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Quasi-Characters for three-character Rational Conformal Field Theories

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Abstract

We revisit $(3, 0)$ and $(3, 3)$ admissible solutions obtained using the MLDE method. We show that all $(3, 0)$ solutions can be written in terms of a universal formula involving the ${}_3F_2$ hypergeometric function that takes into account the monodromy at the elliptic points. We construct $(3, 3)$ admissible solutions from $(3, 0)$ CFTs using a duality due to Bantay and Gannon. This enables us to compute their modular properties such as the S-matrix and the fusion rules. We find that only 7 of the 15 known $(3, 3)$ admissible solutions have proper fusion rules.

Using the theory of matrix MLDE, starting with a known $(3, 0)$ and $(3, 3)$ solutions, we construct two other solutions, that are typically quasi-characters that share the same multiplier as the original solution. We then construct linear combinations that lead to new admissible solutions. We observe that admissible solutions arise as integer points that lie on a polytope. We construct all possible $(3, 6)$ and $(3, 9)$ admissible solutions that arise in this fashion. In some cases, we identify RCFT that arise from our $(3, 6)$ admissible solutions. In addition, we obtain a large family of admissible solutions with higher Wronskian index.

Contents

1 Introduction

The holomorphic modular bootstrap aims to classify two-dimensional Rational Conformal Field Theories (RCFTs) – these are two-dimensional CFTs with a finite number of primaries. It can happen that the distinct primaries have the same character. Thus the number of characters of an RCFT (which we denote by n) can be less than the number of primaries. This approach began with the work of Mathur, Mukhi and Sen (MMS) in the late 80s where they mapped this problem to one of solving Modular Linear Differential Equations(MLDE) [1–3]. A second parameter, the Wronskian index which we denote by ℓ, appears in this approach. The difficulty of this classification increases as the number of characters as well as the Wronskian index increases. However, when the number of characters, Wronskian index as well as central charge are all small, this problem is somewhat tractable.

Not unexpectedly, the case of one and two character theories has seen the most progress. For one-character theories, the central charge is necessarily a multiple of 8. There is a unique RCFT at $c = 8$, the E_8 Kac-Moody Lie algebra at level one, two at $c = 16$ and 71 at $c = 24$ [4]. The numbers increase tremendously when $c > 24$ [5]. For two-character theories, the Wronskian index is even and non-negative. The ones with $\ell = 0, 2, 4$ have been completely classified. More recently, the ones with two primaries, $\ell \geq 6$ and $c < 25$ have also been classified [6, 7].

The MLDE approach has been used to classify three-character theories. In particular, admissible characters with Wronskian index $\ell = 0, 3$ have been constructed [8, 9]. For central charges $c \leq 24$, the unitary RCFTs with vanishing Wronskian index have also been determined [10]. However, when the Wronskian index is 3, it is not known whether the admissible characters that have been found are those of RCFTs.

The method of quasi-characters has been used to construct new admissible characters with Wronskian index $\ell \geq 6$ [11–13]. Quasi-characters are also solutions to an MLDE for which the q-series has integral coefficients that are not necessarily non-negative. For suitable choices of coefficients, adding these quasi-characters to known admissible characters that share the same multiplier system can lead to admissible solutions, albeit with higher Wronskian index. For instance, linear combinations to two solutions of $(2, 0)$ MLDE's can lead to an admissible character with Wronskian index 6.

In another approach, Kaidi et al. use the representation theory of $PSL(2, \mathbb{Z}_N)$ for low values of N and determine possible exponents (modulo one) of all theories with up to 5 characters [14]. This can be used as an input to the MLDE approach. In particular, the MLDE is uniquely determined by the exponents when the Wronskian index vanishes and the number of characters is less than six. This needs one to fix the modulo one ambiguity in the Kaidi et al. list which leads to a plethora of choices of exponents even after fixing the Wronskian index.

Given a known RCFT, one can ask whether one can construct new characters from its characters. One such method uses Hecke operators, slightly generalized to act on vector valued modular forms (VVMFs), on the characters of the given RCFT [15,16]. This method has been utilised to construct new admissible characters for a large class of examples with up to six characters [17]. This method has the advantage that we know the modular data of the new characters.

Gaberdiel, Hampapura, Mukhi (GHM) propose a coset-like construction that leads to potentially new RCFTs [18]. Let $\chi_i(\tau)$ $(i = 0, \ldots, (n-1))$ be the characters of a known RCFT (with central charge c less than 24). Let m_i denote the multiplicities of the characters in the RCFT. Then, the diagonal torus partition function is given by (with $m_0 = 1$)

$$Z = \sum_{i=0}^{n-1} m_i \, |\chi_i(\tau)|^2 \ . \tag{1.1}$$

Let the GHM coset dual have character $\widetilde{\chi}_i(\tau)$. Form the vector consisting of the characters of given RCFT and its GHM dual. Call them, $\mathbb{X}$ and $\widetilde{\mathbb{X}}$, respectively. Further, let $M = \text{Diag}(m_0, \ldots, m_{n-1})$ be the matrix of multiplicities. Then, one has the relation

$$\widetilde{\mathbb{X}}^T \cdot M \cdot \mathbb{X} = (J(\tau) + \mathcal{N}) \ M \ . \tag{1.2}$$

Here $\mathcal{N}$ is a constant and the coset dual has central charge $(24 - c)$. One then 'solves' this equation to obtain the q-series for $\widetilde{\mathbb{X}}$.

The last approach that is pursued in this paper is to use the theory of vector valued modular forms developed by Bantay and Gannon [19, 20]. The main idea is to start with a known RCFT with VVMF $\mathbb{X}$ and construct a basis of VVMFs with the same multiplier as the known RCFT. Taking linear combinations of the basis with coefficients that are polynomials in J leads to new solutions [21](see also [22, see appendix C]). This is in the spirit of quasi-characters but leads to *all* possible admissible characters that share the same multiplier as the known RCFT. This paper pursues this approach to systematically construct new admissible solutions with higher Wronskian index starting from $(3, 0)$ and $(3, 3)$ theories.

The main results of this paper are as follows.

1. We provide a uniform expression for all $(3, 0)$ characters and quasi-characters in terms of the ${}_3F_2$ hypergeometric function. While this is not necessarily a new result, it incorporates the monodromy data from the MLDE in a natural way. Proposition 3.1 summarises the result.

2. Given a $(3, 0)$ RCFT with VVMF that we denote by $\mathbb{X}$, we construct two additional solutions that we call $\mathbb{Y}_1$ and $\mathbb{Y}_2$ that provide a basis for VVMF with this multiplier. Given the exponents of $\mathbb{X}$ and two *degeneracies*, y_{21} and y_{31}, we give an explicit formula, Eq. (3.13), for the solution to the matrix MLDE, Eq. (2.4), that appears in the theory of Bantay and Gannon. Multiple tables present explicit results for all known $(3, 0)$ RCFTs.

3. Starting with known $(3,0)$ RCFTs with central charge $c < 24$, we construct all possible $(3,6)$ admissible characters. Admissible characters appear as integral points inside a convex polytope, that can be non-compact with $(3,6)$ theories appearing at co-dimension one facets and occasionally $(3,0)$ theories appears at corners (co-dimension ridges) of the polytope. Interior points correspond to higher Wronskian index. We show that some of the (3.6) admissible characters are those of an RCFT. Our study is not comprehensive but done only for a few cases.

4. Bantay and Gannon propose a duality that maps one basis of VVMF with a given multiplier to the basis of a VVMFs with a multiplier derived from the original one. As an application, we show that this duality maps $(3,0)$ theories to $(3,3)$ theories. We obtain all the 15 known solutions that were obtained using the MLDE method in this fashion. This enables us to obtain the S-matrices for the 15 solutions and we find only 7 of them have fusion coefficients that are 0 or 1.

5. Like for the $(3,0)$ examples, we obtain $(3,9)$ admissible solutions starting from the $(3,3)$ admissible solutions.

The organisation of the paper is as follows. Section 1 provides an introduction to the problem being studied as well as a summary of the results. Section 2 provides the necessary background that connects VVMFs and RCFTs. Section 3 focuses on $(3,0)$ RCFTs We first show that all characters and quasi-characters that arise from $(3,0)$ MLDE have a uniform presentation in terms of the ${}_3F_2$ hypergeometric function. Then, we extend this presentation to the basis of VVMFs, $\varXi$, associated with any $(3,0)$ RCFT. These bases are presented in multiple tables that provide the information needed to construct the q-series for all characters. In section 4, we move to the study of $(3,3)$ theories. In section 4.1, we provide a new construction of all known $(3,3)$ admissible solutions using a duality due to Bantay and Gannon [19]. Using this construction, we are able to show that only one class containing 7 characters of $(3,3)$ theories lead to consistent fusion rules. Section 4.2 obtains quasi-characters and admissible characters for each of these seven theories. In section 5, we look for and find RCFTs among the $(3,6)$ admissible solutions constructed in section 3. We conclude with some remarks. In appendix A, we discuss some of the modular forms that appear in the paper. Appendix B consists of a large number of tables that provide details of the $(3,6)$ theories obtained from $(3,0)$ theories.

