KIRILLOV-RESHETIKHIN DUAL EQUIVALENCE GRAPHS

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Abstract. Let U be a tensor product of highest weight modules of $GL_n(\mathbb{C})$ corresponding to multiples of fundamental weights (i.e. rectangles). We consider three ways to stratify $U^{\otimes k}$ into components: using isotypic components of the cyclic action on tensor factors, using a generalization of the charge statistic, and using certain generalizations of Assaf's dual equivalence graphs. We conjecture that all three ways coincide, and we prove that the latter two ways coincide. The Kirillov-Reshetikhin dual equivalence graphs (KR DEGs) we introduce for this purpose are defined on 0-weight spaces of tensor products of Kirillov-Reshetikhin crystals. They generalize Kazhdan-Lusztig dual equivalence graphs (KL DEGs) that previously appeared in the study of Kazhdan-Lusztig cells in affine type A. While the tensor products of Kirillov-Reshetikhin crystals are connected as affine crystals, the KR DEGs in general are not.

1. Introduction

Let $V = \mathbb{C}^n$ and let $\phi_U : GL(V) \to GL(U)$ be a polynomial representation of $GL_n(\mathbb{C})$. Consider the $GL(V)$ representation $U^{\otimes k} = U \otimes U \otimes \cdots \otimes U$ and the action of cyclic group C_k on it by cycling tensor factors. One can split $U^{\otimes k}$ into isotypic components of C_k, and since the two actions commute, each component carries a representation of $GL(V)$. Thus, we stratify $U^{\otimes k}$ into k components as a $GL_n(\mathbb{C})$ representation. To a representation ϕ_U, one can associate a character $\mathrm{char}_U(x) = \mathrm{tr}(\phi_U(\mathrm{diag}(x)))$, which is a symmetric function given by the trace of the image of a diagonal matrix with diagonal entries x_i for each $i \in [n]$. Recall that cyclic characters $\ell_k^{(m)} = ch(\mathrm{Ind}_{C_k}^{S_k}(\chi^m))$ are Frobenius characters of the inductions of irreducible characters χ^m of C_k to S_k. Cyclic characters were studied by Foulkes [10], and further work connecting them to the charge statistic was done by Kraskiewicz and Weyman [15] as well as Lascoux, Leclerc and Thibon [16]. It can be shown that characters of the above components of $U^{\otimes k}$ can be obtained as plethysms $\ell_k^{(m)}[\mathrm{char}_U(x)]$.

Assume now U itself is a tensor product of highest weight modules of $GL_n(\mathbb{C})$, where each factor is given by a multiple of a fundamental weight. In other words, U is a tensor product of highest weight modules of rectangular shapes. By a slight abuse of terminology, we shall also refer to them as Kirillov-Reshetikhin modules. There are two other ways to stratify $U^{\otimes k}$ into k components - one using a generalization of the charge statistic introduced by Shimozono [28] modulo k, and one using a certain generalization of Assaf's dual equivalence graphs. We conjecture that all three ways to stratify $U^{\otimes k}$ coincide, and we prove that the latter two ways do. In special cases, the fact that the first two ways coincide is known due to Lascoux, Leclerc, and Thibon [16], and Iijima [12]. A related question of splitting a tensor square of a representation into symmetric and antisymmetric parts has also been studied by Carré and Leclerc [6] and Maas-Gariépy and Tétreault [22].

Kirillov-Reshetikhin crystals (KR crystals) and their tensor products (which we denote $\otimes$KR crystals) are an important family of affine crystals, which arise in the representation theory of quantum groups [14]. The $\otimes$KR crystals are indexed by sequences of rectangular Young diagrams, and can be realized as the corresponding type A crystal with the extra operators f_n and e_n (often denoted f_0 and e_0). These action of these operators can be computed explicitly using promotion on tableaux [28]. Extensive work has been done connecting $\otimes$KR crystals to Demazure crystals, see [28, 24, 25, 19, 20, 21]. $\otimes$KR crystals have a grading by the global energy function, which is computed using combinatorial R-matrices and the local energy function between tensor factors. Nakayashiki and Yamada [23] proved that the energy function on tensor products of single columns, considered as $\otimes$KR crystals, equals the cocharge of the corresponding semistandard tableau (originally defined by Lascoux and Schützenberger [17, 18]) obtained by insertion via the Robinson-Schensted-Knuth (RSK) correspondence. This result was generalized to more general KR crystals by [27, 28, 25, 19, 20].

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Now we take U as a tensor product of KR modules over $GL_n(\mathbb{C})$. The character $\mathrm{char}_{U^{\otimes k}}(x)$ is a product of rectangular Schur functions. Associating with $U^{\otimes k}$ a $\otimes$KR crystal, one can define the energy/charge function on it, and further stratify $U^{\otimes k}$ into k disjoint components by the residue of charge modulo k.

Dual equivalence graphs (DEGs) were introduced by Assaf [2] as a tool to prove Schur positivity of symmetric functions. They have deep connections to crystal graphs [1, 4, 8]. A DEG is a finite graph with vertices given by the standard Young tableaux of a fixed shape, and edges given by dual Knuth moves. One can also construct DEGs via dual Knuth moves on permutations, which preserve the recording tableau of the permutation under the RSK correspondence. Certain generating functions over DEGs are the Schur functions. Assaf's work then provides axiomatization of DEGs, allowing one to prove Schur positivity by verifying a set of local axioms on graphs. A W-graph is a combinatorial graph encoding the representations of Hecke algebra associated with a Coxeter group W. W-graphs were introduced in the pioneering work of Kazhdan and Lusztig, further notion of molecules as certain connected components of W-graphs was introduced by Stembridge in [30], [31]. Chmutov [7] showed that in type A molecules are exactly the DEGs as described above. Further, Chmutov, Pylyavskyy, and Yudovina [9] introduced KL DEGs - an affine analogue of the DEGs, using affine permutations and tabloids.

The main combinatorial object we introduce in this paper is the Kirillov-Reshetikhin dual equivalence graph (KR DEG). they are defined on the 0-weight space of a tensor product of KR crystals, in a manner analogous to Assaf's construction in the type A case. We show that, similarly to Assaf's work, the edges of KR DEGs can be realized by commutators of consecutive crystal operators. Perhaps surprisingly, KR DEGs are not always connected, even though the corresponding $\otimes$KR crystals are. This was observed in [8] for KL DEGs, which are a special case of KR DEGs, see Remark 3.8. The connected components of the corresponding KR DEG provide another natural stratification of $U^{\otimes k}$. The main result of this paper, Theorem 4.1, shows that this stratification agrees with the stratification based on charge modulo k. We note here that, as far as we can tell, KR DEGs are unrelated to the affine dual equivalence graphs defined in [3], which used starred strong tableaux to study k-Schur functions.

Our paper is organized as follows: In Section 2, we introduce definitions and notations, as well as important algorithms and facts on representation of $GL(V)$, tableaux, crystal graphs and dual equivalence graphs. In Section 3, we construct the KR DEGs from the 0 weight space of $\otimes$KR crystals, and then establish their connections to the affine crystals. In Section 4, we study the structure of the KR DEGs and prove that the number of connected components is exactly the gcd of the multiplicities of each rectangle. Furthermore, the elements in each connected component are determined by the charge statistic. In Section 5 we discuss the characters associated to connected components of KR DEGs, and conjecture a formula for them involving plethysms with cyclic characters.

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2. Background

2.1. **Tableau Combinatorics.** For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, a *semistandard Young tableau* (SSYT) T of shape λ is a filling of the cells of the Young diagram of λ by elements of $[n] = \{1, \ldots, n\}$ that increases weakly in every row and increases strictly in every column. If T uses each number in $[n]$ only once, we say T has a standard filling. For a pair of partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$, satisfying $\mu_i \leq \lambda_i$ for each i, the *skew* semistandard (resp. standard) tableaux are defined in the natural way in terms of the skew shape λ/μ. Throughout the paper, we use English notation for partitions and tableaux.

Let T be a (skew) tableau. The *row-reading word* of T, denoted by $\mathsf{rw}(T)$, is the word obtained by concatenating the entries of T row by row from bottom to top, reading from left to right in each row. The *column-reading word* of T, denoted by $\mathsf{cw}(T)$, is the word obtained by concatenating the entries of T column by column from left to right, reading each column from bottom to top.

Example 2.1. Let $\lambda/\mu = (4,3,2)/(1)$. For $T = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 6 \\ \hline 3 & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-3} 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$, $\mathsf{rw}(T) = 56334126$, and $\mathsf{cw}(T) = 53631426$.

Let T be a skew tableau of shape λ/μ, and let a be a box that shares at least one edge with T such that adding a to T results in a valid skew shape. We call a an inner (resp. outer) box if a shares lower or right (resp. upper or left) edges with T. The *jeu de taquin slide* of T in terms of a, denoted by $\mathrm{jdt}_a(T)$, is defined in the following steps: First, mark box a with $*$. If the inner (resp. outer) box a shares only one edge with the box a_1 of T, then slide a_1 into $*$. If a shares edges with two boxes of T, choose the one with a smaller (resp. larger) entry and slide the entry into $*$. If the two adjacent entries are equal, then slide the lower one upward (resp. slide the higher one downward). Third, continue the same procedure until $*$ is moved to the outer (resp. inner) boundary. Lastly, remove $*$ to obtain a valid skew tableau $\mathrm{jdt}_a(T)$.

Example 2.2. For $T = \begin{ytableau} a & 1 & 2 & 6 \\ 1 & 3 & 4 \\ 5 & 6 & b \end{ytableau}$ with the inner box a and the outer box b marked. Then $\mathrm{jdt}_a(T)$ is obtained by the sequence of jeu de taquin slides

$$\begin{ytableau} * & 1 & 2 & 6 \\ 1 & 3 & 4 \\ 5 & 6 \end{ytableau} \to \begin{ytableau} 1 & 1 & 2 & 6 \\ * & 3 & 4 \\ 5 & 6 \end{ytableau} \to \begin{ytableau} 1 & 1 & 2 & 6 \\ 3 & * & 4 \\ 5 & 6 \end{ytableau} \to \begin{ytableau} 1 & 1 & 2 & 6 \\ 3 & 4 \\ 5 & 6 \end{ytableau}.$$

Similarly, $\mathrm{jdt}_b(T) = \begin{ytableau} \none & 1 & 2 & 6 \\ \none & 3 & 4 \\ 1 & 5 & 6 \end{ytableau}$.

Definition 2.3. Let T be a semistandard tableau with entries at most n. The *promotion operator* pr on T with respect to n is a method of obtaining a new semistandard tableau $\mathrm{pr}(T)$, consisting of the following steps:

(1) Mark all n's in T as outer boxes $*$.
(2) Compute jdt_n from bottom to top, and add the entry 0 in the empty spaces on the inner boundary.
(3) Increase all entries by 1.

Example 2.4. Let $n = 6, T = \begin{ytableau} 1 & 1 & 2 & 6 \\ 3 & 4 \\ 5 & 6 \end{ytableau}$. Then $\mathrm{pr}(T)$ is obtained as follows:

$$\begin{ytableau} 1 & 1 & 2 & 6 \\ 3 & 4 \\ 5 & * \end{ytableau} \to \begin{ytableau} 1 & 1 & 2 & 6 \\ 3 & 4 \\ * & 5 \end{ytableau} \to \begin{ytableau} 1 & 1 & 2 & 6 \\ * & 4 \\ 3 & 5 \end{ytableau} \to \begin{ytableau} 0 & 1 & 2 & 6 \\ 1 & 4 \\ 3 & 5 \end{ytableau} \to$$

$$\begin{ytableau} 0 & 1 & 2 & * \\ 1 & 4 \\ 3 & 5 \end{ytableau} \to \begin{ytableau} 0 & 1 & * & 2 \\ 1 & 4 \\ 3 & 5 \end{ytableau} \to \begin{ytableau} 0 & 0 & 1 & 2 \\ 1 & 4 \\ 3 & 5 \end{ytableau} \to \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 5 \\ 4 & 6 \end{ytableau}.$$

Now, we briefly recall the *row insertion* algorithm of the RSK correspondence, which we will use later to define the *charge* of two tableaux. Let T be a semistandard tableau. Row inserting a positive integer k into T, denoted by $T \leftarrow k$, is defined as follows: Find the left-most cell of the first row of T containing a value strictly larger than k. If no such cell exists, append a cell filled with k to the end of the row. Otherwise, replace that cell with k, and insert the value that used to be there into the next row using the same procedure.

Example 2.5. Let $T = \begin{ytableau} 1 & 1 & 2 & 4 & 5 \\ 3 & 3 & 4 & 6 \\ 5 & 7 \end{ytableau}$. Then $T \leftarrow 2 = \begin{ytableau} 1 & 1 & 2 & \mathbf{2} & 5 \\ 3 & 3 & 4 & \mathbf{4} \\ 5 & \mathbf{6} \\ 7 \end{ytableau}$, where the bumped positions are highlighted in bold.

2.2. **Crystal Graphs.**

Definition 2.6. Consider a word w on the alphabet $[n]$. For $i \in [n-1]$, define the *raising* and *lowering* operators e_i and f_i on w as follows:

(1) In the subword of w consisting of only i's and $(i+1)$'s, replace $i+1$ with " (" and i with ") ".
(2) Remove matched parentheses.

(3) If applying f_i, replace the right-most unmatched i with $i+1$. If applying e_i, replace the left-most unmatched $i+1$ with i. The word remains unchanged if no such unmatched entries exist.

It is clear that e_i and f_i are inverse operations when they are defined.

Example 2.7. Let $w = 1345264453455$ and $i = 4$, we replace all 4 with ") " and 5 with " (" to obtain the sequence of parentheses **)**())()**((**. Then we match the parentheses in the usual way, and the unmatched ones are highlighted in bold. Thus, $f_i(w) = 1345264\mathbf{5}53455$ and $e_i(w) = 13452644534\mathbf{4}5$.

We can extend this definition to a semistandard tableau T by applying the raising and lowering operators to $\mathsf{rw}(T)$ (resp. $\mathsf{cw}(T)$), and changing the cell of T that corresponds to the letter that was changed. For simplicity, we write $f_i(T)$ and $e_i(T)$. It is well known (see, for instance, [5, Chapter 3]) that the set of row (resp. column) reading words of semistandard tableaux is closed under e_i and f_i for $i \in [n-1]$.

Definition 2.8. Let λ be a partition on alphabet $[n]$. The type A_{n-1} *crystal graph* B_λ is a finite directed graph with vertices given by $\mathrm{SSYT}(\lambda)$ and edges colored by $\{1, 2, \cdots, n-1\}$. In particular, we have an i-colored edge pointing from tableau T to T' if $f_i(T) = T'$. See Figure 1 for an example.

