ and $B' \neq B_R$, then by $j \geq l$, we have $\mu(B) > \max(B_L)$ and $\mu(B') > \max(B_R)$. Then we have

$$\begin{aligned}\mathrm{dist}(p_j, c) &\geq \mathrm{dist}(\max(B_L), c) > \mathrm{dist}(\mu(B), c) > \mathrm{dist}(\mu(B'), c) > \mathrm{dist}(\max(B_R), c)\\ &\geq \mathrm{dist}(p_{j+n-m}, c),\end{aligned}$$

which contradicts the inequality $\mathrm{dist}(p_j, c) < \mathrm{dist}(p_{j+n-m}, c)$. Thus, either $B = B_L$ or $B' = B_R$ holds. This indicates that the total number of points satisfying the above conditions is at most $|B_L| + |B_R|$. Formally, denote $B^{(i)}$ as the bucket containing p_i, we have

$$\mathbb{I}_1 = \sum_{j\in[m]} \left|\mathbb{I}(p_j \in P_I^{(c)}, \mathrm{dist}(\mu(B^{(j)}), c) > \mathrm{dist}(\mu(B^{(j+n-m)}), c))\right| \leq |B_L| + |B_R|. \tag{14}$$

Symmetrically, consider the points in P_R, we have

$$\sum_{j\in[n-m+1,n]} \left|\mathbb{I}(p_j \in P_I^{(c)}, \mathrm{dist}(\mu(B^{(j-n+m)}), c) < \mathrm{dist}(\mu(B^{(j)}), c))\right| \leq |B_L| + |B_R|. \tag{15}$$

Since $|P_I^{(c)}| = n - m$, for any $i \in [1, m]$, exactly one point in $\{p_i, p_{i+n-m}\}$ is in $P_I^{(c)}$. This means that $p_i \in P_I^{(c)}$ if and only if $p_{i+n-m} \notin P_I^{(c)}$. Thus, Inequality (15) is equivalent to the following form:

$$\mathbb{I}_2 = \sum_{j\in[m]} \left|\mathbb{I}(p_j \notin P_I^{(c)}, \mathrm{dist}(\mu(B^{(j)}), c) < \mathrm{dist}(\mu(B^{(j+n-m)}), c))\right| \leq |B_L| + |B_R|. \tag{16}$$

Moreover, we'll show that $\mathbb{I}_1$ and $\mathbb{I}_2$ cannot both be greater than 0. Suppose $\mathbb{I}_1 > 0$. In this case, there exists a point $p_j \in P_L \cap P_I^{(c)}$ such that $\mathrm{dist}(\mu(B^{(j)}), c) > \mathrm{dist}(\mu(B^{(j+n-m)}), c)$. Consider a point $p_{j'} \in P_L \setminus P_I^{(c)}$, obviously $j' < j$. Then we have

$$\mathrm{dist}(\mu(B^{(j')}), c) \geq \mathrm{dist}(\mu(B^{(j)}), c) > \mathrm{dist}(\mu(B^{(j+n-m)}), c) \geq \mathrm{dist}(\mu(B^{(j'+n-m)}), c).$$

Thus $\mathbb{I}_2 = 0$. Conversely, it still holds due to symmetry. Based on the above analysis, we have

$$\mathbb{I}_1 + \mathbb{I}_2 \leq |B_L| + |B_R|. \tag{17}$$

By $p_i \in P_I^{(c)} \iff p_{i+n-m} \notin P_I^{(c)}$, it is easy to prove that $\sum_i |m_i - m_i'|$ in P_L and P_R are exactly the same. Then it follows that

$$\begin{aligned}
\sum_{i\in[q]} |m_i - m_i'| &= 2 \sum_{i\in[q], B_i\subseteq P_L} |m_i - m_i'|\\
&= 2 \sum_{i\in[q], B_i\subseteq P_L} \left|\sum_{p_j\in B_i} \mathbb{I}(p_j \in P_I^{(c)}) - \mathbb{I}(\mathrm{dist}(\mu(B_i), c) < \mathrm{dist}(\mu(B^{(j+n-m)}), c))\right|\\
&\leq 2 \sum_{j\in[m]} \left|\mathbb{I}(p_j \in P_I^{(c)}) - \mathbb{I}(\mathrm{dist}(\mu(B^{(j)}), c) < \mathrm{dist}(\mu(B^{(j+n-m)}), c))\right|\\
&\leq 2(\mathbb{I}_1 + \mathbb{I}_2)\\
&\leq 2(|B_L| + |B_R|) && \text{(Inequality (17))}\\
&\leq \frac{\varepsilon n}{4}. && (|B| \leq \frac{\varepsilon n}{16})
\end{aligned}$$

□

3.4 Proof of Lemma 3.7: Error analysis for $c \notin P_M$

Proof of Lemma 3.7. W.L.O.G., we assume that $c \geq p_{n-m}$. Recall that P is divided into buckets $B_1, \ldots, B_q$, from left to right. Suppose B_t ($t \in [q]$) contains the center c, i.e., $\min_{p \in B_t} p \leq c \leq \max_{p \in B_t} p$. We define function $f_P(c) = \mathrm{cost}^{(m)}(P, c)$ and $f_S(c) = \mathrm{cost}^{(m)}(S, c)$ for every $c \in \mathbb{R}$.

Note that $f_P(c) = f_P(p_{n-m}) + \int_{p_{n-m}}^{c} f'_P(x)$ and $f_S(c) = f_S(p_{n-m}) + \int_{p_{n-m}}^{c} f'_S(x)$ holds for any $c > p_{n-m}$. Next, we first show that $f_P(p_{n-m}) = f_S(p_{n-m})$ by Line 8 in Algorithm 1. Then we verify that $|f'_P(c) - f'_S(c)|$ is bounded by $\sum\limits_{i \in [q]} |m_i - m'_i|$, which suffices to prove the lemma. The main idea is similar to that of the proof of Theorem 2.1 in [37].

By Line 8 in Algorithm 1, there is no bucket that partially intersect with $P_I^{(p_{n-m})}$. In this case, for each bucket $B \subset P_I^{(p_{n-m})}$, $\mathrm{dist}(\mu(B), c) \leq \max_{p \in P_I^{(p_{n-m})}} \mathrm{dist}(p, c)$; for each bucket $B \not\subset P_I^{(p_{n-m})}$, $\mathrm{dist}(\mu(B), c) > \max_{p \in P_I^{(p_{n-m})}} \mathrm{dist}(p, c)$. This indicates that $S_I^{(p_{n-m})}$ contains $\mu(B)$ with weight $|B|$ for each $B \subset P_I^{(p_{n-m})}$, which maintains consistent weights with $P_I^{(p_{n-m})}$. Then by Lemma 3.1, we have $f_P(p_{n-m}) = f_S(p_{n-m})$. Note that $f_p(c)$ is a linear function of c, and we have

$$
\begin{aligned}
f'_P(c) &= |\{p \mid p \in P_I^{(c)}, p \leq c\}| - |\{p \mid p \in P_I^{(c)}, p > c\}| \\
&= \sum_{i<t} (|B_i| - m_i) + |B_t| \cap (-\infty, c]| - |B_t \cap (c, \infty)| - \sum_{i>t} (|B_i| - m_i).
\end{aligned}
$$

Similarly, when $\mu(B_t) \leq c$, we have $f'_S(c) = \sum_{i<t}(|B_i| - m'_i) + |B_t| - \sum_{i>t}(|B_i| - m'_i)$; when $\mu(B_t) > c$, we have $f'_S(c) = \sum_{i<t}(|B_i| - m'_i) - |B_t| - \sum_{i>t}(|B_i| - m'_i)$. Thus

$$
\begin{aligned}
|f'_P(c) - f'_S(c)| &\leq \sum_{i \in [q]} |m_i - m'_i| + 2|B_t| \\
&\leq \Gamma + 2|B_t| && \text{(Definition of } \Gamma) \\
&\leq \Gamma + \frac{\varepsilon n}{8}. && (|B| \leq \frac{\varepsilon n}{16})
\end{aligned}
$$

Then by $f_P(c) = f_P(p_{n-m}) + \int_{p_{n-m}}^{c} f'_P(x)\,dx$ and $f_S(c) = f_S(p_{n-m}) + \int_{p_{n-m}}^{c} f'_S(x)\,dx$, we have

$$
\begin{aligned}
|f_P(c) - f_S(c)| &= \left| f_P(p_{n-m}) + \int_{p_{n-m}}^{c} f'_P(x)\,dx - f_S(p_{n-m}) - \int_{p_{n-m}}^{c} f'_S(x)\,dx \right| \\
&= \left| \int_{p_{n-m}}^{c} f'_P(x) - f'_S(x)\,dx \right| && (f_P(p_{n-m}) = f_S(p_{n-m})) \\
&\leq \int_{p_{n-m}}^{c} |f'_P(x) - f'_S(x)|\,dx \\
&\leq \int_{p_{n-m}}^{c} (\Gamma + \frac{\varepsilon n}{8})\,dx \\
&= (c - p_{n-m})(\frac{\varepsilon n}{8} + \Gamma) \\
&= (\frac{\varepsilon n}{8} + \Gamma) \cdot \mathrm{dist}(c, P_M). && \text{(Definition of } P_M)
\end{aligned}
$$

Moreover, since $|P_R| = m$, at least $n - 2m$ inliers w.r.t. c are on the left of p_{n-m}. These points satisfy that $\mathrm{dist}(p, c) \geq \mathrm{dist}(p_{n-m}, c)$, which implies that $\mathrm{cost}^{(m)}(P, c) \geq (n - 2m) \cdot \mathrm{dist}(p_{n-m}, c)$.

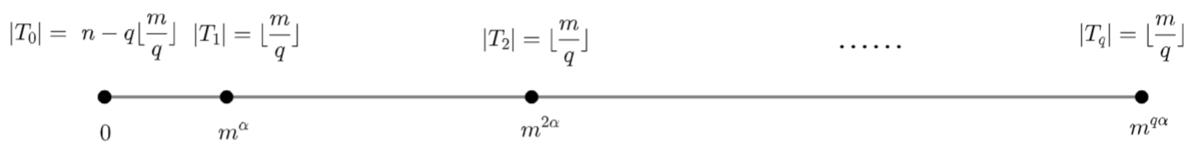


Figure 2: A case for demonstrating the coreset lower bound for robust 1D geometric median. T_i contains $\lfloor\frac{m}{q}\rfloor$ points where each point $p \in T_i$ satisfies $p = m^{i\alpha}$. T_0 contains the remaining points where each point $p \in T_0$ satisfies $p = 0$.

Then we have

$$\begin{aligned}
|f_P(c) - f_S(c)| &\le (c - p_{n-m})(\frac{\varepsilon n}{8} + \Gamma) \\
&\le (\frac{3\varepsilon n}{8})(c - p_{n-m}) && \text{(Lemma 3.6)} \\
&< \varepsilon(n - 2m)(c - p_{n-m}) && (n \ge 4m) \\
&\le \varepsilon \cdot \mathrm{cost}^{(m)}(P, c).
\end{aligned}$$

This completes the proof. □

3.5 Proof of the lower bound in Theorem 1.2

Next we show that for $n \ge 4m$, the size lower bound of ε-coreset for robust 1D geometric median is $\Omega(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$. This lower bound matches the upper bound in the above discussion, which completes the proof of Theorem 1.2. In the following discussion, we assume that the size of dataset P is sufficiently large such that $\varepsilon n > 1$, which holds in nearly all practical scenarios.

Proof of Theorem 1.2 (lower bound). For vanilla 1D geometric median, [26] shows that the size lower bound of ε-coreset is $\Omega(\varepsilon^{-\frac{1}{2}})$, which is obviously also the coreset size lower bound for robust 1D geometric median. Thus, it remains to show the coreset size is $\Omega(\frac{m}{n}\varepsilon^{-1})$ when $\frac{m}{n} > \varepsilon^{\frac{1}{2}}$.

We first construct the dataset P of size n. Let $q = \lfloor\frac{m}{2n\varepsilon}\rfloor$. The dataset P is a union of $1+q$ disjoint sets $\{T_0, T_1, \ldots, T_q\}$. For each $i \in \{1, \ldots, q\}$, T_i contains $\lfloor\frac{m}{q}\rfloor$ points, and every point $p \in T_i$ satisfies $p = m^{i\alpha}$, where $\alpha = 2 + \log_m(\varepsilon^{-2})$. T_0 contains $n - q\lfloor\frac{m}{q}\rfloor$ points where each point $p \in T_0$ satisfies $p = 0$. Correspondingly, define q disjoint intervals $\{I_1, \ldots, I_q\}$, where $I_i = \left[m^{(i-1)\alpha+1}, m^{i\alpha+1}\right]$.

Suppose S is a ε-coreset of P with size $|S| < q$. Then by the pigeonhole principle, there exists an interval I_j ($j \in [1, q]$) such that S does not include any points located in I_j. This implies that for any point $p \in S$, we have $p \le m^{(j-1)\alpha+1}$ or $p \ge m^{j\alpha+1}$. Fix the center $c = m^{j\alpha}$, then we have

$$\begin{aligned}
\mathrm{cost}^{(m)}(S, c) &\ge (n - m) \cdot \min_{p \in S} \mathrm{dist}(p, c) \\
&\ge (n - m) \cdot (m^{j\alpha} - m^{(j-1)\alpha+1}) && (S \cap I_j = \emptyset) \\
&> [(1 - m^{1-\alpha})n - m]m^{j\alpha} \\
&> [(1 - \varepsilon^2)n - m]m^{j\alpha} && (\alpha = 2 + \log_m(\varepsilon^{-2})) \\
&> [(1 - \varepsilon - 2\varepsilon^2)n + \varepsilon n - m]m^{j\alpha} \\
&> (1 + \varepsilon)[(1 - 2\varepsilon)n - (m - 1)]m^{j\alpha}. && (\varepsilon > \frac{1}{n})
\end{aligned}$$

Since $n \geq 4m$, we have $|T_0| = n - q \cdot \lfloor \frac{m}{q} \rfloor \geq n - m$. Consider the points in T_0 and T_j as inliers, we have

$$\begin{aligned}
\mathrm{cost}^{(m)}(P, c) &\leq \mathrm{cost}(T_j, c) + (n - m - |T_j|)m^{j\alpha} \\
&= (n - m - \lfloor \frac{m}{q} \rfloor)m^{j\alpha} && \text{(Definition of } T_j\text{)} \\
&\leq (n - m - \frac{m}{q} + 1)m^{j\alpha} \\
&\leq [(1 - 2\varepsilon)n - (m - 1)]m^{j\alpha}. && \text{(Definition of } q\text{)}
\end{aligned}$$

It follows that $\mathrm{cost}^{(m)}(S, c) > (1 + \varepsilon)\mathrm{cost}^{(m)}(P, c)$, thus S is not a ε-coreset of P, which leads to a contradiction. This implies that any ε-coreset of P contains $\Omega(\frac{m}{n}\varepsilon^{-1})$ points when $\frac{m}{n} > \varepsilon^{\frac{1}{2}}$. Considering the discussion above, the lower bound of the coreset size is $\Omega(\varepsilon^{-\frac{1}{2}} + \frac{m}{n}\varepsilon^{-1})$.

□

Note that the dataset P we construct is a multiset, which is slightly different from the definition. However, this does not affect the proof, because we can move each point by a sufficiently small and distinct distance, making the cost value almost unchanged.

When $\varepsilon n < 1$, we can show that the coreset size is $\Omega(\varepsilon^{-\frac{1}{2}} + m)$ by the above discussion. Moreover, consider a trivial method that applies algorithm $\mathcal{A}$ on P_M and keeps all points not in P_M. It's easy to prove that this method constructs an ε-coreset of the original dataset under the assumption. In this case, the coreset size is $\tilde{O}(\varepsilon^{-\frac{1}{2}} + m)$, which matches the above lower bound.

4 Improved coreset sizes for general $d \geq 1$

We present Algorithm 2 for Theorem 1.3. In Line 1, we construct $L^\star$ as the set of outliers of P w.r.t. $c^\star$ and $P_I^\star = P - L^\star$ as the set of inliers. We construct coresets for $P_I^\star$ and $L^\star$ separately. In Line 2, we take a uniform sample S_O from $L^\star$ as the coreset of $L^\star$. This step is the key for eliminating the $O(m)$ dependency in the coreset size. In Line 3, we use the following theorem by [40] to construct a coreset S_I for $P_I^\star$. The coreset S is the union of S_O and S_I (Line 5). In Section 5, we show how to generalize this algorithm to robust (k, z)-clustering (Algorithm 3).

