

NONLINEAR SCHRÖDINGER EQUATION ON A UNIT BALL IN ONE AND TWO DIMENSIONS

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ABSTRACT. We consider the nonlinear Schrödinger equation on a unit ball in one and two dimensions with Dirichlet boundary conditions, which have stabilizing effect on solutions behavior. In particular, we confirm that the ground state solutions are stable in subcritical and critical cases, and in the supercritical case the ground state solutions split into a stable and an unstable branch. Perturbations of a ground state on the stable branch keep solutions near a corresponding ground state with very small oscillation around it, while perturbations of the unstable branch make solutions either blow up in finite time, if perturbations have an amplitude large than the height of the ground state, or oscillate between two states, if perturbations have an amplitude smaller than the original ground state. We also observe that this equation does not have any scattering or radiation, and thus, the soliton resolution holds for all data, splitting solutions into coherent structures such as ground state solutions even for very small initial data.

1. INTRODUCTION

We consider the nonlinear Schrödinger (NLS) equation on a bounded domain:

$$i u_t + \Delta u + |u|^\alpha u = 0, \quad x \in \Omega \subset \mathbb{R}^d, \quad t \in \mathbb{R}, \quad (1.1)$$

where $u(t, x)$ is a complex-valued function, Δ is the standard Laplacian, the nonlinearity power is $\alpha > 0$, and Ω is a bounded domain, which for this paper we take as a unit ball $B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$ in 1 or 2 dimensions. We are specifically interested in the Dirichlet boundary condition, namely,

$$u(0, x) = 0, \quad x \in \partial\Omega, \quad (1.2)$$

and its influence on the solution, including solitary waves and their stability.

The Cauchy problem for the equation (1.1) is locally well-posed in H_0^1 : for any $u_0 \in H_0^1(B_1)$ there exists $T = T(\|u_0\|_{H^1}) > 0$ and a unique solution $u \in C([0, T), H_0^1(B_1))$ with the given initial condition u_0 . The existence and continuous dependence (as well as mass and energy conservations) follow from [2, Thm. 3.3.9]. The uniqueness is addressed in dimension $d = 1$ for any α in [2, Thm. 3.5.1], in dimension $d = 2$ when $0 < \alpha \leq 2$ in [14], [10], and when $2 < \alpha < \infty$ with an additional assumption $\int_{B_1} \exp(\kappa|u_0|^p) dx < \infty$ for some $\kappa > 0$ in [15, Thm. 5].¹ As a consequence, during their lifespan, solutions $u(t)$ to (1.1) conserve mass and energy

Key words and phrases. nonlinear Schrödinger equation, Dirichlet boundary condition, bounded domain, unit ball, ground states, branching, stability, blow-up.

¹While this condition may seem to impose an additional assumption, it will not affect our analysis, as we consider perturbations of ground states, which decay exponentially, and thus, are bounded on B_1 for any positive κ .

(or Hamiltonian):

$$M[u(t)] = \int_{B_1} |u(t)|^2 dx \equiv M[u(0)] \quad (1.3)$$

and

$$E[u(t)] = \frac{1}{2} \int_{B_1} |\nabla u(t)|^2 dx - \frac{1}{\alpha + 2} \int_{B_1} |u(t)|^{\alpha+2} dx \equiv E[u(0)]. \quad (1.4)$$

While the scaling invariance is absent in the equation (1.1) due to the boundary, one can still refer to the criticality of the equation according to the scaling of the NLS equation on the whole space:

$$s_c = \frac{d}{2} - \frac{2}{\alpha}.$$

Thus, the NLS equation is (L^2 or mass)-critical if $\alpha = \frac{4}{d}$, subcritical when $\alpha < \frac{4}{d}$ and supercritical otherwise.

From the conservation laws and the Gagliardo-Nirenberg inequality, it follows that solutions for $u_0 \in H_0^1(B_1)$ exist globally in the subcritical case and under the mass assumption, $\|u_0\|_{L^2(B_1)} < \|Q\|_{L^2(B_1)}$, in the critical case.

1.1. Motivation. The cubic NLS on the unit ball in 2D with Dirichlet boundary condition describes the propagation of the laser beam in a hollow-core fiber (opposed to a solid-core, thus, reducing drastically light-matter interaction). A hollow waveguide filled with a noble gas is used as an effective technique for extending the interaction length between nonlinear optical materials and high-energy laser pulses. The electric field outside the core is negligible and almost all of the laser beam reflects on the interface between the core and the cladding, implying the Dirichlet boundary condition (1.2), [4], [6], [12]. This setting influences the stability of ground states (and excited states) in the NLS on a unit ball as was indicated in [5] and [6], which differs from the NLS considered on the whole space $\mathbb{R}^d$. The purpose of this paper is exactly to investigate further solutions behavior and dynamics of ground states in this model and the difference with the whole space.

1.2. Solitary waves in a ball. The NLS equation (1.1) on a bounded domain has a family of standing solitary waves,

$$u(t, x) = e^{ibt} Q_b(x), \quad (1.5)$$

with $b \in \mathbb{R}$ and $Q_b(x)$ a *ground state* solution² in $H_0^1(B_1)$ with the Dirichlet boundary condition on B_1 , i.e.,

$$\begin{cases} -\Delta Q + bQ - |Q|^\alpha Q = 0, & x \in B_1, \\ Q(x) = 0, & x \in \partial B_1. \end{cases} \quad (1.6)$$

The ground states are defined as the energy minimizers, which we discuss next, following [6]. For $b \in \mathbb{R}$ define a Weinstein functional $S_b(u) = E(u) + \frac{b}{2}M(u)$. Then $S'_b(u) = 0$ if and only if $u = u_b \in H_0^1(B_1)$ is a weak solution of (1.6) for a given b . The *ground state* solution $Q_b \in H_0^1(B_1)$ for a given b is the *least-energy* solution of (1.6) if Q_b is a weak solution of (1.6) and it minimizes S_b among all non-trivial solutions of (1.6). The least-energy solution exists for $b > -\lambda_1$, where $\lambda_1 > 0$ is the *first (lowest) eigenvalue* of the operator $(-\Delta)$ in B_1 ; this

²On the whole space $\mathbb{R}^d$ the standing waves Q_b are defined for $b > 0$ and $Q_b(x) \rightarrow 0$ as $|x| \rightarrow +\infty$.

solution is unique (up to a phase), radially symmetric and strictly decreasing in the radial variable $r = |x|$, see [6] and references therein.

The standing wave $e^{ibt}Q_b$ is said to be *orbitally stable* in $H_0^1(B_1)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in H_0^1(B_1)$ with $\|u_0 - Q_b\|_{H^1} < \delta$, it implies that the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $\|u(t) - e^{it\theta}Q_b\|_{H^1} < \varepsilon$. Otherwise, the standing wave is *unstable*³.

It was found in [5] ($d = 2, \alpha = 2$) and [6] (general dimension and power) that the Dirichlet boundary has a *stabilizing* effect: standing solitary waves $e^{ibt}Q_b$ of the equation (1.1) are orbitally stable in $H_0^1(B_1)$ not only in the subcritical case $0 < \alpha < \frac{4}{d}$, but also for some range in the critical case $\alpha = \frac{4}{d}$. Furthermore, in the supercritical case, the ground state is also stable in the neighborhood of $-\lambda_1$; this shows a significant difference with the behavior of the ground state being unstable in the critical case on the whole $\mathbb{R}^d$ as well as in the supercritical case. To be precise, it was shown

Theorem 1.1 ([5], [6]). *For $d \geq 1$ and Q_b being the least-energy (ground state) solution, the following holds:*

I. Orbital stability. (a.1) Let $0 < \alpha \leq \frac{4}{d}$. The standing wave $e^{ibt}Q_b$ is orbitally stable in $H_0^1(B_1)$ for any $b \in (-\lambda_1, -\lambda_1 + \epsilon) \cup (b_1, \infty)$, for some $b_1 > 0$ and $\epsilon > 0$.

(a.2) When $d = 1$, $0 < \alpha \leq 4$, the standing wave $e^{ibt}Q_b$ is orbitally stable for all $b \in (-\lambda_1, \infty)$.

(b) For $\frac{4}{d} < \alpha < \frac{4}{d-2}$, $d \geq 2$, there exists $\epsilon > 0$ such that standing wave $e^{ibt}Q_b$ is orbitally stable in $H_0^1(B_1)$ for any $b \in (-\lambda_1, -\lambda_1 + \epsilon)$.

II. Instability. Let $\frac{4}{d} < \alpha < \frac{4}{d-2}$. There exists $b_2 > 0$ such that the standing wave $e^{ibt}Q_b$ is unstable for any $b \in (b_2, \infty)$.

We confirm the stability of the standing wave in dimension 1 in the critical and subcritical cases on the whole interval $(-\lambda_1, \infty)$ as indicated in Theorem 1.1 part I (a). Furthermore, in dimension 2 in subcritical and critical cases we confirm stability on the whole interval $(-\lambda_1, \infty)$, thus, eliminating the gap in part I(a) in dimension $d = 2$. In the supercritical cases in dimensions 1 and 2 we show the (complimentary) ranges of stability and instability. What is surprising is that in the supercritical case the *branching* of ground states occurs, where a stable and an unstable branches appear (similar to the branching we discovered in the bi-harmonic NLS in our work [9], see also [1]).

One way to prove Theorem 1.1 (see Prop. 5 in [6]) is to check the *slope* condition (or Vakhitov-Kolokolov criteria adapted to the nonlinear case), consisting of checking the sign of the derivative with respect to b of the ground state mass, i.e., $\partial_b M(Q_b)$, which is an application of the well-known Grillakis-Shatah-Strauss method [7]. (We note that the differentiability of Q_b with respect to b is ensured for any $\alpha > 0$, see [11, Thm. 18]), provided that the spectral properties of the linearized around Q_b operator $S_b''(Q_b)$ hold, which is indeed the case as discussed in [6, Prop.5].) We examine the slope condition in various settings and confirm stability or instability, accordingly. In fact, we confirm numerically the following conjecture.

³Sometimes if the instability produces a finite time blow-up, then it is referred to as strong instability.