For a rectangular partition $\lambda = R = (s^r)$, the type A_{n-1} crystal graph B_λ admits a type $\widetilde{A}_{n-1}$ crystal structure. The underlying vertices are the same, as are the normal crystal operators e_i and f_i for $i \in [n-1]$. The crystal operators e_n, f_n for the affine node were given explicitly by Shimozono [28] as

$$e_n = \mathrm{pr}^{-1} \circ e_1 \circ \mathrm{pr} \quad \text{and} \quad f_n = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}$$

The cyclic symmetry of the $\widetilde{A}_{n-1}$ Dynkin diagram also implies that, more generally, $e_{i+1} = \mathrm{pr}^{-1} \circ e_i \circ \mathrm{pr}$ and $f_{i+1} = \mathrm{pr}^{-1} \circ f_i \circ \mathrm{pr}$. The classical crystal B_R with this added structure is the *Kirillov-Reshetikhin (KR) crystal* $B^{r,s}$, which we will call B^R.

Example 2.9. Let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array}$. We have

$$f_3(T) = \mathrm{pr}^{-1} \circ f_1 \circ \mathrm{pr}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array}\right) = \mathrm{pr}^{-1} \circ f_1\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array}\right) = \mathrm{pr}^{-1}\left(\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}.$$

Similarly, $e_3(T) = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array}$.

Figure 1 shows an example of KR crystal graphs.

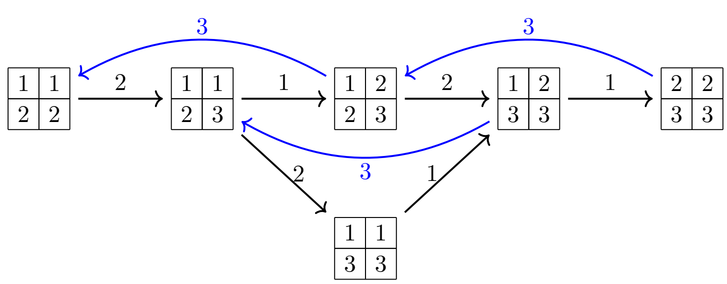


FIGURE 1. The above graph with only black edges is the crystal graph $B_{(2,2)}$ for $n = 3$. Adding in the blue edges, we get the corresponding KR crystal graph $B^{(2,2)}$.

It is easy to see that the set of row (or column) reading words of semistandard tableaux is also closed under e_n and f_n.

Let $R = (R_1, \ldots, R_k)$ be a tuple of rectangular partitions, that is, for each $i \in [k]$, $R_i = (s_i^{r_i})$ for some $s_i, r_i > 0$. We consider the tensor product of KR crystals $B^R = B^{R_1} \otimes \cdots \otimes B^{R_k}$. The vertex set of this graph consists of tuples of SSYT, $T = T_1 \otimes \cdots \otimes T_k$ where $T_i \in SSYT(R_i)$. It is sometimes useful to think of T as a disconnected skew tableau, where the northeast corner of T_i touches the southwest corner of T_{i+1}. Unlike their type A analogs (which split into connected components according to the Littlewood-Richardson rule), tensor products of KR crystals are connected.

Following [13, 28], given rectangles R_1, R_2, there is a unique affine crystal graph isomorphism $\sigma : B^{R_1} \otimes B^{R_2} \to B^{R_2} \otimes B^{R_1}$ called the *combinatorial R-matrix*. It can be easily described in terms of tableaux. Given $T = T_1 \otimes T_2 \in B^{R_1} \otimes B^{R_2}$, $\sigma(T)$ is the unique skew-shape $T_2' \otimes T_1'$ of shape $R_2 \otimes R_1$ obtainable by applying jdt moves to T (see Example 2.11 for an example of this computation). For general $\otimes$KR crystals, we let σ_i denote the combinatorial R-matrix which swaps the i and $i+1$ tensor factors. The σ_i satisfy the Yang-Baxter equation, so

$$B^{R_1} \otimes \cdots \otimes B^{R_k} \cong B^{R_1'} \otimes \cdots \otimes B^{R_k'},$$

where $(R_1', \ldots, R_k')$ is any permutation of $(R_1, \ldots, R_k)$.

Definition 2.10. Let $R = (R_1, \ldots, R_k)$ be a sequence of rectangles, with $R_i = (s_i^{r_i})$. The charge function is defined on B^R as follows:

1. If $k = 1$, $\mathrm{charge}(T) = 0$ for every $T \in B^R$.
2. If $k = 2$, so $T = T_1 \otimes T_2$, then $\mathrm{charge}(T)$ is equal to the number of cells with row index greater than $\max(r_1, r_2)$ in the tableau $P(\mathsf{rw}(T_1)\mathsf{rw}(T_2))$ obtained by row insertion.
3. If $k > 2$, then
$$\mathrm{charge}(T) = \sum_{i<j} \mathrm{charge}(T_{j-1}^{(i)} \otimes T_j)$$
where $T_{j-1}^{(i)}$ the $(j-1)$'st component of the image of T under the composition of R-matrices $\sigma_{j-2} \cdots \sigma_i$.

This definition differs slightly from that of [28], due to our usage of row insertion instead of column insertion, but it is equivalent.

Example 2.11. Let $R = ((3^3), (2^2), (2^2))$, and consider

$$T = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 9 & 13 & 14 \\ \hline 10 & 16 & 17 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 11 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 12 & 15 \\ \hline \end{array}$$

To compute $\mathrm{charge}(T)$, we must compute $\mathrm{charge}(T_1 \otimes T_2)$, $\mathrm{charge}(T_2 \otimes T_3)$, and $\mathrm{charge}(T_2^{(1)} \otimes T_3)$, each of which can be computed by row insertion. The first two are straightforward:

$$T_1 \leftarrow \mathsf{rw}(T_2) = \begin{array}{cccc} 2 & 5 & 6 & 8 \\ 3 & 11 & 14 & \\ 4 & 13 & 17 & \\ 9 & 16 & & \\ 10 & & & \end{array} \qquad T_2 \leftarrow \mathsf{rw}(T_3) = \begin{array}{cccc} 1 & 7 & 12 & 15 \\ 2 & 8 & & \\ 3 & 11 & & \end{array}$$

To compute the last term, we first compute $T_2^{(1)}$ via the combinatorial R-matrix σ_1:

$$T_1 \otimes T_2 = \begin{array}{ccccc} & & & 2 & 8 \\ & & & 3 & 11 \\ 4 & 5 & 6 & & \\ 9 & 13 & 14 & & \\ 10 & 16 & 17 & & \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} & & 2 & 6 & 8 \\ & & 3 & 11 & 14 \\ & & 5 & 13 & 17 \\ 4 & 9 & & & \\ 10 & 16 & & & \end{array} = T_1^{(1)} \otimes T_2^{(1)}$$

Finally, we see that

$$T_2^{(1)} \leftarrow \mathsf{rw}(T_3) = \begin{array}{ccccc} 1 & 6 & 7 & 12 & 15 \\ 2 & 8 & 14 & & \\ 3 & 11 & 17 & & \\ 5 & 13 & & & \end{array}$$

Thus, $\mathrm{charge}(T) = 3 + 2 + 2 = 7$, where cells that contribue a $+1$ to charge are highlighted in blue.

Proposition 2.12 ([28])**.** *Let $T_1 \otimes T_2 \in B^R \otimes B^R$. Then*

$$\mathrm{charge}(e_n(T_1 \otimes T_2)) = \mathrm{charge}(T_1 \otimes T_2) + \begin{cases} 1 & e_n(T_1 \otimes T_2) = e_n(T_1) \otimes T_2 \\ -1 & e_n(T_1 \otimes T_2) = T_1 \otimes e_n(T_2) \end{cases}$$

and

$$\mathrm{charge}(f_n(T_1 \otimes T_2)) = \mathrm{charge}(T_1 \otimes T_2) + \begin{cases} -1 & f_n(T_1 \otimes T_2) = f_n(T_1) \otimes T_2 \\ 1 & f_n(T_1 \otimes T_2) = T_1 \otimes f_n(T_2) \end{cases}$$

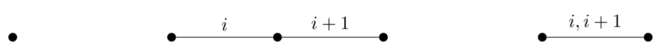
Example 2.13. To illustrate the above proposition, we consider $T = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 7 & 8 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$. Note that $\mathrm{charge}(T) = 2$, but $f_8(T) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 7 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$, and $\mathrm{charge}(f_8(T)) = 1$.

2.3. **Dual Equivalence Graphs.** Dual equivalence graphs were introduced by Assaf in [2] as a general framework to prove that certain sums of quasi-symmetric functions were symmetric and Schur positive. We summarize some results here that will be useful to us. However, we follow the slightly altered conventions of [7, Section 3] as they match our notation better. Fix $n > 0$. A signed color graph is a graph $G = (V, E)$ along with the extra data

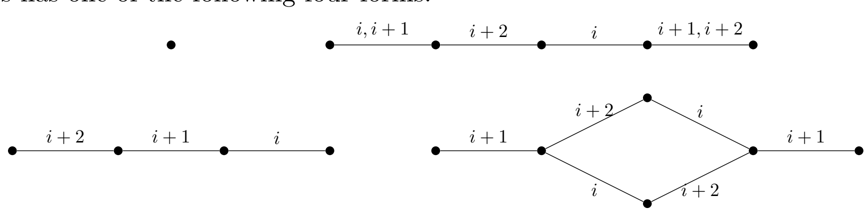
- For each vertex $x \in V$, a set of descents $D(x) \subset [n-1]$.
- For each edge $xy \in E$, a set of colors $\tau(E) \subset [n-2]$.

Let $E_i = \{e \in E \mid i \in \tau(e)\}$. A dual equivalence graph is a signed color graph $G = (V, E, D, \tau)$, satisfying the following axioms.

(1) Suppose that $i \in D(x)$ and $i+1 \notin D(x)$. Then there is a unique $y \in V$ such that $xy \in E_i$, $i \notin D(y)$, $i+1 \in D(y)$.
(2) If $xy \in E_i$, then $i \in D(x)$ iff $i \notin D(y)$, $i+1 \in D(x)$ iff $i+1 \notin D(y)$ and $j \in D(x)$ iff $j \in D(y)$ for any $j < i-1$ or $j > i+2$. In other words, walking along an edge in E_i swaps i and $i+1$, and does not affect any labels besides $i-1, i, i+1$, and $i+2$.
(3) Suppose $xy \in E_i$. If $i-1 \in D(x)\Delta D(y)$ (where Δ is the symmetric difference), then $i-1 \in D(x)$ iff $i+1 \in D(x)$. Similarly, if $i+2 \in D(x)\Delta D(y)$, then $i+2 \in D(x)$ iff $i \in D(x)$.
(4) If $i < n-2$, and we consider the subgraph of G consisting of only i and $i+1$ colored edges, then every connected component has one of the following three forms:



If $i < n-3$, then every connected component of the subgraph of G consisting of $i, i+1$, and $i+2$ colored edges has one of the following four forms:



(5) Suppose $xy \in E_i$ and $yz \in E_j$, with $|i-j| \geq 3$. Then there is some w such that $xw \in E_j$ and $wz \in E_i$.
(6) Consider a connected component of the subgraph of G obtained by restricting to edges with colors $\leq i$. Erasing all i colored edges breaks it down into several components, *any two of which* are connected by an edge in E_i.

Given a partition λ, we can construct a dual equivalence graph G_λ whose vertices are standard Young tableaux of shape λ, where there is an edge between T and T' if we can obtain T' by swapping i and $i+1$ in T, with the added condition that the descent sets of T and T' are incomparable with respect to containment. Edges in this graph correspond to dual-Knuth moves on the row-reading words of the tableaux (so, in particular, G_λ is connected). It turns out that the G_λ are essentially all of the dual equivalence graphs.

Theorem 2.14 ([2, Theorem 3.7]). *Suppose G is a dual equivalence graph, then every connected component of G is isomorphic to G_λ for some λ.*

We will use the following observation of Assaf in the type A case to generalize the notion of a dual equivalence graph to affine type A.

Theorem 2.15 ([1]). *The graph G_λ can be constructed by taking the 0-weight space of the type A crystal graph B_λ, and connecting vertices by commutators of consecutive crystal operators. Explicitly, T and T' are connected by an edge in G_λ if and only if $T' = e_i e_{i+1} f_i f_{i+1}(T)$ or $T' = e_{i+1} e_i f_{i+1} f_i(T)$ for some $i \in [n-1]$.*

Example 2.16. The following is $G_{(3,2)}$, where the descent set of each tableau is highlighted in red.

$$\begin{array}{|c|c|c|}\hline 1&2&{\color{red}3}\\\hline 4&5\\\cline{1-2}\end{array} \overset{23}{\text{———}} \begin{array}{|c|c|c|}\hline 1&{\color{red}2}&{\color{red}4}\\\hline 3&5\\\cline{1-2}\end{array} \overset{1}{\text{———}} \begin{array}{|c|c|c|}\hline {\color{red}1}&3&{\color{red}4}\\\hline 2&5\\\cline{1-2}\end{array} \overset{3}{\text{———}} \begin{array}{|c|c|c|}\hline {\color{red}1}&{\color{red}3}&5\\\hline 2&4\\\cline{1-2}\end{array} \overset{12}{\text{———}} \begin{array}{|c|c|c|}\hline 1&{\color{red}2}&5\\\hline 3&4\\\cline{1-2}\end{array}$$

The left-most edge can be computed using the following sequence of crystal operators on their row words.

$$45123 \xrightarrow{f_4} 55123 \xrightarrow{f_3} 55124 \xrightarrow{e_4} 45124 \xrightarrow{e_3} 35124$$

Remark 2.17. In Example 2.16 we have added the label i to an edge if exactly one of the adjacent vertices has a descent at i and the other a descent at $i+1$. In later sections, we will instead label edges by t_i, where we obtain one tableau from another by swapping i and $i+1$.

2.4. **Representation of $GL(V)$.** We summarize some results in the representation of $GL(V)$ that will be useful in Section 5. We refer the reader to [11, 29] for a more thorough treatment. Let $V = \mathbb{C}^n$, and let $\varphi_U : GL(V) \to GL(U)$ be a polynomial representation. The *character* of φ_U is a symmetric polynomial given by $\mathrm{char}_U(x_1, \ldots, x_n) = \mathrm{tr}(\varphi_U(\mathrm{diag}(x_1, \ldots x_n)))$, where $\mathrm{diag}(x_1, \ldots, x_n)$ is a diagonal matrix with diagonal entries x_i for $i \in [n]$. If U is a $GL(V)$ module, $U^{\otimes k}$ is a $GL(V) \times S_k$ module, where $GL(V)$ acts diagonally and S_k permutes tensor factors.