Algorithm 2 Coreset Construction for General d

Input: A dataset $P \subset \mathbb{R}^d$, $\varepsilon \in (0, 1)$ and an $O(1)$-approximate center $c^\star \in \mathbb{R}^d$
Output: An ε-coreset S

1: $L^\star \leftarrow \arg\min_{L:|L|=m} \mathrm{cost}(P - L, c^\star)$, $P_I^\star \leftarrow P - L^\star$
2: Uniformly sample $S_O \subseteq L^\star$ of size $\tilde{O}(\varepsilon^{-2} \min\{\varepsilon^{-2}, d\})$. Set $\forall p \in S_O$, $w_O(p) \leftarrow \frac{m}{|S_O|}$.
3: Construct $(S_I, w_I) \leftarrow \mathcal{A}_d(P_I^\star)$ by Theorem 4.1.
4: For any $p \in S_O$, define $w(p) = w_O(p)$ and for any $p \in S_I$, define $w(p) = w_I(p)$.
5: Return $S \leftarrow S_O \cup S_I$ and w;

Theorem 4.1 (Restatement of corollary 5.4 in [40]). *There exists a randomized algorithm $\mathcal{A}_d$ that in $O(nd)$ time constructs a weighted subset $S_I \subseteq P_I^\star$ of size $\tilde{O}(\varepsilon^{-2} \min\{\varepsilon^{-2}, d\})$, such that for every dataset P_O of size m, every integer $0 \leq t \leq m$ and every center $c \in \mathbb{R}^d$, $|\mathrm{cost}^{(t)}(P_O \cup P_I^\star, c) - \mathrm{cost}^{(t)}(P_O \cup S_I, c)| \leq \varepsilon \cdot \mathrm{cost}^{(t)}(P_O \cup P_I^\star, c) + 2\varepsilon \cdot \mathrm{cost}(P_I^\star, c^\star)$.*

The theorem tells that S_I serves as an ε-coreset for the combination of $P_I^\star$ and any possible set of outliers P_O. The flexible choice of P_O is useful for our analysis. To estimate the error induced by S_O, we introduce the key lemma below, whose proof can be found in Section 4.1.

Lemma 4.2 (Induced error of S_O)**.** *For any center $c \in \mathbb{R}^d$, we have $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| \leq O(\varepsilon) \cdot \mathrm{cost}^{(m)}(P,c)$.*

Proof of Theorem 1.3. Fix a center $c \in \mathbb{R}^d$. Let $P_O = S_O$ and $t = m$ in Theorem 4.1, we have $|\mathrm{cost}^{(m)}(S_O \cup P_I^\star, c) - \mathrm{cost}^{(m)}(S_O \cup S_I, c)| \leq \varepsilon \cdot \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c) + 2\varepsilon \cdot \mathrm{cost}(P_I^\star, c^\star)$. By Lemma 4.2, we have $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| \leq O(\varepsilon) \cdot \mathrm{cost}^{(m)}(P,c)$. Adding the two inequalities above, we have $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S,c)| \leq O(\varepsilon) \cdot \mathrm{cost}^{(m)}(P,c)$.

The runtime is dominated by Line 1 and Line 3 that costs $O(nd)$ time by Theorem 4.1, making the total overhead $O(nd)$. This completes the proof of Theorem 1.3. ☐

In the following subsections, we list out the proof of Lemma 4.2 and extend Theorem 1.3 to general metric spaces.

4.1 Proof of Lemma 4.2: induced error of S_O

Below we briefly introduce the proof idea. We first observe that the induced error of S_O is primarily caused by points that act as inliers in P but outliers in S, or vice versa, as shown in Lemmas 4.4 and 4.5. The error from a single point is bounded by $O(\frac{\mathrm{cost}^{(m)}(P,c)}{m})$ (see Lemma 4.6). The next task is ensuring that there are $O(\varepsilon m)$ such points in S_O, which is guaranteed when S_O provides an ε-approximation for the ball range space on $L^\star$ (Lemma 4.7). To achieve this, we sample $\tilde{O}(d/\varepsilon^2)$ points for S_O (Lemma 4.3).

Fix a center $c \in \mathbb{R}^d$. Let $L^{(c)} := \arg\min_{L \subseteq P : |L| = m} \mathrm{cost}(P - L, c)$ be the set of outliers of P w.r.t. c and $m_P := |L^\star \cap L^{(c)}|$ represent the number of these outliers contained in $L^\star$. For $S_O \cup P_I^\star$, we first define a family of weight functions $\mathcal{W} := \{w_S : S_O \cup P_I^\star \to \mathbb{R}^+ \mid w_S(p) \leq w(p), \forall p \in S_O;\ w_S(p) \leq 1, \forall p \in P_I^\star;\ \|w_S\|_1 = n - m\}$. Intuitively, $\mathcal{W}$ represents the collection of all possible weight functions for $n - m$ inliers of the weighted dataset $S_O \cup P_I^\star$. Define a weight function w' as follows:

$$w' := \arg\min_{w_S \in \mathcal{W}} \sum_{p \in S_O \cup P_I^\star} w_S(p) \cdot \mathrm{dist}(p,c),$$

i.e., $(S_O \cup P_I^\star, w')$ represents the $n - m$ inliers of $S_O \cup P_I^\star$ with respect to center c. Let $m_S := \sum_{p \in S_O}(w(p) - w'(p))$ denote the number of outliers of $S_O \cup P_I^\star$ w.r.t. c that are contained in S_O.

For preparation, we introduce the concept of ball range space, which facilitates a precise analysis of point distributions in P and S.

Definition 4.1 (Approximation of ball range space, Definition F.2 in [39])**.** *For a given dataset $P \subset \mathbb{R}^d$, the ball range space on P is $(P,\mathcal{P})$ where $\mathcal{P} := \{P \cap \mathrm{Ball}(c,u) \mid c \in \mathbb{R}^d, u > 0\}$ and $\mathrm{Ball}(c,u) := \{p \in \mathbb{R}^d \mid \mathrm{dist}(p,c) \leq u\}$. A subset $Y \subset P$ is called an ε-approximation of the ball range space $(P,\mathcal{P})$ if for every $c, u \in \mathbb{R}^d$,*

$$\left|\frac{|P \cap \mathrm{Ball}(c,u)|}{|P|} - \frac{|Y \cap \mathrm{Ball}(c,u)|}{|Y|}\right| \leq \varepsilon.$$

Based on this definition, we have the following preparation lemma that measures the performance of S_O; which is refined from [39].

Lemma 4.3 (Refined from Lemma F.3 of [39])**.** *Given dataset $P_O \subset \mathbb{R}^d$. Let S_O be a uniform sampling of size $\tilde{O}(\frac{d}{\varepsilon^2})$ from P_O, then with probability at least $1 - \frac{1}{poly(1/\varepsilon)}$, S_O is an ε-approximation of the ball range space on P_O. Define a weight function w: $w(p) = \frac{|P_O|}{|S_O|}$, for any $p \in S_O$. Then for any $c \in \mathbb{R}^d$, $u \in \mathbb{R}^+$,*

$$||P_O \cap \mathrm{Ball}(c,u)| - w(S_O \cap \mathrm{Ball}(c,u))| \le \varepsilon |P_O|.$$

By the iterative method introduced by [48], the factor $O(d)$ of coreset size can be replaced by $\tilde{O}(\varepsilon^{-2})$. Therefore, S_O is an ε-approximation of the ball range space on $L^\star$.

Then we are ready to prove Lemma 4.2. We first analyze where the induced error comes from.

Recall that we fix a center c in this section. Let $T_O := \min_{p \in L^\star} \mathrm{dist}(p,c)$ denote the minimum distance from points in $L^\star$ to c. Let $T_I := \max_{p \in P_I^\star} \mathrm{dist}(p,c)$ denote the maximum distance from points in $P_I^\star$ to c. We have the following Lemma.

Lemma 4.4 (Comparing T_O and T_I)**.** *When $m_P = m$, $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| = 0$ holds. When $m_P < m$, $T_O \le T_I$ holds.*

Proof. If $m_P = m$, for any point $p \in L^\star$, we have $\mathrm{dist}(p,c) \ge \max_{p \in P_I^\star} \mathrm{dist}(p,c)$. Since S_O is sampled from $L^\star$, we know that for any point $p \in S_O$, $\mathrm{dist}(p,c) \ge \max_{p \in P_I^\star} \mathrm{dist}(p,c)$. Therefore, we have $m_P = m_S = m$ and then $\mathrm{cost}^{(m)}(P,c) = \mathrm{cost}(P_I^\star, c) = \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)$.

If $m_P < m$, then there exists a point $\widehat{p} \in P_I^\star$ such that $\mathrm{dist}(\widehat{p}, c) \ge T_O$. By definition, we also know that $\mathrm{dist}(\widehat{p}, c) \le T_I$. Therefore, combining these two conditions, we have $T_O \le T_I$. □

It remains to analyze the case that $m_P < m$. In this setting, we present the following lemma, which provides an upper bound on the estimation error $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)|$. Before stating the lemma, we introduce a notation that will also be used in subsequent lemmas. We sort all distances $\mathrm{dist}(p,c)$ for each point $p \in P_I^\star$ in descending order, w.l.o.g. say $\mathrm{dist}(p_1,c) > \ldots > \mathrm{dist}(p_{|P_I^\star|}, c)$. Here, we can safely assume all distances $\mathrm{dist}(p,c)$ are distinct, given that adding small values to the distances has only a subtle impact on the cost. Let $d_i := \mathrm{dist}(p_i, c)$ for $i \in [m]$. Then $d_1, \ldots, d_m$ represent the distances from the m furthest points in $P_I^\star$ to the center c. Now, we are ready to provide the following lemmas.

Lemma 4.5 (An upper bound of the induced error of S_O)**.** *When $m_P < m$, suppose $||P_O \cap \mathrm{Ball}(c,u)| - w(S_O \cap \mathrm{Ball}(c,u))| \le \Delta$ for any $u > 0$, the following holds: $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| \le 2 \cdot (\Delta + |m_P - m_S|) \cdot (T_I - T_O)$.*

Proof. By Fact F.1 in [39], we have

$$\mathrm{cost}^{(m_P)}(L^\star, c) = \int_0^\infty (m - m_P - |L^\star \cap \mathrm{Ball}(c,u)|)^+ du \tag{18}$$

and

$$\mathrm{cost}^{(m_S)}(S_O, c) = \int_0^\infty (m - m_S - w(S_O \cap \mathrm{Ball}(c,u)))^+ du \tag{19}$$

where $(a)^+ = \max\{a, 0\}$.

Let $T := \max_{p \in L^\star - L^{(c)}} \mathrm{dist}(p,c)$ and $T_S := \max_{p \in S_O : w'(p) > 0} \mathrm{dist}(p,c)$. Then we know that $T_O \le T$, $T_S \le T_I$. By definition, for any distance $u > T$, we have $|L^\star \cap \mathrm{Ball}(c,u)| \ge m - m_P$. Then we can transform the Inequality (18) to

$$\mathrm{cost}^{(m_P)}(L^\star, c) = \int_0^T (m - m_P - |L^\star \cap \mathrm{Ball}(c,u)|) du. \tag{20}$$

Similarly, for any distance $u > T_S$, we have $w(S_O \cap \mathrm{Ball}(c,u)) \geq m - m_S$. We can transform the Inequality (19) to

$$\mathrm{cost}^{(m_S)}(S_O, c) = \int_0^{T_S} (m - m_S - w(S_O \cap \mathrm{Ball}(c,u)))du. \tag{21}$$

Based on the notation of $d_1, ..., d_m$, we know that each point $p \in P_I^\star$ satisfying $\mathrm{dist}(p,c) \leq d_{m-m_P+1}$ is an inlier of $P_I^\star \cup L^\star$, and each point $q \in P_I^\star$ with weight $w'(q) > 0$ satisfying $\mathrm{dist}(q,c) \leq d_{m-\lfloor m_S \rfloor}$ is an inlier of $S_O \cup P_I^\star$. Let $m'_S := \lfloor m_S \rfloor$. Let $l := |m_P - m_S|$ denote the difference in the number of outliers. If $m_P > m_S$, we have

$$l \cdot d_{m-m'_S} \leq \mathrm{cost}^{(m-m_P)}(P_I^\star, c) - \mathrm{cost}^{(m-m_S)}(P_I^\star, c) \tag{22}$$

and

$$\mathrm{cost}^{(m-m_P)}(P_I^\star, c) - \mathrm{cost}^{(m-m_S)}(P_I^\star, c) \leq l \cdot d_{m-m_P+1}. \tag{23}$$

If $m_P < m_S$, we have

$$l \cdot d_{m-m_P} \leq \mathrm{cost}^{(m-m_S)}(P_I^\star, c) - \mathrm{cost}^{(m-m_P)}(P_I^\star, c) \tag{24}$$

s and

$$\mathrm{cost}^{(m-m_S)}(P_I^\star, c) - \mathrm{cost}^{(m-m_P)}(P_I^\star, c) \leq l \cdot d_{m-m'_S}. \tag{25}$$

If $m_P = m_S$, we have

$$\mathrm{cost}^{(m-m_S)}(P_I^\star, c) - \mathrm{cost}^{(m-m_P)}(P_I^\star, c) = 0. \tag{26}$$

By definition, we know that

$$d_m \leq d_{m-1} \leq \ldots \leq d_1 \leq T_I. \tag{27}$$

When $m_S \neq m$, we know the point $p \in P_I^\star$ that has distance $\mathrm{dist}(p,c) = d_{m-m'_S}$ is an outlier on $P_I^\star \cup S_O$. Then there exists a point $q \in S_O$ such that $\mathrm{dist}(q,c) \leq \mathrm{dist}(p,c)$. Then we have

$$d_{m-m'_S} \geq T_O. \tag{28}$$

Similarly, the point $p \in P_I^\star$ with distance $\mathrm{dist}(p,c) = d_{m-m_P}$ is an outlier on P, thus there exists a point $q \in L^\star$ such that $\mathrm{dist}(q,c) \leq \mathrm{dist}(p,c)$. Then we have

$$d_{m-m_P} \geq T_O. \tag{29}$$

By Lemma 4.4, it suffices to discuss the following three cases based on the values of m_P and m_S.

Case 1: $m_S = m$ Recall that $l = m_S - m_P$, we have

$$\begin{aligned}
\mathrm{cost}^{(m_P)}(L^\star, c) = & \int_0^T (m - m_P - |L^\star \cap \mathrm{Ball}(c,u)|)du && \text{(Inequality (20))} \\
= & \int_0^T (l - |L^\star \cap \mathrm{Ball}(c,u)|)du && (l = m_S - m_P) \\
= & \; l \cdot T - \int_{T_O}^T |L^\star \cap \mathrm{Ball}(c,u)|du && (\forall p \in L^\star,\ \mathrm{dist}(p,c) \geq T_O).
\end{aligned}$$

Since $m_S = m$ and $m_P < m_S$, each point $p \in L^\star - L^{(c)}$ satisfies $\mathrm{dist}(p, c) < \min_{q \in S_O} \mathrm{dist}(q, c)$. For each $T_O \le u \le T$, we have $w(S_O \cap \mathrm{Ball}(c, u)) = 0$. Since $|L^\star \cap \mathrm{Ball}(c, u)| - w(S_O \cap \mathrm{Ball}(c, u)) \le \Delta$, we know $|L^\star \cap \mathrm{Ball}(c, u)| \le \Delta$ for $T_O \le u \le T$ and

$$l \cdot T - (T - T_O) \cdot \Delta \le \mathrm{cost}^{(m_P)}(L^\star, c) \le l \cdot T.$$

Since $\mathrm{cost}^{(m_S)}(S_O, c) = 0$, we have

$$\mathrm{cost}^{(m_P)}(L^\star, c) - \mathrm{cost}^{(m_S)}(S_O, c) \le l \cdot T \tag{30}$$

and

$$\mathrm{cost}^{(m_S)}(S_O, c) - \mathrm{cost}^{(m_P)}(L^\star, c) \le (T - T_O) \cdot \Delta - l \cdot T. \tag{31}$$

Adding Inequality (24) and Inequality (30), we have

$$\begin{aligned}
\mathrm{cost}^{(m)}(P, c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c) \le &\quad l \cdot T_I - l \cdot T_O && (T \le T_I \text{ and Inequality (29)})\\
\le &\quad |m_P - m_S| \cdot (T_I - T_O) && (\text{Lemma 4.7}).
\end{aligned}$$

Adding Inequality (25) into Inequality (31), we have

$$\begin{aligned}
\mathrm{cost}^{(m)}(S_O \cup P_I^\star, c) - \mathrm{cost}^{(m)}(P, c) \le &\quad (T - T_O) \cdot \Delta + l \cdot (d_1 - T)\\
\le &\quad (T_I - T_O) \cdot \Delta + l \cdot (T_I - T_O)\\
&\qquad\qquad (T_O \le T \le T_I \text{ and Inequality (27)})\\
\le &\quad (|m_P - m_S| + \Delta) \cdot (T_I - T_O).
\end{aligned}$$

Then we complete the proof of Case 2.

Case 2: $m_S \neq m_P < m$ Without loss of generality, we assume that $m_P > m_S$ and $T \ge T_S$.

