

Regularized products of Gauss and Eisenstein integers and primes

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Abstract. We provide heuristic computations à la Euler of the regularized infinite products of Gauss and Eisenstein integers and primes. Our approach, yielding explicit expressions, is inspired by the work by Muñoz García and Pérez-Marco, who evaluated the product of all natural primes to $4\pi^2$.

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1. Introduction and summary of results

Infinite products of integers are obviously divergent. A broad class of such products may however be given a meaning, i.e., attributed a well-defined finite value, by means of the zeta-regularization approach. This technique dates back to Euler, long before it was given a firm basis in terms of analytical continuation (see [1, 2, 3, 4] for historical accounts, and e.g. [5, 6, 7, 8, 9, 10, 11, 12] for modern references). In the physics literature, zeta-regularized infinite products are often referred to as renormalized products. A prominent instance consists in the determinants, i.e., products of eigenvalues, of various operators arising in quantum mechanics and quantum field theory (see [13] for an overview).

The present note is directly inspired by the work by Muñoz García and Pérez-Marco [14, 15], summarized in section 2, where the product of natural primes was evaluated to

$$\Pi = 4\pi^2. \quad (1)$$

This outcome may be put in perspective with the famous Euler formula for the product of all natural integers,

$$P = \sqrt{2\pi}. \quad (2)$$

We thus have

$$\Pi = P^4. \quad (3)$$

Hereafter we provide heuristic computations à la Euler of the regularized infinite products of Gauss and Eisenstein integers, forming the rings $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, with $\omega = e^{2\pi i/3}$, and of the associated primes. These examples are the most evident generalizations of the natural integers and primes. They are also related to the simplest quantum billiards whose spectra are integrable, namely the square and the equilateral triangle [16, 17]. Our main outcomes are as follows. For Gauss integers (see section 3), we recover that the product P_4 of all non-zero Gauss integers is such that

$$|P_4| = \frac{\Gamma(1/4)^2}{2\sqrt{\pi}} \approx 3.708149, \quad (4)$$

and find that the product Π_4 over Gauss primes obeys

$$|\Pi_4| = |P_4|^8. \quad (5)$$

For Eisenstein integers (see section 4), we find that the product P_3 of all non-zero Eisenstein integers is such that

$$|P_3| = \frac{3^{1/4} \Gamma(1/3)^3}{2\pi} \approx 4.027065, \quad (6)$$

and that the product Π_3 over Eisenstein primes obeys

$$|\Pi_3| = |P_3|^{12}. \quad (7)$$

The striking resemblance between (3), (5) and (7) suggests the existence of a more general underlying property, of which we are unaware.

2. A reminder on the products of natural integers and primes

We begin with a reminder on the products of natural (positive) integers and primes.

2.1. Product of natural integers

The product of all natural integers,

$$P = \prod_{n \geq 1} n, \quad (8)$$

can be regularized by relating it to the Riemann zeta function,

$$\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}. \quad (9)$$

The identity between both rightmost sides is known as the Euler product formula. It expresses that every natural integer n can be decomposed in a unique way as a product of primes. We have formally

$$P = \exp \sum_{n \geq 1} \ln n = e^{-\zeta'(0)}. \quad (10)$$

This expression is the gist of the zeta-regularization, consisting in regularizing divergent products such as (8) by means of analytic continuation. The series and product entering (9) converge for $\operatorname{Re} s > 1$ and can be analytically continued to a meromorphic function in the whole complex s -plane, with a simple pole with unit residue at $s = 1$. In particular, $\zeta(s)$ is analytic in a neighborhood of $s = 0$. We have

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi. \quad (11)$$

Inserting the second of these expressions into (10), we obtain the famous Euler formula

$$P = \prod_{n \geq 1} n = \infty! = \sqrt{2\pi}, \quad (12)$$

sometimes also attributed to Riemann.

2.2. Product of natural primes

The product of all natural primes,

$$\Pi = \prod_{p \text{ prime}} p, \quad (13)$$

has been evaluated by Muñoz García and Pérez-Marco, first by a heuristic computation à la Euler [14], and then rigorously [15]. These investigations were motivated by a question raised in [8, p. 101]. Remarkably enough, their approach circumvents the obstruction that the prime zeta function $Z(s)$, to be introduced in (16), has a natural boundary along the whole imaginary axis [18], and is therefore not analytic in a neighborhood of $s = 0$.