Let C_k be a cyclic group (viewed as a subgroup of S_k generated by a k-cycle), and let $\chi : C_k \to \mathbb{C}^\times$ be the irreducible character which maps a generator of C_k to a primitive k'th root of unity, so that $\chi, \chi^2, \ldots \chi^k$ are all of the irreducible characters of C_k. For $m \geq 0$, let $e_m = \frac{1}{k} \sum_{\sigma \in C_k} \chi^{-m}(\sigma) \cdot \sigma \in \mathbb{C}[C_k]$. A quick calculation shows that e_m is idempotent, and that it projects any C_k representation onto its χ^m-isotypic component. The proofs of the following are quick exercises.

Proposition 2.18. *Viewing $V^{\otimes k}$ as a $GL(V) \times C_k$ module by restriction, we have*

$$\mathrm{char}_{e_m V^{\otimes k}}(x_1, \ldots, x_n) = \ell_k^{(m)}$$

where $\ell_k^{(m)}$ is the Fröbenius image of $\mathrm{Ind}_{C_k}^{S_k}(\chi^m)$.

Corollary 2.19. *Let U be a polynomial representation of $GL(V)$. Then*

$$\mathrm{char}_{e_m U^{\otimes k}}(x_1, \ldots, x_n) = \ell_k^{(m)} \circ \mathrm{char}_U(x_1, \ldots, x_n)$$

3. KR dual equivalence graphs

Recall that we have $R = (R_1, \ldots, R_k)$ be a sequence of rectangular partitions such that for each $i \in [1, k], R_i = (s_i^{r_i})$ for some $s_i, r_i > 0$. We fix this notation for the rest of the paper. In this section, we introduce a dual equivalence graph defined based on $\otimes$KR crystals B^R. We begin by defining a set of vertices V and the descent set for each vertex of V as follows.

Definition 3.1. Let V be the set consisting of all weight 0 elements of $\otimes$KR crystals $B^R = B^{R_1} \otimes \cdots \otimes B^{R_k}$. For an element $T = T_1 \otimes \cdots \otimes T_k \in V$, the *descent set* $D(T)$ of T is defined by

$$\begin{aligned} D(T) = &\big\{ i \in [n-1] : i+1 \text{ appears to the left of } i \text{ in } \mathsf{rw}(T) \big\} \\ &\cup \big\{ n : \text{whenever 1 is in } D(\mathrm{pr}(T)) \big\}. \end{aligned}$$

Definition 3.2. Following the notation from Definition 3.1, for any $T, T' \in V$ and $i \in \{1, 2, \cdots, n\}$, we connect T and T' with an edge if T' can be obtained from T by doing one of the following sequences of moves.

(1) For $1 \leq i \leq n-1$, we swap i and $i+1$ to obtain T' and $D(T)$ and $D(T')$ are incomparable with respect to inclusion. In this case we write $t_i(T) = T'$.
(2) For $i = n$, if $n \in T_\ell$ and $i+1 = 1 \in T_j$ are not in the same tensor factor, we compute $\mathrm{jdt}_n(T_\ell)$ with replacing the created inner box with 1, and compute $\mathrm{jdt}_1(T_j)$ with replacing the created outer box with n. We obtain $T' = t_n(T)$ if $D(T)$ and $D(T')$ are incomparable.

(3) If $1, 2$ and n are in the same tensor factor T_j of T such that $\mathsf{rw}(T_j) = (n \cdots 1\, 2 \cdots)$ and $n \in D(T)$, then we can find T' by the sequence of moves: (i) Apply $\mathrm{jdt}_n(T_j)$ to get an inner box $*$; (ii) Switch 1 and 2; (iii) Apply jdt_* to the tableau obtained from the last move, and replace the resulting outer box with n. We denote this edge by $\bar{t}_1(T) = T'$.

(4) If $1, n-1$ and n are in the same tensor factor T_j of T such that $\mathsf{rw}(T_j) = (\cdots n \cdots n-1 \cdots 1 \cdots)$ and $n \notin D(T)$, then we can find T' by the sequence of moves: (i) Apply $\mathrm{jdt}_n(T_j)$ and fill the inner box with 1; (ii) Apply jdt_{n-1} to the tableau obtained from the last move and get an inner box $*$; (iii) Apply jdt_1 for the 1 in the first row of the tableau followed by applying jdt_*, and then fill the two outer boxes with $n-1$ and n. We denote this edge by $\bar{t}_{n-1}(T) = T'$.

We note that edges coming from case (1) correspond to classical dual equivalence moves on the skew tableaux of the shape defined by R.

Example 3.3. We use the following examples to illustrate the establishment of the edges between T and T', corresponding to the four cases we described above.

(1) Let $T = T_1 \otimes T_2 = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 4&7\\\hline 6&8\\\hline\end{array}$ with $D(T) = \{2, 4, 7, 8\}$. For $i = 3$, $T' = t_3(T) = \begin{array}{|c|c|}\hline 1&2\\\hline 4&5\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 3&7\\\hline 6&8\\\hline\end{array}$ with $D(T') = \{3, 7, 8\}$. For $i = 5$, T does not have an edge t_5 since after switching 5 and 6, $D\Big(\begin{array}{|c|c|}\hline 1&2\\\hline 3&6\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 4&7\\\hline 5&8\\\hline\end{array}\Big) = \{2, 4, 5, 7, 8\} \supsetneq \{2, 4, 7, 8\} = D(T)$.

(2) For the same T in (1), let $i = 8$. We find $\mathrm{jdt}_8(T_2)$ and replace the created inner box with 1 to obtain $T_2' = \begin{array}{|c|c|}\hline 1&4\\\hline 6&7\\\hline\end{array}$. Also, find $\mathrm{jdt}_1(T_1)$ and replace the created outer box with 8 gives $T_1' = \begin{array}{|c|c|}\hline 2&5\\\hline 3&8\\\hline\end{array}$. Thus $T' = t_8(T) = \begin{array}{|c|c|}\hline 2&5\\\hline 3&8\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&4\\\hline 6&7\\\hline\end{array}$ with $D(T') = \{1, 2, 4, 7\}$.

(3) Let $T = T_1 \otimes T_2 = \begin{array}{|c|c|c|}\hline 1&2&8\\\hline 5&10&12\\\hline 7&11&13\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 3&6\\\hline 4&9\\\hline\end{array}$ with $D(T) = \{3, 4, 6, 9, 10, 12, 13\}$. T falls into Case (3) of Definition 3.2. We take the sequence of moves as

$$T_1 = \begin{array}{|c|c|c|}\hline 1&2&8\\\hline 5&10&12\\\hline 7&11&13\\\hline\end{array} \xrightarrow{\mathrm{jdt}_{13}} \begin{array}{|c|c|c|}\hline *&2&8\\\hline 1&5&10\\\hline 7&11&12\\\hline\end{array} \xrightarrow{\substack{\text{switch}\\ 1,2}} \begin{array}{|c|c|c|}\hline *&1&8\\\hline 2&5&10\\\hline 7&11&12\\\hline\end{array} \xrightarrow{\mathrm{jdt}_*} \begin{array}{|c|c|c|}\hline 1&5&8\\\hline 2&10&12\\\hline 7&11&13\\\hline\end{array},$$

and

$$\bar{t}_1(T) = T' = T_1' \otimes T_2' = \begin{array}{|c|c|c|}\hline 1&5&8\\\hline 2&10&12\\\hline 7&11&13\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 3&6\\\hline 4&9\\\hline\end{array}$$

with $D(T') = \{1, 3, 4, 6, 9, 10, 12\}$.

Now T' falls into Case (4) of Definition 3.2. We take the sequence of moves on T_1' as

$$T_1' = \begin{array}{|c|c|c|}\hline 1&5&8\\\hline 2&10&12\\\hline 7&11&13\\\hline\end{array} \xrightarrow{\mathrm{jdt}_{13}} \begin{array}{|c|c|c|}\hline 1&1&8\\\hline 2&5&10\\\hline 7&11&12\\\hline\end{array} \xrightarrow{\mathrm{jdt}_{12}} \begin{array}{|c|c|c|}\hline *&1&8\\\hline 1&5&10\\\hline 2&7&11\\\hline\end{array} \xrightarrow{\mathrm{jdt}_1} \begin{array}{|c|c|c|}\hline *&5&8\\\hline 1&7&10\\\hline 2&11&12\\\hline\end{array} \xrightarrow{\mathrm{jdt}_*} \begin{array}{|c|c|c|}\hline 1&5&8\\\hline 2&7&10\\\hline 11&12&13\\\hline\end{array},$$

and

$$\bar{t}_{12}(T') = T'' = T_1'' \otimes T_2'' = \begin{array}{|c|c|c|}\hline 1&5&8\\\hline 2&7&10\\\hline 11&12&13\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 3&6\\\hline 4&9\\\hline\end{array}$$

with $D(T'') = \{1, 3, 4, 6, 9, 10, 13\}$.

Now we are ready to define our dual equivalence graph.

Definition 3.4. Let $R = (R_1, \ldots, R_k)$ be a sequence of rectangles. The *Kirillov-Resehtikhin dual equivalence graph* (KR DEG) of R, denoted $\mathcal{T}(R)$, is the graph with vertex set given by weight 0 elements of $B^{R_1} \otimes \cdots \otimes B^{R_k}$. Two vertices T, T' are connected by an edge if $g(T) = T'$, for some $g \in \{t_1, \ldots, t_n, \bar{t}_1, \bar{t}_{n-1}\}$.

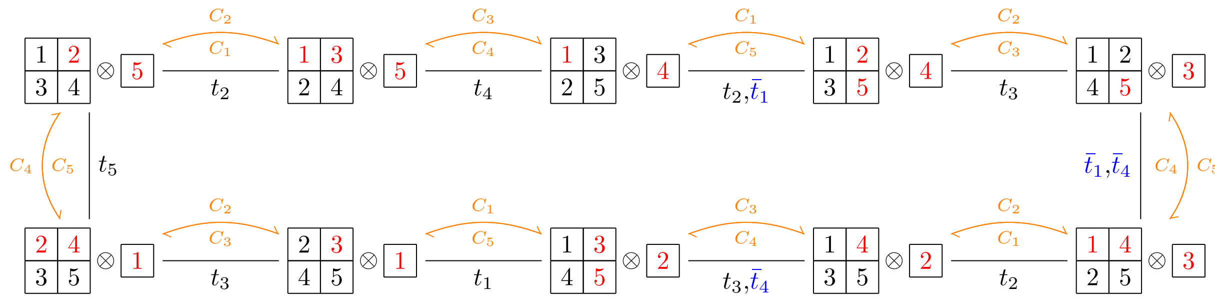


FIGURE 2. The KR DEG $\mathcal{T}((2^2),(1))$. The elements of the descent set of each vertex are shown in red. The edges t_1-t_5 are shown in black and the special edges $\bar{t}_1, \bar{t}_4$ are shown in blue. The edges equivalently obtained by crystal commutators C_1-C_5 are shown in orange.

Figure 2 shows an example of the KR DEG $\mathcal{T}((2^2),(1))$.

Next, we show the connection between our KR DEGs and affine crystals, as a generalization of the result by Assaf [1] on DEGs in type A.

Theorem 3.5. *For any $i \in [n]$, let C_i be the crystal commutator consisting of the composition of the consecutive crystal operators $e_i e_{i+1} f_i f_{i+1}$. The KR DEG $\mathcal{T}(R)$ can be equivalently generated by its vertex set V and the crystal commutators C_i. In other words, for any two vertices T and T' in V, T and T' are connected by an edge of $\mathcal{T}(R)$ if and only if $\mathsf{rw}(T) \neq \mathsf{rw}(T')$ and the two reading words are connected by a crystal commutator C_i for some i.*

Proof. Suppose that T and T' are two vertices of V connected by an edge (t_i, $\bar{t}_1$, or $\bar{t}_{n-1}$) in $\mathcal{T}(R)$. Note that T' is obtained from T by swapping i and $i+1$ or conducting certain jdt moves. In either of the cases specified in Definition 3.2, we can assume that, without loss of generality, $i \notin D(T)$ and $i \in D(T')$. Since $D(T)$ and $D(T')$ are not comparable with respect to inclusion, we have either $i-1 \in D(T), i-1 \notin D(T')$ or $i+1 \in D(T), i+1 \notin D(T')$. It suffices to consider the first case, since the other follows immediately after reindexing and switching the roles of T and T'.

(1) If $2 \leq i \leq n-1$, then the relative positions of $i-1, i, i+1$ in the reading words will be $\mathsf{rw}(T) = (\cdots i \cdots i-1 \cdots i+1 \cdots)$ and $\mathsf{rw}(T') = (\cdots i+1 \cdots i-1 \cdots i \cdots)$. We find $C_{i-1}(T)$ as

$$\begin{aligned}
&e_{i-1}e_i f_{i-1} f_i(\cdots i \cdots i-1 \cdots i+1 \cdots) \\
&= e_{i-1}e_i f_{i-1}(\cdots i+1 \cdots i-1 \cdots i+1 \cdots) \\
&= e_{i-1}e_i(\cdots i+1 \cdots i \cdots i+1 \cdots) \\
&= e_{i-1}(\cdots i+1 \cdots i \cdots i \cdots) \\
&= (\cdots i+1 \cdots i-1 \cdots i \cdots).
\end{aligned}$$

We obtain exactly $\mathsf{rw}(T')$ with all other letters fixed, and $i, i+1$ are swapped. Thus T and T' are connected by the crystal commutator $C_{i-1} = e_{i-1}e_i f_{i-1} f_i$.

(2) If $i = 1$, then $i-1 = n$, so we have $1 \notin D(T), n \in D(T)$ and $1 \in D(T'), n \notin D(T')$. By definition, this means that $1 \in D(\mathrm{pr}(T))$ and $1 \notin D(\mathrm{pr}(T'))$. We discuss four sub-cases below. The sub-cases (b1)-(b3) correspond to the usual edge t_1 where we simply swap 1 and 2 for T and T'. The sub-case (b4) corresponds to the edge $\bar{t}_1$.

(2a) If 1 and n are in distinct tensor factors and n appears to the left of 2 in the reading word of T. It follows that $\mathsf{rw}(T) = (\cdots 1 \cdots n \cdots 2 \cdots)$ and $\mathsf{rw}(T') = (\cdots 2 \cdots n \cdots 1 \cdots)$. After promotion, we have $\mathsf{rw}(\mathrm{pr}(T')) = (\cdots 3 \cdots 1 \cdots 2 \cdots)$ since $1 \notin D(\mathrm{pr}(T'))$. Further, swapping 1 and 2 will not affect the path of jdt_n so that $\mathsf{rw}(\mathrm{pr}(T)) = (\cdots 2 \cdots 1 \cdots 3 \cdots)$. We use different colors on the reading words to represent distinct tensor factors. Note that the computation would be identical regardless of whether 2 and n are in the same tensor factor. We claim that $\mathsf{rw}(T)$ and $\mathsf{rw}(T')$ are connected by the crystal commutator $C_n = e_n e_1 f_n f_1$, since

$$\begin{aligned}
&e_n e_1 f_n f_1(\cdots 1 \cdots n \cdots 2 \cdots)\\
&= e_n e_1(\mathrm{pr}^{-1} f_1 \,\mathrm{pr})(\cdots 2 \cdots n \cdots 2 \cdots)\\
&= e_n e_1(\mathrm{pr}^{-1} f_1)(\cdots 3 \cdots 1 \cdots 3 \cdots)\\
&= e_n e_1(\cdots 2 \cdots 1 \cdots 2 \cdots)\\
&= \mathrm{pr}^{-1} e_1 \,\mathrm{pr}(\cdots 2 \cdots 1 \cdots 1 \cdots)\\
&= \mathrm{pr}^{-1}(\cdots 3 \cdots 1 \cdots 2 \cdots)\\
&= (\cdots 2 \cdots n \cdots 1 \cdots).
\end{aligned}$$

We make a clarification here regarding the above computation, which also applies to all cases discussed below. Although the promotion operator changes the reading words we choose not to show it for clarity, as the inverse promotion will move all these entries back to their original position.

