On the maximum spectral radius of planar graphs*

Guanglong Yu[a] Lin Sun[b†]

[a]School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong, 510665, P.R. China

[b]Department of Mathematics, Lingnan Normal University, Zhanjiang, Guangdong, 524048, P.R. China

Abstract

This paper investigates the maximum spectral radius of planar graphs with concrete fixed number of vertices, providing some tight bounds on the maximum spectral radius of general planar graph resorting to its order, and confirming that among all planar graphs containing dominating vertex with concrete fixed order $n \geq 48$, the join of P_2 and P_{n-2} attains the maximum spectral radius.

AMS Classification: 05C50
Keywords: Spectral radius; Planar graph; Dominating vertex

1 Introduction

The study of planar graphs has a long history because of its good structural properties, topological properties, algebraic properties, and so on. To questions in spectral extremal graph theory asking to maximize or minimize eigenvalues over a fixed family of graphs, Boots and Royle, and independently, Cao and Vince conjectured that the join of P_2 and P_{n-2} attains the maximum spectral radius among all planar graphs on $n \geq 9$ vertices [2,3]. This conjecture has been open for more than 30 years although many researchers tries to solve it. Just up to 2017, the study on this conjecture got a substantial development that Tait and Tobin proved the conjecture holding for graphs of sufficiently large order [6]. Unfortunately, the conjecture for concrete fixed order is still open. This paper investigates the maximum spectral radius of planar graphs with concrete fixed order, providing some tight bounds on the maximum spectral radius of general planar graph resorting to its order, and confirming that among all planar graphs containing dominating vertex with fixed order $n \geq 48$, the join of P_2 and P_{n-2} attains the maximum spectral radius.

1.1 Notions and notations

All graphs considered in this paper are undirected and simple, i.e. neither loop nor multiple edge is allowed. For a set S, $|S|$ is employed to denote its cardinality. And for a *graph* $G = (V, E)$ consisting of a nonempty vertex set $V = V(G)$ and an edge set $E = E(G)$, $n = |V(G)|$ is called the *order*, $m = |E(G)|$ is called the *edge number*.

Recall that in a graph G, for a walk $W = v_0e_0v_1e_1 \cdots e_{k-1}v_k$, where $e_i = v_iv_{i+1}$ for $0 \leq i \leq k-1$ (possibly, $e_i = e_j$ for $0 \leq i \neq j \leq k-1$, $v_i = v_j$ for $0 \leq i \neq j \leq k$), written as $W = v_0v_1 \cdots v_k$ or $W = e_1e_2 \cdots e_k$ for short, the integer k is called the *length*. A *path* is a walk in which the vertices are pairwise different; a *circuit* is a closed walk; a *cycle* is a circuit in which the vertices are pairwise different. A cycle with length k is called a *k-cycle*; *k-walk* and *k-path* are defined similarly. We sometime denote by P_k (C_k) a k-path (a k-cycle) for

*Supported by National natural science foundation of China (NSFC) (Nos. Nos. 12171222, 12101285), Natural science foundation of Guangdong province (No. 2021A1515010254).

†Corresponding authors, E-mail addresses: yglong01@163.com (G. Yu).

convenience. A cycle in a graph G is called *Hamilton cycle* if it contains all vertices of G. Denoted by K_n a *complete graph* of order n, $K_{s,t}$ a *complete bipartite graph* with one part of s vertices, the other one of t vertices.

For two graphs H and G, $H \cup G$ is employed to denote the the graph obtained by $V(H) \cup V(G)$ and $E(H) \cup E(G)$. In a graph G, for two subgraphs G_1, G_2 satisfying $V(G_1) \cap V(G_2) = \emptyset$, $E(G_1, G_2)$ is employed to denote the edge set between $V(G_1)$ and $V(G_1)$. A vertex v is said to be *incident* with graph H if $E(H, v) \neq \emptyset$.

Two graphs H, G are called *disjoint* or *independent* if $V(H) \cap V(G) = \emptyset$, and there is no edge between H and G. Denote by $H \bigtriangledown G$ the *join* of two disjoint graphs H and G obtained from $H \cup G$ by adding edges between every pair of vertices u, v where $u \in V(H)$, $v \in V(G)$.

For a graph G, we denote by $G + e$ $(G - e)$ the graph obtained from G by adding a new edge e if $e \notin E(G)$ (deleting an edge $e \in E(G)$); $G[S]$ the subgraph induced by $S \subseteq V(G)$; $G - S$ the graph obtained from G by deleting all the vertices in S and all the edges incident with the vertices in S; $e(S)$ the number of edges in $G[S]$; $G - v$ for $G - \{v\}$ for short. For two subsets $S_1 \subseteq V(G), S_2 \subseteq V(G)$, denote by $E(S_1, S_2)$ the set of edges between S_1 and S_2, $e(S_1, S_2) = |E(S_1, S_2)|$, $G(S_1, S_2)$ the bipartite subgraph with parts S_1, S_2 and edge set $E(S_1, S_2)$. In a graph G, a vertex v (an edge e) is called a *cut vertex* (*cut edge*) if the component number of $G - v$ $(G - e)$ increases. A 2-connected graph is a connected graph having no cut vertex.

In a graph, we denote by $u \sim v$ for vertex u being adjacent to vertex v. For a graph G, denoted by $N_G(v)$ ($d_G(v) = |N_G(v)|$) the *neighbor set* (degree) of vertex v; $N_G[v] = N_G(v) \cup \{v\}$, $G^v = G[N_G[v]]$; $\Delta(G)$ or Δ for short ($\delta(G)$ or δ for short) the *maximal degree* (*minimal degree*). In a graph G with n vertices, a vertex v is called *dominating vertex* if $d_G(v) = n - 1$.