The heuristic derivation exposed in [14] starts with the Artin-Hasse identity for the exponential function:

$$e^x = \prod_{n \geq 1} (1 - x^n)^{-\mu(n)/n}, \quad (14)$$

where μ is the Möbius function:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0 & \text{else.} \end{cases} \quad (15)$$

The prime zeta function,

$$Z(s) = \sum_{p \text{ prime}} p^{-s}, \quad (16)$$

such that

$$\Pi = e^{-Z'(0)}, \quad (17)$$

therefore obeys

$$\begin{aligned} e^{Z(s)} &= \prod_{p \text{ prime}} e^{p^{-s}} \\ &= \prod_{p \text{ prime}} \prod_{n \geq 1} (1 - p^{-ns})^{-\mu(n)/n} \\ &= \prod_{n \geq 1} \prod_{p \text{ prime}} (1 - p^{-ns})^{-\mu(n)/n} \\ &= \prod_{n \geq 1} \zeta(ns)^{\mu(n)/n}. \end{aligned} \quad (18)$$

It is underlined in [14, 15] that the above expression has been known for long [18, 19]. Taking logarithmic derivatives of both sides in (18) yields

$$Z'(s) = \sum_{n \geq 1} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)}, \quad (19)$$

and in particular

$$Z'(0) = \frac{\zeta'(0)}{\zeta(0)} \sum_{n \geq 1} \mu(n). \quad (20)$$

The sum entering the above expression is divergent, but it can consistently be evaluated by means of zeta-regularization, using the identity

$$\sum_{n \geq 1} \mu(n) n^{-s} = \frac{1}{\zeta(s)}. \quad (21)$$

We thus obtain, using (11),

$$Z'(0) = \frac{\zeta'(0)}{\zeta(0)^2} = -2 \ln(2\pi), \quad (22)$$

and so, using (17),

$$\Pi = 4\pi^2. \quad (23)$$

This is the central result of [14, 15]. Notice the identity

$$\Pi = P^4, \quad (24)$$

announced in (3).

3. The products of Gauss integers and primes

Gauss integers form the ring $\mathbb{Z}[i]$. They can be viewed as the vertices of the unit square lattice, of the form $z = a + bi$, where a and b are usual integers, and so $|z|^2 = a^2 + b^2$.

3.1. Product of Gauss integers

The product of all Gauss integers is

$$P_4 = \prod_{(a,b) \neq (0,0)} (a + bi). \quad (25)$$

The ring $\mathbb{Z}[i]$ is invariant under multiplication by i , every non-zero Gauss integer having an orbit of length 4: $(a, b) \rightarrow (-b, a) \rightarrow (-a, -b) \rightarrow (b, -a)$. In particular, $\mathbb{Z}[i]$ has 4 units, the distinct powers of i , namely 1, i , -1 and $-i$. It therefore seems hopeless to give a meaning to the phase of P_4 . Henceforth we rather consider

$$|P_4|^2 = \prod_{(a,b) \neq (0,0)} (a^2 + b^2). \quad (26)$$

The corresponding Dedekind zeta function,

$$\zeta_4(s) = \sum_{(a,b) \neq (0,0)} (a^2 + b^2)^{-s}, \quad (27)$$

is such that

$$|P_4|^2 = e^{-\zeta'_4(0)}. \quad (28)$$

The theory of Dedekind zeta functions is exposed in detail in [20, 21, 22] for the cases of present interest, namely $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$, and in [23, Sec. 10] for the rings of integers of more general number fields (see also [24] for an overview). In the present situation, we have

$$\zeta_4(s) = 4\zeta(s)L_4(s), \quad (29)$$

where $\zeta(s)$ is the Riemann zeta function, and

$$L_4(s) = \sum_{n \geq 1} \chi_4(n) n^{-s} \quad (30)$$

is the Dirichlet L -function associated with the Dirichlet character χ_4 , i.e., the completely multiplicative function of $n \bmod 4$ such that

$$\chi_4(0) = 0, \quad \chi_4(1) = 1, \quad \chi_4(2) = 0, \quad \chi_4(3) = -1. \quad (31)$$