(**2b**) If 1 and n are in distinct tensor factors and n appears to the right of 2 in the reading word of T (i.e., 2 and n belong to the same tensor factor consisting of a single row), then $\mathsf{rw}(T) = (\cdots 1 \cdots 2 \cdots n \cdots)$ and $\mathsf{rw}(T') = (\cdots 2 \cdots 1 \cdots n \cdots)$. Thus we find $C_n(T)$ as

$$\begin{aligned}
&e_n e_1 f_n f_1(\cdots 1 \cdots 2 \cdots n \cdots)\\
&= e_n e_1(\mathrm{pr}^{-1} f_1 \,\mathrm{pr})(\cdots 2 \cdots 2 \cdots n \cdots)\\
&= \mathrm{pr}^{-1} e_1 \,\mathrm{pr}(\cdots 2 \cdots 1\ 1 \cdots)\\
&= (\cdots 2 \cdots 1 \cdots n \cdots) = \mathsf{rw}(T').
\end{aligned}$$

(**2c**) If 1 and n are in the same tensor factor with 2 being in a different factor, then $\mathsf{rw}(T) = (\cdots n \cdots 1 \cdots 2 \cdots)$ and $\mathsf{rw}(T') = (\cdots n \cdots 2 \cdots 1 \cdots)$. Since $1 \in D(\mathrm{pr}(T))$, we also have $\mathsf{rw}(\mathrm{pr}(T)) = (\cdots 2 \cdots 1 \cdots 3 \cdots)$ and $\mathsf{rw}(\mathrm{pr}(T')) = (\cdots 3 \cdots 1 \cdots 2 \cdots)$. Thus

$$\begin{aligned}
&e_n e_1 f_n f_1(\cdots n \cdots 1 \cdots 2 \cdots)\\
&= e_n e_1(\mathrm{pr}^{-1} f_1 \,\mathrm{pr})(\cdots n \cdots 2 \cdots 2 \cdots)\\
&= \mathrm{pr}^{-1} e_1 \,\mathrm{pr}(\cdots 2 \cdots 1 \cdots 1 \cdots)\\
&= (\cdots n \cdots 2 \cdots 1 \cdots) = \mathsf{rw}(T').
\end{aligned}$$

(**2d**) If $1, 2$ and n are all in the same tensor factor, one can verify that we have exactly the same computation as in (**b3**) with 1 and 2 being adjacent in $\mathsf{rw}(T) = (\cdots n \cdots 1\ 2 \cdots)$. Moreover, by applying $C_n(T)$, we essentially conduct the same sequence of moves on T as in Definition 3.2(3) to derive T'.

(3) If $i = n$, we have $n \notin D(T), n-1 \in D(T)$ and $n \in D(T'), n-1 \notin D(T')$. By definition, $1 \notin D(\mathrm{pr}(T))$ and $1 \in D(\mathrm{pr}(T'))$. Again, We discuss four sub-cases below. The sub-cases (c1)-(c3) correspond to the edge t_n for which we apply jdt moves described in Definition 3.2(2). The sub-case (c4) corresponds to the edge $\bar{t}_{n-1}$.

(**3a**) If n and $n-1$ are in distinct tensor factors and $n-1$ appears to the left of 1 in the reading word of T, then $\mathsf{rw}(T) = (\cdots n \cdots n-1 \cdots 1 \cdots)$ and $\mathsf{rw}(T') = (\cdots 1 \cdots n-1 \cdots n \cdots)$. Similarly, we have $\mathsf{rw}(\mathrm{pr}(T)) = (\cdots 1 \cdots n \cdots 2 \cdots)$ and $\mathsf{rw}(\mathrm{pr}(T')) = (\cdots 2 \cdots n \cdots 1 \cdots)$. Note that the computation would be identical regardless of 1 and $n-1$ belonging to the same or distinct tensor factors. Now $\mathsf{rw}(T)$ and $\mathsf{rw}(T')$ are connected by $C_{n-1}(T)$ as

$$\begin{aligned}
&e_{n-1} e_n f_{n-1}(\mathrm{pr}^{-1} f_1 \,\mathrm{pr})(\cdots n \cdots n-1 \cdots 1 \cdots)\\
&= e_{n-1}(\mathrm{pr}^{-1} e_1 \,\mathrm{pr})(\cdots 1 \cdots n \cdots 1 \cdots)\\
&= e_{n-1}\,\mathrm{pr}^{-1}(\cdots 2 \cdots 1 \cdots 1 \cdots)\\
&= e_{n-1}(\cdots 1 \cdots n \cdots n \cdots)\\
&= (\cdots 1 \cdots n-1 \cdots n \cdots) = \mathsf{rw}(T').
\end{aligned}$$

(**3b**) If n and $n-1$ are in distinct tensor factors and $n-1$ appears to the right of 1 in T (i.e., 1 and $n-1$ belong to the same tensor factor consisting of a single row), then we have $\mathsf{rw}(T) =$

$(\cdots n \cdots 1 \cdots n-1 \cdots)$ and $\mathsf{rw}(T') = (\cdots 1 \cdots n-1\ n \cdots)$, and

$$\begin{aligned} & e_{n-1}e_n f_{n-1} f_n(\cdots n \cdots 1 \cdots n-1 \cdots) \\ &= e_{n-1}e_n(\cdots 1 \cdots 1 \cdots n \cdots) \\ &= e_{n-1}(\cdots 1 \cdots n\ n \cdots) \\ &= (\cdots 1 \cdots n-1\ n \cdots). \end{aligned}$$

(**3c**) If n and $n-1$ are in the same tensor factor with 1 being in a different factor, then $\mathsf{rw}(T) = (\cdots n \cdots n-1 \cdots 1 \cdots)$ and $\mathsf{rw}(T') = (\cdots n-1 \cdots 1 \cdots n \cdots)$. After promotion we have $\mathsf{rw}(\mathrm{pr}(T)) = (\cdots n \cdots 1 \cdots 2 \cdots)$ and $\mathsf{rw}(\mathrm{pr}(T')) = (\cdots n \cdots 2 \cdots 1 \cdots)$. Thus

$$\begin{aligned} & e_{n-1}e_n f_{n-1} f_n(\cdots n \cdots n-1 \cdots 1 \cdots) \\ &= e_{n-1}e_n(\cdots n \cdots 1 \cdots 1 \cdots) \\ &= e_{n-1}(\cdots n \cdots 1 \cdots n \cdots) \\ &= (\cdots n-1 \cdots 1 \cdots n \cdots). \end{aligned}$$

(**3d**) If $1, n-1$ and n are all in the same tensor factor, one can compare that applying $C_{n-1}(T)$, and conduct the sequence of moves on T as in Definition 3.2(4) are essentially the same operations to derive T'.

Now suppose that $\mathsf{rw}(T) \neq \mathsf{rw}(T')$ and $C_{i-1}(T) = T'$ for some $i \in [n]$. We need to check all six relative positions of $i-1, i$ and $i+1$ in $\mathsf{rw}(T)$. For $2 \leq i \leq n-1$, after applying the crystal commutator C_{i-1}, the relative positions of $i-1, i$ and $i+1$ remain the same in all possibilities except for two cases: First, $e_{i-1}e_i f_{i-1} f_i(\cdots i \cdots i-1 \cdots i+1 \cdots) = (\cdots i+1 \cdots i-1 \cdots i \cdots)$, in which case we exchange i and $i+1$; Second, $e_{i-1}e_i f_{i-1} f_i(\cdots i \cdots i+1 \cdots i-1 \cdots) = (\cdots i-1 \cdots i+1 \cdots i \cdots)$, in which case we exchange i and $i-1$. For $i = 1$ (resp. $i = n$), we use the same approach to check all relative positions of $n, 1, 2$ (resp. $n-1, n, 1$) in $\mathsf{rw}(T)$. A brute-force computation with applying $e_n e_1 f_n f_1$ (resp. $e_{n-1} e_n f_{n-1} f_n$) on $\mathsf{rw}(T)$ yields either $\mathsf{rw}(T)$ itself, or $\mathsf{rw}(T')$ for some T' connected with T in $\mathcal{T}(R)$, or some reading word that is not from a valid vertex of $\mathcal{T}(R)$. □

In Figure 2, we illustrate all the equivalent edges obtained by the crystal commutators C_i in orange.

Proposition 3.6. *The map* pr *is a graph automorphism of $\mathcal{T}(R)$, with the additional property that $D(\mathrm{pr}(T)) = \{i+1 \mid i \in D(T)\}$, where addition is taken in $\mathbb{Z}/n\mathbb{Z}$.*

Proof. First, since the crystal operators e_i satisfy $e_{i+1} = \mathrm{pr}^{-1} \circ e_i \circ \mathrm{pr}$ (and similarly for f_i) [28], it is clear that the edges of $\mathcal{T}(R)$ (which are compositions of crystal operators) are preserved under promotion. Second, for $i = n$ the statement about descent sets is true by construction. For other i, it is well known that jeu de taquin slides do not change descents, so $i \in D(T)$ if and only if $i+1 \in D(\mathrm{pr}(T))$. □

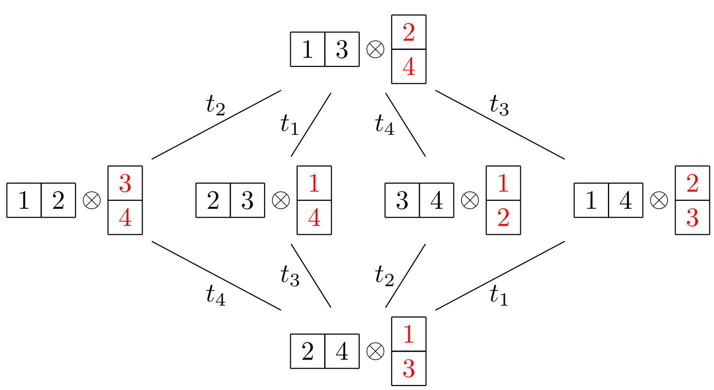


FIGURE 3. The KR DEG $\mathcal{T}((2),(1,1))$.

Example 3.7. Let $R = ((2),(1,1))$. Then $\mathcal{T}(R)$ can be seen in Figure 3. Promotion acts on $\mathcal{T}(R)$ by cycling the middle row once to the right, and exchanging the top and bottom vertices. One can confirm that the descent sets also get cycled appropriately.

Remark 3.8. In [8], Chmutov, Lewis, and Pylyavskyy described the Knuth equivalence classes of affine permutations via the *Affine Matrix Ball Construction* (AMBC) algorithm[9], whose image gives a pair of tabloids P and Q. The Knuth moves on permutations preserve their P tabloids and determine the Knuth moves on their Q tabloids. A modified charge statistic determines the equivalence classes of these Q tabloids, and the induced graph is called the Kazhdan-Lusztig dual equivalence graph (KL DEG). If $R = (R_1, \ldots, R_k)$ with $R_i = (s_i)$, i.e., R is a sequence of single rows, our KR DEG $\mathcal{T}(R)$ reduces to the KL DEG, whose vertex set consists of all Q tabloids of shape $(s_k, \cdots, s_1)$.

4. Connected components of KR DEGs

In this section, we determine the number of connected components of the KR DEG $\mathcal{T}(R)$ (Theorem 4.1). This result can be seen as a generalization of [9, Theorem 8.6] which characterizes the connected components of KL DEGs.

Theorem 4.1. *Let $R = (R_1, \ldots, R_k)$ be a sequence of rectangular partitions. Suppose R_i appears in R with multiplicity m_i, and let $d_R = \gcd(m_1, \ldots, m_k)$. Then $\mathcal{T}(R)$ has d_R connected components, each consisting of all vertices T with a given charge modulo d_R.*

Example 4.2. Let $R = ((1^3),(1^3))$. Then R consists of a single shape with multiplicity 2. Figure 4 shows how $\mathcal{T}(R)$ splits up into two connected components. The left (resp. right) connected component consists of vertices with even (resp. odd) charge.

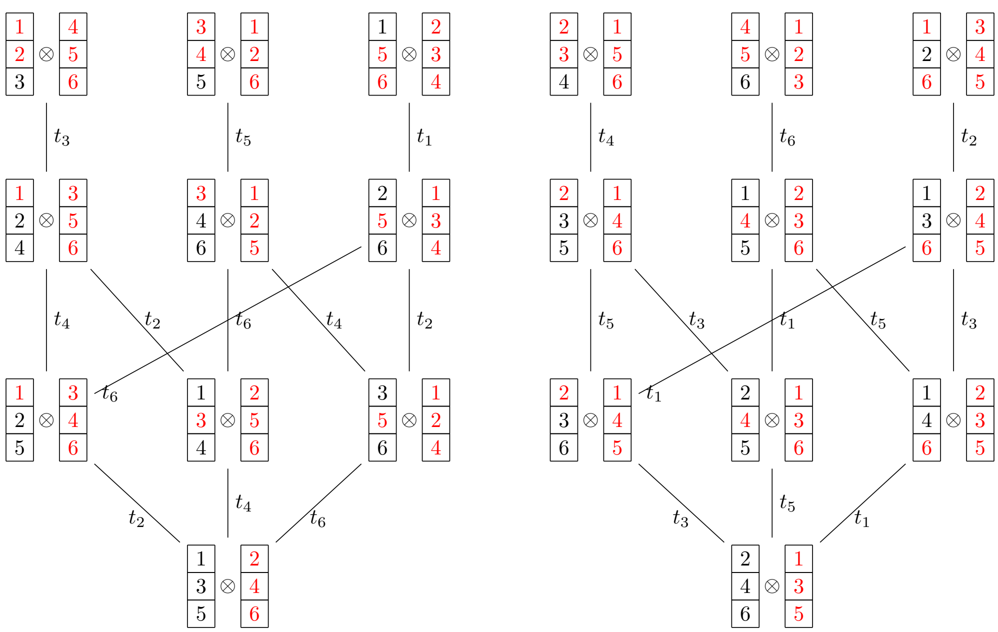


Figure 4. The KR DEG $\mathcal{T}((1^3),(1^3))$ has two connected components. The elements of the descent set of each vertex are shown in red. The elements of the left and right components have charge congruent to 0 and 1 mod 2 respectively.