A graph is said to be *planar* (or embeddable in the plane), if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of the graph (see Fig. 1.1 for example). A planar embedding of a planar graph G partitions the plane into a number of edgewise-connected open sets. These sets are called the *faces* of G. The number of faces in a planar embedding of a planar graph G, denoted by $\mathbb{f}$ or $\mathbb{f}_G$. Among the faces of a planar embedding, the outer one is called the *outer face*, and any one of the other faces is called the *inner face* (see Fig. 1.1 for example). It can be seen that the boundary of a face f in a planar embedding of a planar graph, dented by $B(f)$, is a circuit. For a planar graph G, in its a planar embedding $\widetilde{G}$, we denote by $O_{\widetilde{G}}$ the outer face. As shown in Fig. 1.1, we can see that $f_1 - f_5$ are inner faces, $O_{\widetilde{G}} = f_6$, $B(f_1) = v_1v_4v_5v_1$.

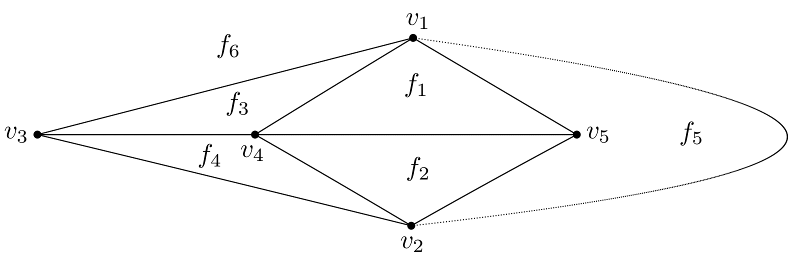


Fig. 1.1. a planar graph G and its a planar embedding $\widetilde{G}$

A simple planar graph is (edge) *maximal* if no edge can be added to the graph without violating its simplicity, or planarity.

A graph H is called a *minor* or H-*minor* of G, or G is called a H-*minor* graph if H can be obtained from G by deleting edges, contracting edges, and deleting isolated (degree zero) vertices. Given a graph H, a graph G is H-*minor free* (or H *minor free*) if H is not a minor of G.

1.2 Overview of main results

Denote by A_G the *adjacency* matrix of a graph G. It is known that A_G is symmetric. The *spectral radius* (or A-*spectral radius*) of graph G, denoted by $\rho(G)$, is defined to be the maximum eigenvalue of A_G. Let $\varrho = \max\{\rho(G)|G$ be a planar graph of order $n\}$. Our study development on the spectral radius of planar graph is as follows:

Theorem 1.1

(1) $1.359 + \sqrt{2n - \frac{15}{4}} < \varrho < 2 + \sqrt{2n-6}$ *for* $n \geq 10$*;*
(2) $1.472 + \sqrt{2n - \frac{15}{4}} < \varrho < 2 + \sqrt{2n-6}$ *for* $n \geq 40$*;*
(3) $1.478 + \sqrt{2n - \frac{15}{4}} < \varrho < 2 + \sqrt{2n-6}$ *for* $n \geq 50$.

Theorem 1.2 *Among all planar graphs containing dominating vertex with concrete fixed order* $n \geq 48$, $P_2 \nabla P_{n-2}$ *attains the maximum spectral radius.*

1.3 Outline of the paper

The layout of this paper is as follows: section 2 introduces some basic knowledge and working lemmas; section 3 represents our results.

2 Preliminary

For the requirements in the demonstrations afterward, we need some prepares.

From graph theory (see [1] for instance), it is known that: (i) a graph is planar graph if and only if it is $K_{3,3}$ and K_5 minor free; (ii) a maximal planar graph can be obtained from a non-maximal graph G by adding new edges to G; (iii) a maximal planar graph G of order $n \geq 3$ is 2-connected and $\delta \geq 2$; (iv) for a planar graph G, $\mathbb{f}$ is invariant, i.e. $\mathbb{f}$ is constant for different planar embeddings of G; (v) in any planar embedding of a maximal planar graph G of order $n \geq 3$, $B(f)$ is a 3-cycle (or called triangle) for every face f, where f is called triangular face consequently; (vi) a planar graph is maximal if and only if in its any planar embedding, $B(f)$ is a 3-cycle for every face f; (vii) for a planar graph G of order $n \geq 3$, $m(G) \leq 3n-6$ with equality if and only if G is maximal; (viii) for a bipartite planar graph G, $m(G) \leq 1$ if $n \leq 2$, and $m(G) \leq 2n-4$ if $n \geq 3$.

For a graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, a vector $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in R^n$ on G is a vector that entry x_{v_i} is mapped to vertex v_i for $i \leq i \leq n$.

Denote by R^n_{++} (R, $R^n_{\geq 0}$) the set of positive real (real, nonnegative real) vectors of dimension n. By the famous Perron-Frobenius theorem [7,8], for A_G of a connected graph G of order n, we know that there is unique one positive eigenvector $Y = (y_{v_1}, y_{v_2}, \ldots, y_{v_n})^T \in R^n_{++}$ corresponding to $\rho(G)$ satisfying $\sum_{i=1}^n y^2_{v_i} = 1$, called *principal eigenvector*; there is unique one positive eigenvector $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in R^n_{++}$ corresponding to $\rho(G)$ satisfying $\max\{x_{v_i}|1 \leq i \leq n\} = 1$, called *noralized eigenvector*.

Let A be an irreducible nonnegative $n \times n$ real matrix with spectral radius $\rho(A)$ which is the maximum modulus among all eigenvalues of A. The following extremal representation (Rayleigh quotient) will be useful:

$$\rho(A) = \max_{X \in R^n, X \neq 0} \frac{X^T A X}{X^T X},$$

and if a vector X satisfies that $\frac{X^T A X}{X^T X} = \rho(A)$, then $AX = \rho(A)X$.

Lemma 2.1

(1) **[4, 5]** *For a connected graph* G, $e \notin E(G)$. *Then* $\rho(G+e) > \rho(G)$.

(2) **[4]** *If* H *is a proper subgraph of a connected graph* G, *then* $\rho(H) < \rho(G)$.

