NON-POLYCONVEX Q-INTEGRANDS WITH LOWER SEMICONTINUOUS ENERGIES

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Abstract. We construct a positive measure on the space of positively oriented 2-vectors in $\mathbb{R}^4$, whose barycenter is a simple 2-vector, yet which cannot be approximated by weighted Gaussian images of Lipschitz Q-graphs for any fixed $Q \in \mathbb{N}$. The construction extends to positively oriented m-vectors in $\mathbb{R}^n$ whenever $n-2 \geq m \geq 2$. This geometric obstruction implies that the approximation result established in [19] is sharp: all $Q \in \mathbb{N}$ are indeed necessary to ensure the density of weighted Gaussian images of Lipschitz multigraphs in the space of positive measures with simple barycenter. As an application, we prove that for every $Q \geq 1$ and $p \geq 2$ there exists a non-polyconvex Q-integrand whose associated energy is weakly lower semicontinuous in $W^{1,p}$. This also provides new insight into the question posed in [19, Remark 1.14].

1. Introduction

Given a function $\psi : \mathbb{R}^{(n-m)\times m} \to [0,\infty)$ and an integer $Q \in \mathbb{N}$, we define the ψ-energy of a weakly differentiable Q-valued map $f : B \subseteq \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{n-m})$ as

$$E_\psi(f) = \int_B \sum_{i=1}^{Q} \psi\big((Df(x))_i\big)\, dx \ =: \ \int_B \bar{\psi}(Df(x))\, dx. \tag{1.1}$$

Although this definition depends on the value of Q, we will omit this dependence from the notation for simplicity. This energy was studied by De Rosa, Lei and Young in [19], and the definition is well posed, as it is invariant under permutations of $\{1,\dots,Q\}$. Geometrically, $E_\psi(f)$ is relevant as it represents the ψ-anisotropic energy of the Q-valued graph of f, interpreted as a current.

The functional E_ψ is a particular case of a broader class of energies associated with Q-integrands, introduced by De Lellis, Focardi, and Spadaro in [11, Section 0.1]. A Q-integrand is a function $F : (\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q \to [0,\infty)$ satisfying for every permutation σ of $\{1,...,Q\}$ the invariance

$$F(a_1,...,a_Q,X_1,...,X_Q) = F(a_{\sigma(1)},...,a_{\sigma(Q)},A_{\sigma(1)},...,A_{\sigma(Q)}).$$

Thus, such integrand induces an anisotropic energy functional on Q-valued maps via

$$\mathcal{F} : f \in Lip(B;\mathcal{A}_Q(\mathbb{R}^{n-m})) \mapsto \int_B F\big((f(x))_1,\dots,(f(x))_Q,(Df(x))_1,\dots,(Df(x))_Q\big)\, dx.$$

While the integrand $\bar{\psi}$ defined in (1.1) is indeed a Q-integrand, the converse is not generally true: not every Q-integrand arises from a scalar function ψ as in in (1.1).

In the study of existence and regularity of minimizers [29, 3, 4, 22, 10, 14, 16]—and more generally, of critical points—for anisotropic geometric variational problems, it is essential to impose suitable ellipticity conditions on the anisotropic integrands; see, for instance, the survey [17].

For $Q = 1$, the notions of *polyconvexity* and *quasiconvexity* for ψ are classical in the nonlinear analysis literature; see [26, Definition 4.2]. Polyconvexity corresponds to convexity in the space of d-vectors, as clarified in [19, Remark 1.4], and plays a central role in the regularity theory of anisotropic minimal graphs [20, 21, 34]. Quasiconvexity, on the other hand, is equivalent to the lower semicontinuity of the energy E_ψ under growth conditions, and is therefore fundamental in establishing the existence of minimizers. We refer the reader to the foundational works by Acerbi and Fusco [1], Fonseca [23], and Fonseca and Müller [24], and to the book [8] for details.

It is by now classical that polyconvexity implies quasiconvexity, but the converse fails in general. In particular, Šverák [32], and independently Alibert and Dacorogna [2], in the $Q = 1$ setting constructed a continuous function $\psi : \mathbb{R}^{2\times 2} \to [0,\infty)$ that is quasiconvex but not polyconvex. Earlier examples of quasiconvex quadratic forms on $\mathbb{R}^{3\times 3}$ that are not polyconvex can be found in [5, 30, 33]. However, it is known that every quasiconvex quadratic form on $\mathbb{R}^{2\times 2}$ is polyconvex [30, 33], which highlights the significance of the counterexamples in [32, 2].

The gap between polyconvexity and quasiconvexity is intimately related to the existence of positive measures on positively oriented 2-vectors in $\mathbb{R}^4$ whose barycenter is a simple 2-vector, but which cannot be approximated by weighted Gaussian images of Lipschitz graphs. We refer the reader to [27, Chapters 4, 5, 6] for an extensive discussion on Young measures that are not gradient Young measures, and the connection with quasiconvexity.

For every $Q \in \mathbb{N}$, the analogues of polyconvexity and quasiconvexity for Q-integrands were introduced and studied by De Lellis, Focardi, and Spadaro [11], building on earlier work by Mattila [25]. The main result of [11] establishes the equivalence between the quasiconvexity of continuous Q-integrands and the lower semicontinuity of the associated energy functionals. As in the classical case, quasiconvexity is thus the appropriate notion to ensure the existence of minimizers among Q-valued maps. Already in [25, Introduction], Mattila raised the question of the relationship between these ellipticity conditions for Q-integrands.

One of the main results of this paper is to show that, analogously to classical 1-integrands, the lower semicontinuity of the energy functional associated to a lower semicontinuous Q-integrand does not imply polyconvexity. We state below the theorem, referring the reader to Section 2 for the relevant notation and terminology:

Theorem 1.1. *Let $m = 2$ and $n = 4$. For every $Q \geq 1$ and $p \geq 2$ there exists a non-polyconvex Q-integrand whose associated energy is weakly lower semicontinuous in $W^{1,p}$.*

We do not claim that the integrand in Theorem 1.1 is quasiconvex. As previously mentioned, in the case of *continuous* Q-integrands, quasiconvexity is equivalent to weak lower semicontinuity of the associated energy [11, Theorem 0.2]. However, continuity of the integrand is not guaranteed in our setting, where the integrand is constructed as an envelope (4.2) that may in general only be lower semicontinuous. For this reason we formulate Theorem 1.1 directly in terms of lower semicontinuity of the associated energy. In particular, the theorem shows that polyconvexity is not necessary for lower semicontinuity of the energy in the Q-valued framework; see also the discussion below and Section 4.5.

Theorem 1.1 relies on our main technical contribution, which we state below, that establishes the existence of a measure on positively oriented 2-vectors in $\mathbb{R}^4$, with simple barycenter, that cannot be approximated by weighted Gaussian images of Lipschitz Q-graphs, for any fixed $Q \in \mathbb{N}$. We again refer the reader to Section 2 for the related notation:

Theorem 1.2. *There exists a measure μ on the positively oriented Grassmannian $\widetilde{Gr^+}(2,4)$ whose barycenter is parallel to the simple 2-vector $e_1 \wedge e_2$, with the following property. For any fixed $Q \in \mathbb{N}$, the measure μ cannot be approximated (in the weak-* topology) by weighted Gaussian images of Q-graphs associated to Q-valued Lipschitz functions $f : B \subseteq \mathbb{R}^2 \to \mathcal{A}_Q(\mathbb{R}^2)$ with $f \llcorner \partial B \equiv Q[0]$.*

Theorem 1.2 should be compared with [19, Theorem 1.5], where De Rosa, Lei, and Young proved that every measure supported on positively oriented m-vectors in $\mathbb{R}^n$, whose barycenter is a simple m-vector, can be approximated by weighted Gaussian images of Lipschitz multigraphs, possibly involving all values of Q. Our result thus shows that [19, Theorem 1.5] is sharp: the full range of values is indeed necessary for the approximation to hold.

The applications of [19, Theorem 1.5] were twofold. First, it was crucial in the proof of [19, Theorem 1.16], establishing that polyconvexity of geometric integrands is necessary for the validity of the rectifiability theorem for anisotropic stationary varifolds proved by De Philippis, De Rosa, and Ghiraldin [15, Theorem 1.2]. This strengthened the earlier result [18, Theorem A], which showed the necessity of quasiconvexity for rectifiability. Second, [19, Theorem 1.5] played a central role in proving that a classical integrand ψ is elliptic for multigraphs—i.e., the function $\bar{\psi}$ defined in (1.1) is a quasiconvex Q-integrand for every $Q \in \mathbb{N}$—if and only if ψ is polyconvex; see [19, Theorem 1.8].

It remains an open question whether there exists a fixed $Q \in \mathbb{N}$ such that every continuous integrand $\psi \in C^0(\mathbb{R}^{(n-m)\times m}, [0,\infty))$ is polyconvex if and only if $\bar{\psi}$ is a quasiconvex Q-integrand; see [19, Remark 1.14]. Although Theorem 1.2 sheds light on this question, a complete answer requires a suitable characterization in the sense of (1.1) of the quasiconvex envelope. In Remark 4.14 we provide compelling evidence that this is a subtle and challenging task.

Structure of the paper. In Section 2, we collect the necessary preliminaries and fix the notation used throughout the paper.

In Section 3, we prove Theorem 1.2, the first main result of the paper. More precisely, we construct a measure μ on the positively oriented Grassmannian $\widetilde{\mathrm{Gr}}^+(2,4)$, whose barycenter is a simple 2-vector, and which cannot be approximated–for fixed $Q \in \mathbb{N}$–by weighted Gaussian images of Lipschitz Q-graphs. While our construction is carried out in dimensions $(2,4)$, it can be extended to $\widetilde{\mathrm{Gr}}^+(m,n)$ for every $n-2 \geq m \geq 2$, as outlined in Remark 3.6.

This result is the multivalued graphical analogue of a theorem by Burago and Ivanov [6, Theorem 4], which demonstrates the existence of measures on the oriented Grassmannian $\widetilde{\mathrm{Gr}}(2,4)$ that cannot be approximated by weighted Gaussian images of surfaces bounded by simple planar curves. In particular, Theorem 1.2 shows that the approximation theorem [19, Theorem 1.5] is sharp: although [19] establishes that every positive measure on $\widetilde{\mathrm{Gr}}^+(m,n)$ with simple barycenter can be approximated by Gaussian images of Lipschitz multigraphs, this approximation requires allowing arbitrary values of Q. Beyond their intrinsic geometric significance, such filling results have key applications to the study of ellipticity conditions for anisotropic integrands. Indeed, [19, Theorem 1.8] depends crucially on [19, Theorem 1.5], just as the equivalence between elliptic and convex hulls in [6, Theorem 3] implies the approximation result [6, Theorem 2].

In Section 4 we construct a non-polyconvex Q-integrand with associated energy that is lower semicontinuous. To this aim, we first define a 1-integrand ψ on $\mathbb{R}^{2\times 2}$, which vanishes on (and only on) the support of the measure μ constructed in Theorem 1.2. Then we would like to characterize the quasiconvex envelope of the associated Q-integrand $\bar{\psi}$. However, carrying this out is technically challenging. The key difficulty is that the usual explicit candidate A for the quasiconvex envelope (4.2)—in the spirit of Dacorogna's formula for quasiconvex envelopes of 1-integrands (see [8, Section 6.3], [7])—produces a Q-integrand that in general may be just lower semicontinuous, rather than continuous. As discussed in greater detail in Remark 4.9, if a sequence of Q-matrices with Q distinct values of multiplicity one converges to a Q-matrix supported on the same value with multiplicity Q, there may be a lower semicontinuity gap. Indeed the class of competitors for the envelope (4.2) evaluated at the limit is strictly larger than the class of competitors for the envelope evaluated along the sequence. As a result, the continuous framework developed in [11] cannot be directly applied. We thus undertake a careful analysis of the envelope A of the integrand ψ, culminating in Theorem 4.10. In this result, we show that the integrand A gives rise to a weakly lower semicontinuous energy on $W^{1,p}$ for every $p \geq 2$.

The potential loss of continuity motivates us to introduce a slight variation of quasiconvexity: Property A. For lower semicontinuous integrands with the same type of dependence as A, this condition is essentially equivalent to weak lower semicontinuity of the associated energy functional, as shown in Proposition 4.11. In general, this condition is weaker than the quasiconvexity introduced in [11]. Nonetheless, when the Q-integrand is continuous, Property A is equivalent to quasiconvexity.

A key technical tool we use in the proofs of Section 4 is the piecewise affine approximation Lemma 5.1, proved in Section 5. Inspired by [9], it provides an approximation of the energy of a Lipschitz Q-valued map by Q-valued functions that are piecewise affine outside a set of arbitrarily small measure. This lemma is instrumental in our study of quasiconvex envelopes for Q-integrands and we believe it is of independent interest.

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2. Notation and preliminaries

Let $n, m \in \mathbb{N}$ with $n > m$. Throughout the paper we will mostly focus on $m = 2$, $n = 4$.

We will use B to denote the unit open ball in $\mathbb{R}^m$, while $B(x, r)$ denotes the open ball of center x and radius r in $\mathbb{R}^n$. We let D denote the unit cube $(-\frac{1}{2}, \frac{1}{2})^m$ in $\mathbb{R}^m$ and $D(x, r) = x + (-\frac{r}{2}, \frac{r}{2})^m$. We will also denote by $\{e_i\}_{i=1,\dots,n}$ the standard basis in $\mathbb{R}^n$, and $e_{ij} = e_i \wedge e_j$.

Furthermore, for every $x \in \mathbb{R}^m$ we denote by $[x]$ the Dirac mass in x.

2.1. **Currents.** We recall some basic definitions of the theory of currents, referring to [31] for additional material.

We denote by $\Lambda^m\mathbb{R}^n$ the set of m-vectors in $\mathbb{R}^n$ and with $\widetilde{Gr}(m, n)$ the oriented Grassmannian of oriented m-planes in $\mathbb{R}^n$. An m-current T is an element in the dual of the space $C^\infty_c(\mathbb{R}^n; \Lambda^m\mathbb{R}^n) =: \mathscr{D}^m(\mathbb{R}^n)$ of differential m-forms on $\mathbb{R}^n$. The boundary of a current T is the $(m-1)$-current defined by $\partial T(\eta) := T(d\eta)$ for every $\eta \in \mathscr{D}^m(\mathbb{R}^n)$. The mass of the m-current T is defined as $\mathbb{M}(T) := \sup\{T(\eta) : \eta \in \mathscr{D}^m(\mathbb{R}^n), |\eta| \leq 1\}$. We define a rectifiable m-current $T \in \mathbf{R}^m(\mathbb{R}^n)$ as a couple $T = (M, \theta\omega_P)$, where M is an m-rectifiable set in $\mathbb{R}^n$, $\theta : M \to \mathbb{R}$ is a locally $\mathcal{H}^m$-integrable positive function (the multiplicity) and $\omega_P : M \to \Lambda^m(\mathbb{R}^n)$ is an $\mathcal{H}^m$-measurable function (the orientation) such that for $\mathcal{H}^m$-a.e. point $x \in M$, $\omega_P(x)$ can be expressed in the form $v_1 \wedge \cdots \wedge v_m$, where $v_1, \dots, v_m$ form an orthogonal basis for $T_xM = P \in \widetilde{Gr}(m, n)$, the approximate tangent space to M at x. The orientation field is $P : M \to \widetilde{Gr}(m, n)$. We will often identify $\widetilde{Gr}(m, n)$ with unit simple m-vectors in $\Lambda^m\mathbb{R}^n$ by identifying P with ω_P. We define an integer-rectifiable m-current (integral current) $T \in \mathbf{I}^m(\mathbb{R}^n)$ as a rectifiable m-current $T = (M, \theta\omega_P)$ where θ takes only integer values. As a particular instance of integral current, for every oriented m-dimensional rectifiable surface Σ in $\mathbb{R}^n$ we denote the fundamental class of Σ, i.e., the multiplicity-one integral current associated to Σ, with $[\![\Sigma]\!] := (\Sigma, \omega_{T_x\Sigma}) \in \mathbf{I}^m(\mathbb{R}^n)$, where $\omega_{T_x\Sigma}(x)$ is an m-vector representing the oriented tangent space to Σ at x.

We also recall the notions of push-forward and slicing of a current. Given an integer-rectifiable m-current $T = (M, \theta\omega_P)$ and a Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$, we define the slice of T by f at $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$ by $\langle T, f, t\rangle := (M_t, \theta_t\omega_t)$ where $M_t = f^{-1}(t)$, $\theta_t = (\theta \llcorner M_t)\chi_{\{\nabla^M f \neq 0\}}$ and $\omega_t = \omega_P \llcorner \frac{\nabla^M f}{|\nabla^M f|}$. For $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$ the slice $\langle T, f, t\rangle$ is well-defined and is an integral $(m-1)$-current. Given a Lipschitz map $f : U \subseteq \mathbb{R}^m \to V \subseteq \mathbb{R}^n$ with V open, if f is proper, i.e. it satisfies $f^{-1}(K) \cap \operatorname{spt}(T)$ is a compact subset of U whenever K is a compact subset of V, the push-forward of $T \in \mathbf{R}^\ell(\mathbb{R}^m)$ by f is the rectifiable ℓ-current on $\mathbb{R}^n$ defined by $f_\sharp T(\eta) = \int_M \theta(x)\langle(\eta(f(x)), (df_\sharp\omega_P)(x)\rangle \, d\mathcal{H}^\ell(x)$ for every $\eta \in \mathscr{D}^\ell(V)$, where $df_\sharp$ denotes the standard push-forward of ℓ-vectors induced by the 1-form df.

Lastly, for a rectifiable m-current $T = (M, \theta\omega_P) \in \mathbf{R}^m(\mathbb{R}^n)$, we define the weighted Gaussian image $\gamma_T \in \mathcal{M}(\widetilde{Gr}(m, n))$ of Σ as the pushforward $P_\sharp(\theta\mathcal{H}^m \llcorner M)$ of the measure $\theta\mathcal{H}^m \llcorner M$ under the orientation field $P : M \to \widetilde{Gr}(m, n)$.

2.2. **Multivalued maps.** For a given integer $Q \geq 1$ we define the space of Q-points in $\mathbb{R}^n$ as

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^{Q} [x_i] \, : \, x_i \in \mathbb{R}^\ell \text{ for every } i = 1, \dots, Q \right\},$$

equipped with the following distance $\mathcal{G}$. Given two Q-points $x = \sum_i [x_i], y = \sum_i [y_i] \in \mathcal{A}_Q(\mathbb{R}^n)$ we denote

$$\mathcal{G}(x, y) = \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_i |x_i - y_{\sigma(i)}|^2},$$

where $\mathcal{P}_Q$ denotes the group of permutations of $\{1, \dots, Q\}$. We can identify $\mathcal{A}_Q(\mathbb{R}^n)$ with the configuration space $(\mathbb{R}^k)^Q/\mathrm{Sym}(Q)$. Whenever clear from the context, we simply write $\mathcal{A}_Q$ instead of $\mathcal{A}_Q(\mathbb{R}^n)$. Translations are well-defined in $\mathcal{A}_Q$ only by points of full multiplicity. Given $x \in \mathbb{R}^n$ we denote by $\tau_x y = \sum_j [y_j + x]$ the translation by the point $Q[x] \in \mathcal{A}_Q$.