2 Background

2.1 Basic theory of Vector Valued Modular Forms

Let $\mathbb{X}(\tau) = (\chi_0(\tau), \chi_1(\tau), \ldots, \chi_{n-1}(\tau))^T$ be a Vector Valued Modular Form (VVMF) of rank n, weight w and multiplier ρ of $PSL(2,\mathbb{Z})$. Then, one has

$$\mathbf{X}\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w\ \rho\left(\tfrac{a\tau+b}{c\tau+d}\right)\ \mathbb{X}(\tau)\ ,$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z})$. Let $\mathcal{M}^!_w(\rho)$ denote the space of weakly holomorphic VVMFs. Our interest will largely be with VVMFs of weight zero. These appear naturally as the vector of characters of an RCFT with the rank being given by the number of characters.

We will make use of some results from the theory of VVMFs largely due to Bantay and Gannon that are of interest to us [19, 20, 23–25]. We will assume that the VVMFs of interest are such that the multipliers are admissible so that the theorems we quote are applicable. Consider the following modular differential operators:

$$\nabla_{1,w} := \frac{E_4(\tau)E_6(\tau))}{\Delta(\tau)}\,\mathcal{D}_w \quad , \quad \nabla_{2,w} := \frac{E_4(\tau)^2}{\Delta(\tau)}\,\mathcal{D}^2_w \quad , \quad \nabla_{3,w} := \frac{E_6(\tau)}{\Delta(\tau)}\,\mathcal{D}^3_w \quad . \tag{2.1}$$

These operators are maps from $\mathcal{M}^!_w(\rho)$ to itself. The following properties are relevant for us.

1. From [20, Thm. 3.3(a)], we have $\mathcal{M}^!_w(\rho)$ is a vector space of dimension equal to the rank. Let $\varXi = (\mathbb{X}, \mathbb{Y}_1, \ldots, \mathbb{Y}_{n-1})$ be a basis for $\mathcal{M}^!_w(\rho)$.

2. Let $\varXi(\tau)$ denote the $n \times n$ matrix whose columns represent the basis for $\mathcal{M}^!_w(\rho)$. One has the following q-expansion that defines $\varLambda$ and $\mathcal{Y}$ as well as fixes the normalisation for the basis.
$$\varXi(\tau) = q^{\varLambda}\Big(\mathbf{1}_n + \mathcal{Y}\; q + O(q^2)\Big) \tag{2.2}$$
We work typically in a basis where $\varLambda$ is the diagonal matrix with entries $(\lambda_1, \ldots, \lambda_n)$ and $\mathcal{Y}$ is a $n \times n$ matrix whose entries we write as follows:
$$\mathcal{Y} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & \ddots & \ddots & y_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix} \quad , \tag{2.3}$$
whose entries will generically be rational. However, we will find them to have integral entries in a large class of examples that we consider. We will indicate situations where we obtain rational entries.

3. From [20, Thm. 3.3(b)] and [19], $\varXi$ is a solution to the following first-order matrix MLDE.
$$\nabla_1 \varXi = \varXi\left((J - 984)\varLambda_w + [\varLambda_w, \mathcal{Y}_w]\right) \ , \tag{2.4}$$
where $\varLambda_w = \varLambda - \frac{w}{12}\mathbf{1}_n$ and $\mathcal{Y}_w = \mathcal{Y} + 2w\mathbf{1}_n$ and $J = \frac{1}{q} + 744 + \cdots$ is the modular J-function.

4. Given a VVMF $\mathbb{X}$, one obtains $\mathcal{M}^!_w(\rho)$ as a $\mathbb{C}(\nabla_1, \nabla_2, \nabla_2, J)$ module over $\mathbb{X}$.

2.2 From RCFTs to VVMFs

Consider an n character RCFT with central charge c, conformal dimensions, h_a and characters $\chi_a(\tau)$ for $a = 0, 1, \ldots, (n-1)$. $\chi_0(\tau)$ is taken to be the identity character and hence $h_0 = 0$. Let ℓ denote the Wronskian index of the RCFT. Form the VVMF $\mathbb{X}$ given by

$$\mathbb{X} = (\chi_0(\tau), \ldots, \chi_{n-1}(\tau))^T \ . \tag{2.5}$$

Define $\alpha_a = -\frac{c}{24} + h_a$. Then, one has the q expansion

$$\chi_a(\tau) = q^{\alpha_a} \left(m_{0,a} + O(q)\right) \ , \tag{2.6}$$

with $m_{0,0} = 1$. The Wronskian index of the RCFT is given by the formula

$$\ell = \frac{n(n-1)}{2} - 6\sum_{a=0}^{n-1} \alpha_a \ . \tag{2.7}$$

We identify the first column of Ξ with $\mathbb{X}$. Then, by comparing Eq. (2.2) and Eq. (2.6), we see that

$$\lambda_1 = \alpha_0 \quad , \quad \lambda_{i+1} = (\alpha_i - 1) \text{ for } i = 1, \ldots, (n-1) \ . \tag{2.8}$$

This provides the relation between the exponents that we associate with Ξ with the ones natural to the RCFT.

Finally, using a procedure implicitly described in item 4 of the previous sub-section, we construct $(n-1)$ other VVMF's $\mathbb{Y}_i$ that share the same multiplier as $\mathbb{X}$.

2.3 Quasi-characters

In our application, we take $\mathbb{X}$ to be the weight zero VVMF constructed out of the n characters of a known RCFT. The multiplier ρ is fixed by the S and T matrices.

$$T = \rho\Big(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \Big) \quad , \quad S = \rho\Big(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \Big) \ . \tag{2.9}$$

Repeated applications of the differential operators ∇_i $(i = 1, 2, 3)$ enables one to generate the complete basis. Let $\mathbb{X}$ and $\mathbb{Y}_i$ for $i = 1, \ldots, (n-1)$ denote the generators for $\mathcal{M}^!_w(\rho)$. The $\mathbb{Y}_i$ will typically lead to quasi-characters as the coefficients of the q-expansion can be negative integers (possibly, rational). For simplicity, we will assume that the $\mathbb{Y}_i$ have integer coefficients.

2.4 From quasi-characters to admissible ones

Given an (n, ℓ) RCFT with characters forming the VVMF $\mathbb{X}$ and quasi-characters $\mathbb{Y}_i$ that generate the basis of weight zero modular forms with multiplier associated with the RCFT. In the following, we will describe three constructions of candidate VVMF's which might lead to admissible characters. A candidate VVMF is said to be *admissible* if all the coefficients in its q-series are non-negative integers. Let A be a subset of $[1, n-1]$ with $|A| = r \leq (n-1)$.

1. To the subset A, associate the VVMF:

$$\mathbb{U}_{r,A} = \mathbb{X} + \sum_{i\in A} b_i \ \mathbb{Y}_i \ , \tag{2.10}$$

with $b_i \in \mathbb{Z}_{>0}$. We scan the values of b_i looking for admissible characters. Looking at the leading exponents of the VVMF $\mathbb{U}_r$ we will generically obtain a $(n, \ell + 6r)$ admissible character. However, for special choices of the b_i, the Wronskian index can be smaller. For the case when $n = 3$, we will have three such VVMF's i.e., $\mathbb{U}_{1,1}$, $\mathbb{U}_{1,2}$, and $\mathbb{U}_2$. Of course, $\mathbb{U}_0 = \mathbb{X}$.

2. Define

$$\mathbb{W}_{r,A} = (\widehat{J} - y_{11} + b) \ \mathbb{X} - \sum_{i=2}^{n} y_{i1} \ \mathbb{Y}_{i-1} + \sum_{i\in A} c_i \ \mathbb{Y}_i \ , \ , \tag{2.11}$$

where $b \in \mathbb{Z}_{\geq 0}$ and $c_i \in \mathbb{Z}_{>0}$ and $\widehat{J} = (J - 744)$. We search for admissible characters by scanning through the allowed values of the coefficients. For the case, when $n = 3$, we will have four classes of VVMFs: $\mathbb{W}_0$, $\mathbb{W}_{1,1}$, $\mathbb{W}_{1,2}$ and $\mathbb{W}_2$.

3. Define

$$\mathbb{V}_0 = (\widehat{J} + b_1)\Big((\widehat{J} - y_{11}) \ \mathbb{X} - \sum_{i=2}^{n} y_{i1} \ \mathbb{Y}_{i-1}\Big) + (b_2^* + b_2) \ \mathbb{X} + \sum_{i=1}^{n-1} s_i^* \ \mathbb{Y}_i \ , \tag{2.12}$$

where b_2^* and s_i^* are chosen so that the leading terms in $\mathbb{V}_0$ take the form

$$\mathbb{V}_0 = \begin{pmatrix} q^{\lambda_1 - 2}\left(1 + b_1 q + b_2 q^2 + O(q^3)\right) \\ q^{\lambda_i}\left(0 + O(q)\right) \ \text{for } i = 2, \ldots, n \end{pmatrix} \ . \tag{2.13}$$

Further we need $b_1, b_2 \in \mathbb{Z}_{\geq 0}$. We do not provide explicit formulae for b_2^* and s_i^* as they are messy and unilluminating to see. From this we see that admissible characters arising in this fashion will generically have Wronskian index $(\ell + 12)$. It may happen that for special values of (b_1, b_2), we might find admissible characters with Wronskian index $(\ell + 6)$ (in co-dimension one) and admissible characters with Wronskian index ℓ at co-dimension zero. The central charge of the potential CFT is $(c + 48)$. Next, define

$$\mathbb{V}_{r,A} = \mathbb{V}_0 + \sum_{i\in A} s_i \ \mathbb{Y}_i \ , \tag{2.14}$$

with $s_i \in \mathbb{Z}_{>0}$. If these lead to admissible characters, the Wronskian index will generically be $(\ell + 12 + 6r)$.