Lemma 2.2 [9] *Let A be an irreducible nonnegative square real matrix with order n and spectral radius ρ. If there exists a nonzero vector $Y = (y_1, y_2, \ldots, y_n)^T \in R^n_{\geq 0}$ and a real coefficient polynomial function f such that $f(A)Y \leq rY$ $(r \in R)$, then $f(\rho) \leq r$. Similarly, if $f(A)Y \geq rY$ $(r \in R)$, then $f(\rho) \geq r$.*

With some modifications of the proof for Lemma 2.2, we get an improved Lemma 2.3.

Lemma 2.3 *Let A be an irreducible nonnegative square real matrix with order n and spectral radius ρ. f is a real coefficient polynomial function and $Y = (y_1, y_2, \ldots, y_n)^T \in R^n_{\geq 0}$ is a nonzero vector.*

(1) If $f(A)Y \leq rY$ $(r \in R)$, then $f(\rho) \leq r$ with equality if and only if $f(A)Y = rY$. Moreover, if there is some y_i such that $(f(A)Y)_i < ry_i$ for some $1 \leq i \leq n$, then $f(\rho) < r$.

(2) If $f(A)Y \geq rY$ $(r \in R)$, then $f(\rho) \geq r$ with equality if and only if $f(A)Y = rY$. Moreover, if there is some y_i such that $(f(A)Y)_i > ry_i$ for some $1 \leq i \leq n$, then $f(\rho) > r$.

Proof. (1) By Lemma 2.2, it is enough to consider only the cases that the equalities hold. We first prove the case for the equality holding in $f(\rho) \leq r$.

Note that $\rho(A^T) = \rho(A) = \rho$, A^T is also irreducible and nonnegative. Denote by $X = (x_1, x_2, \ldots, x_n)^T$ the principal eigenvector of A^T.

Suppose $f(A)Y = rY$. Then $f(\rho)X^TY = (f(\rho)X)^TY = (f(A^T)X)^TY = X^Tf(A)Y = rX^TY$. Hence it follows $f(\rho) = r$. Then the sufficiency follows.

Suppose $f(\rho) = r$. If $f(A)Y \neq rY$, then from $f(A)Y \leq rY$, it follows that there is some y_i such that $(f(A)Y)_i < ry_i$. Then $f(\rho)X^TY = (f(\rho)X)^TY = (f(A^T)X)^TY = X^Tf(A)Y < r\sum_{j\neq i} x_jy_j + rx_iy_i = rX^TY$. Thus it follows that $f(\rho) < r$, which contradicts $f(\rho) = r$. Then the necessity follows.

Furthermore, from the proof for the necessity, we get that if there is some y_i such that $(f(A)Y)_i < ry_i$ for some $1 \leq i \leq n$, then $f(\rho) < r$.

(2) is proved similarly. This completes the proof. □

3 Maximum spectral radius of general planar grpah

In this section, we let $\mathbb{G}_n$ be a planar graph of order n satisfying $\rho(\mathbb{G}_n) = \varrho$; $\mathcal{D} = \{G|\, G$ be a planar graph of order $n \geq 2$ containing dominating vertex$\}$, $\mathcal{G}_n \in \mathcal{D}$ satisfy $\rho(\mathcal{G}_n) = \max\{\rho(G)|\, G \in \mathcal{D}\}$.

Note that a connected planar graph can be obtained from a disconnected planar graph by adding some edges. Using Lemma 2.1 gets the following Lemma 3.1 immediately.

Lemma 3.1

(1) $\mathbb{G}_n$ is connected.

(2) Both $\mathbb{G}_n$ and $\mathcal{G}_n$ are maximal.

Lemma 3.2

(1) In any embedding of $\mathbb{G}_n$, for a vertex v_i, then in $\mathbb{G}_n^{v_i}$, there is a cycle formed by all the vertices in $N_{\mathbb{G}}(v_i)$, denoted by $\mathbb{C}_i = v_{i_1}v_{i_2}\cdots v_{i_j}v_{i_1}$ where $j = d_{\mathbb{G}}(v_i)$, $N_{\mathbb{G}}(v_i) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}$, and $v_i \bigtriangledown \mathbb{C}_i$ is a subgraph.

(2) In any embedding of $\mathcal{G}_n$, for a vertex v_i, then in $\mathcal{G}_n^{v_i}$, there is a cycle formed by all the vertices in $N_{\mathcal{G}_n}(v_i)$, denoted by $\mathbb{C}_i = v_{i_1}v_{i_2}\cdots v_{i_j}v_{i_1}$ where $j = d_{\mathcal{G}_n}(v_i)$, $N_{\mathcal{G}_n}(v_i) = \{v_{i_1}, v_{i_2}, \ldots, v_{i_j}\}$, and $v_i \bigtriangledown \mathbb{C}_i$ is a subgraph.

Proof. Suppose $\widetilde{\mathbb{G}_n}$ is a planar embedding of $\mathbb{G}_n$. For convenience, we use some new labels for the vertices of $\mathbb{G}_n$. In $\widetilde{\mathbb{G}_n}$, along clockwise direction, denote by $w_{i_1}, w_{i_2}, \ldots, w_{i_j}$ the neighbors of v_i. Note that in $\widetilde{\mathbb{G}_n}$, every face of $\mathbb{G}_n$ is a triangular face; every pair of adjacent edges $v_i w_{i_\iota}$ and $v_i w_{i_{\iota+1}}$ are in a common face (a triangular face) for each $1 \le \iota \le j-1$; $v_i w_{i_j}$ and $v_i w_{i_1}$ are in a common face. Thus in $\widetilde{\mathbb{G}_n}$, $w_{i_1} w_{i_2} \cdots w_{i_j} w_{i_1}$ forms a cycle, denoted by $\mathbb{C}_i$, and $v_i \bigtriangledown \mathbb{C}_i$ is a subgraph. Let $v_{i_\iota} = w_{i_\iota}$ for $1 \le \iota \le j$. Then (1) follows. (2) is proved similarly. This completes the proof. $\square$

Let $r_i(A)$ be the ith row sum of a matrix A, $r_v(A_G)$ be the row sum of A_G corresponding to vertex v for a graph G.