A (Lipschitz) Q-valued function is a (Lipschitz) map from an open set $U \subseteq \mathbb{R}^m$ to $\mathcal{A}_Q(\mathbb{R}^{n-m})$. A multigraph T_f is the graph of the Q-valued function f, defined by

$$T_f = \{(x, y) \in U \times \mathbb{R}^{n-m}, y \in \mathrm{spt} f(x)\};$$

note that for all x, $\mathrm{spt}(f(x))$ has cardinality at most Q. A Lipschitz multigraph naturally supports an integral current [13, Definition 1.10]. This can be defined using [12, Lemma 1.1], which ensures that, for every Lipschitz Q-valued function $f : U \to \mathcal{A}_Q(\mathbb{R}^{n-m})$ defined on a measurable subset U of $\mathbb{R}^m$, there exists a countable measurable partition $(U_i)_{i\in\mathbb{N}} \subseteq U$ and Lipschitz functions $f^i_j : U_i \to \mathbb{R}^{n-m}$ for $j = 1, \dots, Q$, such that $f \llcorner U_i = \sum_{j=1}^{Q} [f^i_j]$. Assuming that each f^i_j is proper, one can define the integral current $\mathbf{T}_f$ associated to the multigraph T_f as

$$\mathbf{T}_f := (\cdot, f(\cdot))_\sharp [\![U]\!] := \sum_{i,j} (\cdot, f^i_j(\cdot))_\sharp [\![U_i]\!].$$

We further refer to [13, 12, 11] for the relevant results concerning Q-valued functions.

2.3. **Anisotropic integrands.** We now recall some notions concerning anisotropic energies defined on multigraphs, following [11].

Definition 2.1. A measurable map $F : (\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q \to \mathbb{R}$ is a Q-integrand if for every permutation $\sigma \in \mathcal{P}_Q$ it holds

$$F(a_1, ..., a_Q, X_1, ..., X_Q) = F(a_{\sigma(1)}, ..., a_{\sigma(Q)}, X_{\sigma(1)}, ..., X_{\sigma(Q)}).$$

In particular, we can evaluate a Q-integrand as above on elements $(a, X) \in \mathcal{A}_Q(\mathbb{R}^{n-m} \times \mathbb{R}^{(n-m)\times m})$, meaning that $F(a, X)$ is the value of F at a selection representing (a, X) in $(\mathbb{R}^{n-m} \times \mathbb{R}^{(n-m)\times m})^Q$. With a slight abuse of notation in the following we will not distinguish between elements in $(\mathbb{R}^{n-m} \times \mathbb{R}^{(n-m)\times m})^Q$ and $\mathcal{A}_Q(\mathbb{R}^{n-m} \times \mathbb{R}^{(n-m)\times m})$ when evaluating Q-integrands. In particular, given a weakly differentiable Q-valued map $f : \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{n-m})$, the expression $F(f, \nabla f)$ is well-defined almost everywhere. We then define the notion of quasiconvexity and polyconvexity considered in [11].

Definition 2.2. We say that a locally bounded Q-integrand $F : (\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q \to \mathbb{R}$ is quasiconvex if for every Q-valued affine map $u(x) = \sum_{j=1}^{J} q_j [a_j + X_j \cdot x]$ with $a_j \neq a_i$ for $i \neq j$, it holds

$$F(u(0), \nabla u(0)) \le \fint_D F(\underbrace{a_1, \dots, a_1}_{q_1}, \dots, \underbrace{a_J, \dots, a_J}_{q_J}, \nabla f_1(x), \dots, \nabla f_J(x)) dx, \tag{2.1}$$

for every collection of maps $f_j \in Lip(D, \mathcal{A}_{q_j}(\mathbb{R}^{n-m}))$ satisfying $f_j \llcorner \partial D = q_j [a_j + X_j \cdot] \llcorner \partial D$.

Here and in the following, with a slight abuse of notation, with $f \llcorner \partial D = [a + X \cdot] \llcorner \partial D$ for $a \in \mathbb{R}^{n-m}$ and $X \in \mathbb{R}^{(n-m)\times m}$ we mean that for every $x \in \partial D$ it holds $f(x) = [a + X \cdot x]$.

Definition 2.3. Let $\tau(n - m, m) := \sum_{j=1}^{\min\{n-m,m\}} \binom{n-m}{j} \binom{m}{j}$ and define $ad : \mathbb{R}^{(n-m)\times m} \to \mathbb{R}^{\tau(n-m,m)}$ as $ad(X) = (X, \mathrm{adj}_2(X), \dots, \mathrm{adj}_{\min\{n-m,m\}}(X))$, where $\mathrm{adj}_k X$ stands for the matrix of all $k \times k$ minors of X. A Q-integrand $F : (\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q \to \mathbb{R}$ is said to be polyconvex if for every $a \in (\mathbb{R}^{n-m})^Q$ there exists a convex function $g_a : (\mathbb{R}^{\tau(n-m,m)})^Q \to \mathbb{R}$ such that

$$F(a, X_1, \dots, X_Q) = g_a(ad(X_1), \dots, ad(X_Q)).$$

The definition of polyconvexity in Definition 2.3 is equivalent to require that the integrand $F(a, \cdot)$ can be extended to a convex function on $((\bigwedge^m \mathbb{R}^n)^+)^Q \subseteq (\bigwedge^m \mathbb{R}^n)^Q$, where $(\bigwedge^m \mathbb{R}^n)^+$ denotes the subset of m-vectors whose projection to $\mathbb{R}^m \subseteq \mathbb{R}^n$ is positively oriented. This can be done in the following way. Given a matrix $X \in \mathbb{R}^{(n-m)\times m}$, we can associate to it a simple m-vector considering

$$M(X) := \begin{pmatrix} id_{m\times m} \\ X \end{pmatrix} \in \mathbb{R}^{n\times m}$$

and defining the simple m-vector $\bigwedge M(X) := v_1 \wedge \dots \wedge v_m$, where $v_1, \dots, v_m$ are the columns of $M(X)$. This procedure gives a map $\bigwedge M : \mathbb{R}^{(n-m)\times m} \to \bigwedge^m \mathbb{R}^n$, and one can calculate that the coordinates

$\bigwedge M(X)$ in the standard basis of $\bigwedge^m \mathbb{R}^n$ are the minors of X of any order, which in turn coincide with the $m \times m$ minors of $M(X)$ i.e., $ad(X)$. This procedure can be extended by linearity to elements in $(\mathbb{R}^{(n-m)\times m})^Q$, which can be identified with Q copies of simple m-vectors modulo permutations.

The notion of quasiconvex Q-integrand is particularly relevant, as in the case of continuous integrands satisfying growth conditions it is equivalent to the lower semicontinuity of the associated energy. Indeed, by [11, Theorem 0.2] the following characterization holds.

Theorem 2.4 ([11])**.** *Let $p \in [1, +\infty), U \subseteq \mathbb{R}^m$ be a bounded open set and $F : (\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q \to \mathbb{R}$ be a continuous Q-integrand. If F is quasiconvex and satisfies for some $C > 1$:*

$$0 \le F(a, X) \le C(1 + |a|^q + |X|^p) \qquad \forall (a, X) \in (\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q,$$

where $q = 0$ if $p > m$, $q = p^*$ if $p < m$ and $q \ge 1$ is any exponent if $p = m$, then the associated energy

$$\mathcal{F} : f \in Lip(U, \mathcal{A}_Q(\mathbb{R}^{n-m})) \mapsto \fint_U F(f, \nabla f)$$

is weakly lower semicontinuous in $W^{1,p}(U, \mathcal{A}_Q(\mathbb{R}^{n-m}))$. Conversely, if $\mathcal{F}$ is weakly* lower semicontinuous in $W^{1,\infty}(U, \mathcal{A}_Q(\mathbb{R}^{n-m}))$, then F is quasiconvex.

As in the single-valued case, showing that an integrand is quasiconvex is not easy in general, while polyconvexity may be easier to check. By [11, Theorem 0.5], we have the following implication.

Theorem 2.5 ([11])**.** *Every locally bounded polyconvex Q-integrand is quasiconvex.*

Theorem 1.1 is one of the main results of this paper, and is in spirit a counterexample to the reverse implication in Theorem 2.5. However, as noted in the introduction, in the following we will work with semicontinuous integrands, hence the theory of [11] does not apply. Theorem 1.1 is a Q-integrands counterpart of the classical results for 1-integrands proved by Sverak [32] and Alibert and Dacorogna [2], who showed the existence of a continuous 1-integrand $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ that is quasiconvex (for graphs of single-valued Lipschitz functions) but not polyconvex.

We now compare Definition 2.2 and Definition 2.3 with [19, Definition 1.7] and [19, Definition 1.9] respectively. The latter are tailored to the study of anisotropic integrands on rectifiable currents. Given a 1-integrand $\psi : \mathbb{R}^{(n-m)\times m} \to \mathbb{R}$, we denote

$$\bar\psi(X) = \sum_{j=1}^{Q} \psi(X_j), \qquad \text{for all } X_1, \dots, X_Q \in \mathbb{R}^{(n-m)\times m},$$

which can be extended in the trivial way to $(\mathbb{R}^{n-m})^Q \times (\mathbb{R}^{(n-m)\times m})^Q$. It is easy to see that this defines a Q-integrand.

If $\Psi : \bigwedge^m \mathbb{R}^n \to (0, +\infty)$ is a 1-homogeneous function, an energy E_Ψ on rectifiable currents can be defined as follows: for every $T = (E, \theta\omega_P) \in \mathbf{R}^m(\mathbb{R}^n)$ set

$$E_\Psi(T) = \int_E \theta(x)\Psi(\omega_P(x))\, d\mathcal{H}^m(x).$$

Thanks to the operator $\bigwedge M$ previously introduced, the restriction of Ψ to the positively oriented unit simple m-vectors can be seen as a 1-integrand on $\mathbb{R}^{(n-m)\times m}$. If we let $\psi : \mathbb{R}^{(n-m)\times m} \to (0, \infty)$, $\psi(X) := \Psi(\bigwedge M(X))$ for all X, by the area formula for Q-valued graphs [12, Corollary 1.11] we have

$$\int_U \bar\psi(\nabla f) = E_\Psi(\mathbf{T}_f)$$

for any Lipschitz Q-valued function $f : U \subseteq \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^{n-m})$. It is therefore interesting to study the quasiconvexity of $\bar\psi$ as it relates to the lower semicontinuity of the functional E_Ψ defined on Q-valued graphs. This is indeed the focus of [19]. Hence, the notion of polyconvexity considered in [19] refers to the classical polyconvexity [26, Definition 4.2] of ψ, requiring the existence of a 1-homogeneous convex function $\Psi : (\Lambda^m\mathbb{R}^n)^+ \to \mathbb{R}$ such that $\psi = \Psi \circ \Lambda$.

Surprisingly, in [19, Theorem 1.8] it is proved the following:

Theorem 2.6 ([19]). *Let $\psi \in C^0(\mathbb{R}^{(n-m)\times m}, [0,+\infty))$. Then ψ is polyconvex if and only if $\bar{\psi}$ is a quasiconvex Q-integrand for every $Q \in \mathbb{N}$.*

Intuitively, our Theorem 1.1 indicates that Theorem 2.6 is a peculiarity of "geometric" Q-integrands, and not extendable to general Q-integrands. One could hope to use our non-approximability Theorem 1.2 to show that Theorem 2.6 fails if the value Q is fixed, however there is a fundamental obstruction in doing so, that we discuss in Remark 4.14.

3. A non-approximable Grassmannian measure: Proof of Theorem 1.2

In the rest of the paper, for the sake of exposition we are going to assume $n = 4$ and $m = 2$, though all results generalize to any $n-2 \geq m \geq 2$, as explained in Remark 3.6. We recall that we will often identify $\Lambda^2\mathbb{R}^4$ with the oriented Grassmannian $\widetilde{Gr}(2,4)$. Moreover, for brevity we denote by e_{ij} the elements $e_i \wedge e_j$ of the canonical basis of $\Lambda^2\mathbb{R}^4$.

The goal of this section is to prove Theorem 1.2. The proof is inspired by [6, Theorem 4], where Burago and Ivanov show how to construct a measure μ which cannot be approximated by means of smooth surfaces. We instead study the corresponding question with approximation by means of Lipschitz multigraphs.

The measure will be defined as the sum of three atoms. The choice of the supporting 2-vectors is rather delicate, as we need them to satisfy several constraints: the barycenter must be parallel to e_{12}, the scalar products with e_{12} need to be strictly positive, and they all need to have same norm in order to be able to use the identification of $\Lambda^2\mathbb{R}^4$ with $\widetilde{Gr}(2,4)$. Indeed, this identification requires the 2-vectors to be unitary, and in order to preserve the property of the barycenter, we need to renormalize the 2-vectors with the same constant.

On top of these requirements, in order to use the measure μ in combination with Proposition 3.5 to prove Theorem 1.2, we need to find a linear transformation of $\mathbb{R}^4$ that preserves the barycenter property and the positive orientation, but maps two of the 2-vectors to be almost parallel to e_{12} and one of them to be almost parallel to e_{34}. With all these constraints in mind, we make the following Ansatz, considering the simple 2-vectors:

$$\begin{aligned} v_1 &= \left(\delta e_1 + \tfrac{1}{2}\left(e_3 + \varepsilon^2\delta e_2\right)\right) \wedge \left(\delta e_2 + \tfrac{1}{2}\left(e_4 + \varepsilon^2\delta e_1\right)\right), \\ v_2 &= \left(\delta e_1 - \tfrac{1}{2}\left(e_3 + \varepsilon^2\delta e_2\right)\right) \wedge \left(\delta e_2 - \tfrac{1}{2}\left(e_4 + \varepsilon^2\delta e_1\right)\right), \\ v_3 &= \tfrac{1}{2}\left(e_4 + \varepsilon^2\delta e_1\right) \wedge \left(e_3 + \varepsilon^2\delta e_2\right), \end{aligned} \tag{3.1}$$

where the parameters $\varepsilon, \delta > 0$ will be chosen later, and we assume ε to be small. Note that, for every $\varepsilon, \delta > 0$, the barycenter condition $v_1 + v_2 + v_3 = 2\delta e_{12}$ is satisfied, since

$$v_1 + v_2 = 2\delta^2 e_{12} + \tfrac{1}{2}(e_3 + \varepsilon^2\delta e_2) \wedge (e_4 + \varepsilon^2\delta e_1) = 2\delta^2 e_{12} - v_3.$$

One can also check by an explicit calculation that v_i for $i = 1,2,3$ can be expressed in the form $\Lambda M(X_i)$ for $X_i \in \mathbb{R}^{2\times 2}$, up to a multiplicative constant. We check this on v_1, since for v_2 the computation is analogous and for v_3 it is trivial. We have

$$v_1 = \delta^2(1 - \tfrac{\varepsilon^4}{4})e_{12} + \tfrac{1}{4}\varepsilon^2\delta(-e_{13} + e_{24}) + \tfrac{1}{2}\delta(e_{14} - e_{23}) + \tfrac{1}{4}e_{34}.$$

We define

$$X_1 = \frac{1}{\delta(4-\varepsilon^4)}\begin{pmatrix} 2 & -\varepsilon^2 \\ -\varepsilon^2 & 2 \end{pmatrix},$$

so that computing $\Lambda M(X_1)$ yields

$$\Lambda M(X_1) = e_{12} + \tfrac{\varepsilon^2}{\delta(4-\varepsilon^4)}(-e_{13} + e_{24}) + \tfrac{2}{\delta(4-\varepsilon^4)}(e_{14} - e_{23}) + \tfrac{1}{\delta^2(4-\varepsilon^4)}e_{34},$$

which coincides with $\frac{4}{\delta^2(4-\varepsilon^4)}v_1$. We now aim to find δ as a function of ε such that:

$$\|v_1\| = \|v_2\| = \|v_3\|,$$

where the norm is the Euclidean norm induced on $\Lambda^2(\mathbb{R}^4)$ by the standard orthonormal basis e_{ij}. Since the expressions of v_1 and v_2 differ only by the sign of symmetric terms, we have $\|v_1\| = \|v_2\|$. Moreover we can explicitly compute:

$$\|v_1\|^2 = \Big(1 - \tfrac{\varepsilon^4}{4}\Big)^2 \delta^4 + \Big(\tfrac{1}{2} + \tfrac{\varepsilon^4}{8}\Big)\delta^2 + \tfrac{1}{16}, \tag{3.2}$$

and

$$\|v_3\|^2 = \tfrac{1}{4}\big(\varepsilon^8\delta^4 + 2\varepsilon^4\delta^2 + 1\big). \tag{3.3}$$

We can now equate (3.2) with (3.3), and letting $t = \delta^2$, we get the quadratic equation:

$$(16 - 8\varepsilon^4 - 3\varepsilon^8)\, t^2 \; + \; (8 - 6\varepsilon^4)\, t \; - \; 3 = 0,$$

which admits two distinct real solutions if and only if the discriminant

$$\Delta = 64\big(4 - 3\varepsilon^4\big) > 0, \qquad \Leftrightarrow \qquad 0 < \varepsilon < \big(\tfrac{4}{3}\big)^{\frac{1}{4}}.$$

On the same interval the second order coefficient $16 - 8\varepsilon^4 - 3\varepsilon^8$ is positive, hence we have exactly one positive root.