This L -function reads explicitly

$$\begin{aligned} L_4(s) &= \sum_{m \geq 0} ((4m+1)^{-s} - (4m+3)^{-s}) \\ &= \frac{\zeta(s, 1/4) - \zeta(s, 3/4)}{4^s}, \end{aligned} \quad (32)$$

in terms of the Hurwitz zeta function

$$\zeta(s, x) = \sum_{m \geq 0} (m+x)^{-s}. \quad (33)$$

The analytic structure of the latter function is similar to that of the Riemann zeta function, which is recovered as $\zeta(s) = \zeta(s, 1)$. We have in particular

$$\zeta(0, x) = \frac{1}{2} - x, \quad \zeta'(0, x) = \ln \frac{\Gamma(x)}{\sqrt{2\pi}}, \quad (34)$$

the second expression being known as the Lerch formula [25]. We also mention that $L_4(s)$ has an Euler product representation of the form

$$L_4(s) = \prod_{p \text{ prime} \neq 2} (1 - \chi_4(p) p^{-s})^{-1}, \quad (35)$$

i.e., explicitly

$$L_4(s) = \frac{\zeta_{4,1}(s) \zeta_{4,3}(2s)}{\zeta_{4,3}(s)}, \quad (36)$$

in terms of the partial prime zeta functions

$$\begin{aligned} \zeta_{4,1}(s) &= \prod_{p \text{ prime} \equiv 1 \pmod{4}} (1 - p^{-s})^{-1}, \\ \zeta_{4,3}(s) &= \prod_{p \text{ prime} \equiv 3 \pmod{4}} (1 - p^{-s})^{-1}, \end{aligned} \quad (37)$$

which obey

$$\zeta_{4,1}(s) \zeta_{4,3}(s) = (1 - 2^{-s}) \zeta(s). \quad (38)$$

Using (29) and (32), together with (11) and (34), we obtain

$$L_4(0) = \frac{1}{2}, \quad L'_4(0) = \ln \frac{\Gamma(1/4)}{2\Gamma(3/4)}, \quad (39)$$

$$\zeta_4(0) = -1, \quad \zeta'_4(0) = \ln \frac{2\Gamma(3/4)^2}{\pi\Gamma(1/4)^2}. \quad (40)$$

The value of $\zeta_4(0)$ can be interpreted as counting negatively the point $(0, 0)$ that is excluded from the product (25) and the sum (27). Finally, (28) yields

$$|P_4|^2 = \frac{\pi\Gamma(1/4)^2}{2\Gamma(3/4)^2} = \frac{\Gamma(1/4)^4}{4\pi}, \quad (41)$$

and so

$$|P_4| = \frac{\Gamma(1/4)^2}{2\sqrt{\pi}} \approx 3.708149. \quad (42)$$

This expression, announced in (4), can be found e.g. in [22].

3.2. Product of Gauss primes

The product of all (unnormalized) Gauss primes,

$$\Pi_4 = \prod_{a+bi \text{ prime}} (a + bi), \quad (43)$$

is also invariant under multiplication by i , so that we rather consider

$$|\Pi_4|^2 = \prod_{a+bi \text{ prime}} (a^2 + b^2). \quad (44)$$

This quantity can be evaluated along the lines of [14, 15], as recalled above, up to the replacement of the prime zeta function $Z(s)$ by

$$Z_4(s) = \sum_{a+bi \text{ prime}} (a^2 + b^2)^{-s}, \quad (45)$$

such that

$$|\Pi_4|^2 = e^{-Z'_4(0)}. \quad (46)$$

Accordingly, in (18) to (20), the Riemann zeta function is to be replaced by

$$\mathbf{Z}_4(s) = \sum_{a+bi \text{ prime}} (1 - (a^2 + b^2)^{-s})^{-1}. \quad (47)$$

This function can be derived from the explicit knowledge of Gauss primes, dating back to Gauss himself. The Gauss integer $z = a + bi$ is prime in the following three situations (see e.g. [24, Ch. XV]):

- z is the product of a prime $p = 3 \bmod 4$ by a unit. The natural prime p is said to be inert in $\mathbb{Z}[i]$. It corresponds to 4 distinct Gauss primes, as there are 4 units.
- $|z|^2 = a^2 + b^2$ is a prime $p = 1 \bmod 4$. The natural prime p is said to be split. It corresponds to 8 distinct Gauss primes, namely the products of $a + ib$ and $b + ia$ by units.
- z is the product of $1 + i$ by a unit. The natural prime $|z|^2 = 2$ is said to be ramified. It corresponds to 4 distinct Gauss primes.