Conjecture 1.2. Let $d \geq 1$ and let Q_b be the least-energy (ground state) solution of (1.6) for $b \in (-\lambda_1, \infty)$.

Part I: (Subcritical case)

Let $0 < \alpha < \frac{4}{d}$. Then for any $b \in (-\lambda_1, \infty)$ the slope condition has a positive sign, $\partial_b M(Q_b) > 0$, which implies that the standing wave $e^{ibt}Q_b$ is orbitally stable in $H_0^1(B_1)$.

Part II: (Critical case)

Let $\alpha = \frac{4}{d}$. Then

- for any $b \in (-\lambda_1, \infty)$ the slope condition has a positive sign, $\partial_b M(Q_b) > 0$, which implies that the standing wave $e^{ibt}Q_b$ is orbitally stable in $H_0^1(B_1)$. Furthermore, $M(Q_b)$ increases monotonically with $\lim_{b \rightarrow \infty} M(Q_b) = M(\mathcal{R})$ (see Figure 1 left, the blue line represents $M(Q_b)$, the area around it shows stable perturbations);
- if $M(u_0) > M(\mathcal{R})$, then the solution $u(t)$ with the initial condition u_0 blows up in finite time (see Figure 1 left, the red line represents the mass of the ground state on the whole space, $M(\mathcal{R})$, above which initial data blows up in finite time).

Part III: (Supercritical case)

Let $\alpha > \frac{4}{d}$ (with $\alpha < \frac{4}{d-2}$ if $d \geq 3$). Then there exists $b^* \in (-\lambda_1, \infty)$ such that

- the curve of $M(Q_b)$, depending on b , increases monotonically up to some maximum value at b^* and then decreases down to the value of $M(\mathcal{R})$ (e.g., see Figure 10 (G));
- for any $b \in (-\lambda_1, b^*)$ the slope $\partial_b M(Q_b) > 0$, which implies that the standing wave $e^{ibt}Q_b$ is orbitally stable in $H_0^1(B_1)$ (see blue curve in Figure 1, right, and Remark 1.3).
- for any $b \in (b^*, \infty)$ the slope $\partial_b M(Q_b) < 0$, and thus, the standing wave $e^{ibt}Q_b$ is unstable. The perturbations of the unstable branch either produce blow-up in finite time (for example, for $A > 1$ in $u_0 = AQ_b$) or oscillating behavior between two states (if $A < 1$ in $u_0 = AQ_b$) (see green curve in Figure 1, right, and Remark 1.3).

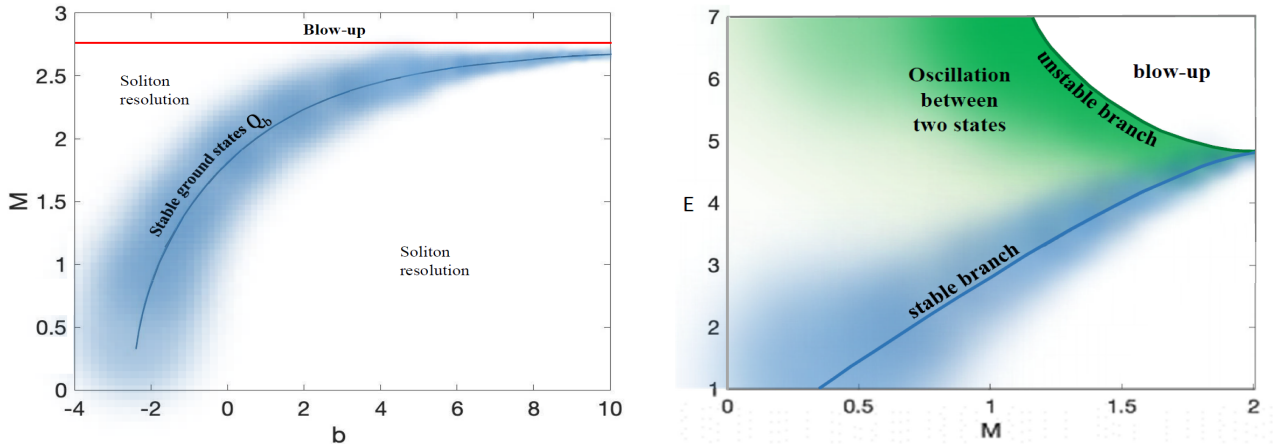


FIGURE 1. Stability of ground states Q_b in critical and supercritical cases. The axes labels on the left are from the example of 1D quintic NLS, Figure 8(D) and on the right are from the example of 2D quintic NLS, Figure 10(I).

Remark 1.3. In Figure 1 we depict the stability of ground states in the critical and subcritical cases as stated in Conjecture 1.2 Part II and III. On the left plot we use an example of the 1D quintic NLS from Figure 8(D) to show the stability of Q_b for different b in the *critical* case: the blue curve indicates the mass of Q_b . Small perturbations of Q_b are stable, which is indicated by a blue cloud area around the blue curve. Note that the shaded perturbation zone shrinks as $b \rightarrow \infty$, since $M(Q_b) \rightarrow M(\mathcal{R})$ (the red line), above which solutions blow up in finite time. In the rest of the plot, solutions are conjectured to satisfy the *soliton resolution*, splitting into a sum of coherent structures (such as solitons), even for the small initial data, as the mass of Q_b covers the entire interval $(0, M(\mathcal{R}))$. There is no radiation forming in this problem, as the Dirichlet boundary condition has a stabilizing effect and reflects solutions completely.

On the right plot we show the *branching* of ground states in the *supercritical* case (here, we use an example of the 2D quintic NLS from Figure 10 (I)): the blue curve indicates the stable branch of Q_b and the green curve shows the unstable branch of Q_b . Small perturbations of the stable branch of Q_b (blue curve) are stable, as indicated by the shaded blue area around the blue curve. The perturbations of the unstable branch either produce a blow-up in finite time (if $A > 1$ in the perturbations $u_0 = AQ_b$) or oscillations between two states (if $A < 1$), shaded in green below the green curve.

Remark 1.4. In dimension 1, Part I of Conjecture 1.2 is proved in Proposition 20 of [6], which is also a part of Theorem 1.1 (I.a.2). In dimensions two and higher Theorems 17, 18 of [6] address the stability either in the neighborhood of $-\lambda_1$ or for sufficiently large b . Part II, in dimension 2, was partially proved in [5], and in all dimensions partially addressed in Theorem 17(i) of [6], which is a part of Theorem 1.1(II) (‘partially’ means that it is proved either in the neighborhood of $-\lambda_1$ or for sufficiently large b , thus, there is a gap in the interval $(-\lambda_1, \infty)$, where the stability is not known.) Part III is also in part addressed in [6] and [5].

We provide confirmation for all three parts of Conjecture 1.2 in dimensions 1 and 2, in particular, showing that the standing waves are *stable* in *all* subcritical and critical cases; in the supercritical case the values of b_1 and b_2 from Theorem 1.1 coincide and are equal to the branching point b^* , i.e., $b_1 = b_2 \equiv b^*$. Furthermore, at b^* the branching of ground states occurs, which not only determines the stability, but also a different type of stable and unstable behavior, depending on the branch as discussed in Remark 1.3 (see further details in Section 4.3.2).

This paper is organized as follows: in Section 2 we review properties of the ground state, including convergence properties. In Section 3 we describe our numerical methods and then in Section 4 we show our numerical finding, first discussing the convergence and then stability, thus, providing numerical confirmations of Conjecture 1.2.

Acknowledgments. The work of C. K. and N.S. was partially supported by the ANR project ANR-17-EURE-0002 EIPHI and by the ANR project ISAAC-ANR-23-CE40-0015-01. S.R. would like to thank the hospitality and support of IMB, where most of this work was done. S.R. was partially supported by the NSF grant DMS-2452782.

2. PROPERTIES OF GROUND STATES

In this section we discuss several useful properties of ground states Q_b from (1.6).

2.1. Pokhozhaev identities. The following Pokhozhaev identities are useful for numerical verification of the ground state Q_b . The first identity is obtained by multiplying (4.24) by Q_b and integrating over B_1 ,

$$\|\nabla Q_b\|_{L^2(B_1)}^2 + b\|Q_b\|_{L^2(B_1)}^2 - \|Q_b\|_{L^{\alpha+2}(B_1)}^{\alpha+2} = 0. \quad (2.7)$$

The second identity is obtained by multiplying (4.24) by $x \cdot \nabla Q_b$ and integrating over B_1 (or in spherical coordinates multiplying by $\sigma(B_1)r^d \partial_r Q_b$, where $\sigma(B_1)$ is the surface area of the unit ball in $\mathbb{R}^d$, and then integrating in r),

$$-\frac{1}{2}\sigma(B_1)|\partial_r Q_b(1)|^2 + \frac{2-d}{2}\|\nabla Q\|_{L^2(B_1)}^2 - \frac{bd}{2}\|Q\|_{L^2(B_1)}^2 + \frac{d}{\alpha+2}\|Q\|_{L^{\alpha+2}(B_1)}^{\alpha+2} = 0. \quad (2.8)$$

Solving for $\|\nabla Q\|_{L^2(B_1)}^2$ and $\|Q\|_{L^{\alpha+2}(B_1)}^{\alpha+2}$ from (2.7) and (2.8), we obtain

$$\|\nabla Q_b\|_{L^2(B_1)}^2 = \frac{1}{\alpha(2-d)+4} (d\alpha b M(Q_b) + (\alpha+2)\sigma(B_1)|\partial_r Q_b(1)|^2), \quad (2.9)$$

$$\|Q_b\|_{L^{\alpha+2}(B_1)}^{\alpha+2} = \frac{(\alpha+2)}{\alpha(2-d)+4} (2b M(Q_b) + \sigma(B_1)|\partial_r Q_b(1)|^2). \quad (2.10)$$

Recalling the energy from (1.4), we obtain the following relation between the mass, energy and the boundary value term:

$$E[Q_b] = \frac{d\alpha-4}{2(\alpha(2-d)+4)} b M(Q_b) + \frac{\alpha}{2(\alpha(2-d)+4)} \sigma(B_1)|\partial_r Q_b(1)|^2. \quad (2.11)$$

Note that in the critical case, $\alpha = \frac{4}{d}$, the energy is

$$E[Q_b] = \frac{1}{4} |\partial_r Q_b(1)|^2 > 0, \quad (2.12)$$

which is different from the NLS on the whole space, where the ground state in the critical case has zero energy.