It suffices to prove Theorem 4.1 in the special case where $R = (R_1^{\otimes m_1}, \ldots, R_\ell^{\otimes m_\ell})$. Any R where partitions of the same shape are not grouped together can be permuted to this form using combinatorial R-matrices without changing the structure of $\mathcal{T}(R)$. The remainder of this section is split into two parts. One showing that $\mathcal{T}(R)$ has at most d_R connected components, and one showing that it has at least d_R connected components. To prove Theorem 4.1, we will define a modified version of the charge statistic (called *semicharge* and denoted $\mathsf{scharge}$) which agrees with charge modulo d_R in this special case, and use it to prove the following statements:

1. For any $T, T' \in \mathcal{T}(R)$, if T and T' are connected by an edge, then $\mathsf{scharge}(T) \equiv \mathsf{scharge}(T') \bmod d_R$ (Proposition 4.7).
2. For any $0 \le i < d_R$, there exists $T \in \mathcal{T}(R)$ such that $\mathsf{scharge}(T) \equiv i \bmod d_R$ (Proposition 4.9).
3. For any $T, T' \in \mathcal{T}(R)$ such that $\mathsf{scharge}(T) \equiv \mathsf{scharge}(T') \bmod d_R$, T and T' are connected in $\mathcal{T}(R)$ (Proposition 4.19 and Proposition 4.25).

4.1. **The lower bound of d_R.** We start by defining the semicharge, and use it to prove the first two statements above.

Definition 4.3. Let $R = (R_1, \dots, R_k)$, and $T \in B^R$. The *semicharge* of T, denoted $\mathsf{scharge}(T)$, is given by the following:

1. If $k = 1$, $\mathsf{scharge}(T) = 0$ for every $T \in B^R$.
2. If $k = 2$, $\mathsf{scharge}(T) = \begin{cases} \mathrm{charge}(T) & R_1 = R_2 \\ 0 & R_1 \neq R_2 \end{cases}.$
3. If $k > 2$, then $\mathsf{scharge}(T) = \sum_{i=1}^{k-1} i \cdot \mathsf{scharge}(T_i \otimes T_{i+1})$.

Example 4.4. Let $R = ((3^3), (2^2), (2^2))$, and

$$T = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline 9 & 13 & 14 \\ \hline 10 & 16 & 17 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 2 & 8 \\ \hline 3 & 11 \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 12 & 15 \\ \hline \end{array},$$

as in Example 2.11. To compute $\mathsf{scharge}(T)$, we see that $\mathsf{scharge}(T_1 \otimes T_2) = 0$ since $R_1 \neq R_2$, and

$$T_2 \leftarrow \mathsf{rw}(T_3) = \begin{array}{|c|c|c|c|}\hline 1 & 7 & 12 & 15 \\ \hline 2 & 8 \\ \cline{1-2} 3 & 11 \\ \cline{1-2} \end{array}$$

so $\mathsf{scharge}(T) = 1 \cdot 0 + 2 \cdot 2 = 4$.

Semicharge is similar to the normal charge statistic, but we ignore any contributions between rectangles of different shapes. This makes it easier to compute, since we only compute the local charge between adjacent tensor factors. A drawback to this simplification is that, unlike charge, semicharge is *not* invariant under the application of combinatorial R-matrices to permute tensor factors. For instance, if $R = ((2^2), (3^3), (2^2))$, then $\mathsf{scharge}$ is identically 0 on B^R, because no adjacent factors are the same shape. However, if all rectangles of the same shape are grouped together in R (as we are assuming), then $\mathsf{scharge}$ and charge agree modulo d_R, as we show in the following lemma.

Lemma 4.5. *Suppose $R = (R_1^{\otimes m_1}, \dots, R_\ell^{\otimes m_\ell})$, and $d_R = \gcd(m_1, \dots, m_\ell)$. Then for any $T \in B^R$ we have $\mathsf{scharge}(T) \equiv \mathrm{charge}(T) \bmod d_R$.*

Proof. Let $m_0 = 0$ and $\alpha_i = m_1 + m_2 + \cdots + m_i$. Since combinatorial R-matrices act by the identity on rectangles of the same shape, the charge computation reduces to the following

$$\begin{aligned} \mathrm{charge}(T) = &\sum_{i<j} m_i m_j \cdot \mathrm{charge}(T_{\alpha_i}^{(\alpha_{j-1})} \otimes T_{\alpha_{j-1}+1}) \\ &+ \sum_{i=0}^{\ell-1} \sum_{j=1}^{m_{i+1}-1} j \cdot \mathrm{charge}(T_{m_i+j} \otimes T_{m_i+j+1}), \end{aligned}$$

and

$$\begin{aligned} \mathsf{scharge}(T) &= \sum_{j=1}^{k-1} j \cdot \mathsf{scharge}(T_j \otimes T_{j+1}) \\ &= \sum_{i=0}^{\ell-1} \sum_{j=1}^{m_{i+1}-1} (\alpha_i + j) \cdot \mathsf{scharge}(T_{m_i+j} \otimes T_{m_i+j+1}). \end{aligned}$$

Since each m_i (and therefore each α_i) is a multiple of d_R, one can directly compare to see that $\mathsf{scharge}(T) \equiv \mathrm{charge}(T) \bmod d_R$. ☐

Lemma 4.6. *Let R be a sequence of rectangles, and $T = T_1 \otimes \cdots \otimes T_k \in B^R$. Then*

$$\mathsf{scharge}(e_n(T)) = \mathsf{scharge}(T) + 1 \bmod d_R,$$

and

$$\mathsf{scharge}(f_n(T)) = \mathsf{scharge}(T) - 1 \bmod d_R.$$

Proof. By definition, we have $e_n(T_1 \otimes \cdots \otimes T_k) = T_1 \otimes \cdots \otimes e_n(T_i) \otimes \cdots \otimes T_k$ for some i. It follows that the only local semicharges that will change are at $T_{i-1} \otimes T_i$ and $T_i \otimes T_{i+1}$. Using Definition 4.3 and Proposition 2.12, we know that

$$\mathsf{scharge}(T_{i-1} \otimes e_n(T_i)) = \begin{cases} \mathsf{scharge}(T_{i-1} \otimes T_i) - 1 & R_{i-1} = R_i \\ 0 & R_{i-1} \neq R_i \end{cases},$$

and

$$\mathsf{scharge}(e_n(T_i) \otimes T_{i+1})) = \begin{cases} \mathsf{scharge}(T_i \otimes T_{i+1}) + 1 & R_i = R_{i+1} \\ 0 & R_i \neq R_{i+1} \end{cases}.$$

If $R_{i-1} = R_i = R_{i+1}$, then

$$\mathsf{scharge}(e_n(T)) = \mathsf{scharge}(T) - (i-1) + i = \mathsf{scharge}(T) + 1.$$

If $R_{i-1} = R_i \neq R_{i+1}$, then $i \equiv 0 \bmod d_R$, and

$$\mathsf{scharge}(e_n(T)) = \mathsf{scharge}(T) - (i-1) \equiv \mathsf{scharge}(T) + 1 \bmod d_R.$$

If $R_{i-1} \neq R_i = R_{i+1}$, then $i \equiv 1 \bmod d_R$, and

$$\mathsf{scharge}(e_n(T)) = \mathsf{scharge}(T) + i \equiv \mathsf{scharge}(T) + 1 \bmod d_R.$$

If $R_{i-1} \neq R_i \neq R_{i+1}$, then $d_R = 1$ so the statement is trivial. Thus in all cases $\mathsf{scharge}(e_n(T)) = \mathsf{scharge}(T) + 1 \bmod d_R$ as desired. The proof of the second claim is analogous. ☐

Proposition 4.7. *Suppose that $T, T' \in \mathcal{T}(R)$ are connected by an edge in $\mathcal{T}(R)$.. Then we have $\mathsf{scharge}(T) \equiv \mathsf{scharge}(T') \bmod d_R$.*

Proof. Without loss of generality, we have $T = e_i e_{i+1} f_i f_{i+1}(T')$ for some $i \in [n]$. If $i, i+1 \in [n-1]$, then $\mathrm{charge}(T) = \mathrm{charge}(T')$ since classical crystal operators do not change charge. Otherwise, one of $i, i+1$ is n, and the net change of $\mathsf{scharge}$ will be $0 \bmod d_R$ by Lemma 4.6. In either case, the claim follows. ☐

Lemma 4.8. *Suppose T_1, T_2 are increasing rectangular tableaux of shape (s^r), such that n appears at most once between them. Then*

$$\mathsf{scharge}(\mathrm{pr}(T_1 \otimes T_2)) = \mathsf{scharge}(T_1 \otimes T_2) + \begin{cases} -1 & n \in T_1 \\ 1 & n \in T_2 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. If n is not in either T_i, then promotion will only increase the value of every cell by 1. This will not change the semicharge, since the relative order of any two cells (and thus the shape of the resulting insertion tableau) remains the same. For the remaining two cases, we introduce the following notation. As in Definition 2.3, for $i \in \{1, 2\}$, we denote T_i' the result of applying the first step of promotion to T_i, and T_i'' the result of applying steps (1)–(3). In each case, we consider how each step of promotion affects the insertion tableau.
($n \in T_1$): Since T_1 is a rectangle shape, n must be in the south-east corner of T_1. No cell in T_2 is bigger than n, so any step of the row insertion algorithm that interacts with n will bump it to a lower row. It follows that removing n from T_1 will decrease the number of cells with row index greater than r by 1. Next, we note that applying jeu de taquin is equivalent to a sequence of Knuth moves, and increasing all values by 1 doesn't change the relative order of any two cells, so steps (2) and (3) don't affect the semicharge either. Finally, let $u = \mathsf{cw}(T_1'')\mathsf{cw}(T_2'')$ and $v = \mathsf{cw}(\mathrm{pr}(T_1))\mathsf{cw}(\mathrm{pr}(T_2)) = u_1 \cdots u_{r-1} 1\, u_r \cdots u_{2rs-1}$. Consider any collection $A_1, \ldots, A_r$ of disjoint increasing subsequences of u of maximal size. At least one A_i must start with a letter u_i with $i > r$. It follows that $A_1, \ldots, A_r$ is maximal in u if and only if $A_1, \ldots, \{1\} \cup A_i, \ldots, A_r$ is maximal

in v. Thus, by Greene's theorem [29, Theorem A1.1.1], the cell in $P(v)/P(u)$ must be in one of the first r rows (and so not contribute to semicharge). Hence $\mathsf{scharge}(\mathrm{pr}(T_1 \otimes T_2)) = \mathsf{scharge}(T_1 \otimes T_2) - 1$ in this case.
($n \in T_2$): We claim after inserting $\mathsf{cw}(T_2)$ into T_1, the cell containing n will be in one of the first r rows of the insertion tableau. Indeed, when n is inserted into T_1, it will be appended to the end of the first row. Since n is in the southeast corner of T_2, there are only $r-1$ cells that are inserted after n that can bump it down. It follows that step (1) doesn't change semicharge. As in the previous case, steps (2) and (3) do not change semicharge either. Finally, adding 1 to T_2'' will increase semicharge by 1, since when it is inserted into T_1 it will shift the first column down by 1, leading to an extra cell with row index greater than r. $\square$

Proposition 4.9. *Let $T \in \mathcal{T}(R)$ for $R = (R_1, \cdots, R_k)$. Then $\mathsf{scharge}(\mathrm{pr}(T)) = \mathsf{scharge}(T) - 1 \bmod d_R$.*

Proof. We may assume $k > 2$ since the case $k = 2$ immediately follows from Lemma 4.8. Since $T = T_1 \otimes \cdots \otimes T_k$ has the standard filling, then $n \in T_i$ for some unique i. By Lemma 4.8, promotion will only change the local semicharge of $T_{i-1} \otimes T_i$ and $T_i \otimes T_{i+1}$. That is,

$$\mathsf{scharge}(\mathrm{pr}(T_{i-1} \otimes T_i)) = \begin{cases} \mathsf{scharge}(T_{i-1} \otimes T_i) + 1 & R_{i-1} = R_i \\ 0 & R_{i-1} \neq R_i \end{cases},$$

and

$$\mathsf{scharge}(\mathrm{pr}(T_i \otimes T_{i+1})) = \begin{cases} \mathsf{scharge}(T_i \otimes T_{i+1}) - 1 & R_i = R_{i+1} \\ 0 & R_i \neq R_{i+1} \end{cases}.$$

Then one can use a similar argument as in Lemma 4.6 to discuss four cases of shapes R_{i-1}, R_i and R_{i+1}. It follows that $\mathsf{scharge}(\mathrm{pr}(T)) = \mathsf{scharge}(T) - 1 \bmod d_R$. $\square$

Corollary 4.10. *$\mathcal{T}(R)$ has at least d_R connected components.*

Proof. Pick any $T \in \mathcal{T}(R)$, then $T, \mathrm{pr}(T), \ldots, \mathrm{pr}^{d_R - 1}(T)$ have distinct semicharges modulo d_R by Proposition 4.9. By Proposition 4.7, they must belong to different components of $\mathcal{T}(R)$. $\square$

4.2. **The upper bound of d_R.** The remainder of this section is spent showing that $\mathcal{T}(R)$ has *at most* d_R connected components. There are many technical propositions and lemmas that we combine to obtain this result, but we will summarize the general strategy here. We need to show that any two vertices with semicharge equivalent modulo d_R are connected by a path in $\mathcal{T}(R)$. The main technical challenge comes from the fact that the edges t_n, $\bar{t}_1$, and $\bar{t}_{n-1}$ are hard to describe, and so any attempt at describing a path using these edges requires lots of extra bookkeeping. To get around this issue, we do the following two things.

1. Find a collection $\mathcal{A} \subset \mathcal{T}(R)$ such that each $A_i \in \mathcal{A}$ is connected to $\mathrm{pr}^{d_i}(A_i)$ for some d_i, with the added condition that $\gcd(\{d_i\}) = d_R$.
2. Find a path consisting of either edges in $\mathcal{T}(R)$ *or* powers of promotion that brings *any* $T \in \mathcal{T}(R)$ to a fixed $T' \in \mathcal{T}(R)$.

These two steps, combined with repeated applications of Proposition 3.6 allow us to prove the second half of Theorem 4.1.