Taking the above multiplicities into account, and using (37), we obtain

$$\mathbf{Z}_4(s) = \zeta_{4,3}(2s)^4 \zeta_{4,1}(s)^8 (1 - 2^{-s})^{-4}. \quad (48)$$

Using (36) and (38), this boils down to

$$\mathbf{Z}_4(s) = \zeta(s)^4 L_4(s)^4 = \left(\frac{\zeta_4(s)}{4} \right)^4. \quad (49)$$

The analogue of (22) therefore reads

$$Z'_4(0) = \frac{1}{\zeta(0)} \frac{\mathbf{Z}'_4(0)}{\mathbf{Z}_4(0)} = \frac{4\zeta'_4(0)}{\zeta(0)\zeta_4(0)}. \quad (50)$$

Using (11) and (40), we obtain

$$Z'_4(0) = 8\zeta'_4(0). \quad (51)$$

We are thus left with the result

$$|\Pi_4| = |P_4|^8, \quad (52)$$

announced in (5).

4. The products of Eisenstein integers and primes

Eisenstein integers form the ring $\mathbb{Z}[\omega]$. They can be viewed as the vertices of the unit triangular lattice, of the form $z = a + b\omega$, where a and b are usual integers, and $\omega = e^{2\pi i/3}$, and so $|z|^2 = a^2 - ab + b^2$.

4.1. Product of Eisenstein integers

The product of all Eisenstein integers is

$$P_3 = \prod_{(a,b) \neq (0,0)} (a + b\omega). \quad (53)$$

The ring $\mathbb{Z}[\omega]$ is invariant under multiplication by $\rho = e^{i\pi/3} = \sqrt{\omega} = 1 + \omega$, every non-zero Eisenstein integer having an orbit of length 6: $(a, b) \rightarrow (a - b, a) \rightarrow (-b, a - b) \rightarrow (-a, -b) \rightarrow (b - a, -a) \rightarrow (b, b - a)$. In particular, $\mathbb{Z}[\omega]$ has 6 units, the distinct powers of ρ , namely $1, 1 + \omega, \omega, -1, -1 - \omega$ and $-\omega$. It again seems hopeless to give a meaning to the phase of P_3 . We rather consider

$$|P_3|^2 = \prod_{(a,b) \neq (0,0)} (a^2 - ab + b^2). \quad (54)$$

The corresponding Dedekind zeta function,

$$\zeta_3(s) = \sum_{(a,b) \neq (0,0)} (a^2 - ab + b^2)^{-s}, \quad (55)$$

such that

$$|P_3|^2 = e^{-\zeta'_3(0)}, \quad (56)$$

reads

$$\zeta_3(s) = 6\zeta(s)L_3(s), \quad (57)$$

where

$$L_3(s) = \sum_{n \geq 1} \chi_3(n) n^{-s} \quad (58)$$

is the Dirichlet L -function associated with the Dirichlet character χ_3 , i.e., the completely multiplicative function of $n \bmod 3$ such that

$$\chi_3(0) = 0, \quad \chi_3(1) = 1, \quad \chi_3(2) = -1. \quad (59)$$

This L -function reads explicitly

$$\begin{aligned} L_3(s) &= \sum_{m \geq 0} ((3m+1)^{-s} - (3m+2)^{-s}) \\ &= \frac{\zeta(s, 1/3) - \zeta(s, 2/3)}{3^s}. \end{aligned} \quad (60)$$