Firstly, we prove that $\mathrm{cost}^{(m_P)}(P, c) - \mathrm{cost}^{(m_S)}(S_O \cup P_I^\star, c) \le 2(\Delta + |m_P - m_S|) \cdot (T_I - T_O)$. We have

$$\begin{aligned}
&\mathrm{cost}^{(m_P)}(L^\star, c) - \mathrm{cost}^{(m_S)}(S_O, c)\\
&\qquad = \quad \int_0^T (m - m_P - |L^\star \cap \mathrm{Ball}(c, u)|) du\\
&\qquad\quad - \int_0^{T_S} (m - m_S - w(S_O \cap \mathrm{Ball}(c, u))) du && (\text{Inequalities (20) and (21)})\\
&\qquad = \quad \int_0^{T_S} (m_S - m_P + w(S_O \cap \mathrm{Ball}(c, u)) - |L^\star \cap \mathrm{Ball}(c, u)|) du\\
&\qquad\quad + \int_{T_S}^T (m - m_P - |L^\star \cap \mathrm{Ball}(c, u)|) du && (T \ge T_S)\\
&\qquad = \quad \int_{T_O}^{T_S} (w(S_O \cap \mathrm{Ball}(c, u)) - |L^\star \cap \mathrm{Ball}(c, u)|) du - l \cdot T_S\\
&\qquad\quad + \int_{T_S}^T (m - m_P - |L^\star \cap \mathrm{Ball}(c, u)|) du\\
&\qquad \le \quad \Delta \cdot (T_S - T_O) - l \cdot T_S + \int_{T_S}^T (m - m_P - |L^\star \cap \mathrm{Ball}(c, u)|) du.
\end{aligned}$$

For $T_S < u < T$, we have $|L^\star \cap \mathrm{Ball}(c,u)| \geq m - \Delta$, thus

$$\begin{aligned}
\mathrm{cost}^{(m_P)}(L^\star, c) - &\mathrm{cost}^{(m_S)}(S_O, c) \\
&\leq \quad \Delta \cdot (T_S - T_O) + \int_{T_S}^{T} (\Delta - m_P) du - l \cdot T_S \\
&\leq \quad \Delta \cdot (T_S - T_O) + \Delta \cdot (T - T_S) - l \cdot T_S \\
&\leq \quad 2 \cdot \Delta \cdot (T_I - T_O) - l \cdot T_S \qquad (T_O \leq T_S \leq T_I \text{ and } T \leq T_I).
\end{aligned}$$

Adding Inequality (23) and the above inequality, we obtain

$$\begin{aligned}
\mathrm{cost}^{(m)}(P, c) - &\mathrm{cost}^{(m)}(S_O \cup P_I^\star, c) \\
&\leq \quad 2 \cdot \Delta \cdot (T_I - T_O) + l \cdot (d_{m-m_P+1} - T_S) \\
&\leq \quad 2 \cdot \Delta \cdot (T_I - T_O) + l \cdot (T_I - T_O) \qquad \text{(Inequality (27))} \\
&\leq \quad (2 \cdot \Delta + |m_P - m_S|) \cdot (T_I - T_O).
\end{aligned}$$

Secondly, we prove that $\mathrm{cost}^{(m_S)}(S_O \cup P_I^\star, c) - \mathrm{cost}^{(m_P)}(P, c) \leq (\Delta + |m_P - m_S|) \cdot (T_I - T_O)$. By Inequality (20) and Inequality (21), we have

$$\begin{aligned}
\mathrm{cost}^{(m_S)}(&S_O, c) - \mathrm{cost}^{(m_P)}(L^\star, c) \\
&= \quad \int_0^{T_S} (m - m_S - w(S_O \cap \mathrm{Ball}(c,u))) du \\
&\quad - \int_0^{T} (m - m_P - |L^\star \cap \mathrm{Ball}(c,u)|) du \\
&= \quad \int_0^{T_S} (m_P - m_S + |L^\star \cap \mathrm{Ball}(c,u)| - w(S_O \cap \mathrm{Ball}(c,u))) du \\
&\quad - \int_{T_S}^{T} (m - m_P - |L^\star \cap \mathrm{Ball}(c,u)|) du \qquad (T_S \leq T) \\
&\leq \quad \int_0^{T_S} (m_P - m_S + |L^\star \cap \mathrm{Ball}(c,u)| - w(S_O \cap \mathrm{Ball}(c,u))) du \\
&\leq \quad \int_{T_O}^{T_S} (|L^\star \cap \mathrm{Ball}(c,u)| - w(S_O \cap \mathrm{Ball}(c,u))) du + l \cdot T_S \\
&\leq \quad \Delta \cdot (T_S - T_O) + l \cdot T_S.
\end{aligned}$$

Adding Inequality (22) and the above inequality, we have

$$\begin{aligned}
\mathrm{cost}^{(m)}(&S_O \cup P_I^\star, c) - \mathrm{cost}^{(m)}(P, c) \\
&\leq \quad \Delta \cdot (T_S - T_O) + l \cdot (T_S - d_{m'_S}) \\
&\leq \quad \Delta \cdot (T_I - T_O) + l \cdot (T_I - T_O) \qquad \text{(Inequality (28) and } T_S \leq T_I) \\
&\leq \quad (\Delta + |m_P - m_S|) \cdot (T_I - T_O).
\end{aligned}$$

In summary, we have

$$|\mathrm{cost}^{(m)}(P, c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| \leq 2 \cdot (\Delta + |m_P - m_S|) \cdot (T_I - T_O).$$

Similarly, we can get the same conclusion when $T < T_S$. Moreover, for the case that $m_P < m_S$, with the help of Inequalities (24)(25)(27)(29), the conclusion still holds.

Case 3: $m_P = m_S \neq m$ By Inequality (26), we have

$$|\mathrm{cost}^{(m)}(S_O \cup P_I^\star, c) - \mathrm{cost}^{(m)}(P, c)| = |\mathrm{cost}^{(m_S)}(S_O, c) - \mathrm{cost}^{(m_P)}(L^\star, c)|.$$

By a similar argument as in Case 2, we have

$$|\mathrm{cost}^{(m_S)}(S_O \cup P_I^\star, c) - \mathrm{cost}^{(m_P)}(P, c)| \leq 2 \cdot (\Delta + |m_P - m_S|) \cdot (T_I - T_O).$$

Overall, we complete the proof of Lemma 4.5.

□

The Lemma 4.5 tells us that at most $2(\Delta + |m_P - m_S|)$ points contribute to the error, with each point inducing at most $T_I - T_O$ error. We now analyze the magnitude of $T_I - T_O$.

Lemma 4.6 (Bounding induced error of each point)**.** $T_I - T_O \leq O(\mathrm{cost}^{(m)}(P, c)/m)$

Proof. Let $r_{\max} := \max_{p \in P_I^\star} \mathrm{dist}(p, c^\star)$ denote the maximum distance from $P_I^\star$ to $c^\star$. Let $d_{\max} := \mathrm{dist}(c, c^\star)$ denote the distance between the approximate center $c^\star$ and the center c. Let $\bar{r} := \frac{\mathrm{cost}(P_I^\star, c^\star)}{n-m}$ denote the average distance from $P_I^\star$ to $c^\star$.

Utilizing the triangle inequality, for any point $p \in L^\star$, we can assert that $\mathrm{dist}(p, c) \geq \mathrm{dist}(p, c^\star) - \mathrm{dist}(c, c^\star) \geq r_{\max} - d_{\max}$, thus $T_O \geq r_{\max} - d_{\max}$. Similarly, for any point $p \in P_I^\star$, we have $\mathrm{dist}(p, c) \leq \mathrm{dist}(p, c^\star) + \mathrm{dist}(c, c^\star) = r_{\max} + d_{\max}$, thus $T_I \leq r_{\max} + d_{\max}$. Consequently, depending on whether $r_{\max}$ is greater than or less than $d_{\max}$, we derive different expressions for $T_I - T_O$: If $r_{\max} \geq d_{\max}$, then $T_I - T_O \leq 2 \cdot d_{\max}$; if $r_{\max} < d_{\max}$, given that $T_O \geq 0$, it follows that $T_I - T_O \leq r_{\max} + d_{\max} < 2 \cdot d_{\max}$. Therefore, we turn to prove $2 \cdot m \cdot d_{\max} \leq O(\mathrm{cost}^{(m)}(P, c))$. We discuss the relationship between $d_{\max}$ and $\bar{r}$ in two cases.

Case 1: $d_{\max} \leq 4\bar{r}$ By definition of $\bar{r}$, we have

$$\begin{aligned} 4 \cdot \mathrm{cost}^{(m)}(P, c^\star) = & \quad 4 \cdot (n-m)\bar{r} \\ \geq & \quad (n-m) d_{\max} \\ \geq & \quad 2 \cdot m \cdot d_{\max} & (n \geq 4m). \end{aligned}$$

since $c^\star$ is an $O(1)$-approximate solution for robust geometric median, we have

$$O(\mathrm{cost}^{(m)}(P, c)) \geq 2 \cdot m \cdot d_{\max}$$

.

Case 2: $d_{\max} > 4\bar{r}$ Let $\widehat{P} := \{p \in P_I^\star | \mathrm{dist}(p, c^\star) \leq 2\bar{r}\}$. By definition, we have $|\widehat{P}| \geq \frac{1}{2}|P_I^\star| = \frac{n-m}{2}$. Since $d_{\max} > 4\bar{r}$, we obtain

$$\begin{aligned} 8 \cdot \mathrm{cost}^{(m)}(P, c) \geq & \quad 8 \cdot ((n-m)/2 - m) \min_{p \in \widehat{P}} \mathrm{dist}(p, c) \\ \geq & \quad 4 \cdot m \cdot \min_{p \in \widehat{P}} \mathrm{dist}(p, c) & (n \geq 4m) \\ \geq & \quad 4 \cdot m \cdot \min_{p \in \widehat{P}} (d_{\max} - \mathrm{dist}(p, c^\star)) & \text{(Triangle Inequality)} \\ \geq & \quad 4 \cdot m \cdot (d_{\max} - 2\bar{r}) \\ > & \quad 2 \cdot m \cdot d_{\max}. \end{aligned}$$

Overall, we have $T_I - T_O \leq O(\mathrm{cost}^{(m)}(P, c)/m)$, which completes the proof. □

Combining Lemmas 4.5 and 4.6, we obtain the bound: $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| \leq O(1)\cdot(\Delta + |m_P - m_S|)\cdot \frac{\mathrm{cost}^{(m)}(P,c)}{m}$. To prove $|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)| \leq O(\varepsilon)\cdot \mathrm{cost}^{(m)}(P,c)$, it remains to ensure that $\Delta = O(\varepsilon m)$ and $|m_P - m_S| = O(\varepsilon m)$. We show that both conditions hold when S_O is an ε-approximation for the ball range space on $L^\star$.

Lemma 4.7 (Bounding misaligned outlier count)**.** *Suppose S_O is an ε-approximation for the ball range space on $L^\star$, we have $|m_P - m_S| \leq 2\cdot\varepsilon\cdot m$.*

Before proving this lemma, we first analyze the properties of w'. Given a point $p \in S_O \cup P_I^\star$, we say p is of partial weight w.r.t. c if $0 < w'(p) < w(p)$ when $p \in S_O$ and $0 < w'(p) < 1$ when $p \in P_I^\star$. Intuitively, such a point is partially an inlier and partially an outlier of $S_O \cup P_I^\star$ to c. The following claim indicates that the number of partial-weight points is at most one for any c.

Claim 4.8 (Properties of partial-weight points)**.** *For every center $c \subset \mathbb{R}^d$, there exists at most one point $v \in S_O \cup P_I^\star$ of partial weight w.r.t. c. Moreover, for any other point $p \in S_O \cup P_I^\star$ with $w'(p) = w(p)$, we have $\mathrm{dist}(p,c) \leq \mathrm{dist}(v,c)$.*

Proof. By contradiction, we assume that there exist two points $v, v' \in S_O \cup P_I^\star$ of partial weight w.r.t. c. W.l.o.g., suppose $\mathrm{dist}(v,c) \leq \mathrm{dist}(v',c)$. We note that transferring weight from v to v' increases $w'(v)$ by any constant $\delta > 0$ and decreases $w'(v')$ by δ does not increase the term $\mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)$. Then by selecting a suitable δ, we can make either v or v' no longer of partial weight. Hence, there exists at most one point $v \in S_O \cup P_I^\star$ of partial weight w.r.t. c.

For any other point $p \in S_O \cup P_I^\star$ with $w'(p) = w(p)$, suppose $\mathrm{dist}(v,c) < \mathrm{dist}(p,c)$. In this case, increasing $w'(v)$ by a small amount $\delta > 0$ and decreasing $w'(p)$ by δ decreases $\mathrm{cost}^{(m)}(S_O \cup P_I^\star, c)$. This contradicts the definition of w', which completes the proof. □

Recall that $d_1, \ldots, d_m$ represent the distances from the m furthest points in $P_I^\star$ to the center c. Now we are ready to prove Lemma 4.7.

Proof of Lemma 4.7. We only need to consider the case of $m_P > m_S$ and $m_P < m_S$. Without loss of generality, we assume that $m_P > m_S$. Let $l_1 := m - m_P + 1$. Based on the definition of $d_1, ..., d_m$, each inlier point p satisfies $\mathrm{dist}(p,c) \leq d_{l_1 - 1}$. There are $m - m_P$ inlier points in $L^\star$, so we have:

$$|L^\star \cap \mathrm{Ball}(c, d_{l_1})| \leq m - m_P. \tag{32}$$

Let $l_2 := m - \lfloor m_S \rfloor + 1$. Since $m_P > m_S$, we have $m_P \geq \lfloor m_S \rfloor + 1$, then we get the inequality

$$l_2 - 1 \geq l_1. \tag{33}$$

Moreover, we claim that

$$m - w(S_O \cap \mathrm{Ball}(c, d_{l_2 - 1})) \leq \lfloor m_S \rfloor + 1. \tag{34}$$

By Claim 4.8, at most one point $v \in S_O \cup P_I^\star$ of partial weight w.r.t. c exists. If $v \in P_I^\star$, we have $\mathrm{dist}(v,c) = d_{l_2}$. Any point $v' \in P_I^\star$ with $\mathrm{dist}(v',c) \geq d_{l_2 - 1}$ is an outlier of $S_O \cup P_I^\star$. Then there are $m - \lfloor m_S \rfloor - 1$ points in $P_I^\star$ that have distance to C at least d_{l_2-1}. By contradiction, we assume that $m - w(S_O \cap \mathrm{Balls}(c, d_{l_2-1})) > \lfloor m_S \rfloor + 1$. Then there are no less than m points in P that have a distance to c greater than d_{l_2-1}, which contradicts the fact that v' is an outlier of $S_O \cup P_I^\star$. Hence, Inequality (34) holds. For the case that $v \in S_O$ or there is no point of partial weight w.r.t. c, the argument is similar.

Since $m_P > m_S$, we have $w(S_O \cap \text{Ball}(c, d_{l_1})) > m - m_P$. Combining with Inequality (32), we conclude that

$$\begin{aligned} |L^\star \cap \text{Ball}(c, d_{l_1})| \leq & \quad m - m_P \\ < & \quad w(S_O \cap \text{Ball}(c, d_{l_1})). \end{aligned} \tag{35}$$

Moreover,

$$\begin{aligned} w(S_O \cap \text{Ball}(c, d_{l_1})) \geq & \quad w(S_O \cap \text{Ball}(c, d_{l_2 - 1})) \\ > & \quad m - \lfloor m_S \rfloor - 1. \end{aligned} \tag{36}$$

Then, we have

$$\begin{aligned} & ||L^\star \cap \text{Ball}(c, d_{l_1})| - w(S_O \cap \text{Ball}(c, d_{l_1}))| \\ = & \quad w(S_O \cap \text{Ball}(c, d_{l_1})) - |L^\star \cap \text{Ball}(c, d_{l_1})| && \text{(Inequality (35))} \\ \geq & \quad m_P - \lfloor m_S \rfloor - 1 && \text{(Inequality (32) and (36))} \\ \geq & \quad m_P - m_S - 1. \end{aligned}$$

Since S_O is an ε-approximation for the ball range space on $L^\star$, by Lemma 4.3, we have

$$||L^\star \cap \text{Ball}(c, d_{l_1})| - w(S_O \cap \text{Ball}(c, d_{l_1}))| \leq \varepsilon \cdot m.$$

Combining the above two inequalities, we have

$$|m_P - m_S| \leq \varepsilon \cdot m + 1 \leq 2 \cdot \varepsilon \cdot m.$$

Similarly, we can get the same conclusion when $m_P < m_S$, which completes the proof of Lemma 4.7. ☐

Now we are ready to prove Lemma 4.2.

Proof of Lemma 4.2. Based on Lemmas 4.4 and 4.5, we have

$$\begin{aligned} & |\text{cost}^{(m)}(P, c) - \text{cost}^{(m)}(P_I^\star \cup S_O, c)| \\ \leq & \quad 2 \cdot (\Delta + |m_P - m_S|) \cdot (T_I - T_O) \\ \leq & \quad O(1) \cdot (\Delta + |m_P - m_S|) \cdot \text{cost}^{(m)}(P, c)/m && \text{(Lemma 4.6)} \\ \leq & \quad O(1) \cdot (\varepsilon m + |m_P - m_S|) \cdot \text{cost}^{(m)}(P, c)/m && \text{(Lemma 4.9)} \\ \leq & \quad O(\varepsilon) \cdot \text{cost}^{(m)}(P, c), && \text{(Lemma 4.3)} \end{aligned}$$

which completes the proof. ☐

4.2 Extension to other metric spaces

In this section, we explore the extension of Algorithm 2, designed for the robust geometric median in Euclidean space, to the metric spaces by leveraging the notions of VC dimension and doubling dimension since the VC dimension is known for various metric spaces [8, 9, 18, 40].

Let $(\mathcal{X}, \text{dist})$ denote the metric space, where $\mathcal{X}$ is the set under consideration, and $\text{dist} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a function that measures the distance between points in $\mathcal{X}$, satisfying the triangle inequality. Specifically, in the Euclidean metric, $\mathcal{X}$ represents Euclidean space $\mathbb{R}^d$ and dist is the Euclidean distance.