We conclude that, for every $0 < \varepsilon < \big(\frac{4}{3}\big)^{1/4}$, there exists a unique $\delta(\varepsilon) > 0$ such that $\|v_1\| = \|v_2\| = \|v_3\|$ and the value is:

$$\delta(\varepsilon) = \sqrt{\frac{6\varepsilon^4 - 8 + \sqrt{(8-6\varepsilon^4)^2 + 12\big(16 - 8\varepsilon^4 - 3\varepsilon^8\big)}}{2\big(16 - 8\varepsilon^4 - 3\varepsilon^8\big)}} = \sqrt{\frac{-8 + 16 + O(\varepsilon^4)}{32 + O(\varepsilon^4)}} = \frac{1}{2} + O(\varepsilon^4).$$

As mentioned above, since v_1 and v_2 are not almost parallel to e_{12}, we will consider the linear transformation $R = diag(1,1,\varepsilon,\varepsilon)$, that maps v_1,v_2,v_3 respectively in

$$\begin{aligned}
w_1 &= \big(\, \delta e_1 + \tfrac{1}{2}\big(\varepsilon e_3 + \varepsilon^2\delta e_2\big)\big) \wedge \big(\, \delta e_2 + \tfrac{1}{2}\big(\varepsilon e_4 + \varepsilon^2 \delta e_1\big)\big),\\
w_2 &= \big(\, \delta e_1 - \tfrac{1}{2}\big(\varepsilon e_3 + \varepsilon^2\delta e_2\big)\big) \wedge \big(\, \delta e_2 - \tfrac{1}{2}\big(\varepsilon e_4 + \varepsilon^2 \delta e_1\big)\big),\\
w_3 &= \tfrac{1}{2}\big(\varepsilon e_4 + \varepsilon^2\delta e_1\big) \wedge \big(\varepsilon e_3 + \varepsilon^2 \delta e_2\big).
\end{aligned} \tag{3.4}$$

It is straightforward to compute the following norms and scalar product, which are going to be useful:

$$\langle w_1, e_{12}\rangle = \delta^2\left(1 - \frac{\varepsilon^4}{4}\right) = \frac{1}{4} + O(\varepsilon^4), \tag{3.5}$$

$$\|w_1\|^2 = \left(1 - \frac{\varepsilon^4}{4}\right)^2 \delta^4 \; + \; \Big(\frac{\varepsilon^2}{2} + \frac{\varepsilon^6}{8}\Big)\delta^2 \; + \; \frac{\varepsilon^4}{16} = \frac{1}{16} + O(\varepsilon^2), \tag{3.6}$$

$$\langle w_3, e_{34}\rangle \; = \; -\frac{\varepsilon^2}{2}, \quad \|w_3\|^2 \; = \; \frac{1}{4}\big(\varepsilon^2 + \varepsilon^4\delta^2\big)^2 = \frac{\varepsilon^4}{4} + O(\varepsilon^6). \tag{3.7}$$

Definition 3.1. The oriented plane associated to e_{12}, respectively e_{34}, will be called horizontal (or h-plane), respectively vertical (or v-plane). We let P_H, P_V denote the orthogonal projection on the h-plane, v-plane respectively. An oriented plane $\pi \in \widetilde{Gr}(2,4)$ is said to be ε-horizontal (resp. ε-vertical) if $P_H \llcorner \pi$ (resp. $P_V \llcorner \pi$) is an orientation-preserving map, and its inverse is $(1+\varepsilon)$-Lipschitz.

We explicitly observe that an ε-horizontal (resp. ε-vertical) oriented plane is positively oriented with respect to the oriented plane e_{12} (resp. the oriented plane e_{43}).

The following lemma will be useful for checking Definition 3.1 on the 2-vectors w_i defined in (3.4):

Lemma 3.2. *Let v be a simple 2-vector in $\mathbb{R}^4$ associated to an oriented plane π.*

If v satisfies

$$\langle v, e_{12}\rangle \geq (1+\varepsilon)^{-1}\|v\|. \tag{3.8}$$

then π is ε-horizontal. Analogously, if v satisfies

$$\langle v, e_{43}\rangle \geq (1+\varepsilon)^{-1}\|v\|, \tag{3.9}$$

then π is ε-vertical.

Proof. Without loss of generality we assume $\|v\| = 1$. We will prove just the first statement, as the second one is equivalent, up to a rotation.

Let $\theta_1 \le \theta_2 \in [0, \pi/2)$ be the principal angles between the oriented planes v and e_{12}. Since v and e_{12} are unit simple 2-vectors, we have:

$$\langle v, e_{12}\rangle = \cos\theta_1 \cos\theta_2, \tag{3.10}$$

which combined with (3.8) implies that $\cos\theta_1\cos\theta_2 \ge (1+\varepsilon)^{-1}$. Recalling that $\cos(\theta_i) \in [0,1]$ for $i = 1, 2$, this in turn implies that $\cos(\theta_i) \ge (1+\varepsilon)^{-1}$ for $i = 1, 2$.

In an appropriate orthonormal basis the restriction $P_H \llcorner \pi$ is represented by the diagonal matrix

$$\begin{pmatrix} \cos\theta_1 & 0 \\ 0 & \cos\theta_2 \end{pmatrix}.$$

Because $\cos\theta_i > 0$, $P_H\llcorner\pi$ is invertible. Moreover $\|(P_H\llcorner\pi)^{-1}\| = \min\{\cos\theta_1, \cos\theta_2\}^{-1} \le 1 + \varepsilon$. $\square$

Thanks to Lemma 3.2, it is easy to observe that, for ε small, the oriented plane associated with w_3 is ε-vertical, while those associated with w_1, w_2 are ε-horizontal. Indeed, by (3.5),(3.6) and (3.7), it holds

$$\langle w_3, e_{43}\rangle = \frac{\varepsilon^2}{2} \ge (1+\varepsilon)^{-1}\|w_3\|, \quad \langle w_1, e_{12}\rangle = \frac{1}{4} + O(\varepsilon^4) \ge (1+\varepsilon)^{-1}\|w_1\|. \tag{3.11}$$

Remark 3.3. Given an ε-horizontal plane π_1, since the map $(P_H\llcorner\pi_1)^{-1}$ is $(1+\varepsilon)$-Lipschitz, then the Jacobian satisfies $J(P_H\llcorner\pi_1)^{-1} \le (1+\varepsilon)^2$.

Given an ε-vertical plane π_2, then the map $P_H\llcorner\pi_2$ is $\sqrt{3\varepsilon}$-Lipschitz, and hence has Jacobian bounded by 3ε. Indeed, for every $x \in \pi_2$ we use the orthogonal decomposition

$$|x|^2 = |P_H x|^2 + |P_V x|^2. \tag{3.12}$$

Since π_2 is ε-vertical, we have that $(P_V\llcorner\pi_2)^{-1}$ is $(1+\varepsilon)$-Lipschitz, which implies

$$|P_V x| \ \ge\ \frac{|x|}{1+\varepsilon}. \tag{3.13}$$

Inserting (3.13) into (3.12) we obtain the desired bound:

$$|P_H x|^2 \ \le\ |x|^2\Big(1 - \tfrac{1}{(1+\varepsilon)^2}\Big) \ =\ |x|^2 \cdot \frac{\varepsilon(2+\varepsilon)}{(1+\varepsilon)^2} \ \le\ 3\varepsilon|x|^2.$$

Analogously, the map $(P_V\llcorner\pi_2)^{-1}$ is $(1+\varepsilon)$-Lipschitz, hence the Jacobian satisfies $J(P_V\llcorner\pi_2)^{-1} \le (1+\varepsilon)^2$, and the map $P_V\llcorner\pi_1$ is $\sqrt{3\varepsilon}$-Lipschitz, and hence has Jacobian bounded by 3ε.

We now fix $f \in Lip(B, \mathcal{A}_Q(\mathbb{R}^2))$ with $f\llcorner\partial B \equiv Q[0]$. We consider $\mathbf{T}_f$ and in this section we drop the subscript f in $\mathbf{T}_f$ as the function f will be fixed. Let $\mathrm{spt}\mathbf{T} \subseteq \mathbb{R}^4$ denote the support of $\mathbf{T}$. Extend f on $\mathbb{R}^2 \setminus B$ as $Q[0]$, and consequently $\mathbf{T}$. Let T_0 denote the subset of points $p \in \mathrm{spt}\mathbf{T}$ of integer density for $\mathbf{T}$ admitting a tangent plane.

Note that $|(\mathrm{spt}\mathbf{T}) \setminus T_0| = 0$ since f is Lipschitz, see [13, Theorem 1.13], and T_pT_0 is well-defined for every $p \in T_0$. We define

$$\begin{aligned} &T_h := \{p \in T_0 : T_pT_0 \text{ is } \varepsilon\text{-horizontal}\}, \quad T_v := \{p \in T_0 : T_pT_0 \text{ is } \varepsilon\text{-vertical}\}, \\ &\mathbf{T}^H := \mathbf{T}\llcorner T_h, \qquad \mathbf{T}^V := \mathbf{T}\llcorner T_v, \qquad \mathbf{T}^M := \mathbf{T} - \mathbf{T}^H - \mathbf{T}^V. \end{aligned} \tag{3.14}$$

We notice that $\mathbf{T}^H$ and $\mathbf{T}^V$ are well-defined, since $\mathbf{T}$ is a rectifiable current and T_h and T_V are measurable subsets of T_0. Moreover, by definition the tangent planes in $\mathbf{T}^V$ are positively oriented with respect to the v-plane. Given $p \in \mathbb{R}^4$ and $r > 0$, we also denote $\mathbf{B}^\square_r(p) = \mathbf{T}^\square\llcorner B_r(p)$, where the symbol $\square \in \{V, M, H\}$.

Lemma 3.4. *There exists $1 \gg \beta = \beta(Q) > 0$ with the following property. For $\varepsilon > 0$ small enough in (3.14), if $\mathbb{M}(\boldsymbol{B}^V_r(p)) \ge \beta r^2$ and $\mathbb{M}(\boldsymbol{B}^V_{2r}(p)) \le \beta(2r)^2$, then $\mathbb{M}(\boldsymbol{B}^M_{2r}(p)) \ge \frac{\beta}{2}r^2$.*

Assuming Lemma 3.4, we prove a global estimate on the mass of the vertical part.

Proposition 3.5. *For $\varepsilon > 0$ small enough in (3.14), there exists a universal constant $C > 0$ such that $\mathbb{M}(\mathbf{T}^M) \ge C\mathbb{M}(\mathbf{T}^V)$.*

Proof. We can assume without loss of generality that $\mathbb{M}(\mathbf{T}^V) > 0$. The proof follows from a covering argument and Lemma 3.4. Fix $0 < \beta \ll 1$ and ε small as in Lemma 3.4. For every $p \in T_v$ let $r(p)$ be the maximum radius r such that $\mathbb{M}(\mathbf{B}^V_r(p)) \geq \beta r^2$. Note that $r(p)$ is uniformly bounded from above (since $\operatorname{spt}\mathbf{T}^V$ is bounded as f is Lipschitz) and, since p is a density point, it is also strictly positive.

By Vitali's 5-times covering lemma [31, Lemma 3.4] applied to the family $\{B_{2r(p)}(p) : p \in T_v\}$, we find a countable set $\mathcal{I}$ of indices and a sub-family $\{B_{2r_i}(p_i)\}_{i\in\mathcal{I}}$ of balls, where $r_i := r(p_i)$, such that

$$\bigcup_{i\in\mathcal{I}} B_{2r_i}(p_i) = \bigcup_{p\in T_v} B_{10r(p)}(p) \quad \text{and} \quad B_{2r_i}(p_i) \cap B_{2r_j}(p_j) = \emptyset, \tag{3.15}$$

By definition of $r(p)$ it holds $\mathbb{M}(\mathbf{B}^V_\rho(p)) \leq \beta\rho^2$ for every $\rho > r(p)$. In particular, we can apply Lemma 3.4 to the balls $\mathbf{B}^V_{r_i}(p_i)$ and obtain

$$\mathbb{M}(\mathbf{B}^M_{2r_i}(p_i)) \geq \frac{\beta}{2} r_i^2 \geq \frac{1}{200}\mathbb{M}(\mathbf{B}^V_{10r_i}(p_i)),$$

where the second inequality follows from the choice of the radii in the cover. Using (3.15) we conclude $\mathbb{M}(\mathbf{T}^M) \geq \frac{1}{200}\mathbb{M}(\mathbf{T}^V)$. $\square$

Proof of Lemma 3.4. In the proof we let c denote a positive constant depending only on Q, whose value may change from line to line. We also assume wlog $p = 0$, since no assumptions on p have been made. Assume by contradiction that

$$\mathbb{M}(\mathbf{B}^M_{2r}) \leq \frac{\beta}{2}\pi r^2. \tag{3.16}$$

Step 1: We estimate from above the mass of the horizontal part of the graph.

We start recalling that the projection P_H restricted to an ε-horizontal plane is a bijection with inverse being an orientation-preserving $(1+\varepsilon)$-Lipschitz map, see Definition 3.1.

Fix $\rho \in (r, 2r)$. By the formula for the pushforward of a current [31], for every $\eta \in \mathscr{D}^2(\mathbb{R}^2)$ it holds

$$(P_H)_\sharp \mathbf{B}^H_\rho(\eta) = \int_{T_h \cap B_\rho} \langle \eta(P_H(x)), (dP_H(x))_\sharp \omega(x)\rangle d\mu_{\mathbf{T}}(x), \tag{3.17}$$

where ω is the orientation of $\mathbf{T}$ and $\mu_{\mathbf{T}}$ is the surface measure associated to the current $\mathbf{T}$. Since on T_h the tangent planes to $\mathbf{T}$ are all ε-horizontal, one can estimate thanks to Remark 3.3 $|(dP_H(x))_\sharp\omega(x)| \geq (1+\varepsilon)^{-2} \geq 1 - 2\varepsilon$ for every $x \in T_h$. Thus, since by duality $\sup_{|\eta|\leq 1, \eta\in\mathscr{D}^2(\mathbb{R}^2)}\langle \eta, \omega\rangle = |\omega|$ for every vector field $\omega : \mathbb{R}^2 \to \Lambda^2(\mathbb{R}^2)$, by the previous estimate and taking the $\sup_{|\eta|\leq 1, \eta\in\mathscr{D}^2(\mathbb{R}^2)}$ in (3.17) we get

$$\mathbb{M}((P_H)_\sharp \mathbf{B}^H_\rho) \geq (1-2\varepsilon)\int_{T_h\cap B_\rho} d\mu_{\mathbf{T}}(x) = (1-2\varepsilon)\mathbb{M}(\mathbf{B}^H_\rho).$$

Note that we do not have cancellations in $(P_H)_\sharp \mathbf{B}^H_\rho$, as all the tangent planes in $\mathbf{T}^H$ are positively oriented with respect to the h-plane. Recalling that $\mathbf{T}$ is the current associated to the Q-valued graph of f, we deduce that $\mathbf{B}^H_\rho$ projects onto the h-plane with multiplicity at most Q. We conclude that

$$Q\pi\rho^2 \geq \mathbb{M}((P_H)_\sharp \mathbf{B}^H_\rho) \geq (1-2\varepsilon)\mathbb{M}(\mathbf{B}^H_\rho). \tag{3.18}$$

Step 2: We estimate from below the mass of the vertical projection of the graph.

By the standing assumption on $\mathbb{M}(\mathbf{B}^V_r)$, we can argue as in Step 1 to show that for every $\rho \in [r, 2r]$ it holds

$$\mathbb{M}((P_V)_\sharp \mathbf{B}^V_\rho) \geq \frac{3}{4}\beta r^2. \tag{3.19}$$

Indeed, we recall that the projection P_V restricted to an ε-vertical plane is a bijection with inverse being an orientation-preserving $(1+\varepsilon)$-Lipschitz map, see Definition 3.1. By the formula for the pushforward of a current we have

$$(P_V)_\sharp \mathbf{B}^V_\rho(\eta) = \int_{T_v\cap B_\rho} \langle \eta(P_V(x)), (dP_V(x))_\sharp\omega(x)\rangle d\mu_{\mathbf{T}}(x),$$

where $\omega, \mu_{\mathbf{T}}$ are as above. Since on T_v the tangent planes to $\mathbf{T}$ are all ε-vertical, by Remark 3.3 one can estimate $|(dP_V(x))_\sharp\omega(x)| \geq (1+\varepsilon)^{-2} \geq 1-2\varepsilon$ for every $x \in T_v$. Thus, taking the sup over forms in $\mathscr{D}^2(\mathbb{R}^2)$ with norm bounded by 1 and using the assumption of the Lemma, we conclude (3.19) as follows:

$$\mathbb{M}((P_V)_\sharp \mathbf{B}^V_\rho) \geq (1-2\varepsilon)\mathbb{M}(\mathbf{B}^V_\rho) \geq (1-2\varepsilon)\frac{\beta}{2}\pi r^2 \geq \frac{3}{4}\beta r^2$$

by taking ε smaller than a universal constant.

Furthermore, we have

$$\mathbb{M}((P_V)_\sharp \mathbf{B}^M_\rho) \leq \mathbb{M}(\mathbf{B}^M_\rho) \leq \mathbb{M}(\mathbf{B}^M_{2r}) \leq \frac{\beta}{2} r^2$$
$$\mathbb{M}((P_V)_\sharp \mathbf{B}^H_\rho) \leq 3\varepsilon\mathbb{M}(\mathbf{B}^H_r) \leq 3\varepsilon\mathbb{M}(\mathbf{B}^H_{2r}) \leq c\varepsilon r^2,$$

where in the last line we used again the formula for the mass of the push-forward and that the Jacobian of P_V restricted to a ε-horizontal plane is bounded by 3ε, see Remark 3.3. This, combined with $\mathbf{T}\llcorner B_\rho - \mathbf{B}^M_\rho - \mathbf{B}^H_\rho = \mathbf{B}^V_\rho$, the subadditivity of the mass and the linearity of the pushforward, yields

$$\mathbb{M}((P_V)_\sharp(\mathbf{T}\llcorner B_\rho)) \geq \mathbb{M}((P_V)_\sharp\mathbf{B}^V_\rho) - \mathbb{M}((P_V)_\sharp\mathbf{B}^M_\rho) - \mathbb{M}((P_V)_\sharp\mathbf{B}^H_\rho) \geq r^2(\tfrac{\beta}{4} - c\varepsilon), \tag{3.20}$$

and the right-hand side is positive taking ε small.

Step 3: We bound from below the mass of the mixed part, deriving a contradiction with (3.16).

For a.e. $\rho \in (r, 2r)$, we have that $\mathbb{M}(\partial(P_V)_\sharp(\mathbf{T}\llcorner B_\rho))$ is finite. This, combined with the fact that $(P_V)_\sharp(\mathbf{T}\llcorner B_\rho)$ is a 2-dimensional integer current in the 2-dimensional v-plane, allows to apply the decomposition result [31, Theorem 27.6] to $(P_V)_\sharp(\mathbf{T}\llcorner B_\rho)$.