2.2. Changes of mass and energy with respect to b . We next record the dependence on b of energy and mass (of the ground state). Recalling the energy from (1.4) and the ground state equation, we have

$$\partial_b E(Q_b) = - \int_{B_1} (\Delta Q_b + Q_b^{\alpha+1}) \partial_b(Q_b) = -b \int_{B_1} Q_b \partial_b(Q_b) = -\frac{b}{2} M(Q_b).$$

As we track the energy and the mass of Q_b in our simulations, we also verify the above dependence, namely,

$$\partial_b E(Q_b) + \frac{b}{2} \partial_b M(Q_b) = 0, \quad (2.13)$$

e.g., see right column in Figure 8.

2.3. Two regimes of convergence of ground states. As b varies in the ground states Q_b , there are two regimes that are typically identified, as $b \rightarrow -\lambda_1$ and as $b \rightarrow \infty$ (called small and large amplitude regimes in [5]). More precisely, the following convergences hold (see Propositions 11, 12 in [6]; additionally for 2D cubic NLS see [5]).

Lemma 2.1. *Let Q_b be the least-energy (ground state) solution of (1.6) for $b \in (-\lambda_1, \infty)$, where λ_1 is the first (lowest) eigenvalue of the Laplacian in the unit ball B_1 .*

I. (Small amplitude regime.) The normalized ground states converge strongly in $H_0^1(B_1)$ to the first (normalized) eigenfunction of the Laplacian operator $(-\Delta)$ on the ball B_1 as b decreases down to $-\lambda_1$:

$$\frac{Q_b}{\|Q_b\|_{L^2}} \rightarrow \chi_1 \quad \text{as } b \rightarrow -\lambda_1,$$

provided $\|\chi_1\|_{L^2} = 1$.

II. (Large amplitude regime.) The rescaled ground states $\tilde{Q}_b(x) = b^\alpha Q_b(\frac{x}{\sqrt{b}})$ for $x \in B_{\sqrt{b}}$ and zero otherwise, converge strongly in $H^1(\mathbb{R}^d)$ to $\mathcal{R}(x)$ as $b \rightarrow \infty$, where $\mathcal{R}$ is the ground state for the NLS equation on the whole space, i.e.,

$$-\Delta \mathcal{R} + \mathcal{R} - |\mathcal{R}|^\alpha \mathcal{R} = 0, \quad x \in \mathbb{R}^d. \quad (2.14)$$

We confirm this behavior in one and two dimensions in Section 4.1. Note that the mass of Q_b is smaller than the mass of $\mathcal{R}$ in subcritical and critical cases, and is increasing monotonically to the value $M(\mathcal{R})$ as $b \rightarrow \infty$, see $M = M(b)$ dependence in Figures 8 and 10 plots (A) & (D).

3. NUMERICAL APPROACH

In this section, we briefly summarize the numerical approaches applied in this paper. These are mainly based on Chebyshev spectral methods in the spatial coordinate and the Crank-Nicolson method with a fix point iteration in time.

3.1. Numerical construction of ground states in 1D. In the 1D case, we work with $x \in [-1, 1]$ and apply standard Chebyshev collocation points $x_n = \cos(n\pi/N)$, $n = 0, 1, \dots, N$, $N \in \mathbb{N}$, see [13] for a comprehensive review of such methods. Using standard Lagrange interpolants on these collocation points and approximating the derivative of a function by the derivative of the interpolant, we get an approximation of derivatives via well known Chebyshev differentiation matrices D , see [13, 16].

With this discretisation, equation (1.6) can be approximated by a system of algebraic equations

$$(D^2 - b)\mathbf{Q} + \mathbf{Q}^{\alpha+1} = 0, \quad (3.15)$$

where $\mathbf{Q} = (Q_0, \dots, Q_N)$ with $Q_n := Q(x_n)$, $n = 0, \dots, N$. The vanishing conditions for $x = \pm 1$ are implemented as in [13] by omitting the columns and lines in (3.15) with the indices 0, N . Denoting $\tilde{\mathbf{Q}} := (Q_1, \dots, Q_{N-1})$ and $\tilde{D}_{nm}^2 = D_{nm}^2$, $n, m = 1, \dots, N-1$, we get for (3.15) the $N-1$ dimensional system

$$F := (\tilde{D}^2 - b)\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}^{\alpha+1} = 0, \quad (3.16)$$

which is solved by a standard Newton iteration. Typically, we take for values of $|b| \sim 1$ initial iterates of the form $Q = \lambda \sin(\pi(x+1)/2)$, where $\lambda > 0$ is chosen in dependence of α and b in

order to get rapid convergence. The iteration is stopped when the residual of F is smaller than some prescribed threshold, generally 10^{-10} . The resulting solution Q is then used in a tracing approach as an initial iterate for a nearby b . This allows to reach values of b on the entire considered interval $(-\lambda_1, \infty)$, i.e., very large values of b and values of b approaching $-\lambda_1$.

The choice of the number N of collocation points is controlled by the *Chebyshev coefficients*: an analytic function can be expanded in terms of Chebyshev polynomials $T_n(x) = \cos(n \arccos(x))$, $n = 0, 1, \dots$, and be approximated by a sum

$$Q(x) \approx \sum_{n=0}^N a_n T_n(x). \quad (3.17)$$

The *Chebyshev coefficients* can be determined by a collocation method, i.e., the condition

$$Q(x_k) = \sum_{n=0}^N a_n T_n(x_k), \quad k = 0, \dots, N. \quad (3.18)$$

The coefficients a_n can be computed with a *Fast Cosine transform*, see [13]. It is known that they decrease exponentially with the index for analytic functions. The numerical error is controlled by the coefficients with the largest values of n . Since the ground states Q are expected to be analytic, we always choose N large enough that $|a_N|$ is on the order of machine precision, here, roughly 10^{16} . The numerical error in approximating smooth functions thus decreases exponentially with N which is called *spectral convergence*.

An advantage of the use of Chebyshev collocation points is that the integrals can be approximated on these points by the Clenshaw-Curtis algorithm [3]:

$$\int_{-1}^1 f(x) dx \approx \sum_{n=0}^N f(x_n) w_n, \quad (3.19)$$

where the w_n , $n = 0, \dots, N$, are known weights, see [13]. The Clenshaw-Curtis algorithm is once more a spectral method.

3.2. Numerical construction of ground states in 2D. In the 2D case, a similar approach is applied for the radial coordinate. We write equation (1.6) in the form

$$Q_{rr} + \frac{1}{r} Q_r - bQ + Q^{\alpha+1} = 0, \quad (3.20)$$

and introduce as in [1] the coordinate $s = r^2$ to obtain a less singular form of the equation (3.20),

$$4sQ_{ss} + 2Q_s - bQ + Q^{\alpha+1} = 0. \quad (3.21)$$

As in the 1D case we apply a Chebyshev collocation method, i.e., we discretize $s \in [0, 1]$ via $s_n = (1 + \cos(n\pi/N))$, $n = 0, \dots, N$. The rest of the approach is as in the 1D case.

We verify our computation of the 2D ground states via the virial and energy identities, as discussed in Section 2.1 in (2.9)-(2.11). In particular, we compute the following errors for the ground states in the cubic $\alpha = 2$ and quintic $\alpha = 4$ cases:

$$\mathcal{E}_1(b) := \|\partial_r Q_b\|_{L^2(B_1)}^2 - \frac{\alpha b}{2} M(Q_b) - \frac{\pi(\alpha+2)}{2} |\partial_r Q_b(1)|^2,$$

$$\mathcal{E}_2(b) := E(Q_b) - \frac{(\alpha - 2)b}{4} M(Q_b) - \frac{\pi \alpha}{4} |\partial_r Q_b(1)|^2.$$

For example, for $b = 1$ when the ground state Q_b is computed up to the resolution of 10^{-9} , we obtain that both errors $\mathcal{E}_1$ and $\mathcal{E}_2$ produce the values of the order 10^{-11} for the $\alpha = 2$ case and 10^{-12} for the $\alpha = 4$ case. For other values of b we find similar precision.

3.3. Time integration. Applying the discretization above in x (and similarly for s), we get for the NLS equation (1.1) an $N - 1$ dimensional system of the form

$$\tilde{u}_t = i\tilde{D}^2\tilde{u} + i|\tilde{u}|^\alpha\tilde{u}. \quad (3.22)$$

We discretise also in time, $t = 0, \dots, t_n, \dots, T$ where T is the final time, and where $N_t > 0$ time steps are used. For simplicity we consider all time steps to be of equal length $h = T/N_t$. The system (3.22) is integrated with the Crank-Nicolson, i.e., we integrate equation (3.22) from t_n to t_{n+1} and apply on the right-hand side the trapezoidal rule. This leads to

$$(1 - \frac{h}{2} i\tilde{D}^2)\tilde{u}(t_{n+1}) = (1 + \frac{h}{2} i\tilde{D}^2)\tilde{u}(t_n) + i\frac{h}{2} |\tilde{u}(t_{n+1})|^\alpha\tilde{u}(t_{n+1}) + i\frac{h}{2} |\tilde{u}(t_n)|^\alpha\tilde{u}(t_n). \quad (3.23)$$

The system (3.23) is solved for $\tilde{u}(t_{n+1})$ with a simplified Newton iteration, i.e., in each iteration not the full Jacobian is inverted, but the operator $(1 - \frac{h}{2} i\tilde{D}^2)$.

The accuracy of the time integration is controlled via the conservation of the numerically computed energy. Since the latter will depend on time due to unavoidable numerical errors, it can be used to control the accuracy. As discussed in [8], the relative conservation of the energy typically overestimates the accuracy by 2-3 orders of magnitude. Except in cases with a blow-up (where we stop the computation when the relative energy becomes larger than 10^{-3}), we choose the time steps in a way that this quantity is always smaller than 10^{-6} .