Similar to the analysis for robust geometric median on Euclidean space, we discuss the induced error of S_I and S_O separately. We first introduce how to bound the error induced by S_O. Since the dist function satisfies the triangle inequality, our Lemmas 4.4-4.5 hold, ensuring that each point in S_O induces at most $O(\mathrm{cost}^{(m)}(P,c)/m)$ error. To ensure the number of points in S_O inducing error is $O(\varepsilon m)$, it suffices for S_O to be an ε-approximation for the ball range space on $L^\star$ (as shown in Lemma 4.7). Next, we illustrate how this condition can be satisfied using the notions: 1) VC dimension; 2) doubling dimension.

VC dimension. We begin by introducing the concept of the VC dimension of the ball range space, which serves as a measure of the complexity of this ball range space.

Definition 4.2 (VC dimension of ball range space)**.** *Let $M = (\mathcal{X}, \mathrm{dist})$ be the metric space and define $\mathrm{Balls}(\mathcal{X}) := \{\mathrm{Ball}(c,u) \mid c \in \mathcal{X}, u > 0\}$ as the collection of balls in the space. The VC dimension of the ball range space $(\mathcal{X}, \mathrm{Balls}(\mathcal{X}))$, denoted by $d_{VC}(\mathcal{X})$, is the maximum $|P|$, $P \subseteq \mathcal{X}$ such that $|P \cap \mathrm{Balls}(\mathcal{X})| = 2^{|P|}$, where $P \cap \mathrm{Balls}(\mathcal{X}) := \{P \cap \mathrm{Ball}(c,u) \mid \mathrm{Ball}(c,u) \in \mathrm{Balls}(\mathcal{X})\}$.*

The VC dimension of $(\mathcal{X}, \mathrm{Balls}(\mathcal{X}))$ aligns to the pseudo-dimension of $(\mathcal{X}, \mathrm{Balls}(\mathcal{X}))$ used by [45]. Based on this observation, we give out a refined lemma from [45].

Lemma 4.9 (Refined from [45])**.** *Let $M = (\mathcal{X}, \mathrm{dist})$ be the metric space with VC dimension $d_{VC}(\mathcal{X})$. Given dataset $P_O \subseteq \mathcal{X}$, assume S_O is a uniform sample of size $\tilde{O}(\frac{d_{VC}(\mathcal{X})}{\varepsilon^2})$ from P_O, then with probability $1 - 1/poly(1/\varepsilon)$, S_O is an ε-approximation of the ball range space on P_O.*

The Lemma 4.3, used earlier for the Euclidean space case, is a direct corollary, since in Euclidean space $\mathbb{R}^d$, we have $d_{VC}(\mathcal{X}) = O(d)$. This lemma illustrates the number of samples required from $L^\star$.

Doubling dimension. Another notion to measure the complexity of the ball range space is the doubling dimension.

Definition 4.3 (Doubling dimension [31, 3])**.** *The doubling dimension $\mathrm{ddim}(\mathcal{X})$ of a metric space $(\mathcal{X}, \mathrm{dist})$ is the least integer t such that for any $\mathrm{Ball}(c,u)$ with $c \in \mathcal{X}$, $u > 0$, it can be covered by 2^t balls of radius $u/2$.*

We denote the metric space with bounded doubling dimension as **doubling metric**. Based on [38], it is known that the VC dimension of a doubling metric $(\mathcal{X}, \mathrm{dist})$ may not be bounded. Therefore, the previous Lemma 4.9 does not apply for an effective bound. We want to directly relate the ball range space bound to the doubling dimension. This goal can be achieved by the following refined lemma.

Lemma 4.10 (Ball range space approximation in doubling metrics [38])**.** *Let $M = (\mathcal{X}, \mathrm{dist})$ be the metric space with doubling dimension $\mathrm{ddim}(\mathcal{X})$. Given $P_O \subseteq \mathcal{X}$, assume S_O is a uniform sample of size $\tilde{O}(\mathrm{ddim}(\mathcal{X})\varepsilon^{-2})$ from P_O, then with probability $1 - 1/poly(1/\varepsilon)$, S_O is an ε-approximation of the ball range space on P_O.*

Proof. Suppose the distorted distance function $\mathrm{dist}' : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ satisfies: for any $x, y \in \mathcal{X}$, we have $(1-\varepsilon)\mathrm{dist}(x,y) \leq \mathrm{dist}'(x,y) \leq (1+\varepsilon)\mathrm{dist}(x,y)$. By Lemma 3.1 of [38], we know that with probability $1 - poly(1/\varepsilon)$, S_O is an ε-ball range space approximation of P_O w.r.t. dist', when $S_O = \tilde{O}(\mathrm{ddim}(\mathcal{X})\varepsilon^{-2})$. By setting of dist', we know S_O is an $O(\varepsilon)$-ball range space approximation of P_O w.r.t. dist, which completes the proof. □

This lemma illustrates the number of points needed to be sampled in order to bound the induced error of S_O when the metric space is a doubling metric.

The remaining problem is to bound the induced error of S_I. We give out a generalized version of Theorem 4.1 as below.

Theorem 4.11 (Refined from Corollary 5.4 in [40]). *Let $M = (\mathcal{X}, \mathrm{dist})$ be the metric space. There exists a randomized algorithm that in $O(n)$ time constructs a weighted subset $S_I \subseteq P_I^\star$ of size:*

- $\tilde{O}(d_{VC}(\mathcal{X}) \cdot \varepsilon^{-4})$*, w.r.t. the VC dimension* $d_{VC}(\mathcal{X})$*,*
- $\tilde{O}(\mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$*, w.r.t. the doubling dimension* $\mathrm{ddim}(\mathcal{X})$*,*

such that for every dataset P_O of size m, every integer $0 \le t \le m$ and every center $c \in \mathbb{R}^d$, $|\mathrm{cost}^{(t)}(P_O \cup P_I^\star, c) - \mathrm{cost}^{(t)}(P_O \cup S_I, c)| \le \varepsilon \cdot \mathrm{cost}^{(t)}(P_O \cup P_I^\star, c) + 2\varepsilon \cdot \mathrm{cost}(P_I^\star, c^\star)$.

By combining the discussions of the errors induced by S_O and S_I, we present the main theorem for constructing a coreset for the robust geometric median across various metric spaces. We study the shortest-path metric $(\mathcal{X}, \mathrm{dist})$, where $\mathcal{X}$ is the vertex set of a graph $G = (V, E)$, and $\mathrm{dist}(\cdot, \cdot)$ measures the shortest distance between two points in the graph. The treewidth of a graph measures how "tree-like" the graph is; see Definition 2.1 of [4] for formal definition. A graph excluding a fixed minor is one that does not contain a particular substructure, known as the minor.

Theorem 4.12 (Coreset size for robust geometric median in various metric spaces). *Let $\varepsilon \in (0, 1)$. For a metric space $M = (\mathcal{X}, \mathrm{dist})$ and a dataset $X \subset \mathcal{X}$ of size $n \ge 4m$, let $S = S_O \cup S_I$ be a sampled set of size*

- $\tilde{O}(\log(|\mathcal{X}|) \cdot \varepsilon^{-4})$ *if M is general metric space.*
- $\tilde{O}(\mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$ *if M is a doubling metric with doubling dimension* $\mathrm{ddim}(\mathcal{X})$*.*
- $\tilde{O}(t \cdot \varepsilon^{-4})$ *if M is a shortest-path metric of a graph with bounded treewidth t.*
- $\tilde{O}(|H| \cdot \varepsilon^{-4})$ *if M is a shortest-path metric of a graph that excludes a fixed minor H.*

Then S is an ε-coreset for robust geometric median on X.

This theorem illustrates the coreset size with respect to different metric spaces, improving upon previous results for robust coresets by eliminating the $O(m)$ dependency.

Proof of Theorem 4.12. If $M = (\mathcal{X}, \mathrm{dist})$ is a general metric space, then $d_{VC}(\mathcal{X}) = O(\log |\mathcal{X}|)$. Thus, we have $|S_O| = \tilde{O}(\log(|\mathcal{X}|) \cdot \varepsilon^{-2})$ by Lemma 4.9 and $|S_I| = \tilde{O}(\log(|\mathcal{X}|) \cdot \varepsilon^{-4})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(\log(|\mathcal{X}|) \cdot \varepsilon^{-4})$.

If $M = (\mathcal{X}, \mathrm{dist})$ is a doubling metric space, then by definition, $\mathrm{ddim}(\mathcal{X})$ is bounded. Thus we have $|S_O| = \tilde{O}(\mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$ by Lemma 4.10 and $|S_I| = \tilde{O}(\mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(\mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$.

If $M = (\mathcal{X}, \mathrm{dist})$ is a shortest-path metric of a graph with bounded treewidth t, we have $d_{VC}(\mathcal{X}) = O(t)$ by [8]. Thus, we have $|S_O| = \tilde{O}(t \cdot \varepsilon^{-2})$ by Lemma 4.9 and $|S_I| = \tilde{O}(t \cdot \varepsilon^{-4})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(t \cdot \varepsilon^{-4})$.

If $M = (\mathcal{X}, \mathrm{dist})$ is a shortest-path metric of a graph that excludes a fixed minor H, we have $d_{VC}(\mathcal{X}) = O(|H|)$ by [8]. Thus, we have $|S_O| = \tilde{O}(|H| \cdot \varepsilon^{-2})$ by Lemma 4.9 and $|S_I| = \tilde{O}(|H| \cdot \varepsilon^{-4})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(|H| \cdot \varepsilon^{-4})$.

$\square$

5 Improved coreset sizes for robust (k, z)-clustering

In this section, we provide a coreset construction for robust (k, z)-clustering when $d \geq 1$. We first extend the definition of the robust geometric median and its coreset to the robust (k, z)-clustering and its corresponding coreset.

Definition 5.1 (Robust (k, z)-clustering). *Given a dataset $P \subset \mathbb{R}^d$ of size $n \geq 1$ and an integer $m \geq 0$, the robust (k, z)-clustering problem is to find a center set $C \subset \mathbb{R}^d$, $|C| = k$ that minimizes the objective function below:*

$$\mathrm{cost}_z^{(m)}(P, C) := \min_{L \subset P : |L| = m} \sum_{p \in P \setminus L} (\mathrm{dist}(p, C))^z,$$

where L represents the set of m outliers w.r.t. C and $\mathrm{dist}(p, C) = \min_{c \in C} \mathrm{dist}(p, c)$ denotes the Euclidean distance from p to the closest center among C.

Before we define the coreset for robust (k, z)-clustering, we introduce the generalized cost function in the context of the robust (k, z)-clustering.

Definition 5.2 (Generalized cost function for robust (k, z)-clustering). *Let m be an integer. Let $S \subseteq \mathbb{R}^d$ be a weighted dataset with weights $w(p)$ for each point $p \in S$. Let $w(S) := \sum_{p \in S} w(p)$. Define a collection of weight functions $\mathcal{W} := \{w' : S \to \mathbb{R}^+ \mid \sum_{p \in S} w'(p) = w(S) - m \wedge \forall p \in S, w'(p) \leq w(p)\}$. Moreover, we define the following cost function on S:*

$$\forall C \subset \mathbb{R}^d, |C| = k \quad \mathrm{cost}_z^{(m)}(S, C) := \min_{w' \in \mathcal{W}} \sum_{p \in S} w'(p) \cdot (\mathrm{dist}(p, C))^z.$$

With this definition of the generalized cost function, we now define the notion of a coreset for robust (k, z)-clustering.

Definition 5.3 (Coreset for robust (k, z)-clustering). *Given a point set $P \subset \mathbb{R}^d$ of size $n \geq 1$, integer $m \geq 1$ and $\varepsilon \in (0, 1)$, we say a weighted subset $S \subseteq P$ together with a weight function $w : S \to \mathbb{R}^+$ is an ε-coreset of P for robust (k, z)-clustering if $w(S) = n$ and for any center set $C \subset \mathbb{R}^d$, $|C| = k$, $\mathrm{cost}_z^{(m)}(S, C) \in (1 \pm \varepsilon) \cdot \mathrm{cost}_z^{(m)}(P, C)$.*

Finally, we extend the concepts of inliers, outliers, and the ball range space.

Inliers and outliers. Throughout this section, we denote the approximate center set for robust (k, z)-clustering by $C^\star = \{c_1^\star, \ldots, c_k^\star\} \subset \mathbb{R}^d$, which is an $O(1)$-approximation of the optimal solution. We then define the inliers and outliers with respect to this approximation center set.

Let $L^\star := \arg\min_{L \subset P, |L| = m} \mathrm{cost}_z(P - L, C^\star)$ denote the set of m outliers with respect to $C^\star$, and define $P_I^\star := P \setminus L^\star$ as the corresponding inlier set. The inlier set $P_I^\star$ is naturally partitioned by $C^\star$ into k clusters $\{P_1^\star, \ldots, P_k^\star\}$, where each cluster $P_i^\star$ contains the points in $P_I^\star$ closest to its corresponding center $c_i^\star$. Additionally, define $r_{\max} := \max_{p \in P_I^\star} \mathrm{dist}(p, C^\star)$ which represents the maximum distance from the points in $P_I^\star$ to this center set $C^\star$. Let $\bar{r} := \sqrt[z]{\frac{\mathrm{cost}_z(P_I^\star, C^\star)}{n - m}}$ denote the average distance from $P_I^\star$ to $C^\star$. Under these notations, the second condition of Assumption 1.4 can be rewritten as $(r_{\max})^z \leq 4k\,(\bar{r})^z$.

Ball range space. We introduce the concept of the k-balls range space, which is a direct extension of the ball range space defined in Definition 4.1.

Definition 5.4 (Approximation of k-balls range space, Definition F.2 in [39])**.** *Let* $\mathrm{Balls}(C,u) := \cup_{c\in C}\mathrm{Ball}(c,u)$*. For a given dataset* $P \subset \mathbb{R}^d$*, the* k*-balls range space on* P *is* $(P,\mathcal{P})$ *where* $\mathcal{P} := \{P \cap \mathrm{Balls}(C,u) \mid C \subset \mathbb{R}^d, |C| = k, u \in \mathbb{R}^+\}$*. A subset* $Y \subset P$ *is called an* ε*-approximation of the* k*-balls range space* $(P,\mathcal{P})$ *if for every* $C \subset \mathbb{R}^d$*,* $|C| = k$*,* $u \in \mathbb{R}^+$*,*

$$|\frac{|P\cap \mathrm{Balls}(C,u)|}{|P|} - \frac{|Y\cap \mathrm{Balls}(C,u)|}{|Y|}| \le \varepsilon.$$

Based on this definition, we have the following lemma that measures the performance of S_O for robust (k,z)-clustering.

Lemma 5.1 (Refined from Lemma F.3 of [39])**.** *Given dataset* $P_O \subset \mathbb{R}^d$*. Let* S_O *be a uniform sampling of size* $\tilde{O}(\frac{kd}{\varepsilon^2})$ *from* P_O*, then with probability at least* $1 - \frac{1}{poly(k/\varepsilon)}$*,* S_O *is an* ε*-approximation of the* k*-balls range space on* P_O*. Define a weight function* w*:* $w(p) = \frac{|P_O|}{|S_O|}$*, for any* $p \in S_O$*. Then for any* $C \subset \mathbb{R}^d$*,* $|C| = k$*,* $u \in \mathbb{R}^+$*,*

$$||P_O \cap \mathrm{Balls}(C,u)| - w(S_O \cap \mathrm{Balls}(C,u))| \le \varepsilon |P_O|.$$

Then we are ready to give out our main result.

5.1 Result for robust (k,z)-clustering

We first recall the following theorem.

Theorem 5.2 (Restatement of Corollary 5.4 in [40])**.** *There exists a randomized algorithm* $\mathcal{A}_{kd}$ *that in* $O(nkd)$ *time constructs a weighted subset* $S_I \subseteq P_I^\star$ *of size* $\tilde{O}(k^2\varepsilon^{-2z}\min\{\varepsilon^{-2},d\})$*, such that for every dataset* P_O *of size* m*, every integer* $0 \le t \le m$ *and every center set* $C \subset \mathbb{R}^d$*,* $|C| = k$*,* $|\mathrm{cost}_z^{(t)}(P_O \cup P_I^\star, C) - \mathrm{cost}_z^{(t)}(P_O \cup S_I, C)| \le \varepsilon \cdot \mathrm{cost}_z^{(t)}(P_O \cup P_I^\star, C) + 2\varepsilon \cdot \mathrm{cost}_z(P_I^\star, C^\star)$*.*

This theorem is a generalization of Theorem 4.1, describing the number of points that need to be sampled from $P_I^\star$ for robust (k,z)-clustering.

To adapt Algorithm 2 for Theorem 1.5, we only need to adjust the size of S_O to $\tilde{O}(k\varepsilon^{-2z}\min\{\varepsilon^{-2},d\})$ in Line 2 and modify the algorithm $\mathcal{A}_d$ to $\mathcal{A}_{kd}$ in Line 3 (see Algorithm 3). Consequently, the runtime remains dominated by Line 1 and Line 3, resulting in an overall complexity of $O(ndk)$ according to Theorem 5.2.