We consider the current $(P_V)_\sharp(\mathbf{T}\llcorner B_\rho)$. Denoting $\tilde\theta$ the (non-negative) density of such current, we define θ as the signed multiplicity of the current with orientation fixed to e_{43}, i.e. $\theta = \tilde\theta\,\omega_{(P_V)_\sharp\mathbf{T}}\cdot e_{43}$, and we extend it to zero on the whole v-plane outside $\operatorname{spt}(P_V)_\sharp(\mathbf{T}\llcorner B_\rho)$. We consider the following 2-dimensional finite perimeter sets in the v-plane:

$$U_j := \{\theta \geq j\}, \quad V_j := B_R \setminus U_j = \{\theta \leq j-1\}, \quad \text{for } j \in \mathbb{Z},$$

where B_R is a large ball containing the support of $(P_V)_\sharp(\mathbf{T}\llcorner B_\rho)$. Hence $\partial U_j \subset\subset B_R$ for every $j \geq 1$ and $\partial V_j \subset\subset B_R$ for every $j \leq 0$. Moreover, up to choosing R even bigger, we can guarantee that

$$|U_j| \leq |V_j| \ \text{ for every } j \geq 1, \qquad \text{and} \qquad |U_j| \geq |V_j| \ \text{ for every } j \leq 0.$$

In particular, denoting with $P_j = P(U_j, B_R) = P(V_j, B_R)$ the relative perimeter of the set U_j in B_R, we conclude from the isoperimetric inequality that

$$P_j^2 \geq c|U_j| \ \text{ for every } j \geq 1, \qquad \text{and} \qquad P_j^2 \geq c|V_j| \ \text{ for every } j \leq 0. \tag{3.21}$$

By [31, Theorem 27.6] it holds:

$$\partial((P_V)_\sharp(\mathbf{T}\llcorner B_\rho)) = \sum_{j\geq 1}\partial[\![U_j]\!] - \sum_{j\leq 0}\partial[\![V_j]\!], \quad \text{and} \quad \mathbb{M}(\partial(P_V)_\sharp(\mathbf{T}\llcorner B_\rho)) = \sum_{j\in\mathbb{Z}} P_j,$$

Moreover[1]

$$\mathbb{M}((P_V)_\sharp(\mathbf{T}\llcorner B_\rho)) = \int_{B_R}\sum_{j\geq 1}\chi_{\{\theta\geq j\}} + \chi_{\{\theta\leq -j\}}\,\mathrm{d}\mathcal{H}^2 = \sum_{j\geq 1}|U_j| + \sum_{j\leq 0}|V_j|.$$

Combining the previous two displayed equations with (3.21), we thus estimate

$$\mathbb{M}(\partial(P_V)_\sharp(\mathbf{T}\llcorner B_\rho))^2 \geq \sum_{j\in\mathbb{Z}} P_j^2 \geq c\left(\sum_{j\geq 1}|U_j| + \sum_{j\leq 0}|V_j|\right) = c\,\mathbb{M}((P_V)_\sharp(\mathbf{T}\llcorner B_\rho)). \tag{3.22}$$

Let $L^\square_r(p) = \mathbb{M}(\langle \mathbf{T}^\square, \mathrm{d}_p, \rho\rangle)$, for $\square \in \{H, V, M\}$, be the mass of the slice of $\mathbf{T}^\square$ by the (Euclidean) distance function to p. By the coarea formula [31, Lemma 28.5] it holds

$$\int_r^{2r} L^\square_\rho(p)\,\mathrm{d}\rho \leq \mathbb{M}(\mathbf{B}^\square_{2r}(p)). \tag{3.23}$$

[1] We recall that the characteristic function χ_E of a set E is defined as $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.

Moreover we explicitly observe that, since $\mathbf{T} = \mathbf{T}_f$, then $\partial\mathbf{T} = Q[\![\partial B]\!]$, which is entirely contained in the h-plane. Hence $(P_V)_\sharp(\partial\mathbf{T}) = 0$ and we deduce that

$$(P_V)_\sharp(\partial(\mathbf{T}\llcorner B_\rho)) = (P_V)_\sharp((\partial\mathbf{T})\llcorner B_\rho) + \langle\mathbf{T}, \mathrm{d}_p, \rho\rangle) = (P_V)_\sharp\langle\mathbf{T}, \mathrm{d}_p, \rho\rangle. \tag{3.24}$$

Combining (3.20), (3.22), (3.24), and using the sublinearity of the mass, we get

$$\begin{aligned} cr\sqrt{\beta/4 - c\varepsilon} \le \mathbb{M}(\partial P_{V\sharp}(\mathbf{T}\llcorner B_\rho)) &= \mathbb{M}((P_V)_\sharp(\partial(\mathbf{T}\llcorner B_\rho))) = \mathbb{M}((P_V)_\sharp\langle\mathbf{T}, \mathrm{d}_p, \rho\rangle) \\ &\le \sum_{\square\in\{V,H,M\}} \mathbb{M}((P_V)_\sharp\langle\mathbf{T}^\square, \mathrm{d}_p, \rho\rangle) \le L_\rho^V + L_\rho^M + \sqrt{3\varepsilon}L_\rho^H, \end{aligned} \tag{3.25}$$

where in the last line we used again the fact that P_V restricted to an ε-horizontal plane is $\sqrt{3\varepsilon}$-Lipschitz (as proved in Remark 3.3) and reasoned as in (3.18). Integrating this inequality from r to $2r$ and using (3.23), we get

$$\mathbb{M}(\mathbf{B}_{2r}^V) + \mathbb{M}(\mathbf{B}_{2r}^M) + \sqrt{3\varepsilon}\mathbb{M}(\mathbf{B}_{2r}^H) \ge cr^2\sqrt{\beta/4 - c\varepsilon}.$$

On the other hand, by (3.18) we have $\mathbb{M}(\mathbf{B}_{2r}^H) \le cr^2$, and by assumption $\mathbb{M}(\mathbf{B}_{2r}^V) \le 4\beta r^2$. Therefore, for β, ε small enough, we arrive at

$$\mathbb{M}(\mathbf{B}_{2r}^M) \ge r^2(c\sqrt{\beta/4 - c\varepsilon} - c\sqrt{\varepsilon} - 4\beta) \ge \frac{\beta}{2}r^2,$$

which contradicts (3.16) and thus concludes the proof. □

Remark 3.6. We remark that Theorem 1.1, and hence the consequent Theorem 1.2, can be generalized to elements in $\widetilde{Gr}(m,n)$ for $n-2 \ge m \ge 2$. Indeed, in this case, for any $\pi_1 \in \widetilde{Gr}(m,n)$ there exists an orthogonal m-plane $\pi_2 \in \widetilde{Gr}(m,n)$ such that $\pi_1 \cap \pi_2$ is at most $(m-2)$-dimensional. Thus, by replacing the h-plane and v-plane with π_1, π_2, one can still obtain (3.24), and one can show analogous bounds for the Jacobians as in Remark 3.3. Although the Lipschitz bounds of the vertical projection of an ε-horizontal plane will hold just on a 2-dimensional subspace of the ε-horizontal plane, this is enough to deduce the required smallness of the Jacobian and in the estimate (3.25). For the latter, we observe that $\langle\mathbf{T}^H, \mathrm{d}_p, \rho\rangle$ will now be an $(m-1)$-dimensional integer current, hence the tangential Jacobian will always be small, as every tangent $(m-1)$-plane of $\langle\mathbf{T}^H, \mathrm{d}_p, \rho\rangle$ will have at least one direction which is orthogonal to $\pi_1 \cap \pi_2$. On the contrary, this argument clearly fails in the codimension-one case $\widetilde{Gr}(n-1,n)$. This is not a surprise, as in codimension one Minkowski existence Theorem [28] guarantees that weighted Gaussian images of Lipschitz graphs are dense in the space of positive measures on $\widetilde{Gr}(n-1,n)$. Hence Theorem 1.2 fails, and the notions of polyconvexity and quasiconvexity coincide with the notion of convexity [8]. In particular, Theorem 1.1 fails as well.

With the previous results in hand we are able to prove Theorem 1.2.

Proof of Theorem 1.2. We first show a similar result for an atomic measure supported on the 2-vectors w_1, w_2, w_3 defined in (3.4), and then conclude the proof of the theorem via an affine transformation. Throughout the proof, we will denote with $P_{v_i}, P_{w_i} \in \widetilde{Gr}(2,4)$ the oriented 2-plane associated to v_i, w_i for $i = 1,2,3$.

Consider the atomic measure on $\widetilde{Gr}(2,4)$ defined by

$$\mu_0 := \sum_{i=1}^3 \|w_i\|[P_{w_i}].$$

Consider $U_V \subseteq \widetilde{\mathrm{Gr}}^+(2,4)$ the interior of the set of ε-vertical planes and U_M the complement of the ε-horizontal and ε-vertical planes. Note that, since P_{w_1}, P_{w_2} are ε-horizontal and $P_{w_3} \in U_V$ as computed in (3.11), it holds $\mu_0(U_M) = 0$ while $\mu_0(U_V) > 0$. For a given Q-valued Lipschitz map $f : B \to \mathcal{A}_Q(\mathbb{R}^2)$ with $f\llcorner\partial B \equiv Q[0]$, let $\mathbf{T}_f$ be the Q-valued graph associated to f and denote with $\gamma_{\mathbf{T}_f}$ the weighted Gaussian image of $\mathbf{T}_f$. Then Proposition 3.5 implies that $\gamma_{\mathbf{T}_f}(U_M) \ge c\gamma_{\mathbf{T}_f}(U_V)$, thus μ_0 cannot be approximated by Gaussian images of the form $\gamma_{\mathbf{T}_f}$.

Then consider the atomic measure μ on $\widetilde{Gr}(2,4)$:

$$\mu := \sum_{i=1}^{3} \|v_i\| [P_{v_i}] = \|v_1\| \sum_{i=1}^{3} [P_{v_i}].$$

Note that the barycenter of the measure μ is the horizontal plane. We aim to show that μ cannot be approximated by Gaussian images of Lipschitz Q-valued maps, which would conclude the proof of the theorem.

Let $R = diag(1,1,\varepsilon,\varepsilon)$, identified with the linear map $\mathbb{R}^4 \to \mathbb{R}^4$, which we recall satisfies $R(P_{v_i}) = P_{w_i}$ for $i = 1,2,3$. More precisely, recalling that $v_i = u_i^1 \wedge u_i^2$, with $u_i^1, u_i^2 \in \mathbb{R}^4$ defined in (3.1), and the definition of w_i in (3.4), we have

$$w_i = (Ru_i^1) \wedge (Ru_i^2).$$

This in particular implies that the tangential Jacobian of the linear map R with respect to P_{v_i} is:

$$J_{P_{v_i}} R = \frac{\|(Ru_i^1) \wedge (Ru_i^2)\|}{\|u_i^1 \wedge u_i^2\|} = \frac{\|w_i\|}{\|v_i\|}. \tag{3.26}$$

In the following, we warn the reader that we will compute push-forwards sometimes in the sense of currents and sometimes in the sense of measures. There is no risk of confusion, as for every instance the correct interpretation depends on the object we push-forward. This will also determine whether we obtain tangential Jacobians in the equations.

For every rectifiable current $\mathbf{T} = (T, \theta\omega_P)$, we claim that

$$\gamma_{R_\sharp \mathbf{T}} = R_\sharp \gamma_{(J_T R)\mathbf{T}}, \tag{3.27}$$

where $J_T R$ denotes the tangential Jacobian of the linear map R on T. Indeed, recall that $R_\sharp \mathbf{T} = (R(T), (\theta \circ R^{-1})\omega_{R(P\circ R^{-1})})$. By the definition of Gaussian image and the area formula we have

$$\begin{aligned}\gamma_{R_\sharp \mathbf{T}} &= (R(P\circ R^{-1}))_\sharp((\theta\circ R^{-1})\,\mathcal{H}^2 \llcorner R(T)) = (R(P\circ R^{-1}))_\sharp(R_\sharp((J_T R)\,\theta\,\mathcal{H}^2\llcorner T)) = \\ &= R_\sharp\Big(((P\circ R^{-1})\circ R)_\sharp((J_T R)\,\theta\,\mathcal{H}^2\llcorner T)\Big) = R_\sharp P_\sharp((J_T R)\,\theta\,\mathcal{H}^2\llcorner T) = R_\sharp \gamma_{(J_T R)\,\mathbf{T}}.\end{aligned}$$

Moreover, in the case of a current $\mathbf{T}_f$ associated to the Q-valued graph of f, it holds

$$R_\sharp \mathbf{T}_f = \mathbf{T}_{\varepsilon f} \tag{3.28}$$

Indeed, if $f = \sum_{j=1}^{Q} [f_j]$ is a local selection of f at $y \in B$ then $\operatorname{spt}\mathbf{T}_f$ is locally around y given by $\bigcup_{j=1}^{Q} \{(y, f_j(y))\} \subseteq \mathbb{R}^4$ and its tangent space (defined for a.e. $y \in B$) is given by the union of the planes identified by $\Lambda M(\nabla f_j(y))$. Thus, the tangent planes in $R_\sharp \mathbf{T}_f$ are locally the union of planes associated to the 2-vectors $\Lambda M(\nabla(\varepsilon f))$ and thus one can explicitly compute $R_\sharp \mathbf{T}_f$.

Finally, by the very definition of push-forward, for every $P \in \widetilde{Gr}(2,4)$ it holds

$$R_\sharp[P] = [R(P)]. \tag{3.29}$$

We now assume by contradiction that there exists a sequence of Lipschitz functions $f^k : B \to \mathcal{A}_Q(\mathbb{R}^2)$ with $f^k \llcorner \partial B \equiv Q[0]$ and such that

$$\gamma_{\mathbf{T}_{f^k}} \rightharpoonup \mu = \sum_{i=1}^{3} \|v_1\| [P_{v_i}].$$

In turn, by (3.26) this implies that

$$\gamma_{(J_{T_{f^k}} R)\,\mathbf{T}_{f^k}} \rightharpoonup \sum_{i=1}^{3} \|v_1\| \cdot (J_{P_{v_i}} R)\,[P_{v_i}] = \sum_{i=1}^{3} \|v_i\| \cdot (J_{P_{v_i}} R)\,[P_{v_i}] = \sum_{i=1}^{3} \|w_i\| [P_{v_i}],$$

where the first convergence holds since the tangential Jacobian is continuous as the tangent plane varies continuously. Combining this convergence with (3.27), (3.28) and (3.29), we deduce the following contradiction with the non-approximability of μ_0:

$$\gamma_{\mathbf{T}_{\sqrt{\varepsilon}f^k}} = \gamma_{R_\sharp \mathbf{T}_{f^k}} = R_\sharp \gamma_{(J_{T_{f^k}} R)\,\mathbf{T}_{f^k}} \rightharpoonup R_\sharp\Big(\sum_{i=1}^{3} \|w_i\|[P_{v_i}]\Big) = \sum_{i=1}^{3} \|w_i\| R_\sharp [P_{v_i}] = \sum_{i=1}^{3} \|w_i\|[P_{w_i}] = \mu_0.$$

□

4. A non-polyconvex Q-integrand defining a lower semicontinuous energy

The aim of this section is to prove Theorem 1.1. In the rest of the paper, we adopt the following convention: indices of sequences are written as superscripts (e.g., u^k), while indices of elements of decompositions are written as subscripts (e.g., a_j).

4.1. **Definition of the candidate integrand** A. We consider the 2-vectors v_1, v_2, v_3 (of equal norm) introduced in (3.1) and define $\psi : \mathbb{R}^{2\times 2} \to \mathbb{R}$ as follows

$$\begin{cases} \psi(X) = 0 & \text{if } \Lambda M(X) \text{ is positively proportional to } v_i \text{ for some } i = 1,2,3, \\ \psi(X) = \|\Lambda M(X)\| & \text{otherwise.} \end{cases}$$

Note that $\Lambda M(X)$ is positively proportional to v_i for some $i = 1,2,3$, if and only if $\Lambda M(X) = \lambda \Lambda M(X_i)$ for some $i = 1,2,3$ and $\lambda > 0$, where the matrices X_i have been introduced in Section 3.

Definition 4.1. Given $p \in \mathcal{A}_Q(\mathbb{R}^\ell)$ for $\ell \in \mathbb{N}$, we define a maximal decomposition of p as:

$$p = \sum_{j=1}^{J} q_j [p_j], \quad \text{with } p_j \neq p_i \text{ for every } j \neq i.$$

The tuple $(q_1, \dots, q_J)$ is unique up to permutations. We say that two points $p, p' \in \mathcal{A}_Q(\mathbb{R}^\ell)$ have the same maximal multiplicities if they admit a maximal decomposition of the form

$$p = \sum_{j=1}^{J} q_j [p_j] \quad \text{and} \quad p' = \sum_{j=1}^{J} q_j [p'_j].$$

Definition 4.2. Consider $(a, X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ with maximal decomposition $\sum_{j=1}^{J} q_j[(a_j, X_j)]$. For a given bounded open set $U \subseteq \mathbb{R}^2$ we let

$$\mathcal{T}(a, X, U) := \sum_{j=1}^{J} \tau_{(a_j + X_j\,\cdot)} W_0^{1,\infty}(U, \mathcal{A}_{q_j}(\mathbb{R}^2)) \tag{4.1}$$

denote the family of all the functions $\varphi \in Lip(U, \mathcal{A}_Q(\mathbb{R}^2))$ which admit a decomposition $\varphi = \sum_{j=1}^{J} \varphi_j$ with $\varphi_j \in Lip(U, \mathcal{A}_{q_j}(\mathbb{R}^2))$ and satisfy

$$\varphi_j \llcorner \partial U = q_j [a_j + X_j\,\cdot] \llcorner \partial U.$$

In particular, for every $j = 1, \dots, J$ it holds $\varphi_j \in \mathcal{T}(q_j[(a_j, X_j)], U)$.

Consider the following Q-integrand, in the sense of Definition 2.1, defined on $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$:

$$A(a, X) := \inf\left\{ \fint_D \bar{\psi}(\nabla\varphi) \,:\, \varphi \in \mathcal{T}(a, X, D) \right\}, \tag{4.2}$$

where we recall that $\bar{\psi}(Y) = \sum_{i=1}^{Q} \psi(Y_i)$ if $Y = \sum_{i=1}^{Q} [Y_i] \in \mathcal{A}_Q(\mathbb{R}^{2\times 2})$.

Note that it is not clear whether A can be written in the form $\bar{\psi}'$ for a 1-integrand $\psi' : \mathbb{R}^{2\times 2} \to \mathbb{R}$. Thus, A does not fall in the framework studied in [19]. It would be interesting to show that A (or a suitable modification) falls in the framework considered in [19]. We discuss this in more detail in Remark 4.14.

Remark 4.3. We note that the value of A is invariant under translations and rescalings of the domain D, that is, for every $(a,X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2}), x_0 \in \mathbb{R}^2$ and $\lambda > 0$ it holds

$$A(a,X) = \inf\left\{ \fint_{x_0+\lambda D} \bar{\psi}(\nabla\varphi) \,:\, \varphi \in \mathcal{T}(a,X,x_0+\lambda D) \right\}.$$

Indeed, if $f \in W^{1,\infty}_0(x_0+\lambda D, \mathcal{A}_Q)$ then $\tilde{f} := \lambda^{-1} f(x_0+\lambda\cdot) \in W^{1,\infty}_0(D,\mathcal{A}_Q)$. A change of variable in (4.2) allows to conclude.

Remark 4.4. Thanks to the definition of the family $\mathcal{T}$, it is possible to link the definition (4.2) with the classical Dacorogna's formula for the quasiconvex envelope of a functional, see [8]. Indeed, for a given $(a,X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ with maximal decomposition $(a,X) = \sum_{j=1}^J q_j[(a_j,X_j)]$, we can write

$$A(a,X) := \inf\left\{ \fint_D \sum_{j=1}^J \bar{\psi}\Big(\tau_{X_j}\nabla\varphi_j\Big) \,:\, \varphi_j \in W^{1,\infty}_0(D,\mathcal{A}_{q_j}) \text{ for } j=1,\dots,J \right\}.$$

The dependence of the integrand on a is hidden into the choice of the maximal multiplicity for (a,X).