Remark: We caution the reader that the notation used here is slightly different from the one used in [21].

3 $(3,0)$ theories and the hypergeometric ODE

Let $D = w\frac{d}{dw}$. The ${}_3F_2$ hypergeometric Ordinary Differential Equation (ODE) is given by

$$w(D+\beta_0)(D+\beta_1)(D+\beta_2)\ F = (D-\alpha_0)(D-\alpha_1)(D-\alpha_2)\ F\ . \tag{3.1}$$

The local exponents in the Frobenius power series expansion about the regular singular points are [26]

$$\begin{aligned} (\alpha_0,\alpha_1,\alpha_2) \quad & \text{at } w=0\ , \\ (\beta_0,\beta_1,\beta_2) \quad & \text{at } w=\infty\ , \\ \Big(0,1,2-\sum_i(\alpha_i+\beta_i)\Big) \quad & \text{at } w=1\ . \end{aligned} \tag{3.2}$$

A basis about $w=0$ is given by (with $(i=0,1,2)$)

$$w^{\alpha_i}\ {}_3F_2\left(\alpha_i+\beta_0,\ \alpha_i+\beta_1,\ \alpha_i+\beta_2\ ;\ \alpha_i-\alpha_0+1, \alpha_i\widehat{-\alpha_1}+1, \alpha_i-\alpha_2+1\ ;\ w\right) \quad ,$$

where $\widehat{\quad}$ indicates that we delete the $(\alpha_i - \alpha_i + 1)$ term.

In order to connect the ${}_3F_2$ hypergeometric ODE with the MLDE for $(3,0)$ RCFTs, we make the following change of variables. Let $w = \frac{1728}{J(\tau)}$. One has the following properties of the $(3,0)$ MLDE.

1. The cusp $\tau \to i\infty$ gets mapped to $w=0$ and the elliptic points $\tau = e^{\frac{2\pi i}{3}}$ gets mapped to $w=\infty$ and $\tau = i$ gets mapped to $w=1$.

2. The monodromy about $w=0$ is given by the T-matrix, monodromy about $w=1$ by the S matrix and the monodromy about $w=\infty$ with the U-matrix.

3. For a unitary RCFT with central charge c and non-zero conformal weights (h_1,h_2), the exponents of the T-matrix are

$$\alpha_0 = -\frac{c}{24} \quad , \quad \alpha_1 = -\frac{c}{24}+h_1 \quad , \quad \alpha_2 = -\frac{c}{24}+h_2\ .$$

4. For a $(3,\ell)$ RCFT, the Wronskian index ℓ is given by

$$\ell = 3 - 6\sum_{i=0}^{n-1}\alpha_i\ . \tag{3.3}$$

   We see that $\ell=0$ implies that $\sum_i \alpha_i = \frac{1}{2}$.

5. Since $U^3 = I$, the eigenvalues of U are cube-roots of unity. For $j = 0, 1, 2$, let b_j denote the number of eigenvalues of U with eigenvalue $\exp(\frac{2\pi i j}{3})$. Similarly, $S^2 = I$ implies that the eigenvalues of S are ± 1. Let a_1 denote the number of eigenvalues of S with eigenvalue -1. Then,

$$\ell = -9 + 3a_1 + 2(b_1 + 2b_2) \tag{3.4}$$

$\ell = 0$ implies only two possible solutions. (i) $a_1 = 3$, $b_1 = b_2 = 0$ and (ii) $a_1 = b_1 = b_2 = 1$. Solution (i) implies that $S = -I$ which would correspond to theories that are tensor product of three one-character RCFTs. We assume that this is not the case. Then, we are left with only solution (ii). In particular this implies that $b_0 = b_1 = b_2 = 1$ and $a_0 = 2$ and $a_1 = 1$.

Proposition 3.1. *Assuming that the local exponents of the MLDE about $w = 0$ do not differ by an integer, the solutions to the $(3, 0)$ MLDE are given by solutions to the ${}_3F_2$ hypergeometric ODE. The three solutions are given by*

$$G(\alpha_0, \alpha_1 - 1, \alpha_2 - 1) \quad , \quad G(\alpha_1, \alpha_0 - 1, \alpha_2 - 1) \quad , \quad G(\alpha_2, \alpha_0 - 1, \alpha_1 - 1) \ . \tag{3.5}$$

where

$$G(a, b, c; w) := \left(\frac{w}{1728}\right)^a \ {}_3F_2\Big(a, a + 1/3, a + 2/3; a - b, a - c; w\Big) \ ,$$

Proof. When the local exponents do not differ by an integer, the monodromy matrix (the T-matrix) about $w = 0$ is diagonal. If the local exponents differ by an integer, one may have logarithmic solutions that lead to non-diagonal T-matrix. We call such cases exceptional and discuss them later.

The $(3, 0)$ MLDE in the coordinate $z = \frac{1}{w}$ is given by

$$\left[\partial_z^3 + \left(\tfrac{2}{z} + \tfrac{3}{2(z-1)}\right)\partial_z^2 + \left(\tfrac{2}{9z^2} + \tfrac{4}{3z(z-1)} + \tfrac{\mu_1}{z(z-1)}\right)\partial_z - \left(\tfrac{\mu_2}{z^2(z-1)}\right)\right] \chi = 0 \tag{3.6}$$

The indicial equation about $z = 0$ has exponents $\beta = (1, \frac{1}{3}, \frac{2}{3})$ and the indicial equation about $z = 1$ has exponents $(0, 1, \frac{1}{2})$. The indicial equation at $z = \infty$ identifies

$$\mu_1 = (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_0\alpha_2) - \tfrac{1}{18} \text{ and } \mu_2 = -\alpha_0\alpha_1\alpha_2 \ .$$

This is consistent with the exponents obtained using Eq. (3.2). □

Remarks:

1. A similar formula can be written for $(2, 0)$ RCFTs. The characters are now given in terms of the ${}_2F_1$ hypergeometric functions. Explicitly, one has

$$\left(\frac{w}{1728}\right)^{\alpha_0} {}_2F_1(\alpha_0, \alpha_0 + \tfrac{1}{3}; \alpha_0 - \alpha_1 + 1; w) \quad , \quad \left(\frac{w}{1728}\right)^{\alpha_1} {}_2F_1(\alpha_1, \alpha_1 + \tfrac{1}{3}; \alpha_1 - \alpha_0 + 1; w) \ .$$

Here $a_1 = b_1 = 1$ and $b_2 = 0$. This agrees with the formula given in [11].

2. A similar formula can be written for $(2,2)$ RCFTs. Explicitly, one has

$$\left(\frac{w}{1728}\right)^{\alpha_0} {}_2F_1(\alpha_0,\alpha_0+\tfrac{2}{3};\alpha_0-\alpha_1+1;w) \quad , \quad \left(\frac{w}{1728}\right)^{\alpha_1} {}_2F_1(\alpha_1,\alpha_1+\tfrac{2}{3};\alpha_1-\alpha_0+1;w) \ .$$

Here $a_1 = b_2 = 1$ with $b_1 = 0$. This agrees with the formula given in [11].

3. Franc and Mason have shown a large number of $(3,0)$ RCFT's are given by solutions to the ${}_3F_2$ hypergeometric ODE [27,28]. However, they fixed the exponents $\alpha_i \mod 1$ in their study. To the extent that we checked, their formulae match ours.

4. All solutions that appeared in the $(3,0)$ modular bootstrap fit the given formula [8]. We will discuss the case when the exponents about $w = 0$ differ by an integer separately.

5. The infinite families of quasi-characters constructed by Mukhi, Poddar and Singh also fit this formula [12].

3.1 The exceptional cases

We consider the examples obtained from the $(3,0)$ modular bootstrap [8]. There are four families of examples, two with $c = 8$ and another two with $c = 16$ that appear in the table.

1. First, consider the two examples of $(c = 8, h_1 = h_2 = 3/4)$ and $(c = 16, h_1 = h_2 = 5/4)$. Here the two of the exponents are equal, $\alpha_1 = \alpha_2$. The Frobenius power series method gives two solutions that are given by the hypergeometric functions given in the conjecture. They are

$$\begin{aligned} F_0 &= \left(\frac{w}{1728}\right)^{\alpha_0} {}_3F_2(\alpha_0, \alpha_0+\tfrac{1}{3}, \alpha_0+\tfrac{2}{3}; \alpha_0-\alpha_1+1; w) \ , \\ F_1 &= \left(\frac{w}{1728}\right)^{\alpha_1} {}_3F_2(\alpha_1, \alpha_1+\tfrac{1}{3}, \alpha_1+\tfrac{2}{3}; \alpha_1-\alpha_0+1; w) \ . \end{aligned} \tag{3.7}$$

The third solution is given by

$$F_2 = F_1 \log w + \text{power series in } q \ .$$

The solution F_1 turns out to be unstable in both cases – the coefficients are rational but there is no single factor that makes them integral to all orders.

2. Next, consider the case of $(c = 8, h_1 = 1/2, h_2 = 1)$. Solving the MLDE leads to an infinite family of solutions (parametrised by b) given by

$$\mathbb{X} = \begin{pmatrix} G(-\frac{1}{3},-\frac{5}{6},-\frac{1}{3};w) + b\ G(\frac{2}{3},-\frac{1}{3},-\frac{4}{5};w) \\ G(\frac{1}{6},-\frac{4}{5},-\frac{1}{3};w) \\ G(\frac{2}{3},-\frac{1}{3},-\frac{4}{5};w) \end{pmatrix} \ , \tag{3.8}$$

Positivity of the q-series requires $b \geq -248$. The RCFT $D_{4,1}^{\otimes 2}$ appears when $b = -192$ and the RCFT $D_{8,1}$ appears when $b = -128$.