We also mention that $L_3(s)$ has an Euler product representation of the form

$$L_3(s) = \prod_{p \text{ prime} \neq 3} (1 - \chi_3(p) p^{-s})^{-1}, \quad (61)$$

i.e., explicitly

$$L_3(s) = \frac{\zeta_{3,1}(s)\zeta_{3,2}(2s)}{\zeta_{3,2}(s)}, \quad (62)$$

in terms of the partial prime zeta functions

$$\begin{aligned}\zeta_{3,1}(s) &= \prod_{p \text{ prime} \equiv 1 \pmod{3}} (1 - p^{-s})^{-1}, \\ \zeta_{3,2}(s) &= \prod_{p \text{ prime} \equiv 2 \pmod{3}} (1 - p^{-s})^{-1},\end{aligned}\tag{63}$$

which obey

$$\zeta_{3,1}(s)\zeta_{3,2}(s) = (1 - 3^{-s})\zeta(s).\tag{64}$$

Using (57) and (60), together with (11) and (34), we obtain

$$L_3(0) = \frac{1}{3}, \quad L'_3(0) = \ln \frac{\Gamma(1/3)}{3^{1/3}\Gamma(2/3)},\tag{65}$$

$$\zeta_3(0) = -1, \quad \zeta'_3(0) = \ln \frac{3\Gamma(2/3)^3}{2\pi\Gamma(1/3)^3}.\tag{66}$$

The value of $\zeta_3(0)$ can again be interpreted as counting negatively the point $(0, 0)$ that is excluded from the product (53) and the sum (55). Finally, (56) yields

$$|P_3|^2 = \frac{2\pi\Gamma(1/3)^3}{3\Gamma(2/3)^3} = \frac{\sqrt{3}\Gamma(1/3)^6}{4\pi^2},\tag{67}$$

and so

$$|P_3| = \frac{3^{1/4}\Gamma(1/3)^3}{2\pi} \approx 4.027065.\tag{68}$$

This expression is announced in (6).

4.2. Product of Eisenstein primes

The product of all (unnormalized) Eisenstein primes,

$$\Pi_3 = \prod_{a+b\omega \text{ prime}} (a + b\omega),\tag{69}$$

is also invariant under multiplication by ρ , so that we rather consider

$$|\Pi_3|^2 = \prod_{a+b\omega \text{ prime}} (a^2 - ab + b^2).\tag{70}$$

This quantity can again be evaluated along the lines of [14, 15], up to the replacement of the prime zeta function $Z(s)$ by

$$Z_3(s) = \sum_{a+b\omega \text{ prime}} (a^2 - ab + b^2)^{-s},\tag{71}$$

such that

$$|\Pi_3|^2 = e^{-Z'_3(0)}.\tag{72}$$

Accordingly, in (18) to (20), the Riemann zeta function is to be replaced by

$$\mathbf{Z}_3(s) = \sum_{a+b\omega \text{ prime}} (1 - (a^2 - ab + b^2)^{-s})^{-1}.\tag{73}$$

This function can be derived from the explicit knowledge of Eisenstein primes. The Eisenstein integer $z = a + b\omega$ is prime in the following three situations (see e.g. [24, Ch. XV]):

- z is the product of a prime $p = 2 \bmod 3$ by a unit. The natural prime p is said to be inert in $\mathbb{Z}[\omega]$. It corresponds to 6 distinct Eisenstein primes, as there are 6 units.
- $|z|^2 = a^2 - ab + b^2$ is a prime $p = 1 \bmod 3$. The natural prime p is said to be split. It corresponds to 12 distinct Eisenstein primes.
- z is the product of $1 - \omega$ by a unit. The natural prime $|z|^2 = 3$ is said to be ramified. It corresponds to 6 distinct Eisenstein primes.

Taking the above multiplicities into account, and using (63), we obtain

$$\mathbf{Z}_3(s) = \zeta_{3,2}(2s)^6 \zeta_{3,1}(s)^{12} (1 - 3^{-s})^{-6}. \quad (74)$$

Using (62) and (64), this boils down to

$$\mathbf{Z}_3(s) = \zeta(s)^6 L_3(s)^6 = \left(\frac{\zeta_3(s)}{6} \right)^6. \quad (75)$$

The analogue of (22) therefore reads

$$Z'_3(0) = \frac{1}{\zeta(0)} \frac{\mathbf{Z}'_3(0)}{\mathbf{Z}_3(0)} = \frac{6\zeta'_3(0)}{\zeta(0)\zeta_3(0)}. \quad (76)$$

Using (11) and (66), we obtain

$$Z'_3(0) = 12\zeta'_3(0). \quad (77)$$

We are thus left with the result

$$|H_3| = |P_3|^{12}, \quad (78)$$

announced in (7).