Lemma 4.11. *Let T be a rectangular tableau with $r+1$ rows and s columns. Suppose T has consecutive entries in each of its columns such that*

$$T = \begin{array}{|c|c|c|c|c|} \hline l_1 & l_2 & l_3 & \cdots & l_s \\ \hline l_1{+}1 & l_2{+}1 & l_3{+}1 & \cdots & l_s{+}1 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline l_1{+}r & l_2{+}r & l_3{+}r & \cdots & l_s{+}r \\ \hline \end{array}$$

with $\ell_i + r < \ell_{i+1}$. Then $\mathrm{pr}^{n-(l_s+r)+(r+1)}(T)$ has consecutive entries $\{1, 2, \cdots, r+1\}$ in its first column. Furthermore, the rest of its columns are obtained by shifting all entries in column 1 to column $s-1$ of T to the right by one cell, and adding $n - (l_s + r) + (r+1)$ to each entry.

Proof. First, note that $\mathrm{pr}^{n-(l_s+r)}(T)$ is obtained by increasing the values of every cell by $n-(l_s+r)$, so the last column has consecutive entries $n-r, n-r+1, \ldots, n$. The next $r+1$ applications of promotion can be computed as follows. The first application of pr will have the effect of shifting the first row to the right (with a one in the northwest corner), and adding one to every cell. This is easy to see, since the jeu de taquin slides will follow the northeast boundary of T. Similarly, the i'th application of pr will have the same effect, with the i'th row of T getting shifted to the right and i getting placed in the left-most cell of that row. After $r+1$ applications of pr, the net effect is for all columns to get shifted one to the right, with the first column now being $1, \ldots, r+1$. Taking into account the first $n-(l_s+r)$ applications of pr, all elements not in the first column will have increased by $n-(l_s+r)+r+1$, as desired. $\square$

Remark 4.12. In Lemma 4.11, $\mathrm{pr}^{n-(l_s+r)+(r+1)}(T)$ can also be obtained by the following steps:

(1) Find jdt_{l_s+r} and add 1 to each cell, $r+1$ times.
(2) Add $n-(l_s+r)$ to each entry except those in column 1.

Example 4.13. Let $n=22, r=2, s=3$, and $T = \begin{array}{|c|c|c|}\hline 4 & 8 & 14 \\ \hline 5 & 9 & 15 \\ \hline 6 & 10 & 16 \\ \hline \end{array}$. Then we obtain $\mathrm{pr}^9(T)$ as

$$T = \begin{array}{|c|c|c|}\hline 4 & 8 & 14 \\ \hline 5 & 9 & 15 \\ \hline 6 & 10 & 16 \\ \hline \end{array} \xrightarrow{\text{Shift the first two columns to the right by one cell.}} \begin{array}{|c|c|c|}\hline * & 4 & 8 \\ \hline * & 5 & 9 \\ \hline * & 6 & 10 \\ \hline \end{array} \xrightarrow{\text{Fill the first column with 1,2,3.}} \begin{array}{|c|c|c|}\hline 1 & 4 & 8 \\ \hline 2 & 5 & 9 \\ \hline 3 & 6 & 10 \\ \hline \end{array} \xrightarrow{\text{Add 9 to each entry except the first column.}} \begin{array}{|c|c|c|}\hline 1 & 13 & 17 \\ \hline 2 & 14 & 18 \\ \hline 3 & 15 & 19 \\ \hline \end{array} = \mathrm{pr}^9(T).$$

By Remark 4.12, denoting step (1) by $\star$ we can also obtain $\mathrm{pr}^9(T)$ as

$$T = \begin{array}{|c|c|c|}\hline 4 & 8 & 14 \\ \hline 5 & 9 & 15 \\ \hline 6 & 10 & 16 \\ \hline \end{array} \xrightarrow{\star} \begin{array}{|c|c|c|}\hline 1 & 5 & 9 \\ \hline 6 & 10 & 15 \\ \hline 7 & 11 & 16 \\ \hline \end{array} \xrightarrow{\star} \begin{array}{|c|c|c|}\hline 1 & 6 & 10 \\ \hline 2 & 7 & 11 \\ \hline 8 & 12 & 16 \\ \hline \end{array} \xrightarrow{\star} \begin{array}{|c|c|c|}\hline 1 & 7 & 11 \\ \hline 2 & 8 & 12 \\ \hline 3 & 9 & 13 \\ \hline \end{array} \xrightarrow{\text{Add 6 to each entry except the first column}} \begin{array}{|c|c|c|}\hline 1 & 13 & 17 \\ \hline 2 & 14 & 18 \\ \hline 3 & 15 & 19 \\ \hline \end{array} = \mathrm{pr}^9(T).$$

Now we introduce a variation of the jeu de taquin slide, which can be applied to any cell of a tableau. Here we focus on rectangular partitions for our purposes.

Definition 4.14. Let T be a rectangular tableau. The *partial jeu de taquin slide* of T in terms of any cell a is defined as follows:

(1) Remove all the boxes below and to the right of a so that a becomes an outer box of the rest of the tableau T'.
(2) Find $\mathrm{jdt}_a(T')$ using the classical jeu de taquin slide, and add the entry 0 in the empty spaces on the inner boundary.
(3) Add the removed boxes back to their original positions.

If a is an outer box of T, then this partial version is exactly the same as the classical jeu de taquin slide. Using an abuse of notation, we also write it as $\mathrm{jdt}_a(T)$.

Example 4.15. Let $T = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 8 \\ \hline 3 & 7 & 9 \\ \hline \end{array}$, then $\mathrm{jdt}_7(T) = \begin{array}{|c|c|c|}\hline 0 & 1 & 5 \\ \hline 2 & 4 & 8 \\ \hline 3 & 6 & 9 \\ \hline \end{array}$ with the last column fixed. Also, $\mathrm{jdt}_6(T) =$ $\begin{array}{|c|c|c|}\hline 0 & 1 & 5 \\ \hline 2 & 4 & 8 \\ \hline 3 & 7 & 9 \\ \hline \end{array}$ with all entries below and to the right of 6 (i.e., the last row and last column) fixed.

Definition 4.16. For $R = (R_1, \ldots, R_k)$ with R_i be a rectangular partition of shape $s_i^{r_i}$ for all $i \in [1,k]$, let $T = T_1 \otimes \cdots \otimes T_k \in \mathcal{T}(R)$. Without loss of generality, we can assume that $s_1 \leq s_2 \leq \cdots \leq s_k$ and let $r' = \max\{r_i\}$. We define the i'th row (resp. column) of R as the union of the cells in the i'th row (resp. column) of each R_j. If a given rectangle has less than i rows or columns, we ignore it.

Next, we define some distinguished elements of $\mathcal{T}(R)$, and prove that they are connected to certain elements in their orbit under promotion. This ends up being helpful in the proof of Theorem 4.1.

Definition 4.17. Let $R = (R_1, \ldots, R_k)$, where each $R_i = (s_i^{r_i})$. Let $r' = \max(r_i)$. For $0 \leq m \leq r'$, we denote by $\mathsf{row}_m(R)$ the element of $\mathcal{T}(R)$ filled in the following way.

- Fill the first m rows of every rectangle from left to right, top to bottom with consecutive entries, starting at 1.
- For the remaining cells, fill the first column of each factor from top to bottom (now reading the factors from right to left), with consecutive entries starting where the last step left off.
- Continue filling one column at a time, skipping rectangles if they do not have enough columns.

We call $\mathsf{row}_0(R)$ the *column superstandard* filling, and $\mathsf{row}_{r'}(R)$ the *row superstandard* filling, and denote them $\mathsf{col}(R)$ and $\mathsf{row}(R)$ respectively.

Later we will want to refer to the values of specific cells in $\mathsf{row}_m(R)$. Let $c_{m,i}$ denote the maximum entry in column i, and $a_{m,i}$ denote the maximum entry in row i. For simplicity, if we set $c_{0,i} = c_i$.

Example 4.18. Let $R = ((2^2), (3^3), (3^3))$. The two special fillings of T in Example 4.21 (a) and (b) give examples for $\mathsf{col}(R) = \mathsf{row}_0(R)$ and $\mathsf{row}_2(R)$, respectively. Also,

$$\mathsf{row}(R) = \mathsf{row}_3(R) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 9 & 10 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 11 & 12 & 13 \\ \hline 17 & 18 & 19 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 6 & 7 & 8 \\ \hline 14 & 15 & 16 \\ \hline 20 & 21 & 22 \\ \hline \end{array}.$$

Proposition 4.19. *Let $T = \mathsf{row}_m(R)$, and fix $d \in \{c_{m,1}, \ldots, c_{m,s_k}, a_{m,m}\}$. Let S_j denote the sequence of edges in $\mathcal{T}(R)$ given by $S_j := t_{j-(d-1)} t_{j-(d-2)} \cdots t_{j-1} t_j$ for $d \le j \le n-1$. Skipping any swaps that are not applicable, we have $S_{n-1} S_{n-2} \cdots S_{d+1} S_d(T) = \mathrm{pr}^{n-d}(T)$. In other words, T and $\mathrm{pr}^{n-d}(T)$ are connected in $\mathcal{T}(R)$.*

Proof. We show the case where $m = 0$. Suppose $T = \mathsf{col}(R)$, and suppose $d = c_i$. By definition, d is the largest entry in column i, and $d+1$ is the smallest entry of column $i+1$. We want to partition the sequence $S_{n-1} S_{n-2} \cdots S_{d+1} S_d$ into intervals in the following way. Suppose that $T_{h_0}, T_{h_0+1}, \cdots, T_k$ are factors with more than i columns and with number of rows $r_{h_0}, r_{h_0+1}, \cdots, r_k$. Then partition $S_{d+(\sum_{h_0}^k r_i)-1} \cdots S_{d+1} S_d$ from right to left into intervals with lengths $r_k, r_{k-1}, \cdots, r_{h_0}$. Similarly, for $\ell \in [0, s_k - i - 1]$, suppose that $T_{h_\ell}, T_{h_\ell+1}, \cdots, T_k$ are factors with more than $i+\ell$ columns and with number of rows $r_{h_\ell}, r_{h_\ell+1}, \cdots, r_k$. Then partition the untouched part of the sequence from right to left into intervals with lengths $r_k, r_{k-1}, \cdots, r_{h_\ell}$.

Now let $T^{(j)}$ denote the result of applying $S_{d+j-1} \cdots S_d$ to T (so we claim that $T^{(n-d)} = \mathrm{pr}^{n-d}(T)$). Restricting $\mathsf{rw}(T)$ to the entries in $[c_{i-1}+1, c_i+1]$, we get $c_i, c_i - 1, \ldots, c_{i-1}+1, c_i+1$ for each i. As a consequence, when we apply S_d to T, all swaps will be applicable except for t_{c_j+1}, for each $j \in [0, i-1]$. Thus, $T^{(1)}$ can be obtained from T by applying jdt_{d+r_k} to T_k and increasing all entries in T with value less than $d + r_k$ by one, leaving every other cell untouched. Similarly, for each $j \in [r_k - 1]$, we can compute $S_{d+j}(T^{(j)})$ by applying jdt_{d+r_k} to $T_k^{(j)}$ and increasing all entries with value less than $d + r_k$ by one. Using Remark 4.12, we can completely characterize $T^{(r_k)}$. Column $i+1$ of T_k has been removed as a consequence of continuously applying jdt_{d+r_k} for r_k times. Column 1 to column i of T_k gets shifted to the right by one cell. All entries of T with values less than $d+1$ are increased by r_k, and every other cell is fixed since none of the transpositions applied have effects on them. Finally, the first column of $T_k^{(r_k)}$ and $T_k^{(n-d)}$ agree by Lemma 4.11.

We continue by applying S_{d+r_k} to $T^{(r_k)}$. Similar to the previous sequence of swaps, $T^{(r_k+1)}$ can be obtained from $T^{(r_k)}$ by the followings steps. Apply $\mathrm{jdt}_{d+r_k+r_{k-1}}$ to $T_{k-1}^{(r_k)}$, and increase all entries with values in $(r_k, d + r_k + r_{k-1})$ by one, then add $r_k + 1$ to the first entry of the $(k-1)$'st factor, and leave every other cell untouched. Similarly, for each $j \in [r_k, r_k + r_{k-1} - 1]$, we can compute $S_{d+j}(T^{(j)})$ by applying $\mathrm{jdt}_{d+r_k+r_{k-1}}$ to $T_{k-1}^{(j)}$ and increasing all entries with value between r_k and $d + r_k + r_{k-1}$ by one, and adding $r_k + 1$ to the first entry of the $(k-1)$'st factor. Thus, we can characterize $T^{(r_k+r_{k-1})}$ as follows. Column $i+1$ of $T_{k-1}^{(r_k)}$ has been removed as a consequence of applying $\mathrm{jdt}_{d+r_k+r_{k-1}}$ for r_{k-1} times. Column 1 to column i of $T_{k-1}^{(r_k)}$ gets shifted to the right by one cell. All entries of $T^{(r_k)}$ with values between r_k and $d + r_k + 1$ are increased by r_{k-1}, and every other cell is fixed since none of the transpositions applied have effects on them. Finally, the first columns of $T_{k-1}^{(r_k+r_{k-1})}$ and $T_k^{(r_k+r_{k-1})}$ agree with that of $T_{k-1}^{(n-d)}$ and $T_k^{(n-d)}$.

Similar argument follows for applying any intervals of $S_{d+(\sum_{h_0}^k r_i)-1} \cdots S_{d+1} S_d$, and we can completely characterize $T^{(\sum_{h_0}^k r_i)}$. Column $i+1$ of T has been removed. Column 1 to column i of $T_{h_0}, \cdots, T_k$ gets shifted to the right by one cell. All entries with values less than $d+1$ are increased by $\sum_{h_0}^k r_i$, and every other cell is fixed. Finally, match the first column of $T^{(\sum_{h_0}^{k} r_i)}$ exactly as $T^{(n-d)}$. The path $T \to T^{(3)} \to T^{(6)} \to T^{(8)}$ in Example 4.21 (a) shows an example when $n = 22, d = 8, h_0 = 1, k = 3$ and $r_1+r_2+r_3 = 8$. Similarly, for any $\ell \in [0, s_k - i - 1]$, we can obtain $T^{(\sum_{j=h_0}^{h_\ell} \sum_{i=j}^{k} r_i)}$ as follows. Take $T^{(\sum_{j=h_0}^{h_{\ell-1}} \sum_{i=j}^{k} r_i)}$, remove column $i+\ell+1$ of it, and for its h_ℓ-th,$\cdots$, k-th factors, shift column $1+\ell$ to column $i+\ell$ to the right by one cell. Then add $\sum_{h_\ell}^{k} r_i$ to all entries (after column ℓ) whose values are less than the smallest entry of the removed column $i+\ell+1$. Finally, match the $(\ell+1)$-th column of the h_ℓ-th,$\cdots$, k-th factors exactly as $T^{(n-d)}$. Note that everything before column $1+\ell$ has been matched with $\mathrm{pr}^{n-d}(T)$ and will be fixed throughout the algorithm. We can eventually obtain $\mathrm{pr}^{n-d}(T)$ by inductively applying through all ℓ. See Example 4.21 (a).