3. Next, consider the case of $(c = 16, h_1 = 1, h_2 = 3/2)$. Solving the MLDE leads to an infinite family of solutions (parametrised by b) given by

$$\mathbb{X} = \begin{pmatrix} G(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{6}; w) + b\ G(\frac{1}{3}, -\frac{5}{3}, -\frac{1}{6}; w) \\ G(\frac{1}{3}, -\frac{5}{3}, -\frac{1}{6}; w) \\ G(\frac{5}{6}, -\frac{5}{3}, -\frac{2}{3}; w) \end{pmatrix} , \tag{3.9}$$

Positivity of the q-series requires $b \geq -496$. The RCFT given by $\mathcal{E}_1(D_{4,1}^{\otimes 4})$, which is a one-character extension of $D_{4,1}^{\otimes 4}$, appears when $b = -384$.

4. Next consider the case of $(c = 16, h_1 = 1/2, h_2 = 2)$. Solving the MLDE leads to an infinite family of solutions (parametrised by b) given by

$$\mathbb{X} = \begin{pmatrix} G(-\frac{2}{3}, -\frac{7}{6}, \frac{1}{3}; w) + b\ G(\frac{4}{3}, -\frac{5}{3}, -\frac{7}{6}; w) \\ G(\frac{1}{6}, -\frac{5}{3}, \frac{1}{3}; w) \\ G(\frac{4}{3}, -\frac{5}{3}, -\frac{7}{6}; w) \end{pmatrix} , \tag{3.10}$$

Positivity of the q-series requires $b \geq -69752$. The RCFT associated with $D_{16,1}$ appears when $b = -32768$.

3.2 $(3,0)$ Quasi-characters from VVMFs

Define $\lambda_1 = \alpha_0$ and $\lambda_{i+1} = \alpha_i - 1$ for $i = 1, 2$. These are the exponents that appear in the Matrix MLDE associated with a $(3,0)$ admissible character. The VVMF associated with an RCFT with central charge c and non-zero conformal weights h_1 and h_2 is given by

$$\mathbb{X} = \begin{pmatrix} G(\lambda_1, \lambda_2, \lambda_3; z) \\ y_{21}\ G(\lambda_2 + 1, \lambda_1 - 1, \lambda_3; z) \\ y_{31}\ G(\lambda_3 + 1, \lambda_1 - 1, \lambda_2; z) \end{pmatrix} , \tag{3.11}$$

where $\lambda_1 = -\frac{c}{24}$, $\lambda_2 = -\frac{c}{24} + h_1 - 1$ and $\lambda_3 = -\frac{c}{24} + h_2 - 1$. The degeneracies y_{21} and y_{31} are determined by details of the RCFT when they are known. In other cases, we can choose the smallest values such that the coefficients of the q-series are integral.

The matrix of solutions is then are determined by the data

$$\mathcal{Y} = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix} , \tag{3.12}$$

and

$$\Xi(\tau) = \begin{pmatrix} G(\lambda_1, \lambda_2, \lambda_3; z(\tau)) & y_{21} G(\lambda_1 + 1, \lambda_2 - 1, \lambda_3; z(\tau)) & y_{31} G(\lambda_1 + 1, \lambda_2, \lambda_3 - 1; z(\tau)) \\ y_{12} G(\lambda_2 + 1, \lambda_1 - 1, \lambda_3; z(\tau)) & G(\lambda_2, \lambda_1, \lambda_3; z(\tau)) & y_{32} G(\lambda_2 + 1, \lambda_1, \lambda_3 - 1; z(\tau)) \\ y_{13} G(\lambda_3 + 1, \lambda_1 - 1, \lambda_2; z(\tau)) & y_{23} G(\lambda_3 + 1, \lambda_1, \lambda_2 - 1; z(\tau)) & G(\lambda_3, \lambda_1, \lambda_2; z(\tau)) \end{pmatrix} \tag{3.13}$$

$$
\left(\nabla_1 - (\alpha_0 J + b_1)\right)\mathbb{X} = y_{21}(\alpha_1-\alpha_0)\mathbb{Y}_1 + y_{31}(\alpha_2-\alpha_0)\mathbb{Y}_2\ , \tag{3.14}
$$

where

$$
b_1 = \frac{192\alpha_0\left(-9\alpha_1\left(\alpha_2-1\right)+9\alpha_2+9\alpha_0\left(\alpha_1+\alpha_2-1\right)-7\right)}{\left(\alpha_0-\alpha_1+1\right)\left(\alpha_0-\alpha_2+1\right)}\ .
$$

$$
\left(\nabla_2 - (a_0 J + b_2)\right)\mathbb{X} = y_{21}(a_1-a_0)\mathbb{Y}_1 + y_{31}(a_2-a_0)\mathbb{Y}_2\ , \tag{3.15}
$$

where $a_i = (\alpha_i^2 - \frac{\alpha_i}{6})$ for $i = 0, 1, 2$ and

$$
b_2 = \frac{32\alpha_0\left(3\alpha_0+1\right)\left(18\alpha_0^2+3\left(6\alpha_1+6\alpha_2+1\right)\alpha_0-18\alpha_1\left(\alpha_2-1\right)+18\alpha_2-8\right)}{\left(\alpha_0-\alpha_1+1\right)\left(\alpha_0-\alpha_2+1\right)}\ .
$$

The entries of the $\mathcal{Y}$ are determined in terms of the exponents α_i and y_{21} and y_{31}.

$$
\begin{aligned}
y_{ii} &= \frac{1728\,\lambda_i\left(\lambda_i+\frac{1}{3}\right)\left(\lambda_i+\frac{2}{3}\right)}{\prod_{j\neq i}(\lambda_i-\lambda_j)} - 744\,\lambda_i \\
\frac{y_{23}y_{31}}{y_{21}} &= \frac{192\left(\alpha_0-\alpha_1\right)\left(9\alpha_2^3-18\alpha_2^2+11\alpha_2-2\right)}{\left(\alpha_0-\alpha_2\right)\left(\alpha_0-\alpha_2+1\right)\left(-\alpha_1+\alpha_2-1\right)\left(\alpha_2-\alpha_1\right)} \\
\frac{y_{32}y_{21}}{y_{31}} &= \frac{192\left(\alpha_0-\alpha_2\right)\left(9\alpha_1^3-18\alpha_1^2+11\alpha_1-2\right)}{\left(\alpha_0-\alpha_1\right)\left(\alpha_0-\alpha_1+1\right)\left(\alpha_1-\alpha_2-1\right)\left(\alpha_1-\alpha_2\right)} \\
y_{12}y_{21} &= \frac{18432\alpha_0\left(9\alpha_0^2+9\alpha_0+2\right)\left(18\alpha_0^3-9(2\alpha_1-2\alpha_2-5)\alpha_0^2-9\left(2\alpha_1^2+(4\alpha_2-2)\alpha_1-6\alpha_2-1\right)\alpha_0+(\alpha_1-1)(-36\alpha_2+9\alpha_1(2\alpha_2-3)+14)\right)}{(\alpha_0-\alpha_1)(\alpha_0-\alpha_1+1)^2(\alpha_0-\alpha_1+2)(\alpha_0-\alpha_2+1)(\alpha_1-\alpha_2)} \\
y_{13}y_{31} &= -\frac{18432\alpha_0\left(9\alpha_0^2+9\alpha_0+2\right)\left(18\alpha_0^3+9(2\alpha_1-2\alpha_2+5)\alpha_0^2-9\left(2\alpha_2^2-2\alpha_2+\alpha_1(4\alpha_2-6)-1\right)\alpha_0+(18\alpha_1(\alpha_2-2)-27\alpha_2+14)(\alpha_2-1)\right)}{(\alpha_0-\alpha_1+1)(\alpha_0-\alpha_2)(\alpha_0-\alpha_2+1)^2(\alpha_0-\alpha_2+2)(\alpha_1-\alpha_2)}
\end{aligned}
$$

Remarks: 1. y_{21} and y_{31} are not fixed here and have to be given as input.
2. One has the relation $y_{11} + y_{22} + y_{33} = 252$.

3.3 Examples

In this subsection, we will present the basis of quasi-characters that arise from $(3,0)$ RCFTs that have been discussed in [10] (for the unitary examples) and [17] (for the non-unitary examples). The results are presented mostly as a series of tables and organised by RCFTs that share the same S-matrix and potentially belong to the same Modular Tensor Category (MTC). The tables further give the multiplicities m_1 and m_2 for each type. Each row gives the following data: (c, h_1, h_2) and $\mathcal{Y}$ matrix. This data is sufficient to write out the associated Ξ using Eq. (3.13).

Interestingly, $(3,6)$ admissible characters of type W_0 arise for $b > 0$ in all $(3,0)$ examples. So we do not include this in our tables.