References

- [1] P. J. Davis. Leonhard Euler's integral: A historical profile of the gamma function: In memoriam: Milton Abramowitz. *Amer. Math. Monthly*, 66:849–869, 1959.
- [2] H. M. Edwards. *Riemann's Zeta Function*. Academic Press, New York, 1974.
- [3] V. S. Varadarajan. Euler and his work on infinite series. *Bull. Amer. Math. Soc. (N.S.)*, 44:515–539, 2007.
- [4] V. S. Varadarajan. *Euler Through Time: A New Look at Old Themes*. American Mathematical Society, Providence, RI, 2007.
- [5] D. B. Ray and I. M. Singer. R -Torsion and the Laplacian on Riemannian manifolds. *Adv. Math.*, 7:145–210, 1971.
- [6] S. W. Hawking. Zeta function regularization of path integrals in curved spacetime. *Commun. Math. Phys.*, 55:133–148, 1977.
- [7] A. Voros. Spectral functions, special functions and the Selberg zeta function. *Commun. Math. Phys.*, 110:439–465, 1987.
- [8] C. Soulé, D. Abramovich, J. F. Burnol, and J. Kramer. *Lectures on Arakelov Geometry*. Cambridge Studies in Advanced Mathematics. Cambridge Univ. Press, Cambridge, 1992.
- [9] J. R. Quine, S. H. Heydari, and R. Y. Song. Zeta regularized products. *Trans. Amer. Math. Soc.*, 338:213–231, 1993.
- [10] Yu. Manin. Lectures on zeta functions and motives (according to Deninger and Kurokawa). *Astérisque*, 228:121–163, 1995.
- [11] G. Illies. Regularized products and determinants. *Commun. Math. Phys.*, 220:69–94, 2001.
- [12] J. P. Allouche. Zeta-regularization of arithmetic sequences. *EPJ Web of Conferences*, 244:01008, 2020.
- [13] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini. *Zeta Regularization Techniques with Applications*. World Scientific Publishing, Singapore, 1995.
- [14] E. Muñoz García and R. Pérez-Marco. The product over all prime numbers is $4\pi^2$. Preprint IHES M/03/34, 2003.
- [15] E. Muñoz García and R. Pérez-Marco. The product over all primes is $4\pi^2$. *Commun. Math. Phys.*, 277:69–81, 2008.
- [16] C. Itzykson and J. M. Luck. Arithmetical degeneracies in simple quantum systems. *J. Phys. A: Math. Gen.*, 19:211–239, 1986.
- [17] C. Itzykson. Simple integrable systems, and Lie algebras. *Int. J. Mod. Phys. A*, 1:65–115, 1986.
- [18] E. Landau and A. Walfisz. Über die Nichtfortsetzbarkeit einiger durch Dirichletsche Reihen definierter Funktionen. *Rendiconti del Circolo Matematico di Palermo*, 44:82–86, 1920.
- [19] G. Dahlquist. On the analytical continuation of Eulerian products. *Arkiv för Matematik*, 1:533–554, 1951.
- [20] P. Cartier. An introduction to zeta functions. In M. Waldschmidt, P. Moussa, J. M. Luck, and C. Itzykson, editors, *From Number Theory to Physics*, chapter 1. Springer, Berlin, 1992.
- [21] K. Kato, N. Kurokawa, and T. Saito. Number Theory 2: Introduction to class field theory. volume 240 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2011.
- [22] N. Kurokawa, M. Kurihara, and T. Saito. Number Theory 3: Iwasawa theory and modular forms. volume 242 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2012.
- [23] H. Cohen. *Number Theory Volume II: Analytic and Modern Tools*. Springer, Berlin, 2007.
- [24] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, Oxford, 6th edition, 2008.
- [25] M. Lerch. Další studie v oboru malmsténovských řad. *Rozpravy České Akad.*, 3:1–61, 1894.