Lemma 3.3 *Let G be a planar graph on $n \ge 7$ vertices. Then $\rho(G) \le 2 + \sqrt{2n-6}$.*

Proof. For any vertex v in G, noting that G^v is planar, we have $e(N_G[v]) \le 3(d_v+1) - 6$, implying that $e(N_G(v)) \le 2d_v - 3$. Note that $G(N_G(v), V(G) \setminus N_G(v))$ is bipartite. If $2 \le d_v \le n-2$, then $e(N_G(v), V(G) \setminus N_G(v)) \le 2n-4$. Otherwise, if $d_v = 1$ or $d_v = n-1$, then $e(N_G(v), V(G) \setminus N_G(v)) \le n-1 \le 2n-4$ for $n \ge 3$. Thus

$$r_v(A_G^2) = \sum_{u \sim v} d_u = 2e(N_G(v)) + e(N_G(v), V(G) \setminus N_G(v)) \le 4d_v - 6 + 2n - 4 = 2n + 4d_v - 10.$$

It follows that $r_v(A_G^2) \le 2n + 4d_v - 10$, i.e. $r_v(A_G^2) \le 2n + 4r_v(A_G) - 10$. Thus for vector $Y = (1, 1, \ldots, 1)^T$, we have $(A_G^2 - 4A_G - 2n + 10)Y \le 0Y$. Then by Lemma 2.3, it follows that $\rho^2(G) - 4\rho(G) - 2n + 10 \le 0$, implying that $\rho(G) \le 2 + \sqrt{2n-6}$. This completes the proof. $\square$

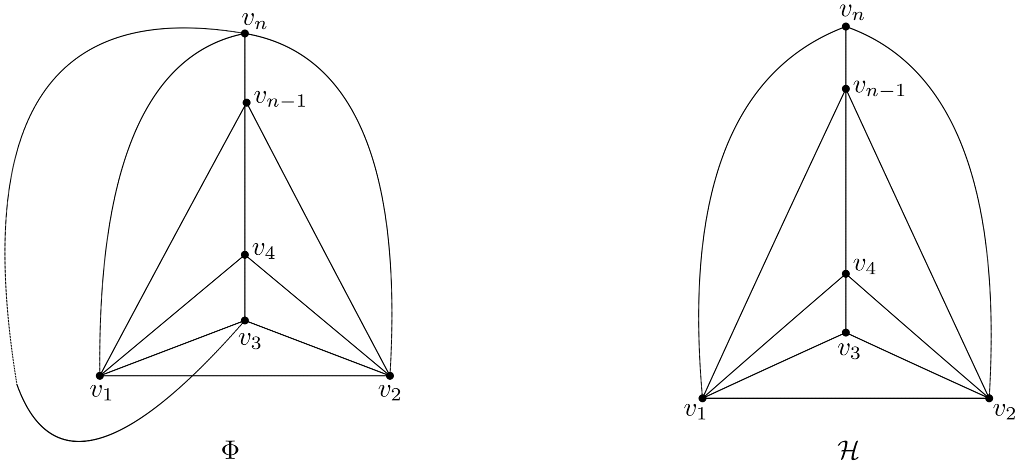


Fig. 3.1. Φ, $\mathcal{H}$

Let $n \ge 4$, $P = v_1 v_2$, $C = v_3 v_4 \cdots v_n v_3$, $\Phi = P \bigtriangledown C$, $\mathcal{H} = \Phi - v_3 v_n$ (see Fig. 3.1).

Lemma 3.4 $\rho(\Phi) = \frac{3}{2} + \sqrt{2n - \frac{15}{4}}$ *for $n \ge 5$; $\rho(\mathcal{H}) \ge \frac{3}{2} + \sqrt{2n - \frac{15}{4}} - \frac{2}{2n - \frac{15}{4} - \frac{\sqrt{2n - \frac{15}{4}}}{2}}$ for $n \ge 5$.*

Proof. Let $Y = (y_{v_1}, y_{v_2}, \ldots, y_{v_n})^T \in R^n_{++}$ be the principal eigenvector of Φ. By symmetry, it follows that $y_{v_1} = y_{v_2}$, $y_{v_3} = y_{v_4} = \cdots = y_{v_n}$. Note that

$$\begin{cases} \rho(\Phi) y_{v_1} = y_{v_2} + (n-2) y_{v_3} = y_{v_1} + (n-2) y_{v_3}, \\ \rho(\Phi) y_{v_3} = y_{v_1} + y_{v_2} + y_{v_4} + y_{v_n} = 2y_{v_1} + 2y_{v_3}, \end{cases}$$

i.e.

$$\begin{cases} (\rho(\Phi) - 1) y_{v_1} = (n-2) y_{v_3}, \\ (\rho(\Phi) - 2) y_{v_3} = 2y_{v_1}. \end{cases}$$

It follows that $\rho^2(\Phi) - 3\rho(\Phi) - 2n + 6 = 0$ and $\rho(\Phi) = \frac{3}{2} + \sqrt{2n - \frac{15}{4}}$.

From $2y_{v_1} = (\rho(\Phi) - 2)y_{v_3} = (\sqrt{2n - \frac{15}{4}} - \frac{1}{2})y_{v_3}$ and $\sum_{i=1}^{n} y_{v_i}^2 = 1$, it follows that $y_{v_3}^2 = \frac{1}{2n - \frac{15}{4} - \frac{\sqrt{2n - \frac{15}{4}}}{2}}$. Then we get

$$\rho(\mathcal{H}) \geq Y^T A_{\mathcal{H}} Y = Y^T A_{\Phi} Y - 2y_{v_3}y_{v_n} = \frac{3}{2} + \sqrt{2n - \frac{15}{4}} - 2y_{v_3}^2 = \frac{3}{2} + \sqrt{2n - \frac{15}{4}} - \frac{2}{2n - \frac{15}{4} - \frac{\sqrt{2n - \frac{15}{4}}}{2}}.$$

This completes the proof. $\square$

Note that if positive integer $n \geq 2$, then $f(n) = \frac{3}{2} + \sqrt{2n - \frac{15}{4}} - \frac{2}{2n - \frac{15}{4} - \frac{\sqrt{2n - \frac{15}{4}}}{2}}$ and $2n - \frac{15}{4} - \frac{\sqrt{2n - \frac{15}{4}}}{2}$ strictly increase with respect to $n \geq 4$ increasing. Thus we get the following Corollary 3.5.