Algorithm 3 Coreset Construction for General d and General k

Input: A dataset $P \subset \mathbb{R}^d$, $\varepsilon \in (0,1)$ and an $O(1)$-approximate center set $C^\star \subset \mathbb{R}^d$
Output: An ε-coreset S

1: $L^\star \leftarrow \arg\min_{L\subset P, |L|=m} \mathrm{cost}_z(P - L, C^\star)$, $P_I^\star \leftarrow P - L^\star$
2: Uniformly sample $S_O \subseteq L^\star$ of size $\tilde{O}(k\varepsilon^{-2z}\min\{\varepsilon^{-2},d\})$. Set $\forall p \in S_O$, $w_O(p) \leftarrow \frac{m}{|S_O|}$.
3: Construct $(S_I, w_I) \leftarrow \mathcal{A}_{kd}(P_I^\star)$ by Theorem 5.2.
4: For any $p \in S_O$, define $w(p) = w_O(p)$ and for any $p \in S_I$, define $w(p) = w_I(p)$.
5: Return $S \leftarrow S_O \cup S_I$ and w;

Similar to the robust geometric median, we present the key lemma used to prove Theorem 1.5 below.

Lemma 5.3 (Induced error of S_O)**.** *For any center set $C \subset \mathbb{R}^d$, $|C| = k$, we have $|\mathrm{cost}_z^{(m)}(P, C) - \mathrm{cost}_z^{(m)}(S_O \cup P_I^\star, C)| \leq O(\varepsilon) \cdot \mathrm{cost}_z^{(m)}(P, C)$.*

Theorem 1.5 follows as a corollary of Theorem 5.2 and this lemma, with a proof similar to that of Theorem 1.3.

Remark 5.4. *The second condition in Assumption 1.4 can be replaced with $\mathrm{dist}(c_i^*, c_j^*)^z \geq \frac{m \cdot r_{\max}^z}{\min\{|P_i^*|, |P_j^*|\}}$ for any $c_i^*, c_j^* \in C^*$, $i \neq j$. Under this modified assumption, Theorem 1.5 becomes a direct generalization of Theorem 1.3, as the second condition vanishes when $k = 1$. This modification ensures that clusters are sufficiently well-separated, which explains why our algorithm performs well on the* ***Bank*** *dataset when $k = 3$, as this modified assumption is satisfied in this case.*

Below we illustrate why this assumption also works. If points in each cluster are mostly assigned to distinct centers, then $\mathrm{cost}_z^{(m)}(P, C) \approx \mathrm{cost}_z^{(m)}(P, C^\star)$, making the error $|\mathrm{cost}_z^{(m)}(P, C) - \mathrm{cost}_z^{(m)}(P_I^\star \cup S_O, C)|$ easy to bound. Alternatively, if two clusters each have at least half of their points assigned to the same center, then by the modified assumption, $\mathrm{cost}_z^{(m)}(P, C) > m r_{\max}^z$, which is large enough to bound the error induced by S_O.

5.2 Proof of Lemma 5.3: Induced error of S_O

In this section, we fix a center set $C \subset \mathbb{R}^d$ of size k and define $T_O := \min_{p \in L^\star} (\mathrm{dist}(p, C))^z$, $T_I := \max_{p \in P_I^\star} (\mathrm{dist}(p, C))^z$. We observe that Lemmas 4.4 and 4.5 can be directly extended to robust (k, z)-clustering. Therefore, the induced error of each point is at most $T_I - T_O$. Next, we show that $T_I - T_O \leq O(\mathrm{cost}_z^{(m)}(P, c)/m)$, supported by a detailed discussion of the sizes of T_I and T_O (see Lemma 5.5). The remaining task is to ensure that the number of points that may induce error is $O(\varepsilon m)$. The result of Lemma 4.7 can be extended to robust (k, z)-clustering when S_O is an ε-approximation of the k-balls range space on $L^\star$. By Lemma 5.1, we sample $\tilde{O}(k \varepsilon^{-2z} \min\{\varepsilon^{-2}, d\})$ points from $L^\star$ to ensure the approximation.

We then demonstrate how to prove that $T_I - T_O \leq O(\mathrm{cost}_z^{(m)}(P, c)/m)$.

Lemma 5.5 (Bounds for T_I and T_O)**.** *When $m_P < m$, we have $T_I - T_O \leq O(\mathrm{cost}_z^{(m)}(P, c)/m)$*

Proof. Let $d_{\max} := \max_{c^\star \in C^\star} \mathrm{dist}(c^\star, C)$ denote the maximum distance of points in $C^\star$ to C. Let $d'_{\max} := \max_{c \in C} \mathrm{dist}(c, C^\star)$ denote the maximum distance of points in C to $C^\star$.

We first claim that $T_I \leq (r_{\max} + d_{\max})^z$. Let $\widehat{p}$ be defined as $\max_{p \in P_I^\star} \mathrm{dist}(p, C)$. Denote $c_p^\star$ as the closest approximate center to $\widehat{p}$, where $c_p^\star := \arg\min_{c^\star \in C^\star} \mathrm{dist}(\widehat{p}, c^\star)$, and c_p as the closest center to $c_p^\star$, where $c_p := \arg\min_{c \in C} \mathrm{dist}(c_p^\star, c)$. This setup allows us to establish an upper bound for T_I:

$$\begin{aligned}
\mathrm{dist}(\widehat{p}, C) \leq &\quad \mathrm{dist}(\widehat{p}, c_p) \\
\leq &\quad \mathrm{dist}(\widehat{p}, c_p^\star) + \mathrm{dist}(c_p^\star, c_p) && \text{(Triangle Inequality)} \\
\leq &\quad r_{\max} + d_{\max},
\end{aligned}$$

thus, $T_I \leq (r_{\max} + d_{\max})^z$.

Next, we discuss the relationship between C and $C^\star$ in two cases: 1) $r_{\max} \leq d_{\max}$; 2) $r_{\max} > d_{\max}$.

Case 1: $d_{\max} \geq r_{\max}$ In this case, we have $T_I - T_O \leq 2^z d_{\max}^z$ since $T_O \geq 0$. Therefore, it suffices to prove $2^z d_{\max}^z \leq O(\mathrm{cost}_z^{(m)}(P, C)/m)$. Suppose $\mathrm{dist}(c_i^\star, C) = d_{\max}$, let $(\bar{r}_i)^z := \mathrm{cost}_z(P_i^\star, C^\star)/|P_i^\star|$. Based on the scale of $d_{\max}$, we discuss in two cases.

Case 1.1: $d_{\max} \leq 8\bar{r_i}$ By definition of $\bar{r_i}$, we have

$$\begin{aligned} 16^z \cdot \mathrm{cost}_z^{(m)}(P_i^\star, C^\star) = & \quad 16^z \cdot |P_i^\star| \cdot (\bar{r_i})^z \\ \geq & \quad 2^z \cdot |P_i^\star| \cdot d_{\max}^z \\ \geq & \quad 2^z \cdot m \cdot d_{\max}^z \qquad \text{(Item 1 of Assumption 1.4).} \end{aligned}$$

Since $C^\star$ is an $O(1)$-approximate solution for robust (k,z)-clustering, we have

$$2^z \cdot (d_{\max})^z \leq 16^z \cdot \mathrm{cost}_z^{(m)}(P, C^\star)/m \leq O(\mathrm{cost}_z^{(m)}(P,C)/m).$$

Case 1.2: $d_{\max} > 8\bar{r_i}$ Let $\widehat{P_i} := \{p \in P_i^\star | \mathrm{dist}(p, C^\star) \leq 4\bar{r_i}\}$. By definition, we have $|\widehat{P_i}| \geq \frac{3}{4}|P_i^\star|$. For any point $p \in \widehat{p_i}$, suppose $c_p := \arg\min_{c\in C} \mathrm{dist}(p,c)$, then we have

$$\begin{aligned} \mathrm{dist}(p,c) \geq & \quad \mathrm{dist}(c_i^\star, c_p) - \mathrm{dist}(p, c_i^\star) \quad \text{(Triangle Inequality)} \\ \geq & \quad \mathrm{dist}(c_i^\star, C) - \mathrm{dist}(p, c_i^\star) \\ = & \quad d_{\max} - \mathrm{dist}(p, c_i^\star) \\ \geq & \quad d_{\max} - 4\bar{r_i} \qquad \text{(Definition of } \widehat{P_i}\text{)} \end{aligned} \tag{37}$$

Since $d_{\max} > 8\bar{r_i}$, we obtain

$$\begin{aligned} 8^z \cdot \mathrm{cost}_z^{(m)}(P,c) \geq & \quad 8^z \cdot (\frac{3}{4}|P_i^\star| - m) \min_{p \in \widehat{P_i}} (\mathrm{dist}(p,C))^z \\ \geq & \quad 8^z \cdot (\frac{3}{4}|P_i^\star| - m)(d_{\max} - 4\bar{r_i})^z \qquad \text{(Inequality (37))} \\ \geq & \quad 4^z \cdot (\frac{3}{4}|P_i^\star| - m) d_{\max}^z \\ \geq & \quad 2^z \cdot m \cdot d_{\max} \qquad \text{(Item 1 of Assumption 1.4).} \end{aligned}$$

Overall, we have $T_I - T_O \leq O(\mathrm{cost}_z^{(m)}(P,C)/m)$.

Case 2: $d_{\max} < r_{\max}$ We have $T_I - T_O < 2^z \cdot r_{\max}^z$ since $T_O \geq 0$. By Item 2 of Assumption 1.4, we have $m \cdot r_{\max}^z \leq n \cdot \bar{r}^z$, thus $T_I - T_O \leq 2^z \cdot \mathrm{cost}_z^{(m)}(P,C)/m$.

Combine Case 1 and Case 2, we complete the proof.

□

5.3 Extension to other metric spaces

In this section, we explore the extension of Algorithm 3 for robust (k,z)-clustering in Euclidean space, to various other metric spaces.

Similar to the analysis for robust geometric median, we discuss the induced error of S_I and S_O separately. For the error induced by S_O, Lemma 5.5 still holds, so it suffices for S_O to be an ε-approximation of the k-balls range space.

VC dimension We begin by introducing the concept of the VC dimension of the k-balls range space.

Definition 5.5 (VC dimension of k-balls range space). *Let $(\mathcal{X}, \mathrm{dist})$ be the metric space and define $\mathrm{Balls}_k(\mathcal{X}) := \{\mathrm{Balls}(C,u) \mid C \subset \mathcal{X}, |C| = k, u > 0\}$. The VC dimension of the k-balls range space $(\mathcal{X}, \mathrm{Balls}_k(\mathcal{X}))$, denoted by $d_{VC}(\mathcal{X})$, is the maximum $|P|$, $P \subseteq \mathcal{X}$ such that $|P \cap \mathrm{Balls}_k(\mathcal{X})| = 2^{|P|}$, where $P \cap \mathrm{Balls}_k(\mathcal{X}) := \{P \cap \mathrm{Balls}(C,u) \mid \mathrm{Balls}(C,u) \in \mathrm{Balls}_k(\mathcal{X})\}$.*

The VC dimension of $(\mathcal{X}, \mathrm{Balls}_k(\mathcal{X}))$ aligns to the pseudo-dimension of $(\mathcal{X}, \mathrm{Balls}_k(\mathcal{X}))$ used by [45]. Based on this observation, we give out a refined lemma from [45].

Lemma 5.6 (Refined from [45])**.** *Let $(\mathcal{X}, \mathrm{dist})$ be the metric space. Given dataset $P_O \subseteq \mathcal{X}$, assume S_O is a uniform sample of size $\tilde{O}(\frac{k \cdot d_{VC}(\mathcal{X})}{\varepsilon^2})$ from P_O, then with probability $1 - 1/poly(k/\varepsilon)$, S_O is a ε-approximation of the k-balls range space on P_O.*

This lemma illustrates the number of points needed to be sampled from $L^\star$.

Doubling dimension Similar to the Lemma 4.10, we give out the Lemma 5.7 that illustrates the number of points needed to be sampled in order to bound the induced error of S_O .

Lemma 5.7 (Balls range space approximation for the doubling dimension)**.** *Let $M = (\mathcal{X}, \mathrm{dist})$ be the metric space with doubling dimension $\mathrm{ddim}(\mathcal{X})$. Given $P_O \subseteq \mathcal{X}$, assume S_O is a uniform sample of size $\tilde{O}(\mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2z} \cdot k)$ from P_O, then with probability $1 - 1/poly(k, 1/\varepsilon)$, S_O is an ε-approximation of the k-balls range space on P_O.*

The remaining problem is to bound the induced error of S_I. We give out a generalized version of Theorem 4.11 as below.

Theorem 5.8 (Refined from Corollary 5.4 in [40])**.** *Let $(\mathcal{X}, \mathrm{dist})$ be the metric space. There exists a randomized algorithm that in $O(nk)$ time constructs a weighted subset $S_I \subseteq P_I^\star$ of size:*

- *$\tilde{O}(k^2 \cdot d_{VC}(\mathcal{X}) \cdot \varepsilon^{-2z-2})$, w.r.t. VC dimension $d_{VC}(\mathcal{X})$,*
- *$\tilde{O}(k^2 \cdot \mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2z})$, w.r.t. doubling dimension $\mathrm{ddim}(\mathcal{X})$,*

such that for every dataset P_O of size m, every integer $0 \le t \le m$ and every center set $C \subset \mathbb{R}^d$, $|C| = k$, $|\mathrm{cost}_z^{(t)}(P_O \cup P_I^\star, C) - \mathrm{cost}_z^{(t)}(P_O \cup S_I, C)| \le \varepsilon \cdot \mathrm{cost}_z^{(t)}(P_O \cup P_I^\star, C) + 2\varepsilon \cdot \mathrm{cost}_z(P_I^\star, C^\star)$.

By combining the discussions of the errors induced by S_O and S_I, we present the main theorem for constructing a coreset for the robust (k, z)-clustering across various metric spaces.

Theorem 5.9 (Coreset size for robust (k, z)-clustering in various metric spaces)**.** *Let $\varepsilon \in (0, 1)$. For a metric space $M = (\mathcal{X}, \mathrm{dist})$ and a dataset $X \subset \mathcal{X}$ satisfying Assumption 1.4, let $S = S_O \cup S_I$ be a sampled set of size*

- *$\tilde{O}(k^2 \cdot \log(|\mathcal{X}|) \cdot \varepsilon^{-2z-2})$ if M is general metric space.*
- *$\tilde{O}(k^2 \cdot \mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2z})$ if M is a doubling metric with doubling dimension $\mathrm{ddim}(\mathcal{X})$.*
- *$\tilde{O}(k^2 \cdot t \cdot \varepsilon^{-2z-2})$ if M is a shortest-path metric of a graph with bounded treewidth t.*
- *$\tilde{O}(k^2 \cdot |H| \cdot \varepsilon^{-2z-2})$ if M is a shortest-path metric of a graph that excludes a fixed minor H.*

Then S is an ε-coreset for robust (k, z)-clustering on X.

This theorem illustrates the coreset size with respect to different metric spaces for robust (k, z)-clustering, improving upon previous results by eliminating the $O(m)$ dependency.

Proof of Theorem 5.9. If $M = (\mathcal{X}, \mathrm{dist})$ is a general metric space, then $d_{VC}(\mathcal{X}) = O(\log |\mathcal{X}|)$. Thus, we have $|S_O| = \tilde{O}(k \cdot \log(|\mathcal{X}|) \cdot \varepsilon^{-2})$ by Lemma 4.9 and $|S_I| = \tilde{O}(k^2 \cdot \log(|\mathcal{X}|) \cdot \varepsilon^{-4})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(k^2 \cdot \log(|\mathcal{X}|) \cdot \varepsilon^{-4})$.

Table 2: Datasets used in our experiments. For each dataset, we report its size, dimension (DIM), number of outliers (m). The number of centers (k) is used in robust (k, z)-clustering. We also provide the values of $\min_i |P_i^\star|$ and $(r_{\max}/\bar{r})^z$ for robust (k, z)-clustering. Our Assumption 1.4 requires that $\min_i |P_i^\star| \geq 4m$ and $(r_{\max}/\bar{r})^z \leq 4k$. The value Y (resp. N) indicates these assumptions are satisfied or not. Note that the **Athlete** dataset is sourced from Kaggle, while the other datasets are from the UCI repository.

DATASET	SIZE	DIM.	m	k	$\min_i \lvert P_i^\star \rvert$ $z=1$	$\min_i \lvert P_i^\star \rvert$ $z=2$	$(\frac{r_{\max}}{\bar{r}})^z$ $z=1$	$(\frac{r_{\max}}{\bar{r}})^z$ $z=2$
TWITTER[11]	10^5	2	2000	5	8968,Y	588,N	3.819,Y	1.083,Y
CENSUS1990 [46]	10^5	68	2000	5	5927,N	5927,N	1.873,Y	3.252,Y
BANK[47]	41188	10	824	5	1361,N	1361,N	4.674,Y	15.731,Y
ADULT[6]	48842	6	977	5	4508,Y	186,N	5.151,Y	5.834,Y
ATHLETE[36]	10000	4	200	5	1018,Y	1018,Y	2.923,Y	7.062,Y
DIABETES[43]	10000	10	200	5	1415,Y	1459,Y	2.551,Y	2.527,Y

If $M = (\mathcal{X}, \mathrm{dist})$ is a doubling metric space, then by definition, $\mathrm{ddim}(\mathcal{X})$ is bounded. Thus we have $|S_O| = \tilde{O}(k \cdot \mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$ by Lemma 4.10 and $|S_I| = \tilde{O}(k^2 \cdot \mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(k^2 \cdot \mathrm{ddim}(\mathcal{X}) \cdot \varepsilon^{-2})$.