4. NUMERICAL RESULTS

We start with showing convergence of ground states as $b \rightarrow \infty$ and as $b \rightarrow -\lambda_1$ as stated in Lemma 2.1. Then we discuss stability of ground states and confirm Conjecture 1.2 in dimensions 1 and 2.

4.1. Convergence as $b \rightarrow \infty$. Recall that Q_b satisfies

$$-\Delta Q + bQ - |Q|^\alpha Q = 0, \quad \text{for } x \in B_1, \quad \text{and } Q|_{\{|x|=1\}} = 0. \quad (4.24)$$

To investigate the large amplitude regime convergence, we define a rescaled $\tilde{Q}_b$ by

$$Q_b(x) \stackrel{\text{def}}{=} b^{1/\alpha} \tilde{Q}_b(\sqrt{b}x). \quad (4.25)$$

Then $\tilde{Q}_b$ satisfies

$$-\Delta \tilde{Q} + \tilde{Q} - |\tilde{Q}|^\alpha \tilde{Q} = 0, \quad \text{for } y \in B_{\sqrt{b}} \quad \text{and } \tilde{Q}|_{\{|y|=\sqrt{b}\}} = 0. \quad (4.26)$$

Notice that there is no b in front of the second term in (4.26), instead the domain of $\tilde{Q}_b$ depends on $\sqrt{b}$ and expands to the whole space as b grows to infinity, thus, giving justification to the convergence of the rescaled ground states $\tilde{Q}_b$ (which are extended to the whole space by zero)

to $\mathcal{R}$ as b grows: $\tilde{Q}_b(x) \rightarrow \mathcal{R}(x)$ as $b \rightarrow \infty$, where $\mathcal{R}$ is the ground state on the whole space $\mathbb{R}^d$, solving (2.14).

We note that, in particular, in 1D the ground state $\mathcal{R} = \mathcal{R}_\alpha$ is explicit

$$\mathcal{R}_\alpha(x) = \left(\frac{\alpha+2}{2}\right)^{\frac{1}{\alpha}} \operatorname{sech}^{\frac{2}{\alpha}}\left(\frac{\alpha}{2}x\right), \quad (4.27)$$

see profiles in the left plot of Figure 2. The mass can also be computed explicitly,

$$M(\mathcal{R}_\alpha) = \frac{2}{\alpha} \left(\frac{\alpha+2}{2}\right)^{\frac{2}{\alpha}} \int_{-\infty}^{\infty} \operatorname{sech}^{\frac{4}{\alpha}} x \, dx = \frac{2\sqrt{\pi}}{\alpha} \left(\frac{\alpha+2}{2}\right)^{\frac{2}{\alpha}} \frac{\Gamma(\frac{2}{\alpha})}{\Gamma(\frac{2}{\alpha} + \frac{1}{2})}, \quad (4.28)$$

values of which are decreasing, see the right plot in Figure 2. Note that both $M(\mathcal{R}_\alpha) \rightarrow 1$ and $\|\mathcal{R}_\alpha\|_{L^\infty} \rightarrow 1$ as $\alpha \rightarrow \infty$.

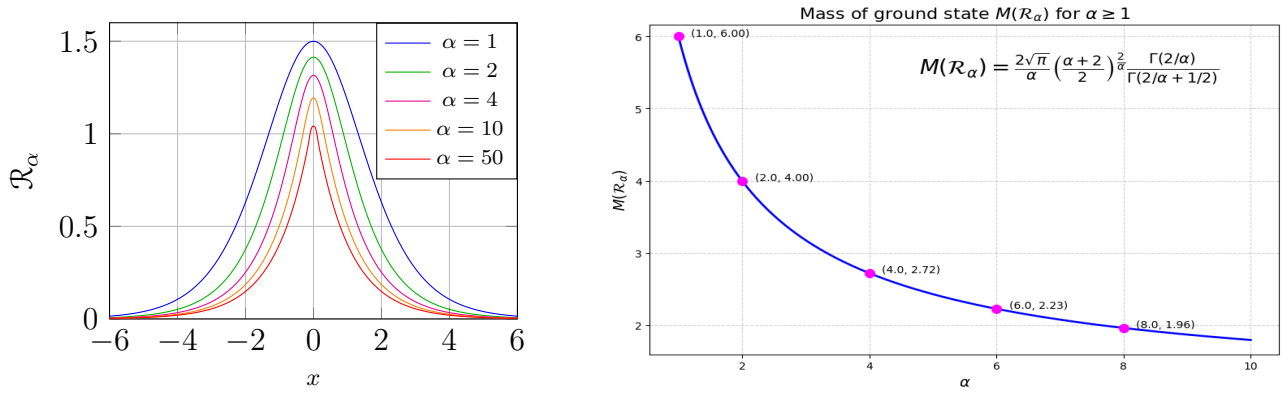


FIGURE 2. The profiles of $\mathcal{R}_\alpha$ for various values of α (left) and mass $M(\mathcal{R}_\alpha)$ (right).

For numerical purposes, since we compute on a finite interval or domain (namely, $[-1, 1]$ or $|r| \leq 1$), conversely, we rescale from $x \in [-\sqrt{b}, \sqrt{b}]$ back to $[-1, 1]$ via $\tilde{Q}_b(x) = b^{-1/\alpha} Q_b(\frac{x}{\sqrt{b}})$, and thus, Q_b is defined on $[-1, 1]$ and solves (4.24). Similarly, we rescale the ground state $\mathcal{R}$ to $\tilde{\mathcal{R}}$ as

$$\tilde{\mathcal{R}}(y) = b^{-1/\alpha} \mathcal{R}\left(\frac{y}{\sqrt{b}}\right). \quad (4.29)$$

In Figure 3 we show convergence in 1D of Q_b as b becomes large to the rescaled explicit ground

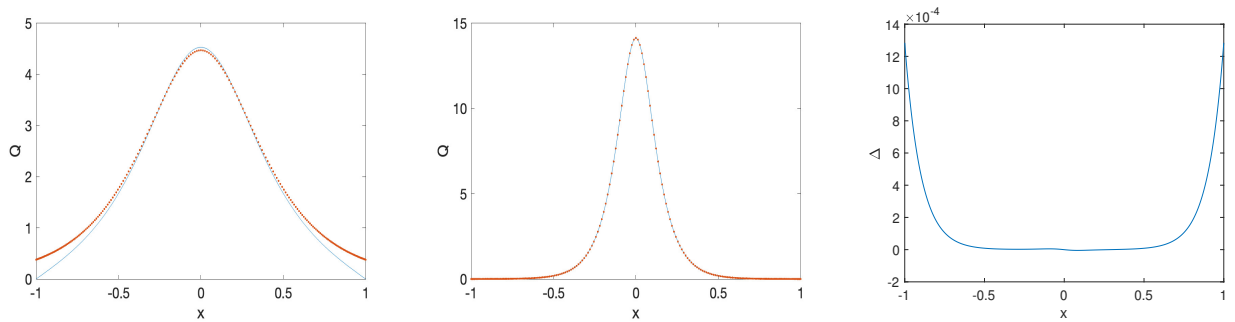


FIGURE 3. 1D cubic NLS. Convergence of the rescaled ground states Q_b (solid blue) to the 1D ground state $\mathcal{R} = \sqrt{2} \operatorname{sech} x$ (dotted red), rescaled to the interval $[-1, 1]$ as in (4.29), as $b \rightarrow \infty$: $b = 10$ (left), $b = 100$ (middle), difference for $b = 100$ (right).

state $\tilde{\mathcal{R}}$ for $b = 10$ and $b = 100$. While there is some difference for $b = 10$, especially on the edges, for $b = 100$ the difference is not visible and the error is plotted on the right of Figure 3, which is on the order of 10^{-4} (at the edges) or smaller.

Similarly, Figure 4 shows the convergence in 2D of Q_b as b becomes large to the rescaled (numerically computed) ground state $\tilde{\mathcal{R}}$ for $b = 10$ and $b = 100$. On the right plot notice the error (the difference) on the order 10^{-4} for $b = 100$.

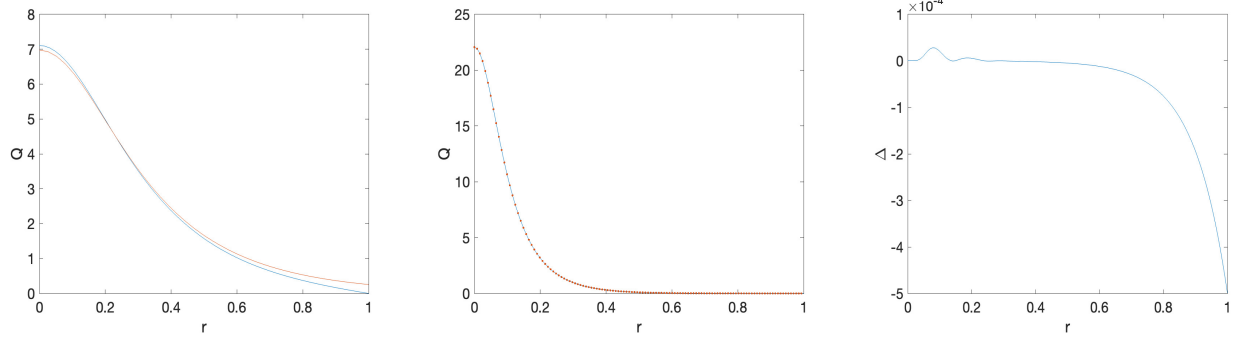


FIGURE 4. 2D cubic NLS. Convergence of the rescaled ground states Q_b (solid blue) to the 2D ground state $\tilde{\mathcal{R}}$ (computed numerically and rescaled to $|r| \leq 1$ as in (4.29), dotted red) as $b \rightarrow \infty$: $b = 10$ (left), $b = 100$ (middle), difference for $b = 100$ (right).

4.2. Convergence as $b \rightarrow -\lambda$. We next investigate the convergence of ground states in the small amplitude regime, in other words, as b decreases.