Corollary 3.5

(1) $\rho(\mathcal{H}) > 1.359 + \sqrt{2n - \frac{15}{4}}$ *for* $n \geq 10$*;*

(2) $\rho(\mathcal{H}) > 1.472 + \sqrt{2n - \frac{15}{4}}$ *for* $n \geq 40$*;*

(3) $\rho(\mathcal{H}) > 1.478 + \sqrt{2n - \frac{15}{4}}$ *for* $n \geq 50$.

Note the maximality of $\rho(\mathbb{G}_n)$ and $\mathcal{H}$ is a planar graph. Combining Lemma 3.3 and Corollary 3.5, we get the following Theorem 3.6.

Theorem 3.6

(1) $1.359 + \sqrt{2n - \frac{15}{4}} < \rho(\mathbb{G}_n) < 2 + \sqrt{2n - 6}$ *for* $n \geq 10$*;*

(2) $1.472 + \sqrt{2n - \frac{15}{4}} < \rho(\mathbb{G}_n) < 2 + \sqrt{2n - 6}$ *for* $n \geq 40$*;*

(3) $1.478 + \sqrt{2n - \frac{15}{4}} < \rho(\mathbb{G}_n) < 2 + \sqrt{2n - 6}$ *for* $n \geq 50$.

Proof of Theorem 1.1. This theorem follows from 3.6. This completes the proof. $\square$

Note that $\mathcal{H}$ is also a planar graph containing dominating vertex. The next Corollary 3.7 follows from Corollary 3.5 immediately.

Corollary 3.7

(1) $\rho(\mathcal{G}_n) > 1.359 + \sqrt{2n - \frac{15}{4}}$ *for* $n \geq 10$*;*

(2) $\rho(\mathcal{G}_n) > 1.472 + \sqrt{2n - \frac{15}{4}}$ *for* $n \geq 40$*;*

(3) $\rho(\mathcal{G}_n) > 1.478 + \sqrt{2n - \frac{15}{4}}$ *for* $n \geq 50$.

Hereafter, we suppose that $V(\mathcal{G}_n) = \{v_1, v_2, \ldots, v_n\}$, and $deg_{\mathcal{G}_n}(v_1) = n - 1$. Let $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in R^n_{++}$ be the normalized eigenvector of $\mathcal{G}_n$.

Lemma 3.8 $x_{v_1} = 1$ *if* $n \geq 3$.

Proof. For v_1, and any vertex v_i where $2 \leq i \leq n$, using $\rho(\mathcal{G}_n)(x_{v_1} - x_{v_i}) = x_{v_i} - x_{v_1} + \sum_{3 \leq j \leq n} x_{v_j} - \sum_{v_j \sim v_i} x_{v_j}$ gets that $(\rho(\mathcal{G}_n) - 1)(x_{v_1} - x_{v_i}) = \sum_{3 \leq j \leq n} x_{v_j} - \sum_{v_j \sim v_i} x_{v_j} \geq 0$. Note that $\mathcal{G}_n$ is maximal and $\mathcal{G}_n$ contains at least one cycle in $\mathcal{G}_n$. Then $\rho(\mathcal{G}_n) \geq 2$. Thus it follows that $x_{v_1} - x_{v_i} = \frac{\sum_{3 \leq j \leq n} x_{v_j} - \sum_{v_j \sim v_i} x_{v_j}}{\rho(\mathcal{G}_n) - 1} \geq 0$. As a result, it follows that $x_{v_1} \geq x_{v_i}$. Note that X is the normalized eigenvector of $\mathcal{G}_n$ and the arbitrariness of v_i. Then we get that $x_{v_1} = 1$. This completes the proof. $\square$

Next, we let $1 = x_{v_1} \geq x_{v_2} \geq \cdots \geq x_{v_n}$ for convenience.

Lemma 3.9 $x_{v_2} > \frac{1}{4}$ *if* $n \geq 10$.

Proof. Let $H = \mathcal{G}_n - v_1$. Note that $\mathcal{G}_n$ is maximal, $x_{v_1} = 1$ and note that for $n \geq 10$, $e(\mathcal{G}_n) = 3n - 6$, $d_{\mathcal{G}_n}(v_1) = n - 1$, $\rho(\mathcal{G}_n) > 1.359 + \sqrt{2n - \frac{15}{4}}$. Then $\sum_{i \geq 2} d_H(v_i) \leq 2(2n - 5)$, and

$$\rho^2(\mathcal{G}_n)x_{v_1} = \sum_{v_j \sim v_1} \rho(\mathcal{G}_n)x_{v_j} = (n-1)x_{v_1} + \sum_{i \geq 2} d_H(v_i)x_{v_i} \leq (n-1)x_{v_1} + 2(2n-5)x_{v_2}$$

$$\Longrightarrow (\rho^2(\mathcal{G}_n)-n+1)x_{v_1} \le (4n-10)x_{v_2}$$
$$\Longrightarrow \frac{\rho^2(\mathcal{G}_n)-n+1}{4n-10}x_{v_1} \le x_{v_2}$$
$$\Longrightarrow \frac{1.846+2.718\sqrt{2n-\frac{15}{4}}+n-\frac{11}{4}}{4n-10}x_{v_1} < x_{v_2} \qquad (*1)$$
$$\Longrightarrow x_{v_2} > \frac{1}{4}$$

This completes the proof. □

Let $f_2(n)=\frac{1.846+2.718\sqrt{2n-\frac{15}{4}}+n-\frac{11}{4}}{4n-10}$. Taking derivation with respect to n gets

$$f_2'(n)=\frac{(\frac{2.718}{2\sqrt{2n-\frac{15}{4}}}+1)(4n-10)-4(1.846+2.718\sqrt{2n-\frac{15}{4}}+n-\frac{11}{4})}{(4n-10)^2}<0.$$

It follows that $f_2(n)=\frac{1.846+2.718\sqrt{2n-\frac{15}{4}}+n-\frac{11}{4}}{4n-10}$ decreases strictly with respect to $n\ge 1$ increasing. Then using inequality $(*1)$ gets the following Lema 3.10.