The argument for $m \neq 0$ is similar so we omit the details. Example 4.21 (b) shows a case when $m = 2$. $\square$

Remark 4.20. For $T = \mathsf{row}_m(R)$, the statement in Proposition 4.19 is also true for any $d \in \{a_{m,1}, a_{m,2}, \cdots, a_{m,m-1}\}$, and the proof follows similarly. For our purposes, knowing $a_{m,m}$ works is sufficient.

We illustrate the technique in Proposition 4.19 with the following examples.

Example 4.21. (a) : Let $T = T_1 \otimes T_2 \otimes T_3 = \mathsf{col}(R) \in \mathcal{T}((2^2),(3^3),(3^3))$. We choose $d = 8$, which is the largest entry in column 1 of T. We consider partitioning part of the sequence $S_{21}S_{20}\cdots S_8$ according to the heights of rectangles with more than one column. That is, $S_{15}S_{14}, S_{13}S_{12}S_{11}, S_{10}S_9S_8$. Applying them on T from right to left, we obtain $T^{(8)}$ as follows. Remove the second column of T and shift the first column of T to the right by one cell, and also fix the third column. Then add $r_3 + r_2 + r_1 = 8$ to every entry with a value less than $d + 1 = 9$. Finally, add back the first column exactly as $T^{(14)}$. To continue, we have intervals of words $S_{21}S_{20}S_{19}, S_{18}S_{17}S_{16}$. Applying them to $T^{(8)}$ from right to left we obtain $T^{(14)}$ as follows. Remove the third column of $T^{(8)}$, and shift the second column of T_2^8 and T_3^8 to the right by one cell. Then add $r_2 + r_3 = 6$ to every entry (after column 1) with a value less than 17 (the smallest entry of column 3). Finally, fill the empties in the second column exactly as $T^{(14)}$. The path from T to $\mathrm{pr}^{14}(T)$ is shown in Figure 5.

(b) : Let $T = T_1 \otimes T_2 \otimes T_3 = \mathsf{row}_2(R) \in \mathcal{T}(2^2, 3^3, 3^3)$. We choose $d = a_2 = 16$. We divide the sequence of words $S_{21}S_{20}\cdots S_{16}$ into intervals according to the heights of rectangles with two rows removed. That is, all heights are 1. Then we apply $S_{21}, S_{20}, \cdots S_{16}$ from right to left to obtain $T^{(1)}, T^{(2)}, \cdots, T^{(6)} = \mathrm{pr}^6(T)$. The path from T to $\mathrm{pr}^6(T)$ is shown in Figure 6.

$$T = \begin{array}{|c|c|}\hline 7 & 15 \\ \hline 8 & 16 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 4 & 12 & 20 \\ \hline 5 & 13 & 21 \\ \hline 6 & 14 & 22 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 9 & 17 \\ \hline 2 & 10 & 18 \\ \hline 3 & 11 & 19 \\ \hline \end{array} \xrightarrow{S_{10}S_9S_8(T)} T^{(3)} = \begin{array}{|c|c|}\hline 10 & 15 \\ \hline 11 & 16 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 7 & 12 & 20 \\ \hline 8 & 13 & 21 \\ \hline 9 & 14 & 22 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 4 & 17 \\ \hline 2 & 5 & 18 \\ \hline 3 & 6 & 19 \\ \hline \end{array}$$

$$\xrightarrow{S_{13}S_{12}S_{11}(T^{(3)})} T^{(6)} = \begin{array}{|c|c|}\hline 13 & 15 \\ \hline 14 & 16 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 4 & 10 & 20 \\ \hline 5 & 11 & 21 \\ \hline 6 & 12 & 22 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 7 & 17 \\ \hline 2 & 8 & 18 \\ \hline 3 & 9 & 19 \\ \hline \end{array}$$

$$\xrightarrow{S_{15}S_{14}(T^{(6)})} T^{(8)} = \begin{array}{|c|c|}\hline 7 & 15 \\ \hline 8 & 16 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 4 & 12 & 20 \\ \hline 5 & 13 & 21 \\ \hline 6 & 14 & 22 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 9 & 17 \\ \hline 2 & 10 & 18 \\ \hline 3 & 11 & 19 \\ \hline \end{array}$$

$$\xrightarrow{S_{18}S_{17}S_{16}(T^{(8)})} T^{(11)} = \begin{array}{|c|c|}\hline 7 & 18 \\ \hline 8 & 19 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 4 & 15 & 20 \\ \hline 5 & 16 & 21 \\ \hline 6 & 17 & 22 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 9 & 12 \\ \hline 2 & 10 & 13 \\ \hline 3 & 11 & 14 \\ \hline \end{array}$$

$$\xrightarrow{S_{21}S_{20}S_{19}(T^{(11)})} T^{(14)} = \begin{array}{|c|c|}\hline 7 & 21 \\ \hline 8 & 22 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 4 & 12 & 18 \\ \hline 5 & 13 & 19 \\ \hline 6 & 14 & 20 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|}\hline 1 & 9 & 15 \\ \hline 2 & 10 & 16 \\ \hline 3 & 11 & 17 \\ \hline \end{array}$$

FIGURE 5. The path from T to $\mathrm{pr}^{14}(T)$ for $T = \mathsf{col}(R) \in \mathcal{T}((2^2),(3^3),(3^3))$.

Lemma 4.22. *We have that* $\gcd\left(\bigcup_m \{c_{m,1}, \ldots, c_{m,s_k}, a_{m,m}\}\right) = d_R$.

$$T = \begin{array}{|c|c|}\hline 1&2\\\hline 9&10\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 3&4&5\\\hline 11&12&13\\\hline 18&20&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 6&7&8\\\hline 14&15&16\\\hline 17&19&21\\\hline\end{array} \xrightarrow{S_{16}(T)} T^{(1)} = \begin{array}{|c|c|}\hline 2&3\\\hline 10&11\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 4&5&6\\\hline 12&13&14\\\hline 18&20&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&8&9\\\hline 7&16&17\\\hline 15&19&21\\\hline\end{array}$$

$$\xrightarrow{S_{17}(T^{(1)})} T^{(2)} = \begin{array}{|c|c|}\hline 3&4\\\hline 11&12\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 2&6&7\\\hline 5&14&15\\\hline 13&20&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&9&10\\\hline 8&17&18\\\hline 16&19&21\\\hline\end{array}$$

$$\xrightarrow{S_{18}(T^{(2)})} T^{(3)} = \begin{array}{|c|c|}\hline 4&5\\\hline 12&13\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 2&7&8\\\hline 6&15&16\\\hline 14&20&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&11\\\hline 9&10&19\\\hline 17&18&21\\\hline\end{array}$$

$$\xrightarrow{S_{19}(T^{(3)})} T^{(4)} = \begin{array}{|c|c|}\hline 5&6\\\hline 13&14\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 2&4&9\\\hline 7&8&17\\\hline 15&16&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&12\\\hline 10&11&20\\\hline 18&19&21\\\hline\end{array}$$

$$\xrightarrow{S_{20}(T^{(4)})} T^{(5)} = \begin{array}{|c|c|}\hline 6&7\\\hline 14&15\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 2&4&10\\\hline 8&9&18\\\hline 16&17&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 11&12&13\\\hline 19&20&21\\\hline\end{array}$$

$$\xrightarrow{S_{21}(T^{(5)})} T^{(6)} = \begin{array}{|c|c|}\hline 7&8\\\hline 15&16\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 9&10&11\\\hline 17&18&19\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 12&13&14\\\hline 20&21&22\\\hline\end{array}$$

FIGURE 6. The path from T to $\mathrm{pr}^6(T)$ for $T = \mathsf{row}_2(R) \in \mathcal{T}((2^2),(3^3),(3^3))$.

Proof. Let g_R denote the gcd of the above set. It's clear that g_R is a multiple of d_R. Now, note that for $i > 1$, $c_{j,i+1} - c_{j,i}$ equals the number of cells in column i with row index greater than j. If $i = 1$, then $c_{j,1} - a_{j,j}$ has the same interpretation. Furthermore, the expression

$$\begin{cases} (c_{j+1,i+1} - c_{j+1,i}) - (c_{j,i+1} - c_{j,i}) & i > 1 \\ (c_{j+1,1} - a_{j+1,j+1}) - (c_{j,1} - a_{j,j}) & i = 1 \end{cases}$$

gives the number of cells across R with row index i and column index j. It's clear that the gcd of the multiplicities of each *cell* across R must be d_R, so it follows that $g_R = d_R$ as desired. $\square$

Now, we show that every $T \in \mathcal{T}(R)$ can be connected to $\mathrm{pr}^j(\mathsf{row}(R))$ for some j. For each i, let a_i be the number of cells across all R_j with row index at most i (by convention, let $a_0 = 0$.). Given $T \in \mathcal{T}(R)$, we want to find a path from T to $\mathrm{pr}^j(\mathsf{row}(R))$ for some j. Equivalently, we want to find a sequence of moves that are either (1) a dual equivalence move or (2) a power of promotion, that brings T to $\mathsf{row}(R)$. To this end, we define the *row-combing* algorithm, rcomb. When each R consists of single row partitions, this algorithm essentially reduces to the Shi combing procedure (see [26, 9]).

First, we define a *partial comb*: Given $T \in \mathcal{T}(R)$, and $i < j$, such that $D(T) \cap \{i, \ldots, j\} = \{j\}$, let $\mathsf{pcomb}_{i,j}(T)$ be the filling obtained by swapping the descent at j for a descent at $j-1$, which is then swapped for $j-2$, and so on, up to i. This results in a new filling T' with $D(T') \cap \{i, \ldots, j\} = \{i\}$. We note that depending on T, $\mathsf{pcomb}_{i,j}$ may affect whether or not $i-1$ or $j+1$ are descents as well. We are now ready to define rcomb:

- Input: $T \in \mathcal{T}(R)$.
- Output: $T' \in \mathcal{T}(R)$ that is "closer" to $\mathsf{row}(R)$.
- Perform the following moves:
  - Let j be the minimal element of $D(T) \setminus \{a_i\}$, and suppose $a_i < j < a_{i+1}$.
  - $T \leftarrow \mathsf{pcomb}_{a_i+1,j}(T)$.
  - For $\ell = i-1, i-2, \ldots, 0$: $T \leftarrow \mathsf{pcomb}_{a_\ell+1,a_{\ell+1}}(T)$
  - $T \leftarrow \mathrm{pr}^{-1}(T)$
- Output T

Example 4.23. We demonstrate one iteration of rcomb for $R = (3^3, 3^3, 2^2)$. Here, $j = 10$.

$$T = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 9&10&15\\\hline 11&16&20\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 4&5&6\\\hline 13&18&21\\\hline 14&19&22\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 7&8\\\hline 12&17\\\hline\end{array} \xrightarrow{\mathsf{pcomb}_{9,10}} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 9&11&15\\\hline 10&16&20\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 4&5&6\\\hline 13&18&21\\\hline 14&19&22\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 7&8\\\hline 12&17\\\hline\end{array}$$

$$\xrightarrow{\mathsf{pcomb}_{1,8}} \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&11&15\\\hline 10&16&20\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 5&6&7\\\hline 13&18&21\\\hline 14&19&22\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 8&9\\\hline 12&17\\\hline\end{array}$$

$$\xrightarrow{\mathrm{pr}^{-1}} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 9&10&14\\\hline 15&19&22\\\hline\end{array} \otimes \begin{array}{|c|c|c|}\hline 4&5&6\\\hline 12&17&20\\\hline 13&18&21\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 7&8\\\hline 11&16\\\hline\end{array}$$

Note that the entries 1 through 10 are already where they should be in T, and that the first cell (reading left to right) that disagrees with $\mathsf{row}(R)$ is decreased by one $(15 \to 14)$.

Lemma 4.24. *Let $T \in \mathcal{T}(R)$, and fix $i \leq j$. Suppose that $D(T) \cap \{i, \dots, j\} = \{j\}$, and that the cells of T containing $\{i, \dots, j+1\}$ are contained with at most two rows of the factors of T. Let c be the smallest value in $\{i, \dots, j+1\}$ that follows $j+1$ in $\mathsf{rw}(T)$. Then $\mathsf{pcomb}_{i,j}(T)$ has the affect of cyclically shifting the cells containing $c+1, \dots, j+1$ up by one (each cell gets increased by 1, except for $j+1$ which wraps around to $c+1$), and cyclically shifting the cells containing $i, \dots, c$ up by one.*

Proof. This can be seen directly by noting that under the assumptions of the Lemma statement, $\mathsf{rw}(T)$ restricted to the alphabet $\{i, \dots, j+1\}$ has the form

$$i\ (i+1)\ \cdots\ (c-1)\ (j+1)\ c\ (c+1)\ \cdots\ (j-1)\ j$$

We can apply dual Knuth moves swapping $j+1$ and j, followed by j and $j-1$, and so on, until we swap $c+1$ and $c+2$, at which point, the row word looks like

$$i\ (i+1)\ \cdots\ (c-1)\ (c+1)\ c\ (c+2)\ \cdots\ j\ (j+1)$$

Since swapping $c, c+1$ is *not* a dual Knuth move for this word, we continue swapping $c-1$ with c, $c-2$ with $c-1$, and so on, to get the word

$$(i+1)\ (i+2)\ \cdots\ c\ (c+1)\ i\ (c+2)\ \cdots\ j\ (j+1)$$

Since elementary dual equivalences on $\mathsf{rw}(T)$ correspond to edges in $\mathcal{T}(R)$, the claim follows. $\square$

We note that under the assumptions of the above Lemma, if the cells containing $\{i, \dots, j+1\}$ lie in two rows of T, then $j+1$ is the unique cell in the lower row, and c is the cell directly above.

Proposition 4.25. *Repeatedly applying rcomb to any $T \in \mathcal{T}(R)$ will eventually reach $\mathsf{row}(R)$ after finitely many iterations.*

Proof. In order to prove that repeated applications of rcomb eventually lead to $\mathsf{row}(R)$, we will use the following strategy. Consider the cells of T that have values strictly larger than their corresponding value in $\mathsf{row}(R)$, and let α be the one with the smallest value. We claim that $\mathsf{rcomb}(T)$ fixes all entries of T that precede α, and decreases the value of α itself.

First, it is clear that no application of pcomb will increase the value of α. Indeed, the only step of the algorithm that could interact with α is the application of $\mathsf{pcomb}_{a_i+1,j}$, but this will only happen if the value of α is the c of Lemma 4.24, or $j+1$. In either case, it will decrease, not increase.