4.2.1. 1D case. Recall that in 1D the Laplace operator is $-\Delta = -\frac{d^2}{dx^2}$, and for $k \in \mathbb{N}$ the eigenvalues and the corresponding eigenfunctions, $-\Delta \chi_k(x) = \lambda_k \chi_k(x)$, $x \in [-1, 1]$, are

$$\lambda_k = \left(\frac{\pi k}{2}\right)^2, \quad \chi_k(x) = \sin\left(\frac{\pi k}{2}(x-1)\right).$$

We consider subcritical and critical cases, $\alpha = 2$ and $\alpha = 4$, respectively, and show the

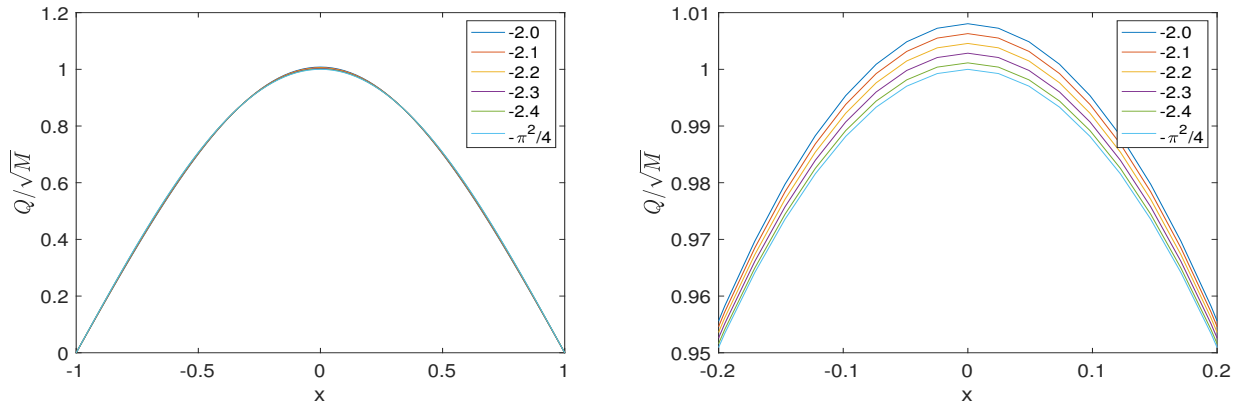


FIGURE 5. 1D cubic NLS (subcritical): convergence of normalized ground states to χ_1 as $b \rightarrow -\lambda_1$, which is indicated by the value $-\pi^2/4$ (light blue); zoom-in of the peak on the right.

convergence of normalized Q_b in each case. In Figures 5 and 6 we show that $\frac{Q_b}{\|Q_b\|_{L^2([-1,1])}}$ numerically converges to the eigenfunction $\chi_1(x) = \sin\left(\frac{\pi}{2}(x-1)\right)$ as $b \rightarrow -\lambda_1 = -\left(\frac{\pi}{2}\right)^2$ (note that $\|\chi_1\|_{L^2([-1,1])} = 1$). On the left of Figure 5 one can see the convergence of the subcritical (cubic NLS) ground states on the whole interval $[-1, 1]$, and on the right, the enlargement of the peak, where the convergence is shown for several values. The limiting profile of χ_1 is the lowest curve indicated by the blue line with the eigenvalue $-\pi^2/4$.

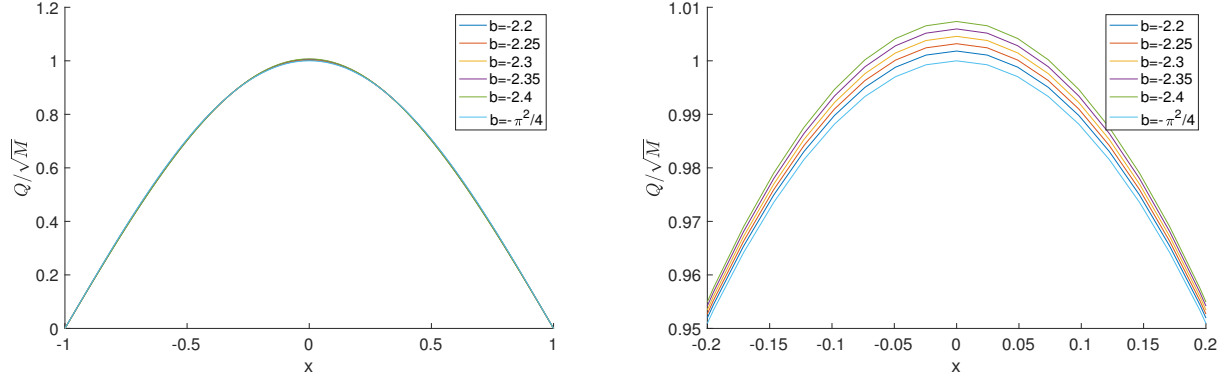


FIGURE 6. 1D quintic NLS (critical): convergence of normalized ground states to χ_1 as $b \rightarrow -\lambda_1$, which is indicated by the value $-\pi^2/4$ (light blue); zoom-in of the peak on the right.

A similar convergence holds in the critical, quintic case, see Figure 6. We have also checked the supercritical cases, and observed a similar convergence.

4.2.2. 2D case. In 2D the Laplacian operator is $-\Delta = -(\partial_{xx} + \partial_{yy}) \equiv -(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta})$. The eigenvalues and eigenfunctions of the Laplacian operator on a unit ball in 2D, $-\Delta u = \lambda u$, can be written in polar coordinates as $u(r, \theta) = R(r)Y(\theta)$, where the radial function R is expressed via the Bessel functions J_m and their set of zeros $\{k_{m,n}\}$, and $Y(\theta) = \cos(m\theta)$ or $\sin(m\theta)$, namely,

$$u_{m,n}(r, \phi) = J_m(k_{m,n}r) \begin{cases} \cos(m\theta) \\ \sin(m\theta) \end{cases} \quad \text{and} \quad \lambda_{m,n} = (k_{m,n})^2.$$

Thus, a full solution can be written as

$$u(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{m,n}r) (A_{m,n} \cos(m\theta) + B_{m,n} \sin(m\theta)).$$

For this work, we only need the smallest eigenvalue, which is known to be $\lambda_{01} = (k_{01})^2 \approx (2.4048)^2 \approx 5.783$. By Lemma 2.1(I), in 2D we have $\frac{Q_b}{\|Q_b\|_{L^2(\mathbb{R}^2)}}$ converge to the normalized first eigenfunction $\frac{\chi_1}{\|\chi_1\|_{L^2}}$, where $\chi_1 = J_0(k_{01}r)$, as $b \rightarrow -\lambda_1 = -(k_{01})^2 \approx -(2.404825557696086)^2$ (note that for computational purposes, a very high precision of the constant k_{01} is needed).

In the 2D cubic case ($\alpha = 2$) in Figure 7 we show the convergence of normalized Q_b to normalized χ_1 as b goes to $-(k_{01})^2$. We take a decreasing sequence by 0.1 starting from $b = -5.35$ and show its convergence to the normalized first eigenfunction, indicated in Figure 7 in light blue via the value -5.78 .

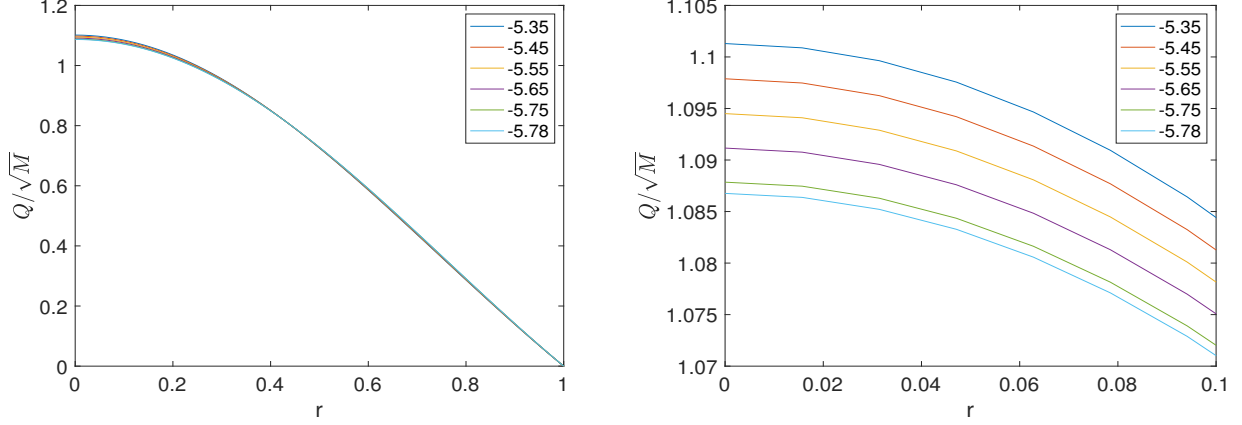


FIGURE 7. 2D cubic NLS: convergence of the 2D normalized ground states Q_b (with b as labeled) to the normalized $J_0(k_{0,1}r)$, labeled by $-(k_{0,1})^2 \approx -5.78$ in light blue.

4.3. Stability of solitary waves. In this section we confirm Theorem 1.1 and investigate stability further, to confirm Conjecture 1.2.