Lemma 3.10

(1) $x_{v_2} > 0.3987$ *if* $n \le 50$*;*

(2) $x_{v_2} > 0.375$ *if* $n \le 80$.

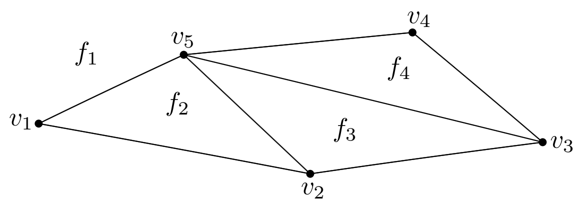


Fig. 3.2. an outerplanar graph G and its an OP-embedding $\widehat{G}$

A simple graph G is *outerplanar* if it has an embedding in the plane, called *outerplane-embedding* (written as *OP-embedding* for short hereafter), denoted by $\widehat{G}$, so that every vertex lies on the boundary of the unbounded (outer) face. Similar to a planar embedding of a planar graph, an OP-embedding of an outerplanar graph G partitions the plane into a number of edgewise-connected $faces$. Among the faces of $\widehat{G}$ for an outerplanar graph G, the outer one is called the $outer\ face$ (see f_1 in Fig. 3.2 for example), and any one of other faces is called the $inner\ face$ (see f_2, f_3, f_4 in Fig. 3.2 for example). Similar to the notations for a planar graph, we denote by $B(f)$ the boundary of a face f in $\widehat{G}$ of an outerplanar graph G. It can be seen that $B(f)$ of a face f is a circuit. For an outerplanar graph G, we denote by $O_{\widehat{G}}$ the outer face of its an OP-embedding $\widehat{G}$. As shown in Fig. 3.2, we can see that $O_{\widehat{G}}=f_1$, $B(f_1)=v_1v_2v_3v_4v_5v_1$.

A simple outerplanar graph is (edge) *maximal* if no edge can be added to the graph without violating its simplicity, or outerplanarity.

Recall some facts (known results in graph theory, see [1] for instance) about outerplanar graph that (i) a maximal outerplanar graph can be obtained from a non-maximal outerplanar graph G by adding new edges to G; (ii) in a maximal outerplanar graph G of order $n\ge 3$, $\delta(G)=2$ and there are at least two vertices with degree 2; (iii) in an OP-embedding $\widehat{G}$ of a maximal outerplanar graph G, the boundary of every inner face is a 3-cycle (or called triangle), where every inner face is called a triangular face similarly, the $B(O_{\widehat{G}})$ is a Hamilton cycle of G; (iv) an outerplanar graph is maximal if and only if there is an OP-embedding $\widehat{G}$ such that the boundary of every inner face is a 3-cycle and the $B(O_{\widehat{G}})$ is a Hamilton cycle of G; (v) for an outerplanar graph G of order n, $e(G) \leq 2n-3$ with equality if and only if G is maximal.

From Lemma 3.2 and definition of outerplanar graph, it follows that both $\mathcal{G}_n - v_1$ and $\mathcal{G}_n - \{v_1, v_2\}$ are outerplanar.

Let $\mathbb{L} = \{v_1, v_2\}$, $\mathbb{S} = V(\mathcal{G}_n) \setminus \mathbb{L}$.

Lemma 3.11 *If $n \geq 81$, then for any $i \geq 3$, we have $x_{v_i} < \frac{x_{v_2}}{2}$.*

Proof. We prove this result by contradiction. Suppose $x_{v_i} \geq \frac{x_{v_2}}{2}$ for $i \geq 3$. Then

$$\begin{aligned}
(n-2)\frac{x_{v_2}}{2} \leq \sum_{v_z \in \mathbb{S}} x_{v_z} &= \frac{1}{\rho(\mathcal{G}_n)} \sum_{v_z \in \mathbb{S}} \rho(\mathcal{G}_n) x_{v_z} \\
&\leq \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} \sum_{v_z \in \mathbb{S}} \rho(\mathcal{G}_n) x_{v_z} \\
&= \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} \sum_{v_z \in \mathbb{S}} \sum_{v_y \sim v_z} x_{v_y} \\
&= \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} \sum_{v_z \in \mathbb{S}} \Big(\sum_{v_y \sim v_z, v_y \in \mathbb{S}} x_{v_y} + \sum_{v_y \sim v_z, v_y \in \mathbb{L}} x_{v_y} \Big) \\
&\leq \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} (x_{v_2} 2e(\mathbb{S}) + (n-2)x_{v_1} + (n-2)x_{v_2}) \qquad (*2)
\end{aligned}$$

Note that $\mathcal{G}_n[\mathbb{S}] = \mathcal{G}_n - \{v_1, v_2\}$ is outerplanar. Then

$$\begin{aligned}
(*2) &\leq \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} (x_{v_2}(2(n-2) - 3) + (n-2)x_{v_1} + (n-2)x_{v_2}) \\
&= \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} (x_{v_2}(3n-9) + n - 2)
\end{aligned}$$

and

$$(n-2)\frac{x_{v_2}}{2} < \frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} (x_{v_2}(3n-9) + n - 2). \qquad (*3)$$

Note that if $n \geq 81$, then $1.478 + \sqrt{2n - \frac{15}{4}} > 14.05$. Combining Lemma 3.9, from $(*3)$, it follows that

$$\frac{x_{v_2}}{2} \leq \frac{\frac{1}{1.478 + \sqrt{2n - \frac{15}{4}}} (x_{v_2}(3n-9) + n - 2)}{n-2} < \frac{3x_{v_2} + 1}{1.478 + \sqrt{2n - \frac{15}{4}}} < \frac{x_{v_2}}{2}. \qquad (*4)$$

It is a contradiction. Thus our result follows. This completes the proof. $\square$

Combining Lemma 3.10 and inequality $(*4)$, and repeating the process in proof of Lemma 3.11, we get the following Lemma 3.13.