3.3.1 Quasi-characters for the B_r series

The $B_{r,1}$ ($r \geq 0$) series with $c = \frac{2r+1}{2}$, $h_1 = \frac{1}{2}$, $h_2 = \frac{2r+1}{16}$. For these theories, one has $y_{21} = 2r+1$ and $y_{31} = 2^r$ leading to the following $\mathcal{Y}$-matrix:

$$\mathcal{Y} = \begin{pmatrix} r(2r+1) & -\frac{1}{3}(2r-31)\left(4r^2+68r+225\right) & 2^{12-r}(2r-23) \\ 2r+1 & -(r-11)(2r+25) & -2^{12-r} \\ 2^r & -2^r(2r+25) & 2r-23 \end{pmatrix} \tag{3.16}$$

Here $r = 0$ corresponds to the Ising model, $r = 1$ is also $A_{1,2}$ and $r = 2$ is also $C_{2,1}$. Note that column three of the $\mathcal{Y}$ matrix is fractional for $r > 12$. Multiplying $\mathbb{Y}_2$ by a suitable power of 2 (2^{r-12}), we obtain quasi-characters with integral entries. Alternately, when we construct admissible characters by taking linear combinations involving $\mathbb{Y}_2$, we take the coefficients to be integer multiples of the same power of two.

All theories share the same S-matrix:

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} .$$

3.3.2 Quasi-characters for the D_r series

For the $D_{r,1}$ series, one has $c = r$, $h_1 = \frac{1}{2}$, $h_2 = \frac{r}{8}$. For these theories, one has $y_{21} = 2r$ and $y_{31} = 2^{r-1}$ leading to the following $\mathcal{Y}$-matrix:

$$\mathcal{Y} = \begin{pmatrix} r(2r-1) & -\frac{8}{3}(r-16)\left(r^2+16r+48\right) & 2^{14-r}(r-12) \\ 2r & -(r+12)(2r-23) & -2^{13-r} \\ 2^{r-1} & -2^r(r+12) & 2(r-12) \end{pmatrix} \tag{3.17}$$

Here $D_{2,1}$ is also the same as $A_{1,1}^{\otimes 2}$, and $D_{3,1}$ is the same as $A_{3,1}$. For D_4, we end up with a two-character theory as $h_1 = h_2 = \frac{1}{2}$. Note that column three of the $\mathcal{Y}$ matrix is fractional for $r > 13$. Multiplying $\mathbb{Y}_2$ by a suitable power of 2, we obtain quasi-characters with integral entries. Alternately, when we construct admissible characters by taking linear combinations involving $\mathbb{Y}_2$, we take the coefficients to be integer multiples of the same power of two.

The character corresponding to $h_2 = \frac{r}{8}$ has multiplicity of two in the RCFT and thus this theory has four primaries. The unresolved and resolved S-matrix are

$$S_{\text{unresolved}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix} \quad , \quad S_{\text{resolved}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-r} & -i^{-r} \\ 1 & -1 & -i^{-r} & i^{-r} \end{pmatrix} . \tag{3.18}$$

3.3.3 Quasi-characters for other unitary examples

S. No.	c	h_1	h_2	$\mathcal{Y}$ matrix
I.	GHM Dual to $B_{r,1}$ models ($0 \leq r \leq 15$)			
	Multiplicities: $m_1 = 1,\ m_2 = 1$ S matrix: $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$			
$\ell = 0$	$\frac{47-2r}{2}$	$\frac{3}{2}$	$\frac{31-2r}{16}$	$\begin{pmatrix} r(47-2r) & 23-2r & 2^{1+r} \\ \frac{1}{3}(47-2r)(31-2r)(9+2r) & (11-r)(23-2r) & -2^{1+r}(1+2r) \\ 2^{11-r}(47-2r) & -2^{11-r} & -(1+2r) \end{pmatrix}$
II.	GHM Dual to $D_{r,1}$ models ($0 \leq r \leq 15$)			
	Multiplicities: $m_1 = 1,\ m_2 = 2$ S matrix: $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$			
ℓ=0	$24-r$	$\frac{3}{2}$	$\frac{16-r}{8}$	$\begin{pmatrix} -(24-r)(2r-1) & 24-2r & 2^{1+r} \\ \frac{8}{3}(24-r)(16-r)(4+r) & (12-r)(23-2r) & -2^{1+r}r \\ 2^{12-r}(24-r) & -2^{11-r} & -2r \end{pmatrix}$

Table 1: $(3,0)$ models that appear as GHM coset duals of the $B_{r,1}$ and $D_{r,1}$ theories.

S. No.	c	h_1	h_2	$\mathcal{Y}$ matrix
III.	Type $A_{2,1}^{\otimes 2}$			
	Multiplicities: $m_1 = 4,\ m_2 = 4$ S matrix: $\begin{pmatrix} \frac{1}{3} & \frac{4}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$			
	4	$\frac{1}{3}$	$\frac{2}{3}$	$\begin{pmatrix} 16 & 34992 & 2916 \\ 3 & 80 & -54 \\ 9 & -2430 & 156 \end{pmatrix}$
	20	$\frac{4}{3}$	$\frac{5}{3}$	$\begin{pmatrix} 80 & 108 & 12 \\ 1215 & 156 & -18 \\ 8748 & -1458 & 16 \end{pmatrix}$

S. No.	c	h_1	h_2	$\mathcal{Y}$ matrix
	12	$\frac{2}{3}$	$\frac{4}{3}$	$\begin{pmatrix} 156 & 4860 & 36 \\ 27 & 80 & -6 \\ 729 & -17496 & 16 \end{pmatrix}$
IV.	Type $A_{4,1}$			
	Multiplicities: $m_1 = 2,\ m_2 = 2$			
	S matrix: $\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \frac{1}{2}\left(\sqrt{5}-1\right) & -\frac{1}{2}\left(\sqrt{5}+1\right) \\ 1 & -\frac{1}{2}\left(\sqrt{5}+1\right) & \frac{1}{2}\left(\sqrt{5}-1\right) \end{pmatrix}$			
	4	$\frac{2}{5}$	$\frac{3}{5}$	$\begin{pmatrix} 24 & 14950 & 3400 \\ 5 & 92 & -119 \\ 10 & -1196 & 136 \end{pmatrix}$
	20	$\frac{7}{5}$	$\frac{8}{5}$	$\begin{pmatrix} 120 & 52 & 14 \\ 2500 & 104 & -49 \\ 8125 & -676 & 28 \end{pmatrix}$

Table 2: $(3,0)$ models of type $A_{2,1}^{\otimes 2}$ and type $A_{4,1}$

3.3.4 Quasi-characters for some non-unitary examples

For non-unitary theories, the quantities (c, h_1, h_2) refer to the unitary presentation. By this, c refers to what is sometimes called c_{eff} which equals $(c - 24h_{\text{min}})$, where c is the true cental charge and h_{min} is the minimal conformal weight. Similarly, (h_1, h_2) are shifted upwards by $-h_{\text{min}}$. Note that $h_{\text{min}} < 0$ non-unitary theories and $h_{\text{min}} = 0$ for unitary theories.

Type LY_2 refers to the class of non-unitary RCFTs in the class of the minimal model $M(7,2)$ and LY_1 refers to the minimal model, $M(5,2)$ associated with the Lee-Yang model [17, 29]. The S-matrices quoted have been obtained using the results of Mukhi et al. [30].

S. No.	c	h_1	h_2	$\mathcal{Y}$ matrix	S matrix
V.	Type LY_2				
	Multiplicities: $m_1 = 1,\ m_2 = 1$				
	S matrix: $\frac{2}{\sqrt{7}} \begin{pmatrix} \cos\left(\frac{\pi}{14}\right) & \cos\left(\frac{3\pi}{14}\right) & \sin\left(\frac{\pi}{7}\right) \\ \cos\left(\frac{3\pi}{14}\right) & -\sin\left(\frac{\pi}{7}\right) & -\cos\left(\frac{\pi}{14}\right) \\ \sin\left(\frac{\pi}{7}\right) & -\cos\left(\frac{\pi}{14}\right) & \cos\left(\frac{3\pi}{14}\right) \end{pmatrix}$ (a)				
	$\frac{2}{\sqrt{7}} \begin{pmatrix} \cos\left(\frac{3\pi}{14}\right) & \cos\left(\frac{\pi}{14}\right) & \sin\left(\frac{\pi}{7}\right) \\ \cos\left(\frac{\pi}{14}\right) & -\sin\left(\frac{\pi}{7}\right) & -\cos\left(\frac{3\pi}{14}\right) \\ \sin\left(\frac{\pi}{7}\right) & -\cos\left(\frac{3\pi}{14}\right) & \cos\left(\frac{\pi}{14}\right) \end{pmatrix}$ (b)				
1.	$\frac{236}{7}$	$\frac{16}{7}$	$\frac{17}{7}$	$\begin{pmatrix} 0 & \frac{5}{17} & \frac{1}{17} \\ 715139 & 220 & 46 \\ 848656 & 145 & 32 \end{pmatrix}$	(b)