Remark 4.5. For any $(a,X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ the value of A is constant in the subset of points $(b,X) \in \mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ having the same maximal multiplicities of (a,X). Indeed, consider $b \in (\mathbb{R}^2)^Q$ such that $(a,X),(b,X)$ admit maximal decompositions $\sum_{j=1}^J q_j[(a_j,X_j)]$ and $\sum_{j=1}^J q_j[(b_j,X_j)]$ respectively. Then for every $j=1,\dots,J$, it holds $\tau_{(b_j-a_j)} q_j[a_j + X_j\,\cdot] = q_j[b_j+X_j\,\cdot]$. Thus, every element in $\mathcal{T}(a,X,D)$ differs from an element in $\mathcal{T}(b,X,D)$ by translations of elements of its maximal decomposition. Since ψ depends only on the gradient variable, by (4.2) we conclude that $A(a,X) = A(b,X)$.

Remark 4.6. For every $q \le Q$, let A^q be the functional defined as in (4.2) on $\mathcal{A}_q$. Consider $(a,X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ with maximal decomposition $\sum_{j=1}^J q_j[(a_j,X_j)]$. Since in the definition of $\mathcal{T}(a,X,D)$ the selections of each map in $\tau_{(a_j+X_j\,\cdot)} W^{1,\infty}_0(U,\mathcal{A}_{q_j}(\mathbb{R}^2))$ are decoupled and the extension $\bar{\psi}$ is linear, it holds

$$A(a,X) = \sum_{j=1}^J A^{q_j}(q_j[(a_j,X_j)]). \tag{4.3}$$

4.2. **Basic properties of the integrand** A. We are going to use the following interpolation result, in the spirit of [13, Lemma 2.15].

Lemma 4.7. *There exists a constant $C(Q) > 0$ with the following property. Let $r > 0$, $\sigma \in (0,1)$ and consider $g \in Lip(\partial D_{(1+\sigma)r}, \mathcal{A}_Q)$ and $f \in Lip(\partial D_r, \mathcal{A}_Q)$ with Lipschitz constant L_f, L_g respectively. Then there exists $h \in Lip(D_{(1+\sigma)r} \setminus D_r, \mathcal{A}_Q)$ with Lipschitz constant L_h such that $h \llcorner \partial D_{(1+\sigma)r} = g, h\llcorner \partial D_r = f$ and*

$$L_h \le C\Big(L_f + L_g + \frac{1}{\sigma r}\sup_{x\in\partial D_r} \mathcal{G}(f(x), g((1+\sigma)x))\Big). \tag{4.4}$$

Proof. By a scaling argument we can consider $r=1$. We denote $\psi = \boldsymbol{\xi}\circ g$ and $\varphi = \boldsymbol{\xi}\circ f$, where $\boldsymbol{\xi} : \mathcal{A}_Q \to \mathbb{R}^N$ is Almgren's bi-Lipschitz embedding, and consider the retraction $\boldsymbol{\rho} : \mathbb{R}^N \to \boldsymbol{\xi}(\mathcal{A}_Q)$. Denote by L_ψ, L_φ the Lipschitz constant of ψ,φ respectively. Consider two Lipschitz extensions $\bar{\psi},\bar{\varphi}$ to the whole $\mathbb{R}^2$ of the functions ψ,φ respectively, which can be chosen to have Lipschitz constants L_ψ, L_φ. For every $x \in D_{1+\sigma}\setminus D$, we define

$$\Phi(x) = \tfrac{\|x\|_\infty - 1}{\sigma}\bar{\psi}(x) + \big(1 - \tfrac{\|x\|_\infty - 1}{\sigma}\big)\bar{\varphi}(x) \tag{4.5}$$

and then $h = \boldsymbol{\xi}^{-1}\circ\boldsymbol{\rho}\circ\Phi$. It is immediate to check that $h\llcorner\partial D = f, h\llcorner\partial D_{(1+\sigma)} = g$, we only need to bound the Lipschitz constant of h. Since $\boldsymbol{\xi}$ is biLipschitz and $\boldsymbol{\rho}$ is a retraction, it is enough to bound the Lipschitz constant of Φ.

For $x \in \mathbb{R}^2$ we denote $x_{in} \in \partial D$ and $x_{out} \in \partial D_{1+\sigma}$ the projection of x on ∂D, and on $\partial D_{1+\sigma}$, respectively. For every $x, y \in D_{1+\sigma} \setminus D$, we estimate

$$\begin{aligned}
|\Phi(x)-\Phi(y)| &= \Big|\tfrac{\|x\|_\infty-1}{\sigma}(\bar\psi(x)-\bar\psi(y))+\tfrac{\|x\|_\infty-\|y\|_\infty}{\sigma}\bar\psi(y)+(1-\tfrac{\|x\|_\infty-1}{\sigma})(\bar\varphi(x)-\bar\varphi(y))-\tfrac{\|x\|_\infty-\|y\|_\infty}{\sigma})\bar\varphi(y)\Big| \\
&\le \tfrac{\|x\|_\infty-1}{\sigma}L_\psi|x-y|+(1-\tfrac{\|x\|_\infty-1}{\sigma})L_\varphi|x-y|+\Big|\tfrac{\|y\|_\infty-\|x\|_\infty}{\sigma}(\bar\psi(y)-\bar\varphi(y))\Big| \\
&\le \Big(L_\psi+L_\varphi\Big)|x-y|+\frac{|x-y|}{\sigma}\Big|\bar\psi(y)-\bar\psi(y_{out})+\bar\psi(y_{out})-\bar\varphi(y_{in})+\bar\varphi(y_{in})-\bar\varphi(y)\Big| \\
&\le \Big(L_\psi+L_\varphi\Big)|x-y|+\frac{|x-y|}{\sigma}((L_\psi+L_\varphi)\sqrt{2}\,\sigma+\max_{z\in\partial D}|\psi((1+\sigma)z)-\varphi(z)|) \\
&\le C\Big(L_g+L_f+\frac{1}{\sigma}\max_{z\in\partial D}\mathcal{G}(f(z),g((1+\sigma)z))\Big)|x-y|,
\end{aligned}$$

where in the last line the constant $C>0$ depends on Almgren's biLipschitz embedding.

$\square$

We now study the continuity properties of A. Crucially, our integrand A turns out to be, in general, only lower semicontinuous and not continuous.

Lemma 4.8. *The integrand A defined in (4.2) is lower semicontinuous on $\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$. Moreover, for every sequence $\{(a^k,X^k)\}_{k\in\mathbb{N}}$ of points in $\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ with the same maximal multiplicities of $(a,X)\in\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ such that $\mathcal{G}((a^k,X^k),(a,X))\to 0$ as $k\to+\infty$, it holds*

$$\lim_{k\to+\infty} A(a^k,X^k)=A(a,X). \tag{4.6}$$

Furthermore, the integrand A satisfies for some $C(Q)>0$

$$A(a,X)\le C(1+\|X\|^2). \tag{4.7}$$

Proof. Fix $(a,X)\in\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ with maximal decomposition $(a,X)=\sum_{j=1}^J q_j[(a_j,X_j)]$. We first observe that for every $f\in Lip(D,\mathbb{R}^2)$ of Lipschitz constant L_f, then

$$\int_D \psi(\nabla f)\le\int_D\|\Lambda M(\nabla f)\|\le C\int_D(1+\|\nabla f\|^2)\le C(1+L_f^2), \tag{4.8}$$

where $C>0$ is a dimensional constant. In particular, (4.7) follows from this estimate testing the definition of A with $\sum_{j=1}^J q_j[a_j+X_j\,\cdot]$.

Consider then a sequence (a^k,X^k) in $\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ converging to (a,X). For k large enough (depending on $\min_{j\neq i}|(a_j,X_j)-(a_i,X_i)|>0$) there exists a maximal decomposition of (a^k,X^k) of the following form

$$(a^k,X^k)=\sum_{j=1}^{J^k} q_j^k[(a_j^k,X_j^k)], \tag{4.9}$$

where $J^k\ge J$ and (up to a permutation) there exists indices $i_0=0,i_1,\dots,i_{J-1},i_J=J^k$ such that $q^k_{i_{j-1}+1}+\dots+q^k_{i_j}=q_j$ for all $j=1,\dots,J$.

Set $\sigma_k:=\mathcal{G}((a,X),(a^k,X^k))\to 0$ as $k\to+\infty$ and denote $u_0:=\sum_{j=1}^J q_j[a_j+X_j\,\cdot]$ and $u_k:=\sum_{j=1}^{J^k} q_j^k[a_j^k+X_j^k\,\cdot]$. In particular we get that $\|X^k\|\le\|X\|+\sigma_k$, and

$$\mathcal{G}(u_0(x),u_k(x))\le\sigma_k,\qquad \forall x\in\partial D. \tag{4.10}$$

For every $j=1,\dots,J$, let $h_j^k\in Lip(D_{1+\sigma_k}\setminus D,\mathcal{A}_{q_j})$ denote the interpolation produced by Lemma 4.7 between

$$q_j[a_j+X_j\,\cdot]\llcorner\partial D_{1+\sigma_k}\quad\text{and}\quad\sum_{i=i_{j-1}+1}^{i_j}q_i^k[a_i^k+X_i^k\,\cdot]\llcorner\partial D, \tag{4.11}$$

and denote $h^k = \sum_{j=1}^J h_j^k$. By (4.10) and the triangular inequality, for $x \in \partial D$ it holds

$$\mathcal{G}(u_k(x), u_0((1+\sigma_k)x)) \leq \sigma_k + \sigma_k \|X\|.$$

Therefore, denoting L_k the Lipschitz constant of h^k, the estimate (4.4) implies

$$L_k \leq C\Big(1 + \|X\|\Big). \tag{4.12}$$

For $\delta > 0$ let $\varphi^k \in \mathcal{T}(a^k, X^k, D)$ be such that $A(a^k, X^k) \geq \fint_D \bar{\psi}(\nabla \varphi^k) - \delta$. By definition of $\mathcal{T}$, the function φ^k admits a decomposition $\varphi^k = \sum_{i=1}^{J^k} \varphi_i^k$, with $\varphi_i^k \in Lip(D, \mathcal{A}_{q_i^k})$. In particular, by (4.9) the function φ^k admits a decomposition

$$\varphi^k = \sum_{j=1}^J \tilde{\varphi}_j^k \quad \text{where} \quad \sum_{i=i_{j-1}+1}^{i_j} \varphi_i^k =: \tilde{\varphi}_j^k \in Lip(D, \mathcal{A}_{q_j}) \text{ for every } j = 1, \ldots, J.$$

We define the following function on $D_{1+\sigma_k}$

$$\eta^k = \sum_{j=1}^J (h_j^k \chi_{D_{1+\sigma_k} \setminus D} + \tilde{\varphi}_j^k \chi_D).$$

By (4.12), the function η^k is Lipschitz and satisfies $\eta^k \llcorner \partial D_{1+\sigma_k} = u_0 \llcorner \partial D_{1+\sigma_k}$, thus $\eta^k \in \mathcal{T}(a, X, D_{1+\sigma_k})$. By the choice of φ^k, (4.8) and (4.12), we can estimate

$$\begin{aligned} A(a^k, X^k) &\geq \fint_D \bar{\psi}(\nabla \varphi^k) - \delta = |D_{1+\sigma_k}| \fint_{D_{1+\sigma_k}} \bar{\psi}(\nabla \eta^k) - \int_{D_{1+\sigma_k} \setminus D} \bar{\psi}(\nabla h^k) - \delta \\ &\geq \fint_{D_{1+\sigma_k}} \bar{\psi}(\nabla \eta^k) - C|D_{1+\sigma_k} \setminus D|\,(1 + L_k^2) - \delta \geq \fint_{D_{1+\sigma_k}} \bar{\psi}(\nabla \eta^k) - C\sigma_k(1 + \|X\|^2) - \delta. \end{aligned} \tag{4.13}$$

Since $\eta^k \in \mathcal{T}(a, X, D_{1+\sigma_k})$, by the definition of A and Remark 4.3, we get

$$A(a^k, X^k) \geq A(a, X) - C\sigma_k(1 + \|X\|^2) - \delta.$$

We conclude the lower semicontinuity of A by letting $k \to +\infty$ and using that δ is arbitrary.

The continuity of A on the subset of $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ consisting of points with the same maximal multiplicities of (a, X) follows essentially the same argument as before. The only difference is that, since the maximal multiplicities of (a, X) and (a^k, X^k) now coincide, we can interpolate between u_0 on D and u_k on $D_{1+\sigma_k}$ by means of a map h^k, which admits a decomposition into J maps in $\mathcal{A}_{q_1}, \ldots, \mathcal{A}_{q_J}$ respectively. This allows to reproduce (4.13), starting from $A(a, X)$, to establish the reverse inequality and thus the desired (4.6).

□

Remark 4.9. We observe that the continuity of the integrand A is not expected in general. For instance $A(Q[(0,0)])$ can be in general strictly smaller than the constant value $A(a, 0)$ evaluated on all $(a, 0) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ with the same maximal multiplicities $(q_1, \ldots, q_J)$, if $J > 1$. Indeed the family of test functions in (4.2) for $A(Q[0,0])$ is strictly larger than the one for $A(a, 0)$, up to a translation. And it is easy to approximate $(0,0) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ with a sequence $(a^k, 0)$ as above.

This obstruction also occurs if one tries to prove upper semicontinuity of A, following the proof of Lemma 4.8, but reversing the roles of the maps in (4.11). More precisely, consider the case where the maximal decomposition of $(a, X), (a^k, X^k)$ satisfies (4.9) and there exists $j \in \{1, \ldots, J\}$ such that $i_j \neq 1$. Assume for simplicity that $i_1 > 1$. Then a test function $\varphi \in \mathcal{T}(a, X, D)$ can be decomposed as J maps $\varphi_1, \ldots, \varphi_J$, each valued in $\mathcal{A}_{q_j}$. But the map φ_1 in general does not admit a Lipschitz decomposition into i_1 Lipschitz maps, each in $\mathcal{A}_{q_1^k}, \ldots, \mathcal{A}_{q_{i_1}^k}$. In particular, the resulting map η^k does not belong to $\mathcal{T}(a^k, X^k, D_{1+\sigma_k})$ since it does not admit the correct decomposition in $D_{1+\sigma_k}$. Therefore we cannot conclude the reverse inequality of (4.13).

4.3. **Lower semicontinuity of the energy associated to A.** The main goal of the section is showing the following result.

Theorem 4.10. *The lower semicontinuous Q-integrand A on $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ defined in (4.2) gives rise to the following energy on $W^{1,p}(D, \mathcal{A}_Q(\mathbb{R}^2))$ for every $p \geq 2$*

$$\mathcal{F}(u) = \fint_D A(u(x), \nabla u(x))\, dx \qquad \forall u \in W^{1,p}(D, \mathcal{A}_Q(\mathbb{R}^2)),$$

which is weakly lower semicontinuous in $W^{1,p}(D, \mathcal{A}_Q(\mathbb{R}^2))$.

A crucial step of the proof of Theorem 4.10 is to show that A satisfies the following property.

Property A. Let F be a Q-integrand on $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$. We say that F satisfies Property A if the following holds. Let $(a, X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ with maximal decomposition $\sum_{j=1}^J q_j[(a_j, X_j)]$ with $a_j \neq a_{j'}$ for $j \neq j'$. In particular, there exists $r > 0$ such that the graph of $x \mapsto \sum_{j=1}^J q_j[a_j + X_j\, x]$ coincides in D_r with the union of J affine planes with multiplicities, at positive distance one from the other in D_r. Then, for every $f = \sum_{j=1}^J f_j \in \mathcal{T}(a, X, D_r)$ such that for every $x \in D_r$ the sets $\mathrm{spt}(f_j(x))$ are pairwise disjoint for different indices j, it holds

$$F(a, X) \leq \fint_{D_r} F(f, \nabla f). \tag{A}$$

In the proof of Theorem 4.10 we will actually show that the integrand A defined in (4.2) enjoys the following slightly stronger property:

Property B. Let A be the Q-integrand on $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ defined in (4.2). We say that A satisfies Property B if the following holds. For every $(a, X) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ and for every $q \leq Q$ and function $f \in \mathcal{T}(q[(a, X)], D)$, it holds

$$A^q(q[(a, X)]) \leq \fint_D A^q(f, \nabla f), \tag{B}$$

where the q-integrand A^q is defined in Remark 4.6.

Thanks to Remark 4.6, Property B implies Property A. Indeed, assuming Property B and reasoning as in Remark 4.3, for every $\lambda > 0$ and $q \leq Q$ it holds

$$A^q(q[(a, X)]) \leq \fint_{D_\lambda} A^q(f, \nabla f), \quad \forall f \in \mathcal{T}(q[(a, X)], D_\lambda).$$

Then, since for every $x \in D_r$, $\mathrm{spt}(f_j(x))$ are disjoint for different indices j, by Remark 4.6 it holds $A(f(x), \nabla f(x)) = \sum_{j=1}^J A^{q_j}(f_j(x), \nabla f_j(x))$ and the same holds for $A(a, X)$. Thus, applying (B) to each A^{q_j} we get (A).

Proof of Theorem 4.10. We divide the proof in two steps.
Step 1: We show that $\mathcal{F}$ is weakly lower semicontinuous in $W^{1,p}(D, \mathcal{A}_Q(\mathbb{R}^2))$, assuming that the integrand A satisfies Property A. The proof of this step essentially follows the proof of [11, Theorem 0.2], specifically the part showing the sufficiency of quasiconvexity for lower semicontinuity of the energy, and is based on the blow-up procedure introduced in [24]. We recall briefly the main steps of the proof, highlighting the main differences with the reference [11].

In order to show the lower semicontinuity of the functional $\mathcal{F}$, we consider $p \in [2, +\infty)$ and a sequence v^k converging to u weakly in $W^{1,p}(U, \mathcal{A}_Q)$. Note that, by (4.7), the integrand A satisfies $A(a, X) \leq C(1 + \|X\|^p)$ for every $p \geq 2$. By [11, Lemma 1.5], one can replace v^k by u^k such that: the functions u^k are in $W^{1,\infty}(U, \mathcal{A}_Q)$ and are converging to u weakly in $W^{1,p}$, there exists $\lim_k \mathcal{F}(u^k)$ and the functions u_k are equi-integrable, meaning that there exists a function $\varphi : [0, +\infty] \to [0, +\infty]$ superlinear at infinity such that

$$\sup_k \int_U \varphi(|\nabla u|^p) < +\infty$$

To conclude, it is sufficient to show that $\mathcal{F}(u) \le \lim_k \mathcal{F}(u_k)$, which is, in turn, implied by showing that

$$A(u(x), \nabla u(x)) \le \frac{d\mu}{d\mathscr{L}^2}(x) \tag{4.14}$$

holds on a set of full measure $\tilde{U}$, where μ is (up to subsequences) the weak-$*$ limit of the measures $A(u_k, \nabla u_k)\mathscr{L}^2 \llcorner U$, see [24]. The set $\tilde{U}$ is chosen in the reference [11] so that some density conditions for u_k, u hold on it. In particular, u is differentiable at every point $x \in \tilde{U}$. The proof of (4.14) follows from two claims, as in [11].