Lemma 3.12 *If $51 \leq n \leq 80$, then for any $i \geq 3$, we have $x_{v_i} < \frac{x_{v_2}}{2}$.*

Combining $\rho(\mathcal{G}_n) > 1.472 + \sqrt{2n - \frac{15}{4}}$ for $n \geq 40$ in Lemma 3.7 and inequality $(*4)$, and repeating the process in proof of Lemma 3.11 by replacing $1.478 + \sqrt{2n - \frac{15}{4}}$ with $1.472 + \sqrt{2n - \frac{15}{4}}$, we get the following Lemma 3.13.

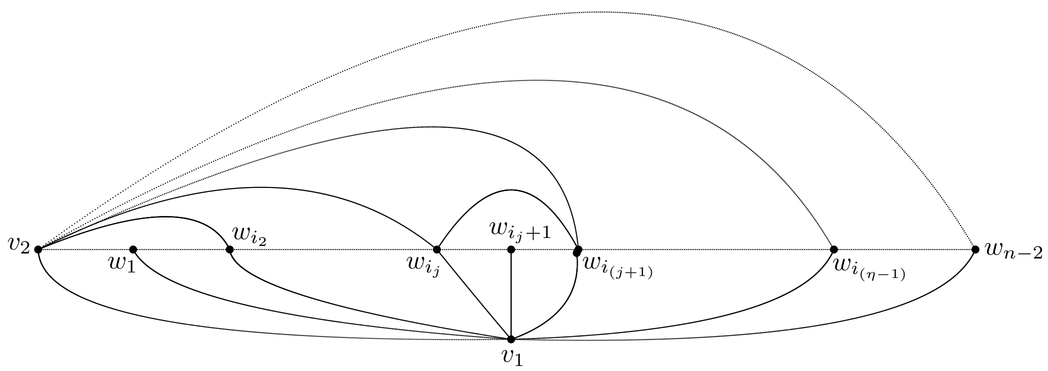
Lemma 3.13 *If $48 \leq n \leq 50$, then for any $i \geq 3$, we have $x_{v_i} < \frac{x_{v_2}}{2}$.*

Lemma 3.14 *If $n \geq 48$, then $d_{\mathcal{G}_n}(v_2) = n-1$.*

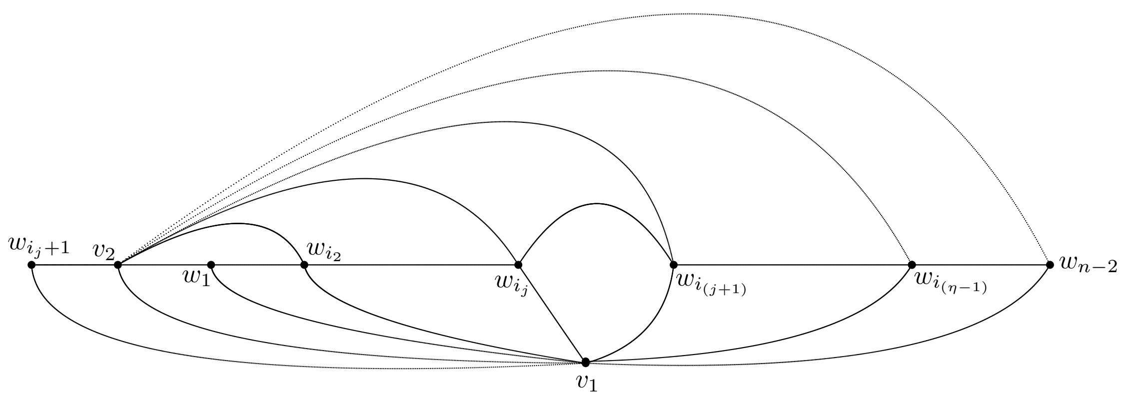
Proof. We prove this result by contradiction. Suppose $d_{\mathcal{G}_n}(v_2) \leq n-2$, $\widetilde{\mathcal{G}_n}$ is a planar embedding of $\mathcal{G}_n$. For convenience, we use some new labels for the vertices of $\mathcal{G}_n$. By Lemma 3.2, in $\widetilde{\mathcal{G}_n}$, suppose $\mathbb{C}_1 = v_2w_1w_2\cdots w_{n-2}v_2$ is a cycle where $N_{\mathcal{G}_n}(v_1) = \{v_2, w_1, w_2, \ldots, w_{n-2}\}$, and $v_1 \bigtriangledown \mathbb{C}_1$ is a subgraph. Suppose v_2, w_1, w_2, $\ldots$, w_{n-2} are distributed along clockwise direction around vertex v_1 in $\widetilde{\mathcal{G}_n}$. Suppose $N_{\mathcal{G}_n}(v_2) = \{v_1, w_{i_1}, w_{i_2}, \ldots, w_{i_\eta}\}$ and $1 = i_1 < i_2 < \cdots < i_\eta = n-2$. For convenience to distinguish $i_t + 1$ and i_{t+1} ($i_t - 1$ and i_{t-1}) easily, we use $w_{i_{(t+1)}}$ for $w_{i_{t+1}}$ ($w_{i_{(t-1)}}$ for $w_{i_{t-1}}$) hereafter. Because $d_{\mathcal{G}_n}(v_2) \leq n-2$, there exist w_{i_j} and $w_{i_{(j+1)}}$ such that $i_{(j+1)} - i_j \geq 2$ where $1 \leq j \leq \eta - 1$. This means that none of vertices w_{i_j+1}, w_{i_j+2}, $\ldots$, $w_{i_{(j+1)}-1}$ is adjacent to v_2.