2.	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\begin{pmatrix} 1 & 45954 & 2925 \\ 1 & -74 & -55 \\ 1 & -1702 & 325 \end{pmatrix}$	(a)
3.	$\frac{164}{7}$	$\frac{13}{7}$	$\frac{11}{7}$	$\begin{pmatrix} 41 & 5 & 17 \\ 50922 & -10 & -782 \\ 4797 & -11 & 221 \end{pmatrix}$	(a)
4.	$\frac{12}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\begin{pmatrix} 6 & 22990 & 3510 \\ 3 & -132 & -117 \\ 2 & -627 & 378 \end{pmatrix}$	(b)
5.	$\frac{156}{7}$	$\frac{12}{7}$	$\frac{11}{7}$	$\begin{pmatrix} 78 & 9 & 10 \\ 27170 & -36 & -285 \\ 5070 & -39 & 210 \end{pmatrix}$	(b)
6.	$\frac{44}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\begin{pmatrix} 88 & 1564 & 1972 \\ 44 & -184 & -725 \\ 11 & -138 & 348 \end{pmatrix}$	(b)
7.	$\frac{124}{7}$	$\frac{9}{7}$	$\frac{10}{7}$	$\begin{pmatrix} 248 & 76 & 13 \\ 2108 & -152 & -78 \\ 2108 & -475 & 156 \end{pmatrix}$	(b)
8.	$\frac{52}{7}$	$\frac{6}{7}$	$\frac{4}{7}$	$\begin{pmatrix} 156 & 475 & 2108 \\ 78 & -152 & -2108 \\ 13 & -76 & 248 \end{pmatrix}$	(a)
9.	$\frac{116}{7}$	$\frac{8}{7}$	$\frac{10}{7}$	$\begin{pmatrix} 348 & 138 & 11 \\ 725 & -184 & -44 \\ 1972 & -1564 & 88 \end{pmatrix}$	(a)
10.	$\frac{60}{7}$	$\frac{8}{7}$	$\frac{3}{7}$	$\begin{pmatrix} 210 & 39 & 5070 \\ 285 & -36 & -27170 \\ 10 & -9 & 78 \end{pmatrix}$	(a)
11.	$\frac{108}{7}$	$\frac{6}{7}$	$\frac{11}{7}$	$\begin{pmatrix} 378 & 627 & 2 \\ 117 & -132 & -3 \\ 3510 & -22990 & 6 \end{pmatrix}$	(a)
12.	$\frac{68}{7}$	$\frac{9}{7}$	$\frac{3}{7}$	$\begin{pmatrix} 221 & 11 & 4797 \\ 782 & -10 & -50922 \\ 17 & -5 & 41 \end{pmatrix}$	(b)
13.	$\frac{100}{7}$	$\frac{5}{7}$	$\frac{11}{7}$	$\begin{pmatrix} 325 & 1702 & 1 \\ 55 & -74 & -1 \\ 2925 & -45954 & 1 \end{pmatrix}$	(b)
14.	$\frac{100}{7}$	$\frac{12}{7}$	$\frac{4}{7}$	$\begin{pmatrix} 380 & -1 & 1247 \\ 11495 & 1 & -260623 \\ 55 & -1 & -129 \end{pmatrix}$	(b)

Table 3: $(3,0)$ models of type LY_2. There are two kinds of S-matrices.

S. No.	c	h_1	h_2	$\mathcal{Y}$ matrix	S matrix
VI.	Type $(LY_1)^{\otimes 2}$				
	Multiplicities: $m_1 = 1,\ m_2 = 2$				
	S matrix: $\frac{1}{\sqrt{5}}\begin{pmatrix} \frac{1}{2}\left(\sqrt{5}+1\right) & \frac{1}{2}\left(\sqrt{5}-1\right) & 2 \\ \frac{1}{2}\left(\sqrt{5}-1\right) & \frac{1}{2}\left(\sqrt{5}+1\right) & -2 \\ 1 & -1 & -1 \end{pmatrix}$ (a)				
	$\frac{1}{\sqrt{5}}\begin{pmatrix} \frac{1}{2}\left(\sqrt{5}-1\right) & \frac{1}{2}\left(\sqrt{5}+1\right) & 2 \\ \frac{1}{2}\left(\sqrt{5}+1\right) & \frac{1}{2}\left(\sqrt{5}-1\right) & -2 \\ 1 & -1 & 1 \end{pmatrix}$ (b)				
	$\frac{164}{5}$	$\frac{12}{5}$	$\frac{11}{5}$	$\begin{pmatrix} 0 & \frac{1}{11} & \frac{10}{11} \\ 615164 & 28 & 220 \\ 254200 & 25 & 224 \end{pmatrix}$	(a)
	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\begin{pmatrix} 2 & 3249 & 47500 \\ 1 & 380 & -1250 \\ 1 & -57 & -130 \end{pmatrix}$	(a)
	$\frac{116}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$\begin{pmatrix} 58 & 11 & 10 \\ 4959 & 220 & -30 \\ 27550 & -275 & -26 \end{pmatrix}$	(a)
	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\begin{pmatrix} 3 & 42483 & 2550 \\ 3 & 27 & -50 \\ 5 & -2295 & 222 \end{pmatrix}$	(b)
	$\frac{108}{5}$	$\frac{9}{5}$	$\frac{7}{5}$	$\begin{pmatrix} 27 & 3 & 50 \\ 42483 & 3 & -2550 \\ 2295 & -5 & 222 \end{pmatrix}$	(b)
	$\frac{28}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\begin{pmatrix} 28 & 676 & 16250 \\ 49 & 104 & -5000 \\ 7 & -26 & 120 \end{pmatrix}$	(b)
	$\frac{92}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$\begin{pmatrix} 92 & 119 & 10 \\ 1196 & 136 & -20 \\ 7475 & -1700 & 24 \end{pmatrix}$	(b)
	$\frac{36}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\begin{pmatrix} 144 & 1452 & 1100 \\ 12 & 336 & -150 \\ 45 & -770 & -228 \end{pmatrix}$	(a)
	$\frac{84}{5}$	$\frac{7}{5}$	$\frac{6}{5}$	$\begin{pmatrix} 336 & 12 & 150 \\ 1452 & 144 & -1100 \\ 770 & -45 & -228 \end{pmatrix}$	(a)

Table 4: $(3,0)$ models of type $(LY_1)^{\otimes 2}$.

3.4 Admissible characters

In the previous sub-section, we obtained explicit formulae for quasi-characters starting from all known $(3,0)$ CFT listed in Das-Gowdigere-Mukhi [10]. The next step is to look for admissible characters that can be obtained using linear combinations of the quasi-characters along with the original $(3,0)$ theory. Our main goal is to list out all $(3,6)$ admissible characters obtained in this fashion. We will also obtain theories with $(3,12)$ and $(3,18)$ along the way. The explicit solutions are listed in the Tables given in Appendix B.

3.4.1 Admissible characters of type U

It is easy to track the change in the Wronskian index of admissible characters. One has $\delta\ell = -6\sum_a \delta\alpha_a$ – in other words, the change in Wronskian idex is determined by the studying the leading exponents of the admissible character. Starting with an RCFT with exponents $(\alpha_1, \alpha_2, \alpha_3)$, one has

$$\mathbb{U}_2(b_1,b_2) = \begin{pmatrix} q^{\alpha_0}(1+O(q)) \\ q^{\alpha_1-1}(b_1 + p_{1,1}(b_1,b_2)\, q + p_{1,2}(b_1,b_2)\, q^2 + O(q^3)) \\ q^{\alpha_2-1}(b_2 + p_{2,1}(b_1,b_2)\, q + p_{2,2}(b_1,b_2)\, q^2 + O(q^3)) \end{pmatrix} \quad , \tag{3.19}$$

where $p_{i,j}$ are linear polynomials in (b_1, b_2). For admissible solutions, one imposes the conditions $b_i \geq 0$ and $p_{i,j} \geq 0$. Since $\sum_a \delta\alpha_a = -2$, we see that $\delta\ell = 12$. Thus, starting from a $(3,0)$ theory, an admissible character obtained as $\mathbb{U}_2$ will have Wronskian index 12.

Next consider the case of $\mathbb{U}_{1,1}$, where $b_2 = 0$ with $b_1 > 0$, then one has $\sum_a \delta\alpha_a = -1$ leading to an admissible character with Wronskian index 6 when $p_{2,1}(b_1, 0) > 0$. However, if $p_{2,1}(b_1,0) = 0$ for an integral value of b_1, then we obtain an admissible character with vanishing Wronskian index.

The set of admissible characters of type U are integral points that lie in the interior of a bounded convex polytope. Interior points are $(3,12)$ characters, while those on the boundaries $b_1 = 0$ and $b_2 = 0$ will be typically $(3,6)$ while those at the corners of the polytope (if they are integral points) will be $(3,0)$. This is illustrated with an example in Fig. 1.

3.4.2 Admissible characters of type W

$$\mathbb{W}_2(b,c_1,c_2) = \begin{pmatrix} q^{\alpha_0-1}(1+b\,q+O(q^2)) \\ q^{\alpha_1-1}(c_1 + p_{1,1}(b,c_1,c_2)\, q + p_{1,2}(b_1,b_2)\, q^2 + O(q^3)) \\ q^{\alpha_2-1}(c_2 + p_{2,1}(b,c_1,c_2)\, q + p_{2,2}(b,c_1,c_2)\, q^2 + O(q^3)) \end{pmatrix} \quad , \tag{3.20}$$

We list the various possibilities that depend on specific choices for the free parameters (b, c_1, c_2).

1. Since $\sum_a \delta\alpha_a = -3$, we have $\delta\ell = 18$ when $c_1, c_2 \neq 0$. These are of type $\mathbb{W}_2$.