If $M = (\mathcal{X}, \mathrm{dist})$ is a shortest-path metric of a graph with bounded treewidth t, we have $d_{VC}(\mathcal{X}) = O(t)$ by [8]. Thus, we have $|S_O| = \tilde{O}(k \cdot t \cdot \varepsilon^{-2})$ by Lemma 4.9 and $|S_I| = \tilde{O}(k^2 \cdot t \cdot \varepsilon^{-4})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(k^2 \cdot t \cdot \varepsilon^{-4})$.

If $M = (\mathcal{X}, \mathrm{dist})$ is a shortest-path metric of a graph that excludes a fixed minor H, we have $d_{VC}(\mathcal{X}) = O(|H|)$ by [8]. Thus, we have $|S_O| = \tilde{O}(k \cdot |H| \cdot \varepsilon^{-2})$ by Lemma 4.9 and $|S_I| = \tilde{O}(k^2 \cdot |H| \cdot \varepsilon^{-4})$ by Theorem 4.11, leading to a coreset of size $\tilde{O}(k^2 \cdot |H| \cdot \varepsilon^{-4})$.

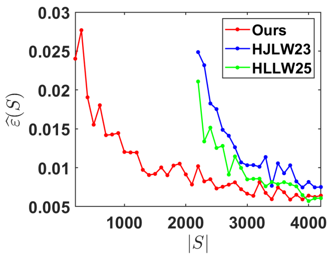
□

6 Empirical results

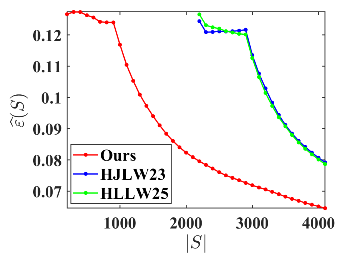
We implement our coreset construction algorithm and compare its performance to several baselines. All experiments are conducted on a PC with an Intel Core i9 CPU and 16GB of memory, and the algorithms are implemented in C++ 11.

Baselines. We compare our algorithm with two baselines: 1) Method **HJLW23** proposed by [39], which directly includes $L^\star$ in the coreset and samples points from $P_I^\star$. 2) Method **HLLW25** proposed by [40], improves the sample size from $P_I^\star$ in [39].

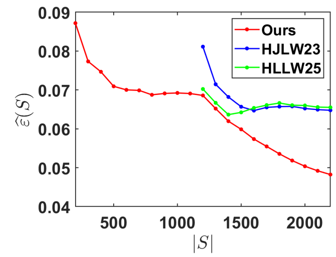
Setup. We conduct experiments on six datasets from diverse domains, including social networks, demographics, and disease statistics, with sample sizes ranging from (10^4) to (10^5) and feature dimensions from 2 to 68, as summarized in Table 2. These datasets cover those used in baseline [39], ensuring fair comparison. In each dataset, numerical features are extracted to create a vector for each record and the outlier number is set to 2% of the dataset size. To simplify computation, we subsample 10^5 points from the **Twitter** and **Census1990** datasets, and 10^4 points from the **Athlete** and **Diabetes** datasets, respectively. We use k-means++ to compute an approximate center $c^\star$.



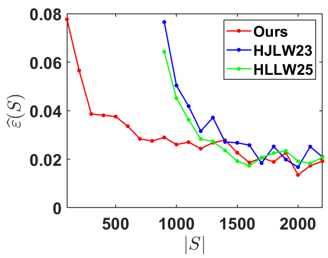
(a) **Census1990**



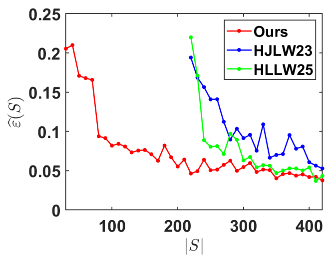
(b) **Twitter**



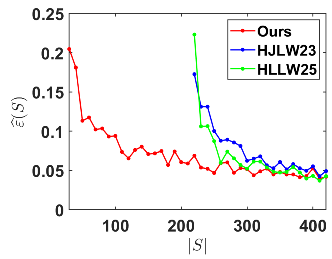
(c) **Adult**



(d) **Bank**



(e) **Athlete**



(f) **Diabetes**

Figure 3: Tradeoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$.

In the following subsections, we implement our algorithms for the robust geometric median ($d \geq 1$), robust 1D geometric median, robust k-median, and k-means, respectively,

6.1 Empirical results for robust geometric median ($d \geq 1$)

Size-error tradeoff. We evaluate the tradeoff between coreset size and empirical error. Given a (weighted) subset $S \subseteq P$ and a center $c \subset \mathbb{R}^d$, we define the empirical error $\widehat{\varepsilon}(S, c) := \frac{|\mathrm{cost}^{(m)}(P,c) - \mathrm{cost}^{(m)}(S,c)|}{\mathrm{cost}^{(m)}(P,c)}$, where lower values indicate better coreset performance for c. It is difficult to estimate the performance for every center. So we sample 500 centers $c_i \in \mathbb{R}^d$, where each c_i is drawn uniformly from P without replacement. Like in the literature [39], we evaluate the empirical error $\widehat{\varepsilon}(S) := \max_{i \in [500]} \widehat{\varepsilon}(S, c_i)$. We vary the coreset size $|S|$ from m to $2m$, and compute the empirical error $\widehat{\varepsilon}(S)$. For each size and each algorithm, we run the algorithm 10 times, compute their empirical errors $\widehat{\varepsilon}(S)$, and report the average of 10 empirical errors.

Figure 3 presents the empirical results illustrating the size-error tradeoff on the six datasets of Table 2. As shown in Figures 3, our coreset algorithm consistently achieves the lowest empirical error among all methods. Moreover, unlike the baselines, which require a coreset of size at least m, our method attains the same level of error with a coreset size **smaller than** m. For example, with the **Census1990** dataset, our method yields a coreset of size 1000 with an empirical error of 0.012, while the best baseline needs size 2300 to achieve a worse error of 0.013.

Statistical test. We also perform statistical tests across six real-world datasets by comparing the ratio of empirical errors between our algorithm and baselines, which further demonstrates that our algorithm consistently outperforms the baselines; see Table 3. The results show that both $\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$ and $\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$ are consistently bigger than 1, demonstrating that our coreset consistently yields lower empirical error than the baselines. This confirms the applicability of our coreset across real-world datasets.

Table 3: Statistical comparison of different coreset construction methods for robust geometric median. The coreset S_1 represents our coreset, S_2 represents the coreset constructed by the baseline **HJLW23**, and S_3 the coreset constructed by baseline **HLLW25**. For each empirical error ratio $\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$ and $\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$, we report the mean value over 20 runs, with the subscript indicating the standard deviation.

Coreset Size	Census1990		Twitter	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
2200	$3.253_{2.063}$	$2.645_{1.458}$	$1.793_{0.644}$	$1.667_{0.479}$
3200	$1.257_{0.842}$	$1.251_{0.632}$	$1.343_{0.234}$	$1.283_{0.197}$
4200	$1.303_{0.692}$	$1.168_{0.739}$	$1.244_{0.152}$	$1.246_{0.148}$

Coreset Size	Bank		Adult	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
1200	$1.647_{0.972}$	$1.360_{1.018}$	$1.467_{0.287}$	$1.094_{0.542}$
1700	$1.010_{0.654}$	$1.028_{0.574}$	$2.149_{0.884}$	$2.416_{1.002}$
2200	$1.010_{0.654}$	$1.026_{0.674}$	$1.089_{0.360}$	$1.172_{0.537}$

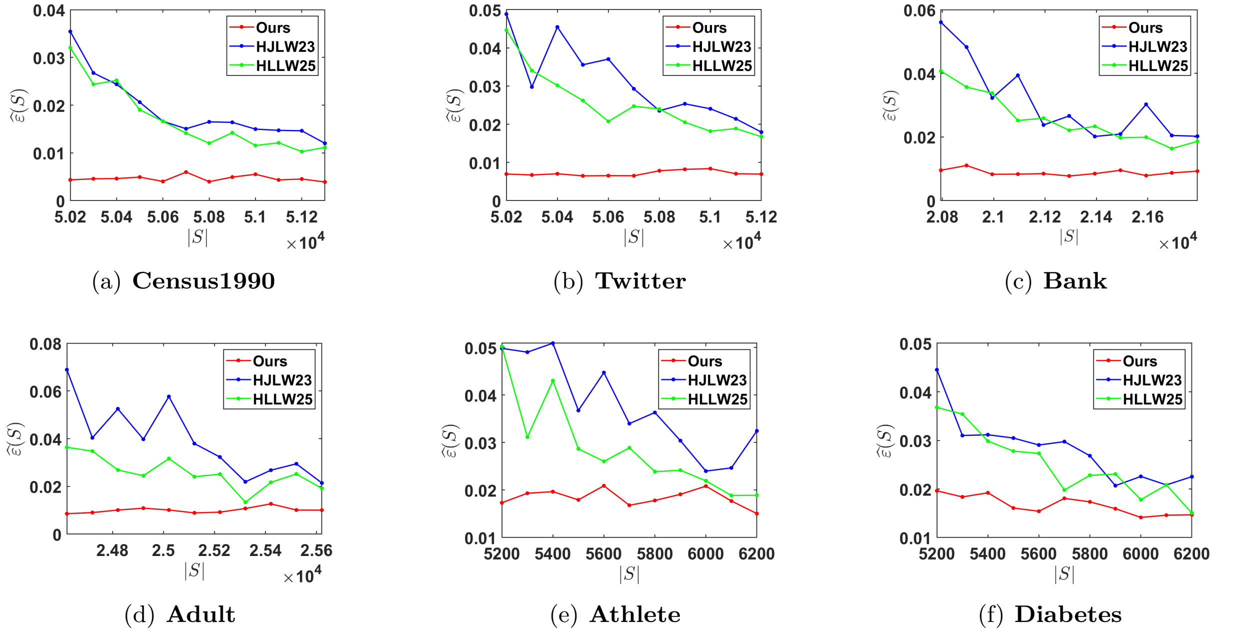
Coreset Size	Athlete		Diabetes	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
210	$5.172_{3.634}$	$4.200_{1.944}$	$5.700_{3.303}$	$5.868_{2.952}$
310	$2.467_{1.564}$	$1.427_{0.660}$	$1.567_{0.800}$	$1.332_{0.653}$
410	$1.658_{0.881}$	$1.045_{0.449}$	$1.360_{1.103}$	$1.216_{0.943}$

Speed-up baselines. In this experiment, we compare the coreset of size $2m$ constructed by the **HLLW25** baseline and coreset of size m constructed by Algorithm 2. We repeat the experiment 10 times and report the averages. The result is listed in Table 8. The construction time of our coreset is similar to that of the baseline **HLLW25**. However, our algorithm achieves a speed-up over **HLLW25** (a 2× reduction in the running time on the coreset), while achieving the same level of empirical error.

Analysis of our algorithm when $n < 4m$. For robust geometric median, we set $m = n/2$ in the six dataset, violating the assumption $n \geq 4m$, and report the corresponding size-error tradeoff in Figure 4.The results show that our method still consistently outperforms the baselines even when $n < 4m$, further highlighting the practical robustness of our algorithms. For instance, in **Census1990** dataset, our method produces a coreset of size 50200 with an empirical error of 0.004, while the best baseline produces a coreset of size 51200 with a much larger empirical error of 0.012.

Analysis under heavy-tailed contamination. For each dataset, we randomly perturb 10% of the points by adding independent Cauchy$(0, 1)$ [2] noise to every dimension, thereby simulating a heavy-tailed data environment. As shown in Figure 5, our method consistently outperforms the baselines on the robust geometric median task, demonstrating that its performance advantage remains stable even under heavy-tailed contamination. For example, on the perturbed **Twitter** dataset, our method achieves a coreset of size 1500 with an empirical error of 0.031, whereas the best baseline, **HLLW25**, requires a coreset twice as large (3000) to attain a higher error of 0.032.

[2] The probability density function of Cauchy$(0, 1)$ is $f(x) = \frac{1}{\pi(1+x^2)}$ for any $x \in \mathbb{R}$.



(a) **Census1990** (b) **Twitter** (c) **Bank**

(d) **Adult** (e) **Athlete** (f) **Diabetes**

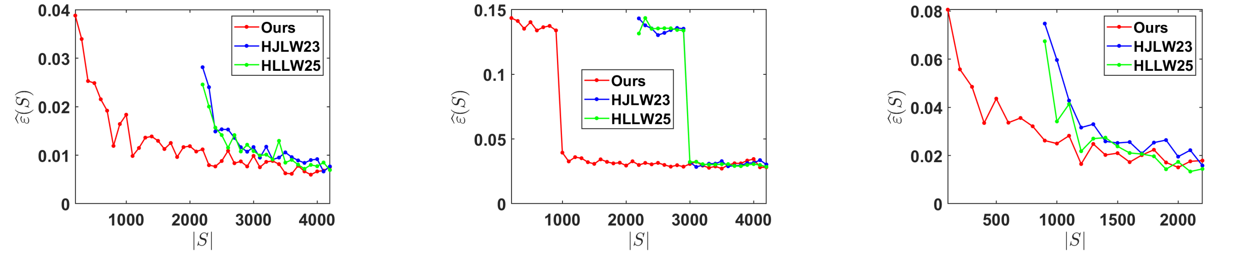
Figure 4: Tradoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust geometric median when we set $m = n/2$. In this scenario, the assumption $n \geq 4m$ is violated.

6.2 Empirical results for robust 1D geometric median

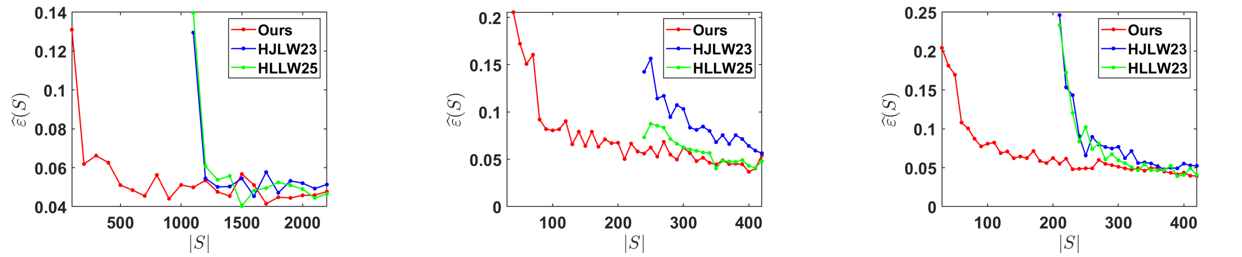
We implement our 1D coreset construction algorithm and compare its performance to the previous baselines and our general dimension method. All experiments are conducted on a PC with an Intel Core i9 CPU and 16GB of memory, and the algorithms are implemented in C++ 11.

Setup. We select the first dimension of the **Twitter** dataset, the second dimension of the **Adult** dataset, and the second dimension of the **Bank** dataset to create the **Twitter1D**, **Adult1D**, and **Bank1D** datasets. We choose these dimensions because other dimensions in the four datasets listed in Table 2 contain no more than 130 distinct points, which would result in an overly small coreset. We conduct experiments on these 1D datasets. To compute an approximate center for each dataset, we use the k-means++ algorithm.

Experiment result. We vary the coreset size from 200 to $m + 1000$, and compute the empirical error $\widehat{\varepsilon}(S)$ w.r.t. the coreset size $|S|$. For each size and each algorithm, we independently run the algorithm 10 times and obtain 10 coresets, compute their empirical errors $\widehat{\varepsilon}(S)$, and report the average of 10 empirical errors. Figure 6 presents our results. This figure shows that our 1D coreset (denoted by **Our1D**) outperforms the previous baselines and our coreset for general dimension. For example, with the **Twitter1D** dataset, our 1D method provides a coreset of size 320 with an empirical error 0.013. The best empirical error achieved by our general dimension method for a coreset size 3500 is much larger, 0.087. Note that the coreset size of **Our1D** method does not grow uniformly, since the number of buckets does not increase linearly with ε^{-1} or $\varepsilon^{-1/2}$.

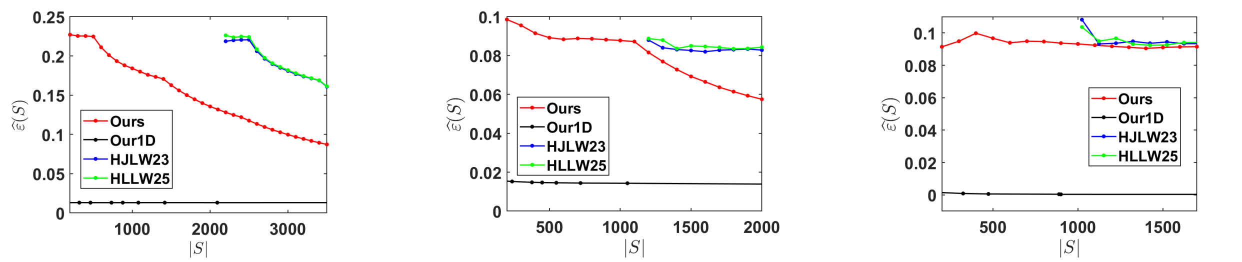


(a) **Census1990** (b) **Twitter** (c) **Bank**



(d) **Adult** (e) **Athlete** (f) **Diabetes**

Figure 5: Tradoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust geometric median when we perturb 10% points.



(a) **Twitter1D** (b) **Adult1D** (c) **Bank1D**

Figure 6: Tradeoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ in 1D datasets.