4.3.1. Mass and energy as functions of b . We start with checking how mass $M(Q_b)$ and energy $E(Q_b)$ of the ground state behave with respect to b . Recalling that $M(Q_b)$ grows monotonically

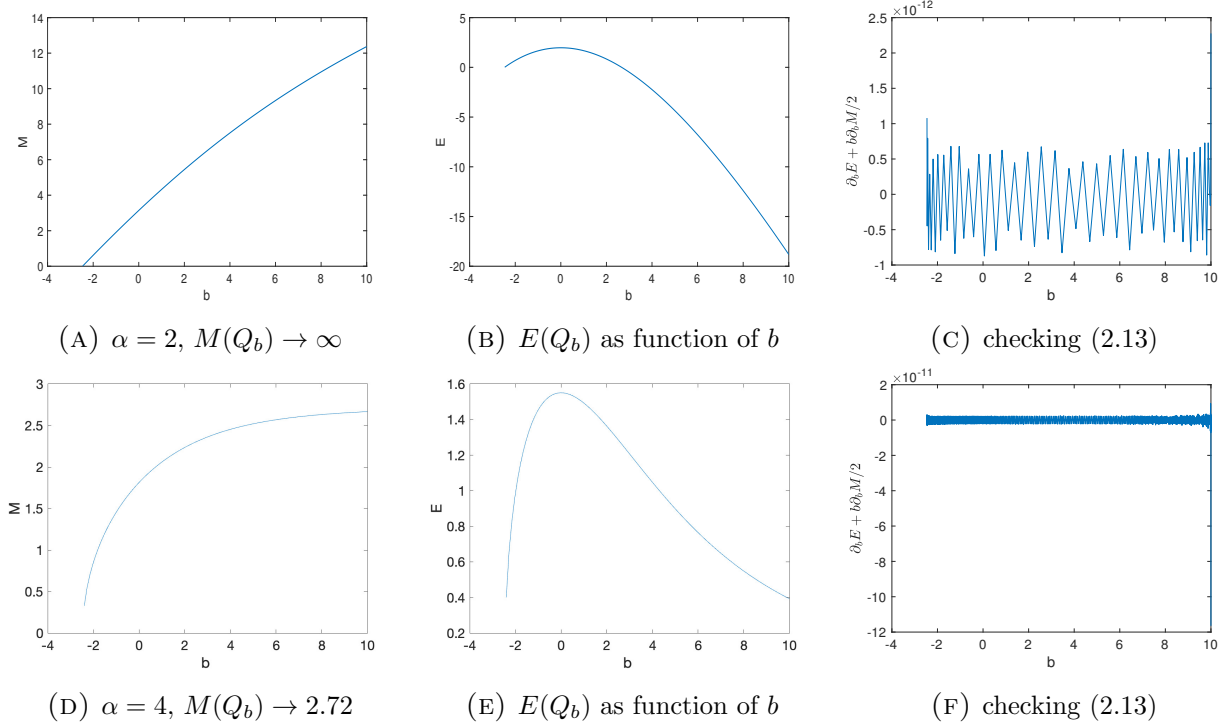


FIGURE 8. 1D NLS: subcritical, $\alpha = 2$ (top) and critical, $\alpha = 4$ (bottom). Mass and energy dependence on b (left and middle columns). Verification of (2.13), which is at least on the order of 10^{-11} (right column).

to $M(\tilde{\mathcal{R}}_\alpha) = b^{\frac{\alpha-4}{2\alpha}} M(\mathcal{R}_\alpha)$ in 1D (or equivalently, $M(\tilde{Q}_b) \rightarrow M(\mathcal{R}_\alpha)$ by Lemma 2.1 (II)), in the

critical case ($\alpha = 4$), the mass of Q_b approaches the mass of $M(\mathcal{R}_4) \approx 2.72$, see (4.28), Fig. 2 and as depicted on the left of Fig. 1; in the subcritical case it grows to the infinity, as one can see in (A) of Fig. 8 (and it goes to zero in the supercritical case as seen in Fig. 9 left column).

In the middle column of Figure 8 we show how energy changes with respect to b and in the right column we provide verification of how the slope of the mass is connected with the slope of energy in (2.13), which are on the order of 10^{-11} or better. In fact, we checked the identity (2.13) in all other computed cases and it holds at least to the same accuracy.

In Figure 9 we consider 1D supercritical cases and show behavior of the mass and energy on b in the left and middle columns, noting that the mass is no longer monotonic: it initially grows and reaches the maximum and then decays. Therefore, we also plot dependence of the energy vs. mass in the right column and observe the appearance of two *branches* in the graph. This means that for the same mass there are **two** ground state solutions with different values of b , and thus, the one with the lower energy should be the *stable* ground state, while the other one would be *unstable*.

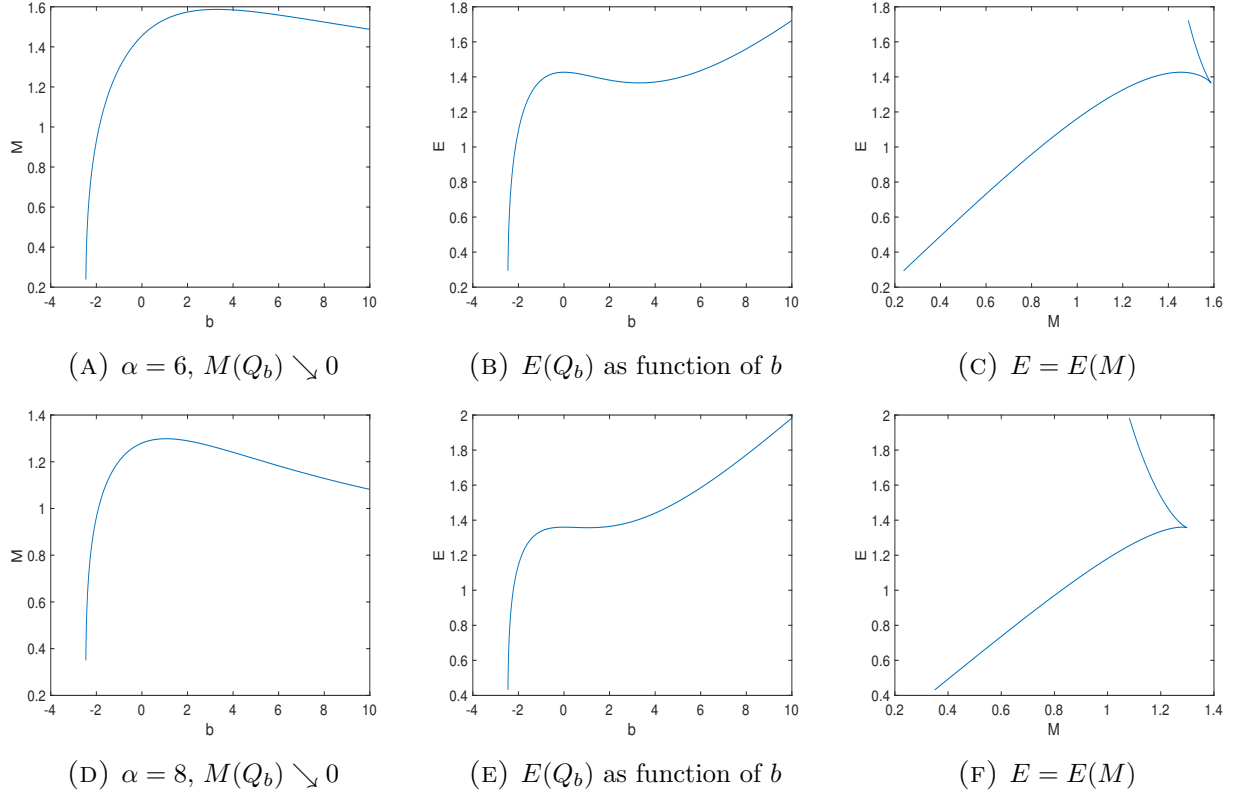


FIGURE 9. 1D NLS, supercritical cases: $\alpha = 6$ (top) and $\alpha = 8$ (bottom). Mass and energy dependence on b (left and middle columns). Dependence of energy as a function of mass, $E = E(M)$ (right column). The maximum values of the mass as a function of b are reached at $b^* \approx 3.3117$ (for $\alpha = 6$) and $b^* \approx 1.0718$ (for $\alpha = 8$). This is the point where the split into two branches occurs in the right column.

We next examine the two dimensional case, and compare the subcritical ($\alpha = 1$), critical ($\alpha = 2$) and supercritical ($\alpha = 4$) cases, see Figure 10. We observe that in all 2D subcritical and critical cases, the mass $M(Q_b)$ as a function of b monotonically increases, see left and