After applying each pcomb, the cycling of entries described in Lemma 4.24 implies that the cells less than α will have increased by 1, except for a single factor of T (the factor where the initial cell that played the role of c in $\mathsf{pcomb}_{a_i+1,j}$ is located). The first i rows of T will look like

1	$\ell_1{+}2$	$\ell_1{+}3$	$\cdots$	$\ell_1{+}h$
$\ell_1{+}1$	$\ell_2{+}2$	$\ell_2{+}3$	$\cdots$	$\ell_2{+}h$
$\ell_2{+}1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\ell_i{+}2$	$\cdots$	α	
$\ell_i{+}1$				

where ℓ_i is the value of the leftmost cell in that row in $\mathsf{row}(R)$. Since 1 is always in this factor of T, the effect of pr^{-1} on the other factors will be to decrease all elements by 1. For this factor, it is clear that the jdt path will go down the left column of the tableau up to at least row i. This will return all values in this factor that were originally less than α back to what they were. If α happens to be in this factor, then it will be in row i but not column one, so pr^{-1} will just decrease the value by 1. $\square$

Example 4.26. Let $R = ((2^2), (2^2))$, then the following is a complete application of the row combing procedure.

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 6 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 7 & 8 \\ \hline\end{array} \xrightarrow{\mathsf{rcomb}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 & 8 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 6 & 7 \\ \hline\end{array} \xrightarrow{\mathsf{rcomb}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 & 6 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 7 & 8 \\ \hline\end{array} = \mathsf{row}(R).$$

Corollary 4.27. *Any vertex $T \in \mathcal{T}(R)$ can be connected to $\mathrm{pr}^j(\mathsf{row}(R))$ for some power j.*

Now we are ready to combine all results and prove the main theorem.

Proof of Theorem 4.1. We know from Corollary 4.10 that $\mathcal{T}(R)$ has at least d_R connected components. What remains to show is that any two T, T' with equal charge are connected in $\mathcal{T}(R)$. By Corollary 4.27, it suffices to show that $\mathsf{row}(R)$ is connected to $\mathrm{pr}^{d_R}\,\mathsf{row}(R)$. Let $A = \{c_{i,j} \mid i \le r', j \le s_k\} \cup \{a_{i,i} \mid i \le r'\}$. We know by Proposition 4.19 that for every d_i in A, there is some $T \in \mathcal{T}(R)$ such that T is connected to $\mathrm{pr}^{n-d_i}(T)$, and since $\mathrm{pr}^n = \mathrm{id}$ on $\mathcal{T}(R)$, this implies that T is also connected to $\mathrm{pr}^{d_i}(T)$. By Corollary 4.27, T is connected to $\mathrm{pr}^{k_i}(\mathsf{row}(R))$ for some k_i. Since pr is a graph automorphism, this means that $\mathrm{pr}^{d_i}(T)$ is connected to $\mathrm{pr}^{k_i+d_i}(\mathsf{row}(R))$. Concatenating the paths, we get that $\mathrm{pr}^{k_i}(\mathsf{row}(R))$ is connected to $\mathrm{pr}^{k_i+d_i}(\mathsf{row}(R))$, which implies that $\mathsf{row}(R)$ is connected to $\mathrm{pr}^{d_i}(\mathsf{row}(R))$ for each d_i in A. Since $\gcd(A) = d_R$ by Lemma 4.22, we can conclude that $\mathsf{row}(R)$ is connected to $\mathrm{pr}^{d_R}(\mathsf{row}(R))$, as desired. $\square$

5. Characters of connected components

Recall that, given a subset $A \subset [n-1]$, the associated fundamental quasisymmetric function F_A is given by

$$F_A(x) = \sum_{\substack{i_1 \le \cdots \le i_n \\ a \in A \Rightarrow i_a < i_{a+1}}} x_{i_1} \cdots x_{i_n}.$$

Theorem 5.1. *Let $R = (R_1, \ldots, R_k)$. For $T \in \mathcal{T}(R)$, let $\overline{D}(T) = D(T) \setminus \{n\}$, then*

$$\sum_{T \in \mathcal{T}(R)} F_{\overline{D}(T)} = s_{R_1} \cdots s_{R_k}$$

where F_A is the fundamental quasisymmetric function corresponding to $A \subset [n-1]$.

Proof. Regard $B^{R_1} \otimes \cdots \otimes B^{R_k}$ as a type A crystal. It decomposes into connected components, each of which is isomorphic to a type A crystal B_λ, where the multiplicity of λ is exactly the the multiplicity of s_λ in $s_{R_1} \cdots s_{R_k}$ [5, Theorem 9.5]. Next, we know from [1] that if we take the 0-weight space of the type A crystal B_λ and regard it as a dual-equivalence graph, then the sum of fundamental quasisymmetric functions indexed by descents is the Schur function s_λ. The claim follows. $\square$

If we consider the same sum of fundamental quasisymmetric functions restricted to a single connected component of $\mathcal{T}(R)$, we get a Schur positive symmetric function, since $\mathcal{T}(R)$ is a dual equivalence graph if we forget about the extra affine edges. Conjecturally, it appears that the symmetric functions that we get come from the plethysm of *cyclic characters* with certain products of rectangular Schur functions. Let

$$\ell_k^{(i)} = \frac{1}{k} \sum_{d|k} c(i,d) p_d^{n/d}$$

where $c(i,d) = \frac{\mu(i')\varphi(d)}{\varphi(i')}$, where $i' = \frac{i}{\gcd(i,d)}$, μ is the (number-theoretic) Möbius function, and φ is Euler's totient function. We call such $\ell_k^{(i)}$ *cyclic characters*, as they were first studied by Foulkes as the Fröbenius images of the symmetric group representations obtained by inducing from an irreducible representation of a maximal cyclic subgroup [10].

Conjecture 5.2. *Suppose that R has distinct rectangles R_j that appear with multiplicity m_j (so $d_R = \gcd(m_j)$). Then for $i \in [d_R]$, the charge i connected component of $\mathcal{T}(R)$ has character given by the plethysm*

$$\ell_{d_R}^{(i)} \left[s_{R_1}^{m_1/d_R} \cdots s_{R_k}^{m_k/d_R} \right].$$

We note that since $\sum_{i=1}^{d_R} \ell_{d_R}^i = p_1^{d_R}$, we can use simple plethystic identites to show that the sum of all such characters is the product of Schur functions from the previous theorem. For some special cases, a Schur expansion of these plethysms is known. If every R_i is a column (so s_{R_i} is an elementary symmetric function) or if every R_i is a row (so S_{R_i} is a homogeneous symmetric function), the expansion is known due to Lascoux, Leclerc, and Thibon [16]. The case of all R_i being a fixed rectangle shape is known due to Iijima [12].

We can interpret the plethysms appearing in Conjecture 5.2 as follows. Let U be a tensor product of $GL_n(\mathbb{C})$ representations corresponding to multiples of fundamental weights, so that the character of U is given by a product of rectangular Schur functions. Then the diagonal action of $GL_n(\mathbb{C})$ on $U^{\otimes k}$ commutes with the action of the cyclic group C_k which cycles tensor factors. Thus, if χ is an irreducible character for C_k corresponding to a primitive k'th root of unity, the corresponding isotypic components of $U^{\otimes k}$ are representations of $GL_n(\mathbb{C})$, and their characters correspond exactly to $\ell_k^{(i)}[\mathrm{char}_U(x_1, \ldots, x_n)]$ for some i.

Example 5.3. Consider $R = ((1^3), (1^3))$ as in Figure 4. The left and right connected components consist of the semicharge 0 and 1 vertices of $\mathcal{T}(R)$ respectively. One can compute that $\ell_2^{(0)} = s_2$ and $\ell_2^{(1)} = s_{1,1}$, and the characters of each connected component are given by

$$2F_{1245} + F_{1234} + F_{2345} + F_{135} + F_{1235} + F_{1345} + F_{134} + F_{235} + F_{24} = s_{2,1,1,1,1} + s_{2,2,2} = s_2[s_{1,1,1}],$$
$$F_{1235} + F_{12345} + F_{1345} + F_{124} + F_{234} + F_{245} + F_{1245} + F_{134} + F_{235} + F_{135} = s_{1,1,1,1,1,1} + s_{2,2,1,1} = s_{1,1}[s_{1,1,1}]$$

Example 5.4. Let $R = ((1^2), (1^2), (2), (2))$, so $d_R = 2$. $\mathcal{T}(R)$ has 2520 vertices and two connected components. The characters of the two components are exactly $\ell_2^{(0)}[s_2 s_{1,1}]$ and $\ell_2^{(1)}[s_2 s_{1,1}]$:

$$\begin{aligned}\ell_2^{(0)}[s_2 s_{1,1}] &= s_{6,2} + 2s_{5,2,1} + 2s_{5,1,1,1} + s_{4,4} + 2s_{4,3,1} + 3s_{4,2,2} + 2s_{4,2,1,1} + 2s_{4,1,1,1,1} + 3s_{3,3,1,1} \\ &\quad + 2s_{3,2,2,1} + 2s_{3,2,1,1,1} + s_{2,2,2,2} + s_{2,2,1,1,1,1}, \\ \ell_2^{(1)}[s_2 s_{1,1}] &= s_{6,1,1} + s_{5,3} + 2s_{5,2,1} + s_{5,1,1,1} + 2s_{4,3,1} + s_{4,2,2} + 4s_{4,2,1,1} + s_{4,1,1,1,1} + 2s_{3,3,2} \\ &\quad + s_{3,3,1,1} + 2s_{3,2,2,1} + 2s_{3,2,1,1,1} + s_{3,1,1,1,1,1} + s_{2,2,2,1,1}.\end{aligned}$$

Example 5.5. Let $R = ((2^2), (2^2), (2^2))$, so $d_R = 3$. $\mathcal{T}(R)$ has 277200 vertices and the characters of its three connected components are exactly $\ell_3^{(i)}[s_{2,2}]$ for each i. Since $\ell_3^{(1)} = \ell_3^{(2)}$, the characters for two of the components coincide.

References

[1] S. Assaf. A combinatorial realization of Schur-Weyl duality via crystal graphs and dual equivalence graphs. *Discrete Math. Theor. Comput. Sci.*, DMTCS Proceedings vol. AJ, 20th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2008), Jan 2008.
[2] S. Assaf. Dual equivalence graphs I: A new paradigm for Schur positivity. *Forum Math. Sigma*, 3:Paper No. e12, 33, 2015.
[3] S. Assaf and S. Billey. Affine dual equivalence and k-Schur functions. *J. Comb.*, 3(3):343–399, 2012.
[4] S. Brauner, S. Corteel, Z. Daugherty, and A. Schilling. Crystal skeletons: Combinatorics and axioms. *arXiv:2503.14782*, 2025.

[5] D. Bump and A. Schilling. *Crystal Bases: Representations And Combinatorics*. World Scientific Publishing Company, 2017.

[6] C. Carré and B. Leclerc. Splitting the square of a Schur function into its symmetric and antisymmetric parts. *J. Algebraic Combin.*, 4(3):201–231, 1995.

[7] M. Chmutov. Type A molecules are Kazhdan-Lusztig. *J. Algebraic Combin.*, 42(4):1059–1076, 2015.

[8] M. Chmutov, J. Lewis, and P. Pylyavskyy. Monodromy in Kazhdan-Lusztig cells in affine type A. *Math. Ann.*, 386(3-4):1891–1949, 2023.

[9] M. Chmutov, P. Pylyavskyy, and E. Yudovina. Matrix-ball construction of affine Robinson-Schensted correspondence. *Selecta Math. (N.S.)*, 24(2):667–750, 2018.

[10] H. O. Foulkes. Characters of symmetric groups induced by characters of cyclic subgroups. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 141–154. Inst. Math. Appl., Southend-on-Sea, 1972.

[11] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[12] K. Iijima. A q-multinomial expansion of LLT coefficients and plethysm multiplicities. *European J. Combin.*, 34(6):968–986, 2013.

[13] S. J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki. Affine crystals and vertex models. *Int. J. Mod. Phys. A*, 7(1):449–484, 1992.

[14] A. N. Kirillov and N. Reshetikhin. Representations of Yangians and multiplicities of occurrence of the irreducible components of the tensor product of representations of simple lie algebras. *J. Sov. Math.*, 52:3156–3164, 1990.

[15] W. Kraśkiewicz and J. Weyman. Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.*, 63:265–284, 2001.

[16] A. Lascoux, B. Leclerc, and J. Thibon. Green polynomials and Hall-Littlewood functions at roots of unity. *European J. Combin.*, 15(2):173–180, 1994.

[17] A. Lascoux and M. Schützenberger. Sur une conjecture de H. O. Foulkes. *C. R. Acad. Sci. Paris Sér. A-B*, 286:A323–A324, 1978.

[18] A. Lascoux and M. Schützenberger. Tableaux and the Littlewood–Richardson rule. In *Combinatorics and Algebra*, volume 34 of *Contemp. Math.*, pages 161–190. Amer. Math. Soc., 1984.

[19] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov–Reshetikhin crystals I: Lifting the parabolic quantum Bruhat graph. *Int. Math. Res. Not.*, 2015(7):1848–1901, 01 2014.

[20] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov–Reshetikhin crystals II. alcove model, path model, and P=X. *Int. Math. Res. Not.*, 2017(14):4259–4319, 07 2016.

[21] C. Lenart, S. Naito, D. Sagaki, A. Schilling, and M. Shimozono. A uniform model for Kirillov–Reshetikhin crystals III: Nonsymmetric Macdonald polynomials at $t = 0$ and Demazure characters. *Transform. Groups*, 22(4):1041–1079, 2017.

[22] F. Maas-Gariépy and E. Tétreault. Splitting the square of homogeneous and elementary functions into their symmetric and anti-symmetric parts. *arXiv:2203.08279*, 2022.

[23] A. Nakayashiki and Y. Yamada. Kostka polynomials and energy functions in solvable lattice models. *Selecta Math.*, 3(4):547–599, 1997.

[24] K. Naoi. Demazure crystals and tensor products of perfect Kirillov–Reshetikhin crystals with various levels. *J. Algebra*, 374:1–26, 2013.

[25] A. Schilling and P. Tingley. Demazure crystals, Krillov-Reshetikhin crystals, and the energy function. *Electron. J. Combin.*, 19, 04 2011.

[26] J. Y. Shi. The generalized Robinson-Schensted algorithm on the affine Weyl group of type $\tilde{A}_{n-1}$. *J. Algebra*, 139(2):364–394, 1991.

[27] M. Shimozono. A cyclage poset structure for Littlewood–Richardson tableaux. *European J. Combin.*, 16(5):447–462, 1995.

[28] M. Shimozono. Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties. *J. Algebraic Combin.*, 15(2):151–187, 2002.

[29] R. P. Stanley. *Enumerative Combinatorics*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.

[30] J. R. Stembridge. Admissible W-graphs. *Represent. Theory*, 12:346–368, 10 2008.

[31] J. R. Stembridge. A finiteness theorem for W-graphs. *Adv. Math.*, 229(4):2405–2414, 2012.

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