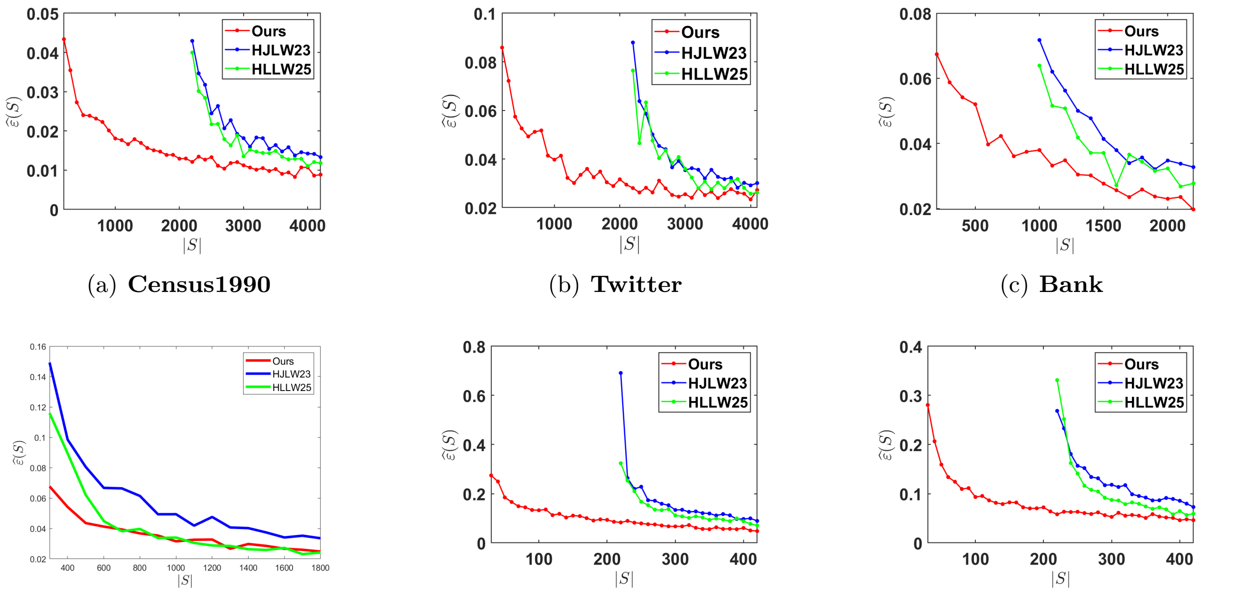
6.3 Empirical results for robust k-median

We implement Algorithm 3 for robust k-median and compare its performance to the previous baselines.

Setup. We do experiments on the six datasets listed in Table 2, and set the number of centers k to be 5. We use Lloyd version k-means++ to compute an approximate center $C^\star$.

Coreset size and empirical error tradeoff for robust k-median. We vary the coreset size from m to $2m$, and compute the empirical error $\widehat{\varepsilon}(S)$ w.r.t. the coreset size $|S|$. For each size and each algorithm, we independently run the algorithm 10 times and obtain 10 coresets, compute their empirical errors $\widehat{\varepsilon}(S)$, and report the average of 10 empirical errors. Figure 7 presents our results. This figure shows that our algorithm (**Ours**) outperforms the previous baselines. For example, with the **Census1990** dataset, our method provides a coreset of size 2200 with empirical error 0.012.

The best empirical error achieved by baselines for a coreset size 3600 is larger, 0.013.



(a) **Census1990** (b) **Twitter** (c) **Bank**

(d) **Adult** (e) **Athlete** (f) **Diabetes**

Figure 7: Tradeoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust k-median on real-world datasets.

Statistical test. Similar to the robust geometric median, we evaluate the statistical performance between our method and baselines for robust k-median on all six real-world datasets. The results, listed in Table 4, further demonstrate that our algorithm consistently outperforms the baselines.

Speed-up baselines. We compare the coreset of size $2m$ constructed by the **HLLW25** baselines and coreset of size m conducted by Algorithm 3. We repeat the experiment 10 times and report the averages. The result is listed in Table 5. Our algorithm achieves a speed-up over **HLLW25**—specifically, a 2× reduction in the running time on the coreset—while maintaining the same level of empirical error.

Validity of Assumption 1.4 for robust k-median. We evaluate the validity of Assumption 1.4 for robust k-median across six datasets. The results in Table 2 show that the assumptions are satisfied by the **Twitter**, **Adult**, **Athlete**, and **Diabetes** datasets, while the condition $r_{\max} \leq 4k\bar{r}$ holds for all six datasets. These results demonstrate that our assumptions are practical in real-world scenarios, and our algorithm performs well even when the assumption $\min_i |P_i^\star| \geq 4m$ is violated. Note that the condition $r_{\max} \leq 4k\bar{r}$ is satisfied by all six real-world datasets, even across different choices of k and m.

Analysis under heavy-tailed contamination. For each dataset, we randomly perturb 10% of the points by adding independent Cauchy$(0, 1)$ noise to every dimension, thereby simulating a heavy-tailed data environment. As shown in Figure 8, our method consistently outperforms the baselines on the robust k-median task, confirming that its superior performance remains stable even under heavy-tailed contamination. For example, on the perturbed **Bank** dataset, our method

Table 4: Statistical comparison of different coreset construction methods for robust k-median. The coreset S_1 represents our coreset, S_2 represents the coreset constructed by the baseline **HJLW23**, and S_3 the coreset constructed by baseline **HLLW25**. For each empirical error ratio $\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$ and $\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$, we report the mean value over 20 runs, with the subscript indicating the standard deviation.

Coreset Size	Census1990		Twitter	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
2200	$3.374_{0.818}$	$3.800_{1.262}$	$2.958_{0.785}$	$3.149_{1.477}$
3200	$1.800_{0.659}$	$1.483_{0.534}$	$1.654_{0.788}$	$1.574_{0.710}$
4200	$1.543_{0.559}$	$1.316_{0.373}$	$1.408_{0.439}$	$1.379_{0.369}$

Coreset Size	Bank		Adult	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
1200	$1.657_{0.415}$	$2.019_{0.740}$	$2.098_{0.551}$	$2.153_{0.772}$
1700	$1.548_{0.582}$	$1.257_{0.543}$	$1.440_{0.619}$	$1.702_{0.584}$
2200	$1.393_{0.558}$	$1.460_{0.737}$	$1.450_{0.677}$	$1.373_{0.545}$

Coreset Size	Athlete		Diabetes	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
210	$11.481_{3.633}$	$8.851_{0.740}$	$12.688_{3.443}$	$8.080_{1.652}$
310	$2.142_{0.525}$	$1.798_{0.553}$	$1.902_{0.512}$	$1.402_{0.395}$
410	$2.104_{0.570}$	$1.523_{0.481}$	$1.691_{0.546}$	$1.257_{0.470}$

achieves a coreset of size 600 with an empirical error of 0.044, whereas the best baseline, **HLLW25**, requires a coreset of size 1300 to reach a higher error of 0.046.

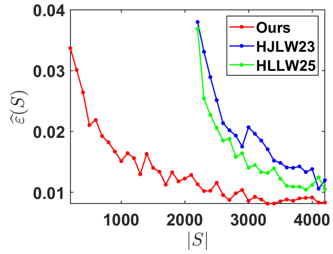
6.4 Empirical results for robust k-means

We implement Algorithm 3 for robust k-means and compare its performance to the previous baselines.

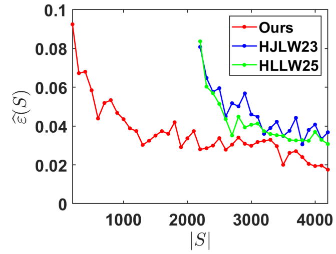
Setup. We do experiments on the six datasets listed in Table 2, and set the number of centers k to be 5. We use k-means++ to compute an approximate center $C^\star$.

Coreset size and empirical error tradeoff for Robust k-means. We vary the coreset size from m to $2m$, and compute the empirical error $\widehat{\varepsilon}(S)$ w.r.t. the coreset size $|S|$. For each size and each algorithm, we independently run the algorithm 10 times and obtain 10 coresets, compute their empirical errors $\widehat{\varepsilon}(S)$, and report the average of 10 empirical errors. Figure 9 presents our results. This figure shows that our algorithm (**Ours**) outperforms the previous baselines. For example, with the **Twitter** dataset, our method provides a coreset of size 2200 with an empirical error 0.039. The best empirical error achieved by our general dimension method for a coreset size 4000 is much larger, 0.056.

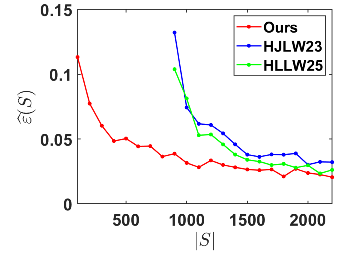
Statistical test. Similar to the previous cases, we evaluate the statistical performance between our method and baselines for robust k-means on all six real-world datasets. The results, listed in Table 6, further demonstrate that our algorithm consistently outperforms the baselines.



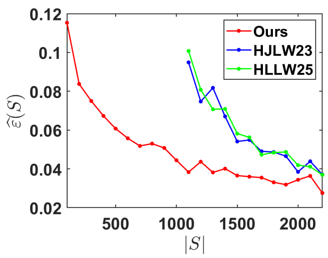
(a) **Census1990**



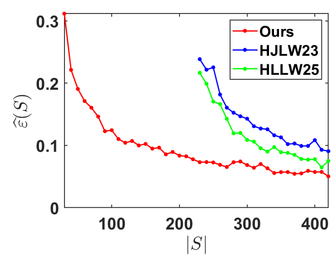
(b) **Twitter**



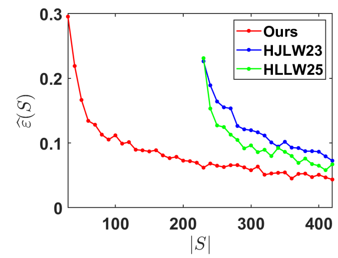
(c) **Bank**



(d) **Adult**



(e) **Athlete**



(f) **Diabetes**

Figure 8: Tradoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust k-median when we perturb 10% points.

Speed-up baselines. We compare the coreset of size $2m$ constructed by the **HLLW25** baselines and coreset of size m conducted by Algorithm 3. We repeat the experiment 10 times and report the averages. The result is listed in Table 7. Our algorithm achieves a speed-up over **HLLW25**—specifically, a $2\times$ reduction in the running time on the coreset—while maintaining the same level of empirical error.

Validity of Assumption 1.4 for robust k-means. We evaluate the validity of Assumption 1.4 for robust k-means across six datasets. The results in Table 2 show that our assumptions are satisfied by datasets **Athlete** and **Diabetes**, while the assumption $r_{\max}^2 \le 4k\bar{r}^2$ is satisfied by all six datasets. These results show that our assumptions are practical in real-world datasets, and our algorithm performs well even when the assumption $\min_i |P_i^\star| \ge 4m$ is violated.

Analysis of our algorithm when Assumption 1.4 violates. We evaluate the applicability of our algorithm when both assumptions $\min_i |P_i^\star| \ge 4m$ and $(r_{\max}/\bar{r})^2 \le 4k$ are violated. In **Bank** dataset, we let $k = 3$, then we have $(r_{\max}/\bar{r})^2 = 15.001 > 4k$ and $\min_i |P_i^\star| = 1432 < 4m$, violating the two assumptions. In **Adult** dataset, we let $k = 8$, $m = 500$, then we have $(r_{\max}/\bar{r})^2 = 36.278 > 4k$ and $\min_i |P_i^\star| = 1592 < 4m$, violating the two assumptions. We present the size-error tradeoff for robust k-means on these datasets in Figure 10. Our results show that our method still outperforms the baselines even when both assumptions are violated.

Analysis under heavy-tailed contamination. For each dataset, we randomly perturb 10% of the points by adding independent Cauchy$(0, 1)$ noise to every dimension, thereby simulating a heavy-tailed data environment. As shown in Figure 11, our method consistently outperforms the baselines on the robust k-means task. For example, on the perturbed **Athlete** dataset, our method

Table 5: Comparison of runtime between our Algorithm 3 and baseline **HLLW25** for robust k-median. For each dataset, the coreset size of baseline **HLLW25** is $2m$ and the coreset size of ours is m. We use Lloyd algorithm given by [7] to compute approximate solutions C_P and C_S for both the original dataset P and coreset S, respectively. "COST$_P$" denotes $\mathrm{cost}_1^{(m)}(P, C_P)$ on the original dataset P. "COST$_S$" denotes $\mathrm{cost}_1^{(m)}(P, C_S)$ on the coreset constructed by METHOD. T_X is the running time on the original dataset. T_S is the running time on coreset. T_C is the construction time of the coreset.

DATASET	COST$_P$	METHOD	COST$_S$	T_X	T_C	T_S
Census1990	1.032×10^6	Ours	1.030×10^6	312.606	38.862	5.532
		HLLW25	1.040×10^6		38.703	10.815
Twitter	1.328×10^6	Ours	1.364×10^6	96.024	12.220	1.621
		HLLW25	1.347×10^6		12.452	2.995
Bank	3.179×10^6	Ours	3.194×10^6	147.324	9.021	1.484
		HLLW25	3.207×10^6		9.210	3.445
Adult	9.221×10^8	Ours	9.153×10^8	144.185	9.324	1.854
		HLLW25	9.166×10^8		9.266	3.517
Athlete	7.251×10^4	Ours	7.503×10^4	41.541	1.825	0.206
		HLLW25	7.389×10^4		1.872	0.421
Diabetes	8.571×10^4	Ours	8.791×10^4	44.733	1.829	0.226
		HLLW25	8.811×10^4		1.850	0.473

yields a coreset of size 110 with an empirical error of 0.181, whereas the best baseline, **HLLW25**, attains a higher error of 0.196 even with a larger coreset of size 290.

7 Conclusion and future work

We investigate coreset construction for robust geometric median problem, successfully eliminating the size dependency on the number of outliers. Specifically, for the 1D Euclidean case, we achieve the first optimal coreset size. Furthermore, our results generalize to robust clustering applications. Empirically, our algorithms achieve a superior size-error balance and a runtime acceleration.

This work opens several intriguing research directions. One immediate problem is to optimize the coreset size for $d > 1$ or $k > 1$, particularly in cases where the size diverges from that of the vanilla setting. Extending robust coresets to the streaming model is a valuable but challenging direction. The primary obstacle is their lack of mergeability, as outlier interactions across different data chunks prevent the compositional updates essential for streaming algorithms. It is also interesting to explore whether our non-component-wise analysis can be applied to other robust machine learning problems, such as robust regression and robust PCA.

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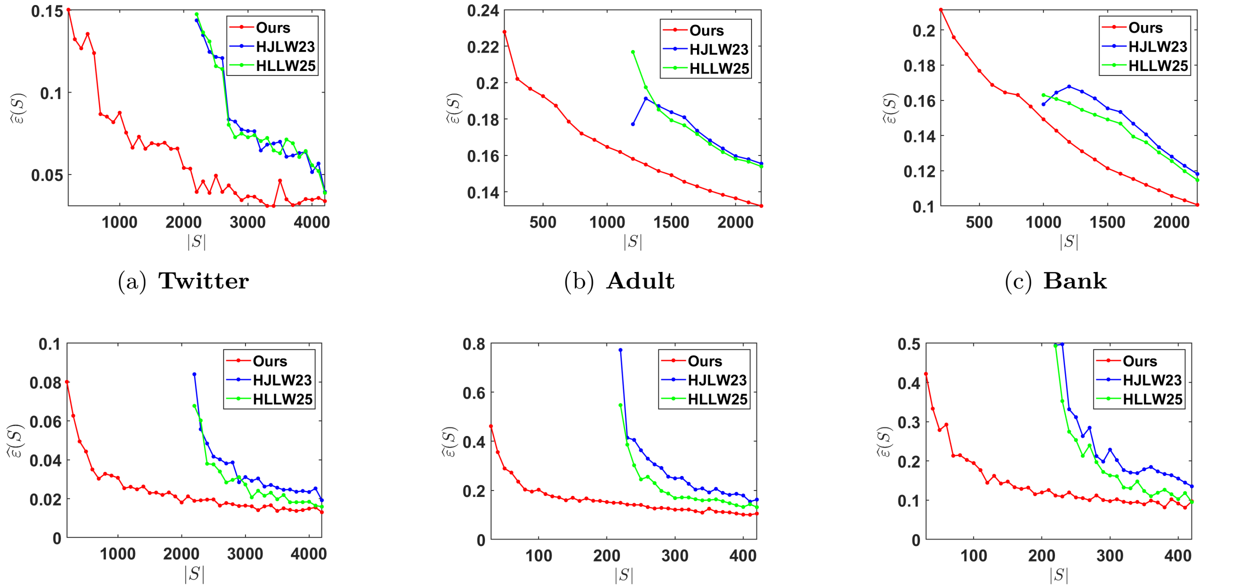

(a) **Twitter** (b) **Adult** (c) **Bank**

(d) **Census1990** (e) **Athlete** (f) **Diabetes**

Figure 9: Tradeoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust k-means.

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Table 6: Statistical comparison of different coreset construction methods for robust geometric median. The coreset S_1 represents our coreset, S_2 represents the coreset constructed by the baseline **HJLW23**, and S_3 the coreset constructed by baseline **HLLW25**. For each empirical error ratio $\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$ and $\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$, we report the mean value over 20 runs, with the subscript indicating the standard deviation.