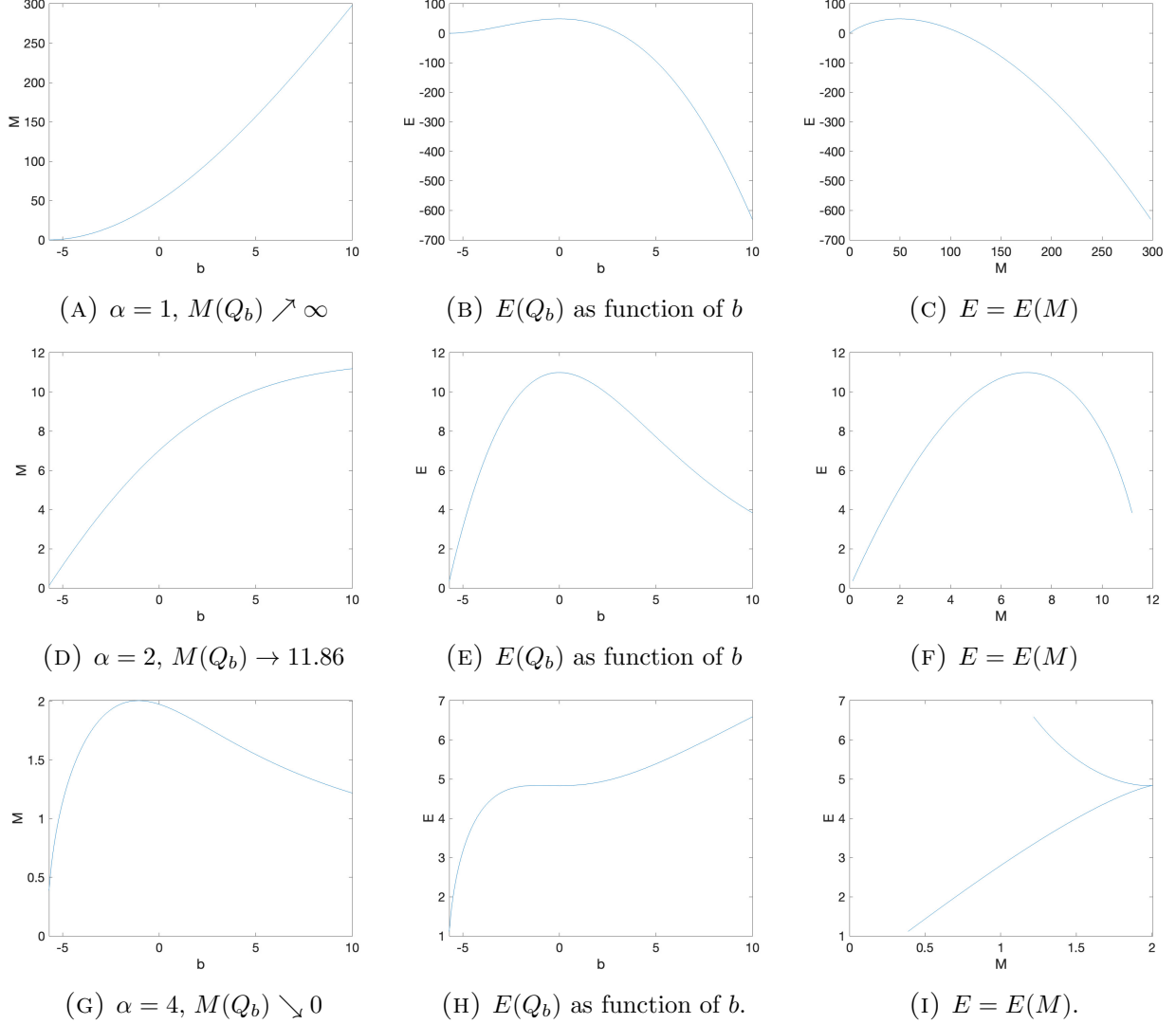


FIGURE 10. 2D NLS: $\alpha = 1$ (top), $\alpha = 2$ (middle) and $\alpha = 4$ (bottom). Mass and energy dependence on b (left and middle columns). Dependence of energy as a function of mass, $E = E(M)$ (right column). The maximum value of the mass as a function of b is reached at $b^* \approx -1.0311$ only in the supercritical case ($\alpha = 4$). This is the point where the split into two branches occurs in the right bottom column.

middle plots in Figure 10, and in the critical case it asymptotically approaches the value of the mass of the whole space ground state $M(\mathcal{R}) \approx 11.86$. In the supercritical case, similarly to the 1D case, the mass reaches the maximum value at $b = b^*$ and then decreases to zero, thus, creating branching in the energy vs. mass graph, see the right bottom plot in Figure 10. We investigate the stability of different branches in the next subsection.

The critical 2D cubic NLS case was initially studied in [5]; we confirm, in particular, that our computations in the middle row ($\alpha = 2$) for mass and energy are in agreement with those computed in [5, Fig. 1 and Fig. 2] up to the factor 2π (which we include in our computations) and corresponding scaling in the energy by a factor $\frac{1}{2}$.

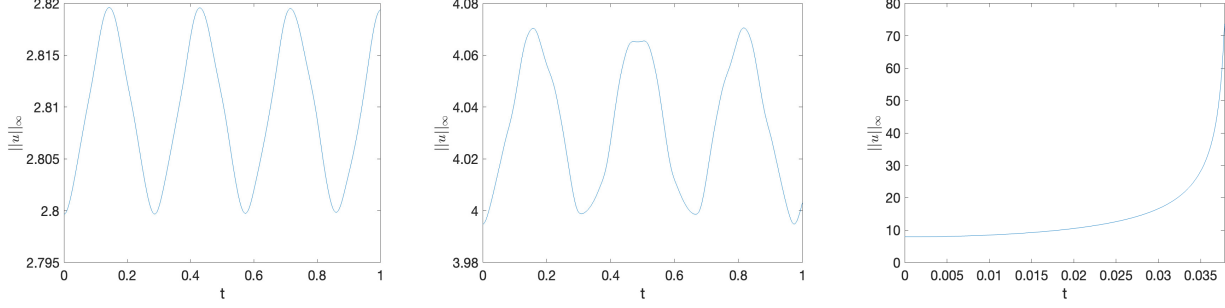


FIGURE 11. 2D cubic NLS: perturbations $u_0 = 1.01Q_b$ with $b = -2.5$ (left) and $b = 1$ (middle); solution behavior with $\|u_0\|_{L^2} > \|\mathcal{R}\|_{L^2} \approx 11.86$ in L^∞ norm (right).

To explain further the behavior of solutions in the critical case, we consider perturbations of the ground state Q_b for various b values as well as solutions with the initial data above the mass of the ground state on the whole space $M(\mathcal{R})$, which we schematically indicated in Figure 1, left plot. In Figure 11 we show that in the 2D cubic NLS small perturbations of Q_b are stable (oscillations in the second or third digit for the perturbations with $A = 1.01$). In fact, we observed that for smaller b , the perturbations (in amplitude A) do not have to be necessarily small, while for larger b , and especially as b increases and the mass of ground states approaches the mass of $\mathcal{R}$, the perturbations of the initial data should be getting smaller, so the mass of the initial data would not become higher than $M(\mathcal{R})$. Above $M(\mathcal{R})$ solutions blow up even with a different type of initial condition, for an example see the right plot in Figure 11.

4.3.2. *Supercritical case: two branches of ground states.* In the supercritical case we consider small perturbations of ground states, of type

$$u_0 = AQ_b \quad \text{with} \quad A \sim 1.$$

For that we take values of b on both sides of b^* to compare dynamics of the perturbations; for A we set either $A = 0.99$ or $A = 1.01$. In Figure 12 in plots (A) and (B) we show the

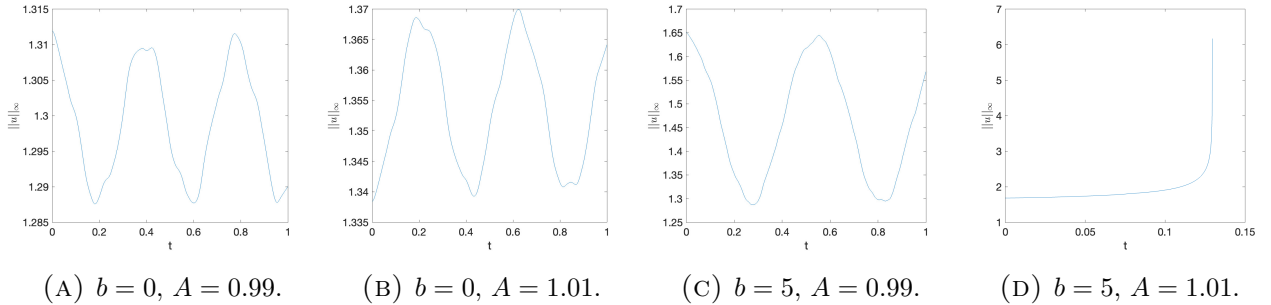


FIGURE 12. 1D septic NLS, $\alpha = 6$. (A) & (B): stable branch, $b = 0 < b^* \approx 3.3$, $u_0 = AQ_0$. (C) & (D): unstable branch, $b = 5 > b^* \approx 3.3$, $u_0 = AQ_5$. The values of A as indicated.

perturbations of the stable branch in 1D septic NLS, $\alpha = 6$, with $b = 0 < b^* \approx 3.3$. Note that both perturbations have tiny oscillations (in the second or third digit) around the Q_b itself. On the other hand, in plots (C) and (D) one can see perturbations of different kind (there we take

$b = 5 > b^*$): solutions with $A = 1.01$ blow up in finite time, while the initial data $u_0 = 0.99 Q_b$ oscillates more drastically (e.g., in (C) between 1.25 and 1.65), indicating that the oscillations are not around one state but rather between two different states. This is indeed the case, as we investigate further in Figure 13, concluding that the branch with $b > b^*$ is unstable.

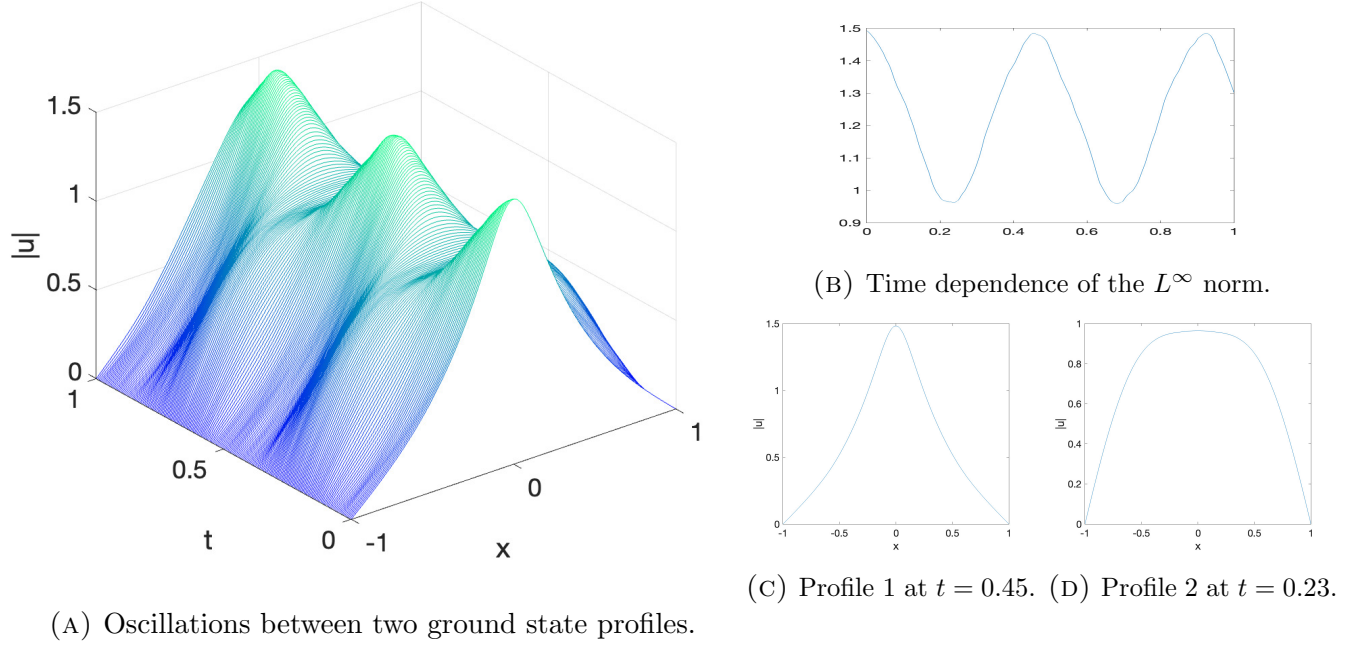


FIGURE 13. 1D nonic NLS, $\alpha = 8$: oscillations of the unstable branch of Q_b with $b = 5 > b^*$, $u_0 = 0.99Q_5$, between the unstable (profile 1) and stable (profile 2) states.