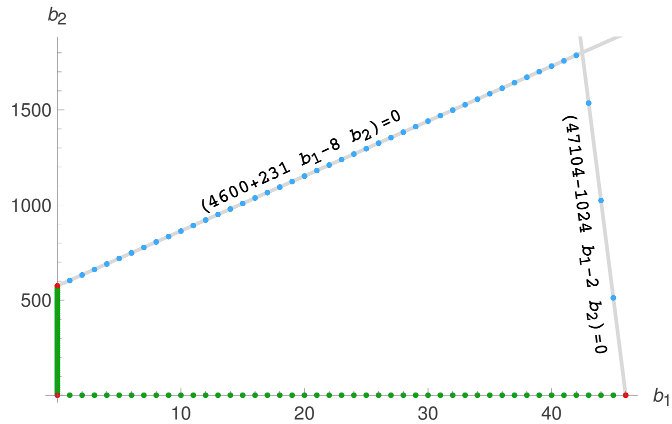

Figure 1: Example of type U admissible characters starting with a $c = 23$, $h_1 = 3/2$, $h_2 = 15/8$ RCFT. They occur as integral points on the interior and boundary of a quadrilateral. We see two other $(3, 0)$ theories at the corners $(b_1, b_2) = (46, 0)$ and $(b_1, b_2) = (0, 575)$. These are shown as red dots and connected by green dots which are $(3, 6)$ admissible characters. We show the $(3, 12)$ points that lie on the boundaries as blue dots. All integral interior points are $(3, 12)$ admissible characters and are not shown in the figure to avoid clutter.

2. However, when one of c_1 or c_2 vanishes, one has $\delta\ell = 12$ generically. However, when $c_2 = 0$ and $p_{2,1}(b, c_1, c_2 = 0) = 0$, we obtain $\delta\ell = 6$. These are of type $\mathbb{W}_{1,1}$.
3. If additionally, we can set $p_{2,2}(b, c_1, c_2 = 0) = 0$, we can obtain theories with $\delta\ell = 0$. This happens only in a limited set of examples.
4. If $c_1 = c_2 = 0$, we obtain admissible solutions for $b \geq b_{\min}$. These are of type $\mathbb{W}_0$ are generically with $\delta\ell = 6$.

An infinite number of admissible theories appear inside a non-compact polytope in the space (b, c_1, c_2) with faces given by $c_1 = 0$, $c_2 = 0$, $b = b_{\min}$ and $p_{i,m} = 0$. This is unlike those of type U where we obtained only a finite number of admissible solutions as the polytope was compact.

3.4.3 Admissible characters of type V

$$\mathbb{V}_2 = \begin{pmatrix} q^{\alpha_0-2}(1 + b_1\, q + b_2 q^2 + O(q^3)) \\ q^{\alpha_1-1}(s_1 + p_{1,1}(b, s_1, s_2)\, q + p_{1,2}(b_1, b_2)\, q^2 + O(q^3)) \\ q^{\alpha_2-1}(s_2 + p_{2,1}(b, s_1, s_2)\, q + p_{2,2}(b, s_1, s_2)\, q^2 + O(q^3)) \end{pmatrix} \quad , \tag{3.21}$$

1. When $s_1, s_2 > 0$, we have $\delta\ell = 24$. This is of type $\mathbb{V}_2$.

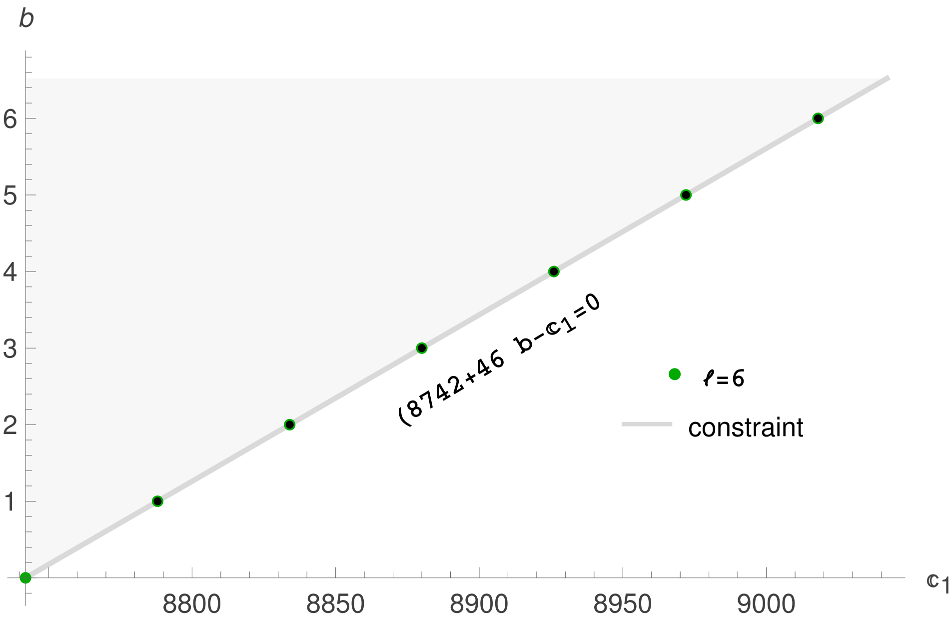

Figure 2: In this type $\mathbb{W}_{1,1}$ example, we obtain $(3,6)$ theories on integral points that lie on the line $\frac{1}{46}(c_1 - 8742)$. All integral points inside the polytope and on the $c_1 = 0$ line are $(3,12)$ admissible solutions. There are no $(3,0)$ solutions.

2. When one of (s_1, s_2) is zero with the other being non-zero, we have $\delta\ell = 18$. When $s_1 > 0$, we look for additional conditions on the constants (s_2, b_1, b_2) such that $p_{2,1} = 0$ which will lead to $\delta\ell = 12$. One can get $\delta\ell = 6, 0$ at higher co-dimension.

3. When $s_1 = s_2 = 0$, we obtain $\delta\ell = 12$. However, by imposing vanishing conditions, at codimension two, we can obtain $\delta\ell = 0$.

4 $(3,3)$ theories

We consider the $(3,3)$ admissible characters obtained using the MLDE method. For a $(3,3)$ CFT, one has $\sum_{i=0}^{2} \alpha_i = 0$. Further, one has

$$3a_1 + 2(b_1 + 2b_2) = 12 \ .$$

$a_1 = 2$ and $b_1 = b_2 = 1$ is the only indecomposable solution. For instance, $a_1 = 0$ is not allowed as the S-matrix is then diagonal and hence the theory is a tensor product of three single-character theories. Thus one has $\det(S) = -1$ and $\det(U) = 1$. Further, $\det(T) = 1$ which is consistent with $\sum_i \alpha_i = 0$.

4.1 A new construction of $(3,3)$ admissible characters

Let $\varXi(\tau)$ represent the basis of VVMFs with a given multiplier system. Further, let $(\varLambda, \mathcal{Y})$ be the data associated with the matrix MLDE associated with $\varXi(\tau)$. Bantay and Gannon introduce the notion of a dual of $\varXi(\tau)$ that is defined by the involutive transformation [19]

$$\varXi^{\vee}(\tau) := \frac{E_4(\tau)^2 E_6(\tau)}{\varDelta(\tau)^{7/6}} \left(\varXi(\tau)^T\right)^{-1} \ . \tag{4.1}$$

The data associated with $\varXi^{\vee}(\tau)$ is the given by

$$\begin{aligned} \varLambda^{\vee} &= -\frac{7}{6} I_n - \varLambda \ , \\ \mathcal{Y}^{\vee} &= 4 I_n - \mathcal{Y}^T \ . \end{aligned} \tag{4.2}$$

Let (n, ℓ) denote the rank and Wronskian index of $\varXi(\tau)$. Then $\varXi^{\vee}(\tau)$ has rank n and Wronskian index given by

$$\ell^{\vee} = (n-5)(n-1) + 7 - \ell \ . \tag{4.3}$$

The S-matrix for $\varXi^{\vee}$ is given by $S^{\vee} = -(S^{-1})^T$, where S is the S-matrix of the original theory. The minus sign arises from the one-dimensional character, $\frac{E_4^2 E_6}{\varDelta^{7/6}}$. Next, let $D = \mathrm{Diag}(1, m_1, \ldots, m_{n-1})$ be the matrix of multiplicities of the original theory. Then, $D^{\vee} \propto D^{-1}$, the proportionality constant is fixed by requiring the vacuum character to have multiplicity one.

This involution relates $(2,0)$ characters to $(2,4)$ theories and maps $(2,2)$ to itself. What is interesting is that this involution maps $(3,0)$ VVMF's to $(3,3)$ VVMFs, thus providing a new construction of $(3,3)$ admissible characters. Not all $(3,3)$ VVMF's obtained this way turn out to be admissible. However, **all** known $(3,3)$ theories arise in this fashion. An additional bonus of this construction is that we can determine the S-matrices of known $(3,3)$ admissible characters along with multiplicities that cannot be obtained from the MLDE approach.

We illustrate this with example of the Ising model. A straightahead computation gives the following result (we show only the first few terms).

$$\varXi^{\vee} = \begin{pmatrix} \frac{1}{q^{55/48}} + \frac{4}{q^{7/48}} & -\frac{1}{q^{7/48}} - 3826 q^{41/48} & -\frac{1}{q^{7/48}} - 53 q^{41/48} \\ -2325 q^{17/48} - 1785755 q^{65/48} & \frac{1}{q^{31/48}} - 271 q^{17/48} & 25 q^{17/48} + 246 q^{65/48} \\ -94208 q^{19/24} - 21581824 q^{43/24} & 4096 q^{19/24} + 208896 q^{43/24} & \frac{1}{q^{5/24}} + 27 q^{19/24} \end{pmatrix}$$

The VVMF corresponding to the third column becomes admissible on switching the sign of the first term. This is the $(3,3)$ admissible solution with $c = 5$ and $h_1 = 1/16$ and $h_2 = 9/16$. One needs to permute the rows to match with solution S1 given in [9]. In computing the S-matrix, we need to account for the changes in sign and permutations. In Table 5, we give the results for the 15 $(3,3)$ admissible characters.