Claim 1: Let $x_0 \in \tilde{U}$ and $(u(x_0), \nabla u(x_0))$ with maximal decomposition $\sum_{j=1}^J q_j[(a_j, X_j)]$. Denote $\delta = \min_{j \ne j'} |a_j - a_{j'}| > 0$ and $\alpha = \min\{1, \delta/(8Q|\nabla u|(x_0))\}$. Then there exist $\rho_k \to 0$ and $\{w^k\}_{k\in\mathbb{N}} \subseteq Lip(D(x_0, \rho_k), \mathcal{A}_Q)$ such that:

(a) w^k admits a decomposition $w^k = \sum_{j=1}^J [w_j^k]$ with $w_j^k \in Lip(D(x_0, \rho_k), \mathcal{A}_{q_j})$, and for every $x \in D(x_0, \rho_k)$ it holds $\mathcal{G}(w^k(x), u(x_0)) \le 2\alpha |\nabla u|(x_0)\, \rho_k$ and $\mathcal{G}(w^k(x), u(x_0))^2 = \sum_{j=1}^J \mathcal{G}(w_j^k(x), q_j[a_j])^2$;
(b) $⨍_{D(x_0,\rho_k)} \mathcal{G}^p(w_k, T_{x_0}u) = o(\rho_k^p)$;
(c) $\lim_{k\to+\infty} ⨍_{D(x_0,\alpha\rho_k)} A\big(w^k, \nabla w^k\big) = \frac{d\mu}{d\mathscr{L}^2}(x_0)$.

Note the difference from the reference [11] in item (c). We recall the main steps of the proof.

As in [11], we can find radii ρ_k which satisfy the following conditions:

$$\begin{aligned} &\sup_k ⨍_{D(x_0,\rho_k)} \varphi(|\nabla u^k|^p) < +\infty, \quad ⨍_{D(x_0,\alpha\rho_k)} A\big(u^k, \nabla u^k\big) \to \frac{d\mu}{d\mathscr{L}^2}(x_0), \\ &⨍_{D(x_0,\rho_k)} \mathcal{G}^p(u_k, u) = o(\rho_k^p) \quad \text{and} \quad ⨍_{D(x_0,\rho_k)} \mathcal{G}^p(u_k, T_{x_0}u) = o(\rho_k^p). \end{aligned} \tag{4.15}$$

Set $r_k = 2\alpha \max\{|\nabla u|(x_0), \delta/(8Q)\}\rho_k$ and consider the retraction $\vartheta_k : \mathcal{A}_Q \to \overline{B}_{r_k}(u(x_0)) \subset \mathcal{A}_Q$ constructed in [13, Lemma 3.7]. We define $w^k := \vartheta_k \circ u^k$, and check that the claim holds.

Indeed, (a) follows since ϑ_k takes values in $\overline{B}_{r_k}(u(x_0)) \subset \mathcal{A}_Q$. As for (b), the choice of r_k implies that $\vartheta_k \circ T_{x_0}u = T_{x_0}u$ on $D(x_0, \alpha\rho_k)$ since

$$\mathcal{G}(T_{x_0}u(x), u(x_0)) \le |\nabla u(x_0)|\, |x - x_0| \le |\nabla u(x_0)|\, \alpha\rho_k \le \frac{r_k}{2}. \tag{4.16}$$

By this, we reason exactly as in [11] to show that $A_k = \big\{w^k \ne u^k\big\}$ satisfies $|A_k| = o(\rho_k^2)$. We thus show item (c), since by the definition of A_k it holds

$$\begin{aligned} &\left| ⨍_{D(x_0,\alpha\rho_k)} A\left(u^k, \nabla u^k\right) - ⨍_{D(x_0,\alpha\rho_k)} A\left(w^k, Dw^k\right) \right| \\ &\qquad \le \frac{1}{\alpha^2\rho_k^2} \int_{A_k} \left(A\left(u^k, \nabla u^k\right) + A\left(w^k, Dw^k\right) \right) \le \frac{C}{\rho_k^2} \int_{A_k} \left(1 + |Dw^k|^p + |\nabla u^k|^p\right). \end{aligned}$$

By the previous estimate and the equi-integrability of ∇u^k, ∇w^k we conclude using [11, Lemma 1.6].

Using Claim 1, one can "blow-up" the functions w^k, as shown in the following claim.

Claim 2: For every $\gamma > 0$, there exist $\{z^k\}_{k\in\mathbb{N}} \subseteq Lip(D_\alpha, \mathcal{A}_Q)$, with the following properties. For each $k \in \mathbb{N}$, z^k admits a decomposition $z^k = \sum_{j=1}^J z_j^k$ in Lipschitz functions $z_j^k \in Lip(D_\alpha, \mathcal{A}_{q_j})$ such that, for every $x \in D_\alpha$, $\mathrm{spt}(z_j^k(x))$ are disjoint for different indices j. Moreover, $z^k \in \mathcal{T}(u(x_0), \nabla u(x_0), D_\alpha)$ and

$$\limsup_{k\to+\infty} ⨍_{D_\alpha} A\big(z^k, \nabla z^k\big) \le \frac{d\mu}{d\mathscr{L}^2}(x_0) + \gamma. \tag{4.17}$$

To show Claim 2, as in [11], one starts with the functions w^k of Claim 1. Since they have the same multiplicity of u at x_0, we can blow-up. Let $\zeta^k := \sum_{j=1}^J \zeta_j^k$ with the maps $\zeta_j^k \in Lip(D_\alpha, \mathcal{A}_{q_j})$ defined by $\zeta_j^k := \tau_{a_j}\rho_k^{-1}\tau_{-a_j} w_j^k(x_0 + \rho_k\cdot)$.

Note that for every $j = 1, \dots, J$ and $x \in D_\alpha$, it holds

$$\mathcal{G}(\zeta_j^k(x), q_j[a_j]) \leq \tfrac{1}{\rho_k}\mathcal{G}\big(w_j^k(x_0 + \rho_k x), q_j[a_j]\big) \leq 2\alpha|\nabla u|(x_0) \leq \frac{\delta}{4}. \tag{4.18}$$

In particular, for every $x \in D_\alpha$, $\mathrm{spt}(\zeta_j^k(x))$ are disjoint for different indices j. By a change of variable, one also gets $\zeta_j^k \to q_j[a_j + X_j \cdot]$ in $L^p(D_\alpha, \mathcal{A}_{q_j})$ and

$$\fint_{D_\alpha} A\big(\zeta^k, \nabla\zeta^k\big) = \fint_{D(x_0,\alpha\rho_k)} A\Big(\sum_{j=1}^J \tau_{a_j}\rho_k^{-1}\tau_{-a_j}[w_j^k], \nabla w^k\Big).$$

For every $x \in D(x_0, \alpha\rho_k)$, the maximal multiplicities of $\tau_{a_j}\rho_k^{-1}\tau_{-a_j}[w_j^k(x)]$ and $w_j^k(x)$ coincide. Since $\mathrm{spt}(w_j^k(x))$ are disjoint for different j, also the maximal multiplicities of $w^k(x)$ and $\sum_{j=1}^J \tau_{a_j}\rho_k^{-1}\tau_{-a_j}[w_j^k(x)]$ coincide for every $x \in D_\alpha$. By Remark 4.5 we get

$$\fint_{D(x_0,\alpha\rho_k)} A\Big(\sum_{j=1}^J \tau_{a_j}\rho_k^{-1}\tau_{-a_j}[w_j^k], \nabla w^k\Big) = \fint_{D(x_0,\alpha\rho_k)} A(w^k, \nabla w^k).$$

Therefore, by item (c) in the previous claim we conclude

$$\lim_{k\to+\infty} \fint_{D_\alpha} A\big(\zeta^k, \nabla\zeta^k\big) = \frac{d\mu}{d\mathscr{L}^2}(x_0).$$

The rest of the proof of Claim 2 follows as in [11], and amounts to employ Lemma 4.7 to match the boundary data $T_{x_0}u\llcorner\partial D$. The only difference is that we first define $z_j^k \in Lip(D_\alpha, \mathcal{A}_{q_j})$ for $j = 1, \dots, J$ interpolating between $\zeta_j^k\llcorner D_{r\alpha}$ and $q_j[a_j + X_j \cdot]\llcorner\partial D_\alpha$, and then $z^k = \sum_{j=1}^J z_j^k \in \mathcal{T}(u(x_0), \nabla u(x_0), D_\alpha)$. Here, the parameter $r \in (0,1)$ is chosen as in [11]. We now check that for every $x \in D_\alpha$, $\mathrm{spt}(z_j^k(x))$ are disjoint for different indices j. If $x \in D_{r\alpha}$, then $z_j^k(x) = \zeta_j^k(x)$ and $\mathrm{spt}(\zeta_j^k(x))$ are disjoint for different indices j. Consider now $x \in D_\alpha \setminus D_{r\alpha}$. By definition of the interpolation Φ in (4.5), we can estimate for every $P \in \mathcal{A}_{q_j}$

$$\begin{aligned}\mathcal{G}(h_j^k(x), P) &\leq Lip(\boldsymbol{\xi}^{-1})|\boldsymbol{\rho}\circ\Phi(x) - \boldsymbol{\xi}(P)| = Lip(\boldsymbol{\xi}^{-1})|\boldsymbol{\rho}\circ\Phi(x) - \boldsymbol{\rho}\circ\boldsymbol{\xi}(P)| \leq Lip(\boldsymbol{\xi}^{-1})|\Phi(x) - \boldsymbol{\xi}(P)| \\ &\leq Lip(\boldsymbol{\xi}^{-1})Lip(\boldsymbol{\xi})\Big(\tfrac{\alpha-\|x\|_\infty}{(1-r)\alpha}\max_{y\in D_\alpha}\mathcal{G}(q_j[a_j + X_j\, y], P) + (1 - \tfrac{\alpha-\|x\|_\infty}{(1-r)\alpha})\max_{y\in D_\alpha}\mathcal{G}(\zeta_j^k(y), P)\Big) \\ &\leq Lip(\boldsymbol{\xi}^{-1})Lip(\boldsymbol{\xi})\max\Big\{\max_{y\in D_\alpha}\mathcal{G}(q_j[a_j + X_j\, y], P), \max_{y\in D_\alpha}\mathcal{G}(\zeta_j^k(y), P)\Big\}.\end{aligned}$$

Considering $P = q_j[a_j]$ and up to choose a smaller α, depending on Q, we conclude that

$$\mathcal{G}(h_j^k(x), q_j[a_j]) \leq C(Q)2\alpha|\nabla u|(x_0) \leq \tfrac{\delta}{4}.$$

By Claim 2, we can test (A) with every z^k, deducing

$$A\big(u(x_0), \nabla u(x_0)\big) \leq \limsup_{k\to+\infty} \int_{D_1} A\big(z^k, \nabla z^k\big) \leq \frac{d\mu}{d\mathscr{L}^2}(x_0) + \gamma,$$

which implies (4.14) by letting $\gamma \to 0$ and concludes the proof.

Step 2: We show that the integrand A defined in (4.2) satisfies Property A, showing Property B, that is (B) holds for every $q \leq Q$ and every function $f \in \mathcal{T}(q[(a, X)], D)$. Since the argument below does not depend on the particular value of q we choose, we fix $q = Q$. We fix $(a, X) \in \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$ and, with an abuse of notation, we denote $(a, X) = Q[(a, X)] \in \mathcal{A}_Q$.

Fix $f \in \mathcal{T}(a, X, D)$ and $\delta > 0$. We first note that the energy of f can be approximated by the energy of almost piecewise affine functions. In fact, denoting with L the Lipschitz constant of f, by Lemma 5.1, we find an $L'(L,Q)-$Lipschitz function $g \in \mathcal{T}(a,X,D)$ for which there exists $K \in \mathbb{N}$ piecewise disjoint cubes $D^\ell \subseteq D$ and points $(a^\ell, X^\ell) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ such that

$$|D \setminus \bigcup_{\ell=1}^{K} D^\ell| \le \delta, \quad g \llcorner D^\ell =: g^\ell \text{ is affine and } g^\ell \in \mathcal{T}(a^\ell, X^\ell, D^\ell),$$

where $X^\ell = \nabla g^\ell$, and g^ℓ is the Taylor expansion of f at a differentiability point for f in D^ℓ. Moreover

$$\int_D A(f, \nabla f) \ge \int_D A(g, \nabla g) - \delta.$$

We are thus left to show (B) for the function g.

We denote $D^\ell = x^\ell + rD$ for $x^\ell \in D$, and recall that $r \in (0,\delta)$. In particular it holds

$$\int_D A(g,\nabla g) \ge \sum_{\ell=1}^{K} \int_{D^\ell} A(g,\nabla g) = \sum_{\ell=1}^{K} \int_{D^\ell} A(g, X^\ell). \tag{4.19}$$

Since g^ℓ is affine, for every $x \in \mathbb{R}^2$, the point $(g^\ell(x), \nabla g^\ell(x))$ has the same maximal multiplicities of (a^ℓ, X^ℓ). Thanks to Remark 4.5 it thus holds $A(g^\ell(x), X^\ell) = A(a^\ell, X^\ell)$ and (4.19) becomes

$$\int_D A(g,\nabla g) \ge \sum_{\ell=1}^{K} |D^\ell| A(a^\ell, X^\ell). \tag{4.20}$$

For every $\ell = 1, \dots, K$ we consider $\varphi^\ell \in \mathcal{T}(a^\ell, X^\ell, D^\ell)$ such that

$$A(a^\ell, X^\ell) \ge \fint_{D^\ell} \bar\psi(\nabla \varphi^\ell) - \delta. \tag{4.21}$$

Note that $g \llcorner \partial D^\ell = \varphi^\ell \llcorner \partial D^\ell$. Therefore, the following Q-valued map is in $Lip(D, \mathcal{A}_Q)$:

$$\varphi := g\, \chi_{D \setminus \bigcup_{\ell=1}^K D^\ell} + \sum_{\ell=1}^{K} \varphi^\ell \chi_{D^\ell}.$$

By (4.20) and (4.21) we get

$$\int_D A(g,\nabla g) \ge \sum_{\ell=1}^{K} \Big(\int_{D^\ell} \bar\psi(\nabla\varphi^\ell) - \delta |D^\ell| \Big) \ge \int_D \bar\psi(\nabla \varphi) - C\delta,$$

where the positive constant C depends on L', and thus on f. Since $(a,X) = Q[(a,X)]$ and $\varphi \llcorner \partial D = Q[a + X\,\cdot] \llcorner \partial D$, then $\varphi \in \mathcal{T}(a,X,D)$. Hence we deduce from the previous inequality and the definition of A in defined in (4.2), that

$$\int_D A(g,\nabla g) \ge A(a,X) - C\delta.$$

Since δ was arbitrary, we conclude (B) and thus the proof.

□

An interesting consequence of the previous result is the following characterization. We consider a bounded, open set $U \subseteq \mathbb{R}^2$.

Proposition 4.11. *Assume that a lower semicontinuous integrand F on $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ satisfies*

$$F(a,X) = F(b,X) \quad \text{for all } (a,X),(b,X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2}) \text{ with same maximal multiplicities.} \tag{4.22}$$

Then, if the associated energy $\mathcal{F}(f) = \int_U F(f,\nabla f)$ is weakly$-*$ lower semicontinuous in $W^{1,\infty}(U,\mathcal{A}_Q)$, the functional F satisfies Property A.

Conversely, if F satisfies Property A and for some $p \ge 2$,

$$F(a,X) \le C(1 + \|X\|^p),$$

then the associated energy $\mathcal{F}$ is weakly lower semicontinuous in $W^{1,p}(U,\mathcal{A}_Q)$.

Proof. Weak$-*$ lower semicontinuity implies Property A. We assume that the functional $\mathcal{F}$ is weakly$-*$ lower semicontinuous in $W^{1,\infty}(U,\mathcal{A}_Q)$. We fix $(a,X)\in\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times 2})$ with a maximal decomposition $\sum_{j=1}^J q_j[(a_j,X_j)]$ and $a_j\neq a_i$ for $j\neq i$. Let $x_0\in U$ and define the affine function $u(x):=\sum_{j=1}^J q_j[a_j+X_j\,x-x_0))]=\sum_{j=1}^J u_j(x)$. We can assume without loss of generality that $x_0=0\in U$. Let $D_r\subseteq U$ for some $r>0$. Note that for r small we can ensure that the graph of u over D_r is the union of J affine planes with multiplicities, at positive distance one from the other. Consider a Lipschitz test function $w=\sum_{j=1}^J w_j\in\mathcal{T}(a,X,D_r)$ such that for every $x\in D_r$ the sets $\mathrm{spt}(w_j(x))$ are pairwise disjoint for different indices j, as in the statement of Property A.

Set $z_j(y):=\tau_{-(a_j+X_j\,y)}w_j(y)$, so that $z_j\llcorner\partial D_r\equiv q_j[0]$, and extend z_j as an r-periodic map to the whole $\mathbb{R}^2$. We consider $v_j^k(y)=\tau_{a_j+X_j\,y}\,(k^{-1}z_j(ky))$ and $v^k=\sum_{j=1}^J v_j^k$ for $k\in\mathbb{N}$. We replace v_j^k by u_j outside D_r.

Note that $v_j^k\rightharpoonup u_j$ weakly$-*$ in $W^{1,\infty}(D_r,\mathcal{A}_{q_j})$ as uniformly Lipschitz and converging in L^∞. Thus, by the lower semicontinuity of $\mathcal{F}$ we get

$$\int_{D_r}F(u,\nabla u)\le\liminf_{k\to+\infty}\int_{D_r}F(v^k,\nabla v^k)\tag{4.23}$$

Since u is affine, the maximal multiplicities of $(u,\nabla u)=(u,X)$ are the same as those of (a,X). By (4.22) and (4.23) we have

$$|D_r|F(a,X)=\int_{D_r}F(u,\nabla u)\le\liminf_k\int_{D_r}F(v^k,\nabla v^k).\tag{4.24}$$

We now estimate the right-hand side in (4.24). First, for every $x\in D_r$, $\mathrm{spt}(v_j^k(x))$ are pairwise disjoint for different indices j. Then, by definition the maximal multiplicities of $v_j^k(x/k)$ and $w_j(x)$ coincide for all $x\in D_r$. Since ∇v^k is $\frac{r}{k}$-periodic in D_r and so are the maximal multiplicities of v^k, by periodicity and a change of variable we get

$$⨍_{D_r}F(v^k,\nabla v^k)=⨍_{D_{r/k}}F(v^k,\nabla v^k)=⨍_{D_{r/k}}F(w(k\,\cdot),\nabla v^k)=⨍_{D_r}F\Big(w,\sum_{j=1}^J\tau_{X_j}\nabla z_j\Big),\tag{4.25}$$

where in the second equality we used (4.22). Since $\tau_{X_j}\nabla z_j=\nabla w_j$ by construction, we conclude

$$\int_{D_r}F(v^k,\nabla v^k)=\int_{D_r}F(w,\nabla w),$$

which, paired with (4.24), yields the conclusion.