In Figure 13 we take 1D nonic NLS, $\alpha = 8$, and consider the unstable branch with $b = 5 > b^* \approx 1.07$. On the left plot (A) the solution behavior with the initial condition $u_0 = 0.99Q_5$ is plotted, which clearly shows oscillation between two states. The L^∞ norm is plotted in (B) and profiles of the two states are given in (C) and (D). Note that since the initial condition has mass less than the ground state Q_b , the solution starts to disperse, however, the Dirichlet boundary conditions do not allow dispersion and reflect the solution so that it concentrates into some more stable profile (D). The height and the mass of this state (D) is below the stable branch ground state, so it starts increasing but passes it and reaches back the unstable profile (C), continuing then this behavior periodically.

Considering the two dimensional supercritical cases, we show in Figure 14 perturbations of the ground states Q_b in the 2D quintic NLS, $\alpha = 4$. Note that perturbations of the stable branch in plots (A) and (B) are around the ground state itself with the difference in the second digit for either $A = 0.99$ or $A = 1.01$ amplifications of Q_b (here $b = -2.5 < b^* \approx -1.03$, while in plot (C) the perturbation of the unstable branch ($b = 5 > b^*$) with $A = 0.99$ clearly oscillate between two different states and with $A = 1.01$ blow up in finite time, see plot (D).

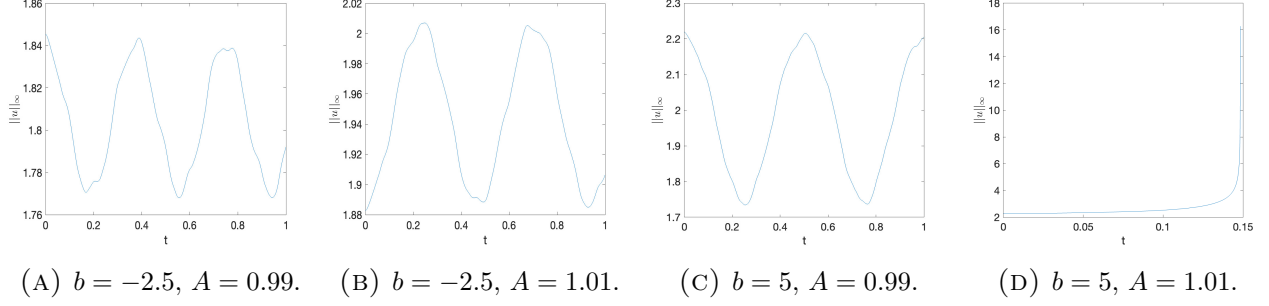
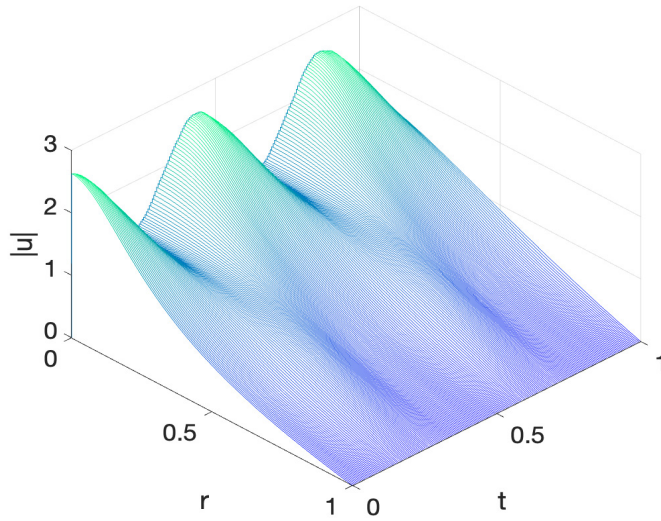
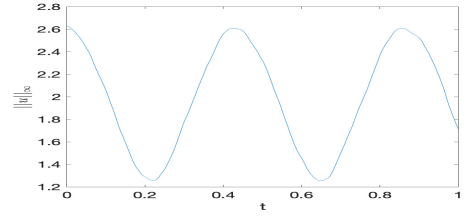


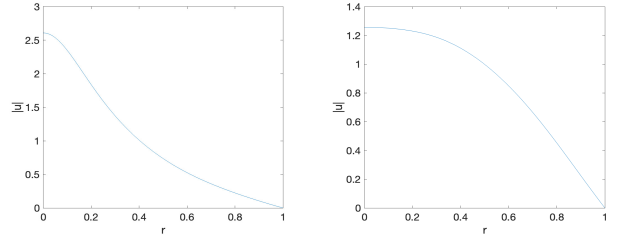
FIGURE 14. 2D quintic NLS, $\alpha = 4$. (A) & (B): stable branch, $b = -2.5 < b^* \approx -1.07$, $u_0 = AQ_{-2.5}$. (C) & (D): unstable branch, $b = 5 > b^*$, $u_0 = AQ_5$. The values of A as indicated.



(A) Oscillations between two ground state profiles.



(B) Time dependence of the L^∞ norm.



(C) Profile 1 at $t = 0.42$. (D) Profile 2 at $t = 0.21$.

FIGURE 15. 2D quintic NLS: oscillations of the unstable branch of Q_b with $b = 2 > b^* \approx -1.03$, $u_0 = 0.99Q_2$, between the unstable (profile 1) and stable (profile 2) states.

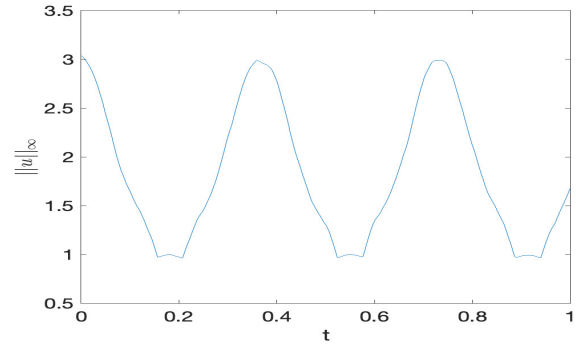
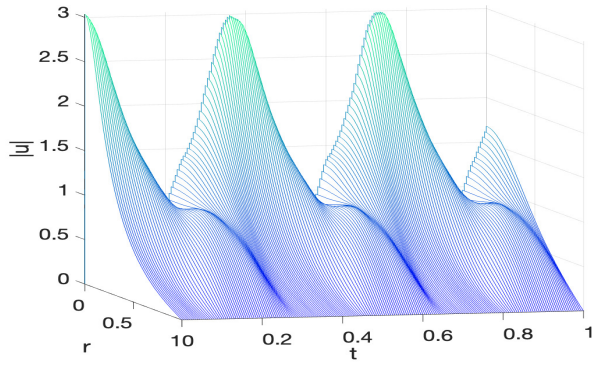


FIGURE 16. 2D quintic NLS. Unstable branch, $b = 5 > b^*$, $u_0 = 0.99Q_5$. Compare oscillations with the case $b = 2$.

The oscillation of the unstable branch between two states are shown in Figures 15 for $b = 2$ and in Figure 16 for $b = 5$, this allows comparison of the oscillations for different values of b on the unstable branch.

The radial profiles of the two states are given in the case of $b = 2$ in plots (C) and (D) of Figure 15. One may notice that the larger b is, the bigger oscillations are between the two states, and it might be that there are more states between which solutions could oscillate.

Finally, we conclude with an example of the solution behavior for small initial data to confirm that no dispersion occurs in this problem, as any solution resolves asymptotically into a sum of coherent structures, in this example, it oscillates around a small soliton (recall that the mass of ground states Q_b cover the entire interval $(0, M(\mathcal{R}))$, see Figure 17.

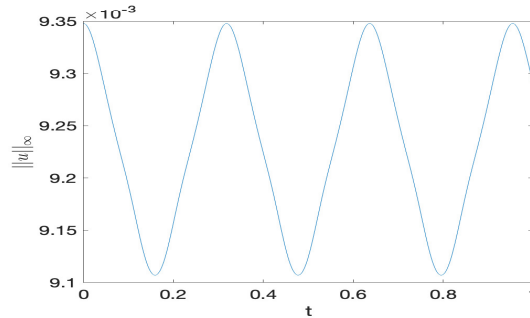


FIGURE 17. 1D quintic NLS, $\alpha = 4$. Small amplitude initial data $u_0 = 0.01Q_{-2}$, $b = -2$, resolves into an oscillation around a small soliton, confirming the soliton resolution.

5. CONCLUSIONS

In this paper, we presented a study of ground state (the least energy) solutions to the non-linear Schrödinger equation on a unit ball in one and two dimensions with Dirichlet boundary condition, which has a stabilizing effect on solutions, in particular, the standing wave or ground state solutions behavior. We confirmed that the ground state is stable in subcritical and critical cases in 1D, and also showed that it is stable in two dimensions for any parameter b which comes from the standing wave solution $e^{ibt}Q_b(x)$, thus, closing the gap in Theorem 1.1 Part I (a.1). Furthermore, in the critical case, we showed that the mass of the ground state $M(Q_b)$ increases to that of the ground state for the whole space, $\mathcal{R}$. While the ground states Q_b are stable under small perturbations in the critical case, this is the case as long as the mass of the initial perturbation is below the mass of $\mathcal{R}$, above which solutions blow up in finite time.

In the supercritical case, we found a new phenomenon of *branching* of ground states, which splits the ground states into a stable and an unstable branch, depending on the size of the parameter b . Perturbations of ground states on a stable branch leave the solutions oscillating around that ground state, while perturbations of the unstable branch force solutions either to blow-up in finite time (if perturbations have an amplitude larger than the solitary state) or oscillate between stable and unstable states (if perturbations have an amplitude smaller than the original ground state). We also observe that this equation does not have any scattering or

radiation, and thus, the soliton resolution holds for all data, splitting any solution into coherent structures such as solitons even for very small initial data.

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