Note that $\mathcal{G}_n$ is maximal, and in $\widetilde{\mathcal{G}_n}$, $v_2w_{i_j}$, $v_2w_{i_{(j+1)}}$ are in a common triangular face. Then $w_{i_j}w_{i_{(j+1)}} \in E(\mathcal{G}_n)$. By the facts about maximal outerplanar graph, $H = \mathcal{G}_n[\{w_{i_j}, w_{i_j+1}, w_{i_j+2}, \ldots, w_{i_{(j+1)}-1}, w_{i_{(j+1)}}\}]$ is a maximal outerplanar graph, $C = w_{i_j}w_{i_j+1}w_{i_j+2}\cdots w_{i_{(j+1)}-1}w_{i_{(j+1)}}w_{i_j}$ is the Hamilton cycle of H.

Claim If $i_{(j+1)} - i_j \geq 3$, then at least one of $d_H(w_{i_j})$, $d_H(w_{i_{(j+1)}})$ is more than 2. With respect to $\widetilde{\mathcal{G}_n}$, noting the maximality $\mathcal{G}_n$ and H, we know that every inner face of H is a triangular face. Thus $w_{i_j}w_{i_{(j+1)}}$ is in a inner triangular face of H. Then in H, there is a vertex w_ς together with w_{i_j}, $w_{i_{(j+1)}}$ forming a 3-cycle $w_{i_j}w_\varsigma w_{i_{(j+1)}}w_{i_j}$, where $i_j + 1 \leq \varsigma \leq i_{(j+1)} - 1$. Because $i_{(j+1)} - i_j \geq 3$, then at least one of $\varsigma - i_j \geq 2$, $i_{(j+1)} - \varsigma \geq 2$ holds. Without loss of generality, suppose $\varsigma - i_j \geq 2$. Note that $\{w_{i_j}w_{i_{(j+1)}}, w_{i_j}w_{i_j+1}, w_{i_j}w_\varsigma\} \subseteq E(\mathcal{G}_n)$. Thus $d_H(w_{i_j}) \geq 3$. Consequently, the claim holds.



$\mathcal{G}_n$



$\mathcal{G}'_n$

Fig. 3.3. $\mathcal{G}_n$, $\mathcal{G}'_n$

Case 1 $i_{(j+1)} - i_j = 2$. Note that H is maximal outerplanar, and note the facts about maximal outerplanar graph. It follows that $w_{i_j} w_{i_j+1} w_{i_{(j+1)}} w_{i_j}$ is a 3-cycle. It means that $d_H(w_{i_j+1}) = 2$. Let $\mathcal{G}'_n = \mathcal{G}_n - w_{i_j} w_{i_j+1} - w_{i_j+1} w_{i_{(j+1)}} + v_2 w_{i_j+1}$ (see Fig. 3.3). Then using Lemmas 3.11-3.13 gets $X^T A_{\mathcal{G}'_n} X - X^T A_{\mathcal{G}_n} X = x_{w_{i_j+1}}(x_{v_2} - x_{w_{i_j}} - x_{w_{i_{(j+1)}}}) > 0$. It follows that $\rho(\mathcal{G}'_n) > \rho(\mathcal{G}_n)$, which contradicts the maximality of $\rho(\mathcal{G}_n)$.

Case 2 $i_{(j+1)} - i_j \geq 3$. By our above Claim, and the fact that a maximal outerplanar graph of order at least 3 has at least 2 vertices with degree 2, there is a vertex w_z with $d_H(w_z) = 2$ where $i_j + 1 \leq z \leq i_{(j+1)} - 1$. Let $\mathcal{G}'_n = \mathcal{G}_n - w_z w_{z+1} - w_z w_{z-1} + v_2 w_z$. Similar to Case 1, we get a contradiction that $\rho(\mathcal{G}'_n) > \rho(\mathcal{G}_n)$ contradicts the maximality of $\rho(\mathcal{G}_n)$.

The above narrations demonstrate that $d_{\mathcal{G}_n}(v_2) = n - 1$. This completes the proof. □

Theorem 3.15 *If $n \geq 48$, then $\mathcal{G}_n \cong P_2 \bigtriangledown P_{n-2}$.*

Proof. By Lemma 3.2, in a planar embedding $\widetilde{\mathcal{G}_n}$ of $\mathcal{G}_n$, suppose $\mathbb{C}_1 = v_2 w_1 w_2 \cdots w_\kappa v_2$ is a cycle where $N_{\mathcal{G}_n}(v_1) = \{v_2,\ w_1,\ w_2,\ \ldots,\ w_{n-2}\}$, and $v_1 \bigtriangledown \mathbb{C}_1$ is a subgraph. Let $\mathbb{P}_2 = v_1 v_2$, $\mathbb{P}_{n-2} = w_1 w_2 \cdots w_{n-2}$. By Lemma 3.14, it follows that $\mathcal{G}_n = \mathbb{P}_2 \bigtriangledown \mathbb{P}_{n-2}$. Thus the result follows as desired. This completes the proof. □

Proof of Theorem 1.2. This theorem follows from Theorem 3.15. This completes the proof. □

References

[1] J. Bondy, U. Murty, Graph theory, London Ltd (LL): Springer Press, 2008.

[2] B. Boots, G. Royle, A conjecture on the maximum value of the principal eigenvalue of a planar graph, Geogr. Anal. 23 (1991) 276-282.

[3] D. Cao, A. Vince, The spectral radius of a planar graph, Linear Algebra Appl. 187 (1993) 251-257.

[4] A. Hoffman, J. Smith, On the spectral radii of topologically equivalent graphs, in: M. Fiedler (Ed.), Recent Advances in Graph Theory, Academic Praha, 1975, pp. 273-281.

[5] Q. Li, K. Feng, On the largest eigenvalues of graphs, Acta Math. Appl. Sinica 2 (1979), 167-175 (in Chinese).

[6] M. Tait, J. Tobin, Three conjectures in extremal spectral graph theory, J. Combin. Theory Ser. B 126 (2017) 137-161.

[7] H. Minc, Nonnegative Matrices, John Wiley & Sons Inc., New York, 1988.

[8] O. Perron, Zur theorie dor matřizan, Math Ann. 64 (1907), 248-263.

[9] G. Yu, Y. Hong, J. Shu, Bounds of spectral radii on edge-most outerplanar bipartite graphs, Linear Multilinear Algebra 61 (2013) 791-800.