Property A implies weak lower semicontinuity. Assume that the integrand F satisfies Property A. The result follows from Step 1 in the proof of Proposition 4.10 once we show that for every sequence $\{(a^k,X^k)\}_{k\in\mathbb{N}}\subseteq\mathcal{A}_Q(\mathbb{R}^2\times\mathbb{R}^{2\times2})$ with the same maximal multiplicities of (a,X), if $\mathcal{G}((a^k,X^k),(a,X))\to 0$ then $F(a^k,X^k)\to F(a,X)$. Indeed, this convergence was used in Step 1 in the proof of Proposition 4.10 to deduce (4.14). Hence we are just left to show this convergence.

Let $\sum_{j=1}^J q_j[(a_j,X_j)]$, $\sum_{j=1}^J q_j[(a_j^k,X_j^k)]$ be a maximal decomposition for $(a,X),(a^k,X^k)$ respectively, and let u_0,u_k be the affine functions defined by $\sum_{j=1}^J q_j[a_j+X_j\,\cdot]$, $\sum_{j=1}^J q_j[a_j^k+X_j^k\,\cdot]$ respectively. Note that, since (4.14) needs to be checked at differentiability points of u, we can assume that $a_j\neq a_i$ for $j\neq i$.

By lower semicontinuity of F, it suffices to show that

$$F(a,X)\ge\limsup_{k\to+\infty}F(a^k,X^k).$$

To do so, we reason as in Lemma 4.8. Note that, for k sufficiently large, there exists $r>0$ such that, for each j, the graph over D_{2r} of the affine map $x\mapsto a_j^k+X_j^kx$ is contained in a tubular neighborhood of the graph of $x\mapsto a_j+X_jx$. Moreover, these neighborhoods are pairwise disjoint for distinct indices j.

Let $\sigma_k=\mathcal{G}((a,X),(a^k,X^k))$. We build an interpolating function h^k between $u_0\llcorner\partial D_{(1+\sigma_k)r}$ and $u^k\llcorner\partial D_r$ by Lemma 4.7 which admits a decomposition $h^k=\sum_{j=1}^J h_j^k$ matching the decomposition of u_0,u_k.

Moreover, the graph of each h_j^k is contained in the tubular neighborhood of the graph of $x \mapsto a_j + X_j x$, and they are thus pairwise disjoint for different j. Reasoning as for (4.12), the Lipschitz constant of h^k is uniformly bounded as $k \to +\infty$. We define on $D_{(1+\sigma_k)r}$ the map $\varphi^k = u_k \chi_{D_r} + h^k \chi_{D_{(1+\sigma_k)r} \setminus D_r}$. Since $\varphi^k \in \mathcal{T}(a, X, D_{(1+\sigma_k)r})$, using (A) we conclude

$$F(a, X) \leq \fint_{D_{(1+\sigma_k)r}} F(\varphi^k, \varphi^k) = |D_r| F(a^k, X^k) + C\sigma_k$$

for a positive constant C. Sending $k \to +\infty$ we conclude.

□

We conclude this section with the following lemma which relates the quasiconvexity introduced in [11] and Property A for integrands satisfying (4.22).

Lemma 4.12. *Consider a Q-integrand F on $\mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ satisfying (4.22). If F is lower semi-continuous, quasiconvexity of F implies Property A. If F is continuous, quasiconvexity and Property A coincide.*

Proof. Let $(a, X) \in \mathcal{A}_Q$ and $(a, X) = \sum_{j=1}^J q_j [\![(a_j, X_j)]\!]$ with $a_j \neq a_i$ for $j \neq i$. We first note the following. Consider $(b, X) \in \mathcal{A}_Q$ admitting a decomposition of the form

$$(b, X) = \sum_{j=1}^{J'} q'_j [\![(b_j, X_j)]\!], \quad b_j \neq b_i \text{ for } j \neq i,$$

where $J' \geq J$ and (up to a permutation) there exists indices $i_0 = 0, i_1, \ldots, i_{J-1}, i_J = J'$ such that $q'_{i_{j-1}+1} + \cdots + q'_{i_j} = q_j$ for all $j = 1, \ldots, J$. Then, if F is lower semicontinuous it holds $F(a, X) \leq F(b, X)$, while if F is continuous it holds $F(a, X) = F(b, X)$. Indeed, let $\{(a^k, X)\}_{k\in\mathbb{N}} \subseteq \mathcal{A}_Q$ be a sequence with the same maximal multiplicities of (b, X) and converging to (a, X). By semicontinuity of F and (4.22) we conclude that $F(a, X) \leq \liminf_k F(a^k, X) = F(b, X)$. If F is continuous we have instead $F(a, X) = F(b, X)$. Consider then $w \in Lip(D_r, \mathcal{A}_Q)$ such that for every $x \in D_r$ the sets $\mathrm{spt}(w_j(x))$ are pairwise disjoint for different indices j. If F is lower semicontinous, for every $x \in D_r$ where w is differentiable it holds $F(w(x), \nabla w(x)) \geq F(a, \nabla w(x))$, from which we can conclude that the quasiconvexity (2.1) implies Property A. If F is continous, $F(w(x), \nabla w(x)) = F(a, \nabla w(x))$ and the two properties are equivalent. □

4.4. **Proof of the main result.** We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We show that the Q-integrand A defined in Theorem 4.10 is not polyconvex. Recall that the simple 2-vectors $v_i = c_i \bigwedge M(X_i)$, $i = 1, 2, 3$ introduced in (3.1), where $c_1 = c_2 = 4\delta^{-2}(4 - \varepsilon^2)^{-1}$ and $c_3 = \frac{\varepsilon^2 \delta^2}{2}$ satisfy $v_1 + v_2 + v_3 = 2\delta^2 e_{12} = 2\delta^2 \bigwedge M(0)$.

We fix $a \in (\mathbb{R}^2)^Q$ and $f \in \mathcal{T}(a, 0, D)$ with Lipschitz constant L. We recall the notation $T_0, \mathbf{T}^H, \mathbf{T}^M, \mathbf{T}^V$ introduced in Section 3, where we again drop the dependence of $\mathbf{T}$ from f. Recall that each point $p \in T_0$ is an integer density point for $\mathbf{T}$ admitting a tangent plane. We also recall that w_1, w_2 are ε-horizontal while w_3 is ε-vertical. Moreover, the 2-vectors v_i are such that $R(P_{v_i}) = P_{w_i}$ for $i = 1, 2, 3$, where $R = diag(1, 1, \varepsilon, \varepsilon)$.

By the definition of ψ we can estimate

$$\int_D \bar{\psi}(\nabla f) \geq \mathbb{M}(\mathbf{T} \llcorner \{p \in T_0 : T_p T_0 \neq P_{v_i} \text{ for } i = 1, 2, 3\}). \tag{4.26}$$

Indeed, consider a point $p \in T_0$ of density $\theta(p)$. We denote $p = (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$. Then

$$f(x) = \theta(p)[\![y]\!] + z$$

with $y \in \mathbb{R}^2$, $z \in \mathcal{A}_{Q-\theta(p)}(\mathbb{R}^2)$ and $d(y, \mathrm{spt} z) > 0$. In particular, choosing a ball B centered at x with a small enough radius, we can ensure

$$d(y, \mathrm{spt} z) > 0 \geq 3(Q-1)L \,\mathrm{diam}(B),$$

so that by [13, Proposition 1.6] the function f admits in B a decomposition into two L-Lipschitz functions $f_1 \in Lip(B, \mathcal{A}_{\theta(p)})$, $f_2 \in Lip(B, \mathcal{A}_{Q-\theta(p)})$ with disjoint graphs. Since there exists the tangent plane to T_0 at p, it also holds $\nabla f_1(x) = \theta(p)X$ for some $X \in \mathbb{R}^{2\times 2}$. If at such p the tangent plane to T_0 is different from P_{v_1}, P_{v_2}, then $\bar{\psi}(\nabla f_1(p)) = \theta(p)\|\Lambda M(X)\|$, which shows (4.26) since $\|\Lambda M(X)\| \geq 1$ for every $X \in \mathbb{R}^{2\times 2}$.

We now claim that a quantitative lower bound can be established for the right-hand side of (4.26). We denote $\mathbf{G} := (R^{-1})_\sharp \mathbf{T}$, which is still a multivalued graph, see (3.28), and $G_0 := R^{-1}(T_0)$. We also denote $\mathbf{G}^{NS} := \mathbf{G} \llcorner \{p \in G_0 : T_pG_0 \neq P_{w_i} \text{ for } i = 1,2,3\}$, and $\mathbf{G}^i = \mathbf{G} \llcorner \{p \in G_0 : T_pG_0 = P_{w_i}\}$ for $i = 1,2,3$, which are well-defined as restrictions of a rectifiable current on measurable subsets of the support. Bounding the tangential Jacobian from above by ε^{-2}, by the area formula

$$\mathbb{M}(\mathbf{G}^{NS}) \leq \varepsilon^{-2}\mathbb{M}(\mathbf{T} \llcorner \{p \in T_0 : T_pT_0 \neq P_{v_i} \text{ for } i = 1,2,3\}).$$

Moreover, since $\mathbf{G} = \mathbf{G}^{NS} + \mathbf{G}^1 + \mathbf{G}^2 + \mathbf{G}^3$, by the linearity of the pushforward it holds $\gamma_{\mathbf{G}} = \gamma_{\mathbf{G}^{NS}} + \sum_{i=1}^3 \gamma_{\mathbf{G}^i}$. In particular, since $e_{12} = \int \omega_P \, d\gamma_{\mathbf{G}}(P)$, we deduce that

$$\sum_{i=1,2} \int \omega_P \cdot e_{34} \, d\gamma_{\mathbf{G}^i}(P) = \int \omega_P \cdot e_{43} \, d\gamma_{\mathbf{G}^{NS}}(P) + \int \omega_P \cdot e_{43} \, d\gamma_{\mathbf{G}^3}(P).$$

Recalling the definition (3.4), we have $w_1 \cdot e_{34} = w_2 \cdot e_{34} = \frac{\varepsilon^2}{4}$ and $w_2 \cdot e_{43} = \frac{\varepsilon^2}{2}$, so that the previous equality reads

$$\frac{\varepsilon^2}{4\|w_1\|} \sum_{i=1,2} \mathbb{M}(\mathbf{G}^i) = \frac{\varepsilon^2}{2\|w_3\|}\mathbb{M}(\mathbf{G}^3) + \int \omega_P \cdot e_{43} \, d\gamma_{\mathbf{G}^{NS}}(P).$$

Since the pushforward of a measure is mass-preserving, we have $|\gamma_{\mathbf{G}^{NS}}|(\widetilde{Gr}(2,4))) = \mathbb{M}(\mathbf{G}^{NS})$, and by $|\omega_P \cdot e_{43}| \leq 1$ for $\gamma_{\mathbf{G}^{NS}}$-a.e. P, we conclude

$$0 \leq \frac{2\|w_1\|}{\|w_3\|}\mathbb{M}(\mathbf{G}^3) - \sum_{i=1,2} \mathbb{M}(\mathbf{G}^i) + \frac{4\|w_1\|}{\varepsilon^2}\mathbb{M}(\mathbf{G}^{NS}). \tag{4.27}$$

On the other hand, since projecting on the h-plane the multivalued graph $\mathbf{G}$ does not produces cancellations, it holds

$$Q \leq \mathbb{M}(\mathbf{G}) = \sum_{i=1,2,3} \mathbb{M}(\mathbf{G}^i) + \mathbb{M}(\mathbf{G}^{NS}). \tag{4.28}$$

Summing left-hand sides and right-hand sides of (4.27) and (4.28), we get

$$\Big(1 + \frac{2\|w_1\|}{\|w_3\|}\Big)\mathbb{M}(\mathbf{G}^3) + \Big(1 + \frac{4\|w_1\|}{\varepsilon^2}\Big)\mathbb{M}(\mathbf{G}^{NS}) \geq Q.$$

Since w_3 is ε-vertical it holds $\mathbb{M}(\mathbf{G}^3) \leq \mathbb{M}(\mathbf{G}^V)$, so that using Proposition 3.5 we arrive at

$$Q \leq \frac{1}{C}\Big(1 + \frac{2\|w_1\|}{\|w_3\|}\Big)\mathbb{M}(\mathbf{G}^M) + \Big(1 + \frac{4\|w_1\|}{\varepsilon^2}\Big)\mathbb{M}(\mathbf{G}^{NS})$$

Finally, since the planes associated to w_1, w_2, w_3 are either ε-horizontal or ε-vertical, it holds $\mathbb{M}(\mathbf{G}^M) \leq \mathbb{M}(\mathbf{G}^{NS})$. All in all, we have concluded a quantitative lower bound for (4.26), which concludes the proof of the claim.

Thanks to the claim, by the very definition of A we conclude that $A(a, 0) > 0$. On the other hand, denoting $\tilde{X}_i = (X_i, \ldots, X_i) \in (\mathbb{R}^{2\times 2})^Q$ for $i = 1,2,3$, we see that $A(a, \tilde{X}_i) = 0$, as follows testing the definition (4.2) with $f = \sum_{j=1}^Q [a_j + X_j \cdot]$. This implies that the integrand A is not polyconvex, as $A(a, \cdot)$ is not convex at 0, concluding the proof. $\square$

Remark 4.13. Referring to the previous proof, we note that, even if the integrand A is degenerate at every point $(a, \tilde{X}_i) \in \mathcal{A}_Q$ with $\tilde{X}_i = (X_i, \ldots, X_i) \in (\mathbb{R}^{2\times 2})^Q$ for $i = 1,2,3$, we can always modify it to produce a nondegenerate Q-integrand $\tilde{A}$, which is not polyconvex but whose associated energy is lower semicontinuous. Indeed, for $(a, X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ we could set $\tilde{A}(a, X) := A(a, X) + \delta \|X\|$. This is a nondegenerate Q-integrand (since $\|X\| = \mathcal{G}(\sum_{j=1}^Q [X_j], Q[0])$) which is not polyconvex for δ small enough, as follows from the fact that $\sum_{i=1}^{3} A(a, \tilde{X}_i) < A(a,0)$. On the other hand, it is immediate to check that $\tilde{A}$ satisfies Property B thanks to the convexity of the norm.

4.5. **Concluding remarks.** In this final subsection we provide some observations and questions that arose during the preparation of the work.

Remark 4.14. The envelope (4.2) provides a nonhomogeneous Q-integrand. It is not clear if it is possible to construct an integrand which depends on the gradient variable only. For all $X \in (\mathbb{R}^{2\times 2})^Q$ one could try to define the space of test functions as $\mathcal{T}(X,D) = \bigcup_{a\in(\mathbb{R}^2)^Q} \mathcal{T}(a,X,D)$, but for such a choice it is not clear whether the quasiconvexity of the envelope (4.2) holds. To be precise, Step 2 in the proof of Theorem 4.10 fails, since it is not clear that $\varphi^\ell \llcorner \partial D^\ell = g \llcorner \partial D^\ell$ and that the global competitor φ can be defined.

Even more interesting would be to strengthen Theorem 1.1 to define a 1-integrand A_Q which is not polyconvex and such that the associated Q-integrand $\bar{A}_Q$ is quasiconvex. If this was the case, Theorem 1.1 would also prove the question posed in [19, Remark 1.14]. One could imagine to define A_Q as follows. For every $X \in \mathbb{R}^{2\times 2}$

$$A_Q(X) := \inf_{q\le Q} \left\{ \frac{1}{q} \fint_B \bar{\psi}(\nabla f) : f \in \mathcal{T}(q[X], B) \right\}. \tag{4.29}$$

It is not difficult to see, following the proof of Theorem 1.1, that for every $Q \in \mathbb{N}$ the 1-integrand A_Q is not polyconvex. Yet, it is not clear whether the energy functional associated to $\bar{A}_Q$ is weakly lower semicontinuous[2], that is the validity of Theorem 4.10. The issue lies again in Step 2 in the proof of Theorem 4.10, as we now sketch. Let $X \in \mathbb{R}^{2\times 2}, Q \in \mathbb{N}$ be such that

$$\inf \left\{ \fint_B \bar{\psi}(\nabla f) : f \in \mathcal{T}(Q[X], B) \right\} < \inf \left\{ \fint_B \bar{\psi}(\nabla f) : f \in \mathcal{T}((Q-1)[X], B) \right\}, \tag{4.30}$$

i.e., $A_Q(X) < A_{Q-1}(X)$. If such X, Q did not exist, by [19, Theorem 1.8] the function ψ would be polyconvex. Now, consider a piecewise affine function g, and suppose that at a Lebesgue point x for ∇g it holds $\nabla g = q[X] + \sum_{j=1}^{J} q_j[X_j]$, with $q < Q$ and $X_j \neq X$. We need to approximate the energy $\bar{A}_Q(\nabla g(x)) = qA_Q(X) + \sum_j q_j A_Q(X_j)$, as we did in (4.21). Since $A_Q(X) < A_{Q-1}(X)$, a Q-valued function f is needed to approximate the value $A_Q(X)$ using (4.29). As there is left the term $\sum_j q_j A_Q(X_j)$ to approximate, we need to add to f at least one more value. Therefore, in order to approximate $\bar{A}_Q(\nabla g(x))$, we need a function φ taking at least $Q+1$ values. However, since in (4.29) we are only allowed to use q-valued maps with $q \le Q$, φ is not an admissible competitor. Hence we cannot conclude the last inequality of the proof of Theorem 4.10. This argument also highlight that we cannot exclude that for every 1-integrand ψ the envelope

$$\sup\{\xi \le \psi : \text{the associated } Q\text{-integrand } \bar{\xi} \text{ is quasiconvex}\}$$

coincides with the polyconvex envelope of ψ.

Remark 4.15. We lastly present a comment on Property A, which replaces the quasiconvexity property of Definition 2.2 introduced in [11]. As observed in Remark 4.9, we cannot expect in general the continuity of A. Hence the weak lower semicontinuity of the associated energy functional is not equivalent to quasiconvexity of A. As shown in Proposition 4.11, under the assumption (4.22), the weak lower semicontinuity of the energy turns out to be equivalent to Property A, which thus seems to be the natural ellipticity condition for the existence of energy minimizers in this context. Moreover, when the integrand A is in addition continuous, Property A is equivalent to the quasiconvexity of A by Lemma 4.12, and our results match with those of [11].

[2]Note that, if that was true, a standard regularization by convolution would allow to find a smooth 1-integrand ψ_Q, close to A_Q and not polyconvex, but with $\bar{\psi}_Q$ quasiconvex (see [6, pag. 479]).

5. An almost-polyhedral approximation result

The aim of this last section is to establish the following approximation result, which we used in Section 4. Essentially, we show that the energy associated to the integrand A defined in (4.2) for any Lipschitz multivalued function f can be recovered through an approximating sequence of uniformly Lipschitz functions that are piecewise affine outside a set of small measure. Actually, the same proof works for every integrand F satisfying the assumptions of Proposition 4.11 and Property A. Specifically, Property A can be used in (5.18) to conclude the proof of Lemma 5.2 below, instead of using the definition of the integrand A.