Coreset Size	Census1990		Twitter	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
2200	$3.253_{2.063}$	$2.645_{1.458}$	$1.793_{0.644}$	$1.667_{0.479}$
3200	$1.257_{0.842}$	$1.251_{0.632}$	$1.343_{0.234}$	$1.283_{0.197}$
4200	$1.303_{0.692}$	$1.168_{0.739}$	$1.244_{0.152}$	$1.246_{0.148}$
Coreset Size	Bank		Adult	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
1200	$1.647_{0.972}$	$1.360_{1.018}$	$1.467_{0.287}$	$1.094_{0.542}$
1700	$1.010_{0.654}$	$1.028_{0.574}$	$2.149_{0.884}$	$2.416_{1.002}$
2200	$1.010_{0.654}$	$1.026_{0.674}$	$1.089_{0.360}$	$1.172_{0.537}$
Coreset Size	Athlete		Diabetes	
	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_2)/\widehat{\varepsilon}(S_1)$	$\widehat{\varepsilon}(S_3)/\widehat{\varepsilon}(S_1)$
210	$5.172_{3.634}$	$4.200_{1.944}$	$5.700_{3.303}$	$5.868_{2.952}$
310	$2.467_{1.564}$	$1.427_{0.660}$	$1.567_{0.800}$	$1.332_{0.653}$
410	$1.658_{0.881}$	$1.045_{0.449}$	$1.360_{1.103}$	$1.216_{0.943}$

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DATASET	COST$_P$	METHOD	COST$_S$	T_X	T_C	T_S
Census1990	1.172×10^7	Ours	1.170×10^7	358.218	35.513	5.770
		HLLW25	1.170×10^7		35.399	11.043
Twitter	2.657×10^7	Ours	2.664×10^7	106.327	14.084	1.806
		HLLW25	2.662×10^7		13.330	3.494
Bank	3.477×10^8	Ours	3.531×10^8	133.978	7.478	1.283
		HLLW25	3.530×10^8		7.225	2.725
Adult	2.575×10^{13}	Ours	2.646×10^{13}	139.16	8.273	1.564
		HLLW25	2.652×10^{13}		8.048	3.091
Athlete	8.630×10^5	Ours	9.089×10^5	44.208	1.890	0.251
		HLLW25	9.016×10^5		1.909	0.466
Diabetes	6.490×10^5	Ours	6.814×10^5	39.570	1.622	0.196
		HLLW25	6.838×10^5		1.602	0.413

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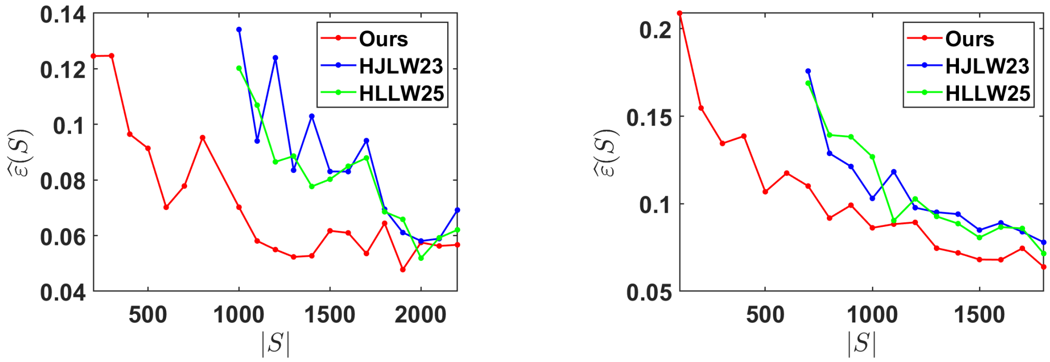

(a) **Bank** (b) **Adult**

Figure 10: Tradeoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust k-means. We set $k = 3$ in the **Bank** dataset and set $k = 8$, $m = 500$ in the **Adult** dataset. In these cases, the values $(r_{\max}/\bar{r})^2$ and $\min_{i\in[k]} |P_i^\star|$ violate both of our assumptions. The goal of this figure is to examine whether our algorithm remains applicable when the assumptions are not satisfied.

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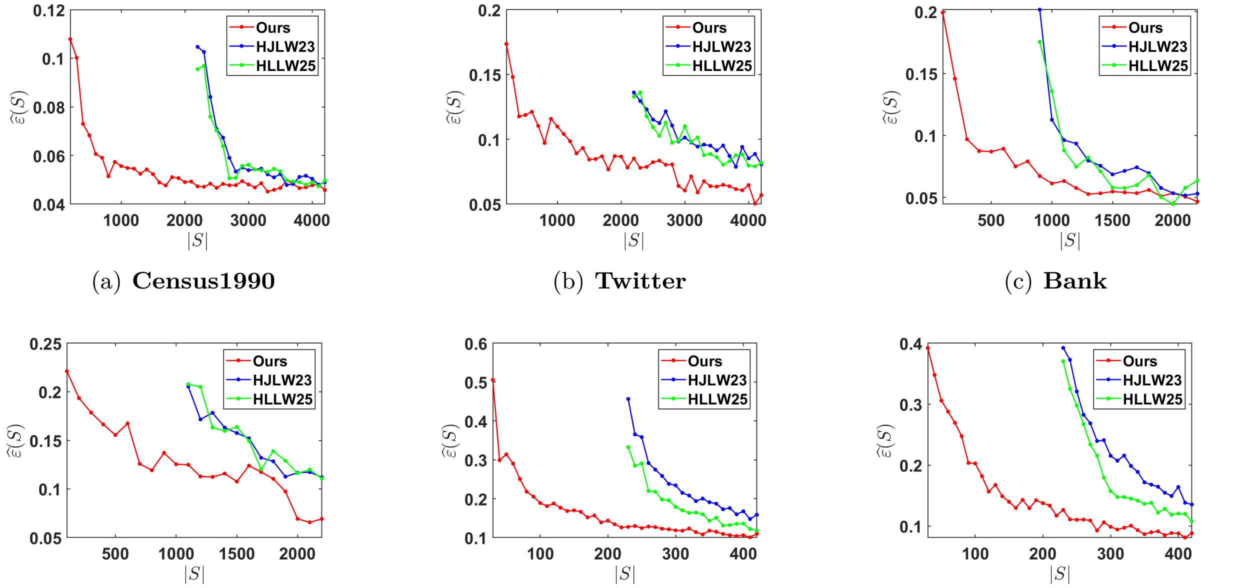

(a) **Census1990** (b) **Twitter** (c) **Bank**

(d) **Adult** (e) **Athlete** (f) **Diabetes**

Figure 11: Tradoff between coreset size $|S|$ and empirical error $\widehat{\varepsilon}(S)$ for robust k-means when we perturb 10% points.

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Table 8: Comparison of runtime between our Algorithm 2 and baseline **HLLW25**. For each dataset, the coreset size of baseline **HLLW25** is $2m$ and the coreset size of ours is m. We use Lloyd algorithm given by [7] to compute approximate solutions c_P and c_S for both the original dataset P and coreset S, respectively. "COST_P" denotes $\mathrm{cost}^{(m)}(P, c_P)$ on the original dataset P. "COST_S" denotes $\mathrm{cost}^{(m)}(P, c_S)$ on the coreset constructed by METHOD. T_X is the running time on the original dataset. T_S is the running time on coreset. T_C is the construction time of the coreset.

DATASET	COST_P	METHOD	COST_S	T_X	T_C	T_S
Census1990	5.099×10^6	Ours	5.100×10^6	63.425	6.876	1.284
		HLLW25	5.099×10^6		6.950	2.629
Twitter	7.307×10^6	Ours	7.310×10^6	41.816	3.233	0.633
		HLLW25	7.307×10^6		3.259	1.278
Bank	7.815×10^6	Ours	7.760×10^6	16.555	1.427	0.308
		HLLW25	7.765×10^6		1.477	0.677
Adult	3.418×10^9	Ours	3.412×10^9	16.907	1.632	0.310
		HLLW25	3.411×10^9		1.596	0.597
Athlete	1.460×10^5	Ours	1.467×10^5	2.92	0.320	0.055
		HLLW25	1.463×10^5		0.321	0.104
Diabetes	1.781×10^5	Ours	1.786×10^5	3.977	0.366	0.062
		HLLW25	1.788×10^5		0.360	0.135

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A An illustrative example for the obstacle of Inequality (3)

Recall that we partition the input dataset $P \subset \mathbb{R}$ into disjoint buckets $\{B_1, \ldots, B_T\}$, constructs a representative point $\mu(B_i)$ with weight $|B_i|$ for each bucket and takes their union as a coreset S of P for robust 1D geometric median. Also, recall that prior work [39, 40] use Inequality (3) for error analysis, i.e., for any center $c \in \mathbb{R}$ and any tuple of outlier numbers $(m_1, \ldots, m_T)$ with $\sum_{i\in[T]} m_i = m$,

$$\sum_{i\in[T]} |\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i)\mathrm{dist}(\mu(B_i), c)| < \varepsilon \cdot \mathrm{cost}^{(m)}(P, c). \tag{38}$$

In this section, we provide an example in which this inequality only holds if $|S| = \Omega(m)$.

Construct the dataset P as follows: for $i = 1, \ldots, n-m$, $p_i = \frac{i}{n}$; for $n-m+1 \le i \le n$, $p_i = n^{3(i-n+m)}$. We show that the Inequality (38) only holds if each p_j forms its own isolated bucket. Assume, for the sake of contradiction, that there exists a bucket $B_q = \{p_i, \ldots, p_j\}$ such that $|B_q| > 1$ and $j > n-m$.

Case 1: If $i \le n-m$, let $c = 0$. Then we have that the inlier set with respect to c is $P_I^{(c)} = \{p_1, \ldots, p_{n-m}\}$. Thus,

$$\mathrm{cost}^{(m)}(P, c) = \mathrm{cost}(P_I^{(c)}, c) = \sum_{t=1}^{n-m} \frac{t}{n} < n.$$

Let $m_1 = \ldots = m_{q-1} = 0$, $m_q = j - n + m$ and $m_t = |B_t|$ for $q+1 \le t \le T$. We have

$$\begin{aligned}
&\sum_{i\in[T]} |\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i)\mathrm{dist}(\mu(B_i), c)| \\
&\quad= |\mathrm{cost}^{(m_q)}(B_q, c) - (|B_q| - m_q)\mathrm{dist}(\mu(B_q), c)| \\
&\quad> (|B_q| - m_q) \cdot \mu(B_q) - n && (\mathrm{cost}^{(m_q)}(B_q, c) \le \mathrm{cost}^{(m)}(P, c) < n) \\
&\quad\ge \mu(B_q) - n && (|B_q| = j - i + 1, i \le n-m) \\
&\quad= \frac{\sum_{p\in B_q} p}{|B_q|} - n \\
&\quad> n^2 - n \\
&\quad> \varepsilon \cdot \mathrm{cost}^{(m)}(P, c), && (\mathrm{cost}^{(m)}(P, c) < n)
\end{aligned}$$

which leads to a contradiction with Inequality (38).

Case 2: If $i > n-m$, let $c = p_i$. Then $P_I^{(c)} = \{p_{i-n+m+1}, \ldots, p_i\}$. In this case, we have

$$\mathrm{cost}^{(m)}(P, c) = \sum_{p\in P_I^{(c)}} d(p, p_i) \le (n-m) \cdot n^{3(i-n+m)}.$$

Note that $|B_q \cap P_I^{(c)}| = 1$, which means $|B_q| - m_q = 1$, then we have

$$\begin{aligned}
& \sum_{i \in [T]} |\mathrm{cost}^{(m_i)}(B_i, c) - (|B_i| - m_i)\mathrm{dist}(\mu(B_i), c)| \\
\geq \;& |\mathrm{cost}^{(m_q)}(B_q, c) - (|B_q| - m_q)\mathrm{dist}(\mu(B_q), c)| \\
= \;& \mathrm{dist}(\mu(B_q), c) && (\mathrm{cost}^{(m_q)}(B_q, c) = \mathrm{dist}(p_i, c) = 0) \\
= \;& \frac{\sum_{p \in B_q} p}{|B_q|} - p_i \\
> \;& n^{3(i-n-m)} \cdot (n^2 - 1) \\
> \;& \varepsilon \cdot \mathrm{cost}^{(m)}(P, c), && (\mathrm{cost}^{(m)}(P, c) \leq (n - m) \cdot n^{3(i-n+m)})
\end{aligned}$$

which leads to a contradiction with Inequality (38). Overall, Inequality (38) does not hold if $\{p_j\}$ does not form a separated bucket.

Non-component-wise analysis breaks the obstacle. We then show how to adapt our new non-component-wise analysis to this example, which allows a more careful bucket decomposition. The key is that in our analysis, the induced error of the aforementioned bucket $B_q = \{p_i, \ldots, p_j\}$ with $j > n - m$ is 0, due to allowing the misaligned outlier numbers in each bucket. Below illustrate the above two cases.

Case 1: If $i \leq n - m$, $c = 0$, then only the bucket B_q may induce an error. Since $j > n - m$, we have $p_j \geq n^3$, thus

$$\mu(B_q) = \frac{\sum_{p \in B_q} p}{|B_q|} > \frac{n^3}{n} = n^2 > \max_{p \in P_I^{(c)}} \mathrm{dist}(p, c).$$

Therefore, B_q will be totally regarded as an outlier which induces 0 error, thus $\mathrm{cost}^{(m)}(S, c) = \mathrm{cost}^{(m)}(P, c)$.

Case 2: If $i > n - m$, $c = p_i$, then $P_I^{(c)} = \{p_{i-n+m+1}, \ldots, p_i\}$. Let the bucket $B_L = \{p_l, \ldots, p_r\}$ with $l \leq i - n + m + 1$, $r \geq i - n + m + 1$ denote the leftmost bucket containing at least one inlier. Therefore, only the buckets B_L and B_q may induce an error. For B_q, we have $\mathrm{dist}(\mu(B_q), c) > (n^2 - 1)\mathrm{dist}(p_1, c)$, thus B_q induces 0 error. For B_L, the induced error is bounded by $\varepsilon n \cdot n^{3(i-n-m)} \leq \varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$, since, in our framework, the size of each bucket in $P - P_M$ is restricted to at most $O(\varepsilon n)$. In summary, the total error is bounded by $\varepsilon \cdot \mathrm{cost}^{(m)}(P, c)$.

Thus, non-component-wise analysis significantly reduces bucket-wise errors in this example, which is crucial for proving S is a coreset.

B Proof of Theorem 1.1: coreset lower bound for robust geometric median

We first construct a bad instance $P := \{p_1, \ldots, p_n\} \subset \mathbb{R}$, where $p_i = i$ for $i \in [m]$ and $p_j = x$ for $m < j \leq n$, $x \to \infty$.

W.l.o.g, we assume $n - m$ is an even number and $n - m \leq (m - 1)/2$. Suppose (S, w) is an ε-coreset of size $|S| < \frac{m}{2(n-m)+1}$ for robust geometric median on P. Define $S_I := S \cap \{p_{m+1}, \ldots, p_n\}$

and $S_O := S \cap \{p_1, \ldots, p_m\}$. There exists $2(n-m)$ consecutive points $p_{i+1}, ..., p_{i+2(n-m)}$ not in S for some $i \in [m-2(n-m)]$. Let the center $c = i + (n-m) + \frac{1}{2}$. Then we have

$$\begin{aligned} \mathrm{cost}^{(m)}(P, c) = \quad & 2((\frac{1}{2}) + (\frac{1}{2}+1) + \ldots + (\frac{1}{2} + \frac{n-m}{2} - 1)) \\ \leq \quad & \frac{(n-m)^2}{2}. \end{aligned} \tag{39}$$

We claim that $w(S_O) + w(S_I) - m \geq (1-\varepsilon)\cdot(n-m)$. Fix a center $c' = x+1$. If $w(S_O) > m$, we have $\mathrm{cost}^{(m)}(P, c') = n-m$ and $\mathrm{cost}^{(m)}(S, c') \to \infty$, leading to an unbounded error. Therefore we only need to consider the case $w(S_O) \leq m$ and our claim can be verified by as follow:

$$\frac{\mathrm{cost}^{(m)}(S, c')}{\mathrm{cost}^{(m)}(P, c')} = \frac{w(S_O) - m + w(S_I)}{n-m} \to (1 \pm \varepsilon). \tag{40}$$

By this claim, we have

$$\begin{aligned} \mathrm{cost}^{(m)}(S, c) \geq \quad & (n-m+0.5)\cdot(w(S_O) + w(S_I) - m) & \\ \geq \quad & (n-m+0.5)\cdot(1-\varepsilon)\cdot(n-m) & \text{(Inequality (40))} \\ > \quad & (1+\varepsilon)\cdot(\frac{(n-m)^2}{2}) & (0 < \varepsilon < 0.5) \\ \geq \quad & (1+\varepsilon)\cdot\mathrm{cost}^{(m)}(P, c) & \text{(Inequality (39))}. \end{aligned}$$

In summary, we have $\mathrm{cost}^{(m)}(S, c) \geq (1+\varepsilon)\cdot\mathrm{cost}^{(m)}(P, c)$, which contradicts the definition of ε-coreset. We conclude that, each ε-coreset of P is of size $\Omega(\frac{m}{n-m})$.