Lemma 5.1. *Let $(a, X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$. Consider $f = \sum_{j=1}^J f_j \in \mathcal{T}(a, X, D_r)$ L-Lipschitz, for $r \in (0,1)$ such that the sets $spt(f_j(x))$ are pairwise disjoint for different indices j. Then there exist a sequence of L'-Lipschitz functions $\{g^k\}_{k\in\mathbb{N}} \subseteq \mathcal{T}(a, X, D_r)$, with $L' = L'(L, Q) > 0$, with the following properties. For every $k \in \mathbb{N}$ there exists $L_k \in \mathbb{N}$ and pairwise disjoint cubes $\{D^k_\ell\}_{\ell=1}^{L_k} \subseteq D_r$ such that*

$$
\begin{aligned}
&|D_r \setminus \bigcup_{\ell=1}^{L_k} D^k_\ell| \le \frac{2}{k}, && (5.1)\\
&g^k \llcorner D^k_\ell = T_z f \llcorner D^k_\ell \ \ \textit{for some point } z \in D^k_\ell \textit{ of differentiability for } f, \quad \textit{for every } \ell = 1, \dots, L_k.
\end{aligned}
$$

Moreover, it holds

$$
\lim_{k\to+\infty} \int_{D_r} A(g^k, \nabla g^k) = \int_{D_r} A(f, \nabla f). \tag{5.2}
$$

In order to prove Lemma 5.1, we first introduce the following notation. Consider an open bounded set $\Omega \subseteq \mathbb{R}^2$. For $r > 0$, we define an r-cubic subdivision of Ω, if it exists, any finite union U of disjoint open cubes contained in Ω whose side lengths are integer multiples of r, and such that $|\Omega \setminus U| = 0$. We can then show the following intermediate result, which is reminiscent of [9, Lemma 4.2, 4.3].

Lemma 5.2. *Let $U \subseteq \mathbb{R}^2$ open and bounded, and $f : U \to \mathcal{A}_Q(\mathbb{R}^2)$ be L-Lipschitz. For every $\delta \in (0, 1)$ there exists $r \in (0, \delta)$ and an r-cubic subdivision $\{D(z_\ell, r)\}_{\ell=1,\dots,N_\delta}$ of a set $D^\delta \subseteq U$ such that*

$$
|U \setminus D^\delta| \le \delta, \tag{5.3}
$$

$$
D(z_\ell, 3r) \subset\subset U, \quad \textit{for every } \ell = 1, \dots, N_\delta \tag{5.4}
$$

and for each $\ell = 1, \dots, N_\delta$ there exists an affine function $g_\ell : \mathbb{R}^2 \to \mathcal{A}_Q(\mathbb{R}^2)$ such that

$$
g_\ell = T_z f \quad \textit{for some point } z \in D(z_\ell, r) \textit{ of differentiability for } f\ , \tag{5.5}
$$

$$
\sup_{D(z_\ell,r)} \mathcal{G}(f, g_\ell) \le \delta r, \tag{5.6}
$$

$$
\Big|\{\mathcal{G}(\nabla f, \nabla g_\ell) > \alpha\} \cap D(z_\ell, r)\Big| \le \frac{\delta}{\alpha} r^2, \quad \forall \alpha \in (0, +\infty). \tag{5.7}
$$

Moreover, if $\sum_{j=1}^J q_j[\![(a_j, X_j)]\!]$ is a maximal decomposition for $(f(z), \nabla f(z))$, then f admits a Lipschitz decomposition in $D(z_\ell, 3r)$ of the form

$$
f = \sum_{j=1}^J f_j, \quad \textit{with } f_j \in Lip(D(z_\ell, 3r), \mathcal{A}_{q_j}) \textit{ for every } j = 1, \dots, J. \tag{5.8}
$$

Proof. Fix $\delta > 0$. Choose $r_0 = r_0(\delta) > 0$ and a set $D_0 \subset\subset U$ such that D_0 is the closure of the union of pairwise disjoint translations of the square $(0, r_0)^2$. Assume in addition that each square composing D_0 satisfies (5.4) and that $|U \setminus D_0| \le \delta/2$. Then, for every r such that $r_0 \in r\mathbb{N}$, there exists a r-cubic subdivision of D_0 made of $I_0(r)$ cubes. We denote $\{D(z_\ell, r)\}_{\ell\in I_0(r)}$ the elements of such a r-cubic subdivision of D_0. We define $I(r) \subseteq I_0(r)$ as the set of indices $\ell \in I_0(r)$ such that there exists an affine function $g_\ell : \mathbb{R}^2 \to \mathcal{A}_Q(\mathbb{R}^2)$ coinciding with $T_z f$ for some $z \in D(z_\ell, r)$ and satisfying

$$\sup_{D(z_\ell,3r)} \mathcal{G}(f, g_\ell) \le \delta r, \quad ⨍_{D(z_\ell,3r)} \mathcal{G}(\nabla f, \nabla g_\ell) \le \delta.$$

We then let

$$D^\delta := \bigcup_{\ell \in I(r)} D(z_\ell, r). \tag{5.9}$$

As (5.4), (5.5) and (5.6) hold by construction, we are left to prove (5.3) and (5.7) for D^δ, and to control the multiplicities of f, g.

We start by choosing radii $r(z)$ depending on the point $z \in \mathring{D}_0$ in the interior of D_0. First of all, for every $z \in \mathring{D}_0$ there exists $r(z)$ such that $D(z, 4r(z)) \subseteq D_0 \subseteq U$ and we can furthermore take $r(z) < \delta$. Then, since f is Lipschitz, by Rademacher's Theorem for multivalued functions [13, Theorem 1.13] for a.e. $z \in D_0$ the Taylor expansion $T_z f$ of f at the point z is well-defined and satisfies

$$\mathcal{G}(f(y), T_z f(y)) = o(|z - y|).$$

In particular, for a.e. $z \in \mathring{D}_0$ there exists $r(z) > 0$ such that $D(z, 4r(z)) \subseteq D_0$ and

$$\sup_{D(z,4r(z))} \mathcal{G}(f, T_z f) \le \delta r. \tag{5.10}$$

We set $r(z) = 0$ for all z at which f is not differentiable. Since almost every point $z \in D_0$ is a Lebesgue point for ∇f, there exists a possibly smaller $r(z) > 0$ such that

$$⨍_{D(z,\rho)} \mathcal{G}(\nabla f, \nabla f(z)) \le \frac{9}{16}\delta, \qquad \forall\, 0 < \rho \le 4r(z). \tag{5.11}$$

Additionally, denoting $\sum_{j=1}^{J(z)} q_j [a_j]$ a maximal decomposition of $f(z)$, for $r(z)$ small we can ensure that

$$|a_j - a_{j'}| \ge 3L(Q-1)\operatorname{diam}(D(z, 4r(z))), \quad \forall j \ne j'. \tag{5.12}$$

We choose r small enough so that the set $A(r) := \{z \in \mathring{D}_0 : r(z) \le r\}$ satisfies

$$|A(r)| \le \frac{\delta}{2}. \tag{5.13}$$

This can be done as $A(r) \searrow \bigcap_{r \ge 0} A(r)$ as $r \to 0$, which has zero measure.

We now prove that, the radius r chosen above satisfies the conclusions of the lemma.

To this end, we first claim that

$$\text{for each } \ell \in I_0(r), \text{ if } D(z_\ell, r) \not\subseteq A(r), \text{ then } \ell \in I(r). \tag{5.14}$$

Indeed, assume there exists $z \in D(z_\ell, r) \setminus A(r)$ and let $g_\ell := T_z f$. In particular, $r(z) > 0$, hence f is differentiable at z. By definition of $A(r)$ and $r(z)$ we get

$$\sup_{D(z_\ell,3r)} \mathcal{G}(f, g_\ell) \le \sup_{D(z,4r(z))} \mathcal{G}(f, T_z f) \le \delta r.$$

Similarly

$$⨍_{D(z_\ell,3r)} \mathcal{G}(\nabla f, \nabla g_\ell) \le \frac{16}{9} ⨍_{D(z,4r(z))} \mathcal{G}(\nabla f, \nabla f(z)) \le \delta.$$

Therefore, $\ell \in I(r)$ and we conclude the proof of (5.14).

Property (5.14) immediately implies that $I(r) \ne \emptyset$, since for $\delta < |U|$ we have $|D_0| > |A(r)|$ by (5.13). The validity of (5.3) for our choice of r follows from $|U \setminus D_0| \le \delta/2$ and the following estimate:

$$|D_0 \setminus D^\delta| \overset{(5.9)}{=} \left| \bigcup_{\ell \in I_0(r) \setminus I(r)} D(z_\ell, r) \right| \overset{(5.14)}{\le} |A(r)| \overset{(5.13)}{\le} \frac{\delta}{2}.$$

Then, using (5.11), for every $\ell \in I(r)$ and $\alpha > 0$ we get

$$\left| \{\mathcal{G}(\nabla f, \nabla g_\ell) > \alpha\} \cap D(z_\ell, r) \right| \le \frac{\delta}{\alpha} r^2,$$

which shows (5.7). As noted before, the properties (5.4), (5.5) and (5.6) hold by construction.

Moreover, by (5.5), the function g_ℓ is affine and $(g_\ell(x), \nabla g_\ell(x))$ has the same maximal multiplicities of $(f(z), \nabla f(z))$ at every point $x \in \mathbb{R}^2$. Denoting $\sum_{j=1}^J q_j[a_j]$ a maximal decomposition for $f(z)$, by definition of $r(z)$ the property (5.12) is satisfied and thus by [13, Proposition 1.6] the function f admits a decomposition in $D(z, 4r(z))$ as $f = \sum_{j=1}^J f_j$, with f_j $L-$Lipschitz taking value in $\mathcal{A}_{q_j}$ and with disjoint supports. In particular, since $D(z_\ell, 3r) \subseteq D(z, 4r(z))$, the same decomposition holds in $D(z_\ell, 3r)$, showing that (5.8) holds.

□

We can now conclude the proof of Lemma 5.1.

Proof of Lemma 5.1. Let $(a, X) \in \mathcal{A}_Q(\mathbb{R}^2 \times \mathbb{R}^{2\times 2})$ and fix an L-Lipschitz function $f \in \mathcal{T}(a, X, D_r)$ with Lipschitz decomposition $\sum_{j=1}^J f_j$. Let $\sum_{j=1}^J q_\ell[(a_j, X_j)]$ be a maximal decomposition for (a, X). Note that by the assumption on the supports of f_j, $a_j \neq a_i$ for $j \neq i$. We divide the proof in two steps.

Step 1: We modify the functions produced by Lemma 5.2 in order to match the boundary data. For each $k \in \mathbb{N}$, let $\{g_\ell^k\}_{\ell=1,\dots,N_k}$ denote the affine functions produced by Lemma 5.2 for the choice $f, \delta = 1/k$, with each g_ℓ^k defined on an element of an r_k-cubic subdivision $\{D(z_\ell^k, r_k)\}_{\ell=1,\dots,N_k}$ of a set $D^k \subseteq D_r$, with $r_k \leq 1/k$. Since $\min_{i\neq j}\min_{x\in D_r} \mathrm{dist}(\mathrm{spt} f_i(x), \mathrm{spt} f_j(x)) > 0$, it holds $T_z f = \sum_{j=1}^J T_z f_j$ at every differentiability point $z \in D_r$ for f. Thus, by (5.5) the affine function g_ℓ^k admits a decomposition into J affine functions $g_{\ell,j}^k = T_z f_j \in Lip(\mathbb{R}^2, \mathcal{A}_{q_j})$. In view of (5.6), for k large enough and for every $x \in D(z_\ell^k, r_k)$, $\mathrm{spt}(g_{\ell,j}^k(x))$ are disjoint for different indices j. In particular, (5.6) implies that for k large enough it holds

$$\sup_{x\in D(z_\ell^k, r_k)} \mathcal{G}^2(f, g_\ell^k) = \sum_{j=1}^J \sup_{x\in D(z_\ell^k, r_k)} \mathcal{G}^2(f_j, g_{\ell,j}^k) \leq (r_k/k)^2. \tag{5.15}$$

For every $j = 1, \dots, J$ we define $h_{\ell,j}^k$ as the function interpolating between f_j on $\partial D(z_\ell^k, r_k)$ and $g_{\ell,j}^k$ on $D(z_\ell^k, (1-1/k)r_k)$ constructed by Lemma 4.7 for the choice $\sigma = \frac{1}{k-1}$, $r = (1-\frac{1}{k})r_k$. In order to estimate the Lipschitz constant of $h_{\ell,j}^k$, we first note that for every $x \in \partial D_{r_k}$ and thanks to (5.15) it holds

$$\begin{aligned}
&\mathcal{G}\Big(f_j(x+z_\ell^k), g_{\ell,j}^k((1-\tfrac{1}{k})x + z_\ell^k)\Big)\\
&\qquad \leq \mathcal{G}\Big(f_j(x+z_\ell^k), f_j((1-\tfrac{1}{k})x+z_\ell^k)\Big) + \mathcal{G}\Big(g_{\ell,j}^k((1-\tfrac{1}{k})x+z_\ell^k), f_j((1-\tfrac{1}{k})x+z_\ell^k)\Big) \leq L\frac{r_k}{k} + \frac{r_k}{k}.
\end{aligned}$$

Thus, by (4.4) we conclude that $h_{\ell,j}^k$ is $L'-$Lipschitz, for a constant L' not depending on k.

We define the following maps on D_r:

$$g^k := \sum_{j=1}^J \left[f_j \chi_{D\setminus D^k} + \sum_{\ell=1}^{N_k} \Big(h_{\ell,j}^k \chi_{D(z_\ell^k, r_k)\setminus D(z_\ell^k,(1-\frac{1}{k})r_k)} + g_{\ell,j}^k \chi_{D(z_\ell^k,(1-\frac{1}{k})r_k)} \Big) \right].$$

By definition the function g^k coincides with g_ℓ^k on $D(z_\ell^k, (1-\frac{1}{k})r_k)$ and it has the same trace of f on $\partial D(z_\ell^k, r_k)$, thus it is L'-Lipschitz in D. Moreover, since g^k admits a Lipschitz decomposition with the same multiplicities as the global decomposition of f, it holds $g^k \in \mathcal{T}(a, X, D_r)$.

By (5.3), and since $N_k \leq \frac{1}{r_k^2}$, it holds

$$\begin{aligned}
\left| D \setminus \bigcup_{\ell=1}^{N_k} D(z_\ell^k, (1-\tfrac{1}{k})r_k) \right| &\leq \frac{1}{k} + \left| \bigcup_{\ell=1}^{N_k} D(z_\ell^k, r_k) \setminus D(z_\ell^k, (1-\tfrac{1}{k})r_k) \right| \\
&\leq \frac{1}{k} + N_k r_k^2 (1-(1-\tfrac{1}{k})^2) \leq \frac{3}{k},
\end{aligned} \tag{5.16}$$

thus the function g^k satisfies the properties listed in (5.1) for k large enough. Lastly, thanks to (5.16) and (5.10) we have $g^k \to f$ in measure. Similarly, by (5.7) we conclude that $\nabla g^k \to \nabla f$ in measure. Passing to a subsequence we also get $(g^k, \nabla g^k) \to (f, \nabla f)$ pointwise a.e. in D_r.

Step 2: We now show (5.2) for the functions g^k built in the previous step. By the pointwise convergence a.e. and lower semicontinuity of A we already have that $\liminf_{k\to+\infty} A(g^k, \nabla g^k) \geq A(f, \nabla f)$ outside a set of zero measure. In particular, Fatou's lemma implies

$$\int_{D_r} A(f, \nabla f) \leq \liminf_{k\to+\infty} \int_{D_r} A(g^k, \nabla g^k).$$

To conclude (5.2) it is sufficient to show the reverse inequality. In turn, by the definition of g^k, to prove the reverse inequality, it is enough to show that there exists $C > 0$ only depending on f such that, for every k large enough and every $\ell = 1, \dots, N_k$ it holds

$$\int_{D^k_\ell} A(g^k_\ell, \nabla g^k_\ell) \leq \int_{D^k_\ell} A(f, \nabla f) + \frac{C}{k}|D^k_\ell|. \tag{5.17}$$

Indeed, by (5.16), (4.7) and the uniform Lipschitz continuity of g^k, f, all the remainders terms are negligible as $k \to +\infty$.

The proof of (5.17) is then very similar to the proof of Lemma 4.8. We fix $k \in \mathbb{N}$ and $\ell \in \{1, \dots, N_k\}$, and denote $g, D' = D(z', r)$ the function g^k_ℓ and the cube $D^k_\ell(z^k_\ell, r_k)$ for simplicity. We consider $z \in D'$ such that $g = T_z f$ as in (5.5). Let $\sum_{j=1}^{\mathcal{J}} q_j[a_j]$ denote a maximal decomposition for $f(z)$. Since f has a Lipschitz decomposition $\sum_{j=1}^{J} f_j$, with $\mathrm{spt}(f_j(z))$ that are pairwise disjoint for different indices j, then $\mathcal{J} \geq J$. On one hand, since z is a differentiability point for f, the point $(f(z), \nabla f(z))$ admits a maximal decomposition with the same multiplicities, i.e. $(f(z), \nabla f(z)) = \sum_{j=1}^{\mathcal{J}} q_j[(a_j, X_j)]$ for some $X_j \in \mathbb{R}^{2\times 2}$. In particular, for every $x \in \mathbb{R}^2$ the maximal multiplicities of $g(x)$ and $f(z)$ are the same, thus $g \in \mathcal{T}(f(z), \nabla f(z), D')$. On the other hand, since f admits a decomposition on D' with the same multiplicities as g by (5.8), we can employ Lemma 4.7 to find a Lipschitz function $\tilde h$ defined on $D' \setminus D(z', (1 - \frac{1}{k})r)$ and interpolating between

$$g \llcorner \partial D', \text{ and } f \llcorner D(z', (1 - \tfrac{1}{k})r).$$

Arguing as in the previous step, the function $\tilde h$ is L'-Lipschitz continuous, with L' not depending on k. We then note that the function $\tilde h$ admits a decomposition into $\mathcal{J}$ multivalued Lipschitz functions in $\mathcal{A}_{q_1}, \dots, \mathcal{A}_{q_{\mathcal{J}}}$ respectively. Hence, we can define a map $\bar h \in \mathcal{T}(f(z), \nabla f(z), D')$ as:

$$\bar h = \tilde h \chi_{D' \setminus D(z', (1-\frac{1}{k})r)} + f \chi_{D(z', (1-\frac{1}{k})r)}.$$

By the definition of A we conclude

$$\int_{D'} A(g, \nabla g) = |D'| A(f(z), \nabla f(z)) \leq \int_{D'} A(\bar h, \nabla \bar h) \leq \int_{D'} A(f, \nabla f) + \frac{C}{k}|D'|, \tag{5.18}$$

where the constant $C > 0$ depends on the Lipschitz constants of f. This shows (5.17) and concludes the proof. □

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