	(3,3) theory			Dual (3,0) Theory		
S.No.	c	h_1	h_2	$\tilde{c}$	$\tilde{h}_1$	$\tilde{h}_2$
$m_1 = m_2 = 1$ $S : \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$,						
1	5	$\frac{1}{16}$	$\frac{9}{16}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}$
				$\frac{25}{2}$	$\frac{3}{2}$	$\frac{9}{16}$
2	7	$\frac{3}{16}$	$\frac{11}{16}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{16}$
				$\frac{27}{2}$	$\frac{3}{2}$	$\frac{11}{16}$
$m_1 = m_2 = 2$ $S : \begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$						
3	6	$\frac{1}{8}$	$\frac{5}{8}$	1	$\frac{1}{2}$	$\frac{1}{8}$
				13	$\frac{3}{2}$	$\frac{5}{8}$
4	8	$\frac{1}{4}$	$\frac{3}{4}$	2	$\frac{1}{2}$	$\frac{1}{4}$
				14	$\frac{3}{2}$	$\frac{3}{4}$
$m_1 = m_2 = 2$ $S : \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{1}{10}\left(\sqrt{5}+5\right) & \frac{1}{10}\left(5-\sqrt{5}\right) \\ \frac{1}{\sqrt{5}} & \frac{1}{10}\left(5-\sqrt{5}\right) & -\frac{1}{10}\left(\sqrt{5}+5\right) \end{pmatrix}$						
5	8	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
				$\frac{84}{5}$	$\frac{6}{5}$	$\frac{7}{5}$
6	16	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{36}{5}$	$\frac{4}{5}$	$\frac{3}{5}$
				$\frac{76}{5}$	$\frac{4}{5}$	$\frac{8}{5}$
7	24	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{44}{5}$	$\frac{6}{5}$	$\frac{2}{5}$
				$\frac{116}{5}$	$\frac{9}{5}$	$\frac{8}{5}$
8	32	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{76}{5}$	$\frac{9}{5}$	$\frac{3}{5}$

$m_1 = m_2 = 1$

$$S : \frac{2}{\sqrt{7}} \begin{pmatrix} \sin\left(\frac{\pi}{7}\right) & \cos\left(\frac{\pi}{14}\right) & \cos\left(\frac{3\pi}{14}\right) \\ \cos\left(\frac{\pi}{14}\right) & -\cos\left(\frac{3\pi}{14}\right) & \sin\left(\frac{\pi}{7}\right) \\ \cos\left(\frac{3\pi}{14}\right) & \sin\left(\frac{\pi}{7}\right) & -\cos\left(\frac{\pi}{14}\right) \end{pmatrix}$$

9	$\frac{48}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{4}{7}$ $\frac{100}{7}$	$\frac{1}{7}$ $\frac{5}{7}$	$\frac{3}{7}$ $\frac{11}{7}$
10	$\frac{64}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{12}{7}$ $\frac{108}{7}$	$\frac{2}{7}$ $\frac{6}{7}$	$\frac{3}{7}$ $\frac{11}{7}$
11	$\frac{104}{7}$	$\frac{5}{7}$	$\frac{8}{7}$	$\frac{44}{7}$ $\frac{116}{7}$	$\frac{4}{7}$ $\frac{8}{7}$	$\frac{5}{7}$ $\frac{10}{7}$
12	$\frac{120}{7}$	$\frac{9}{7}$	$\frac{6}{7}$	$\frac{52}{7}$ $\frac{124}{7}$	$\frac{4}{7}$ $\frac{9}{7}$	$\frac{6}{7}$ $\frac{10}{7}$
13	$\frac{160}{7}$	$\frac{12}{7}$	$\frac{8}{7}$	$\frac{60}{7}$ $\frac{156}{7}$	$\frac{3}{7}$ $\frac{11}{7}$	$\frac{8}{7}$ $\frac{12}{7}$
14	$\frac{176}{7}$	$\frac{9}{7}$	$\frac{13}{7}$	$\frac{68}{7}$ $\frac{164}{7}$	$\frac{3}{7}$ $\frac{11}{7}$	$\frac{9}{7}$ $\frac{13}{7}$
15	$\frac{216}{7}$	$\frac{12}{7}$	$\frac{15}{7}$	$\frac{100}{7}$	$\frac{4}{7}$	$\frac{12}{7}$

Table 5: We provide the S-matrix and multiplicities for all known $(3,3)$ admissible characters. The third column gives the dual $(3,0)$ theories – there can be more than one when Ξ leads to multiple admissible characters.

Now that we know the S-matrix of the $(3,3)$ admissible characters, we can see if they can represent RCFTs. The next step is to determine the fusion rules and see if they lead to positive, integral coefficients. For the cases when the multipicities are one, it is somewhat straightforward. Even in these cases, we have consider two possibilities – the theory is unitary or non-unitary. To account for non-unitary theories, we allow for the identity operator to be the operator with smallest dimension. We find the following:

1. For items 1 and 2 in Table 5, the theories cannot be unitary as $S_{00} = 0$. Allowing for non-unitary theories leads to fusion coefficients being negative. Thus, these two cannot be RCFTs.

2. For items 9-15, we find acceptable fusion rules compatible with a unitary RCFT. The fusion rules are given in Table 6.

3. We next consider the remaining theories. The two familes of S-matrices needs to be first resolved into 5×5 matrices. There are three undetermined coefficients which are determined by the condition that $S_{\text{resolved}} \cdot S_{\text{resolved}} = C$, where C satisfies $C^2 = 1$ with C being the identity matrix being the simplest possibility. It turns out that

	1	ϕ_1	ϕ_2
1	1	ϕ_1	ϕ_2
ϕ_1	ϕ_1	$1+\phi_1+\phi_2$	$\phi_1+\phi_2$
ϕ_2	ϕ_2	$\phi_1+\phi_2$	$1+\phi_1$

Table 6: The three primary operators are labelled $(1, \phi_1, \phi_2)$ and the fusion rules are given as a multiplication table.

none of the solutions lead to sensible fusion coefficients. Thus, we believe that these are not associated with RCFTs.

Remark: Using a different method, Gowdigere-Kala-Rawat have obtained the S-matrix and fusion coefficients for the 15 admissible characters [31]. Their results are consistent with the one discussed here.

4.2 $(3,3)$ Quasi-characters and Admissible characters from VVMFs

Column 3 of Table 7 provides the q-series of the $(3,3)$ admissible characters. The important addition we provide is the degeneracies that we obtain through the duality. For instance, in item 2, the third character has degeneracy 2. In column 4 of Table 7, we list the $\mathcal{Y}$ matrices corresponding to the 15 $(3,3)$ solutions obtained using the MLDE method in [9]. The quasi-characters $\mathbb{Y}_1$ and $\mathbb{Y}_2$ can be obtained by solving the matrix modular differential equation of Gannon [20]. Once these are obtained, we look for admissible characters of types U, W and V. The last column of Table 7, lists the range of values $0 \leq b \leq b_{\text{min}}$ for which we obtain $(3,9)$ admissible characters of type W_0. Notice that there are cases where we obtain *no* solutions. Table 8 lists $(3,9)$ admissible characters of type U_i, Table 9 lists $(3,9)$ admissible characters of type $W_{1,i}$ and Table 10 lists $(3,9)$ admissible characters of type V_0 that arise from the 7 solutions that are potentially RCFTs.

Sl. No.	c	h_1	h_2	$\mathbb{X}$	$\mathcal{Y}$ matrix	bmin$_2$
1	5	$\frac{1}{16}$	$\frac{9}{16}$	$\begin{pmatrix} q^{-\frac{5}{24}}(1+27q+106q^2+\dots) \\ q^{-\frac{7}{48}}(1+53q+404q^2+\dots) \\ q^{\frac{17}{48}}(25+246q+1297q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 27 & 94208 & 4096 \\ 1 & 4 & 1 \\ 25 & 2325 & -271 \end{pmatrix}$	–
2	7	$\frac{3}{16}$	$\frac{11}{16}$	$\begin{pmatrix} q^{-\frac{7}{24}}(1+25q+53q^2+\dots) \\ 2q^{-\frac{5}{48}}(1+106q+1214q^2+\dots) \\ 2q^{\frac{19}{48}}(27+433q+3400q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 25 & 43008 & 2048 \\ 2 & 1 & 3 \\ 54 & 2871 & -266 \end{pmatrix}$	–

3	6	$\frac{1}{8}$	$\frac{5}{8}$	$\begin{pmatrix} q^{-\frac{1}{4}}(1+26q+79q^2+\dots) \\ q^{-\frac{1}{8}}(1+79q+755q^2+\dots) \\ 2q^{\frac{3}{8}}(13+163q+1053q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 26 & 90112 & 4096 \\ 1 & 3 & 2 \\ 26 & 2600 & -269 \end{pmatrix}$	–
4	8	$\frac{1}{4}$	$\frac{3}{4}$	$\begin{pmatrix} q^{-\frac{1}{3}}(1+24q+28q^2+\dots) \\ 2q^{-\frac{1}{12}}(1+134q+1809q^2+\dots) \\ 8q^{\frac{5}{12}}(7+142q+1329q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 24 & 40960 & 2048 \\ 2 & -2 & 4 \\ 56 & 3136 & -262 \end{pmatrix}$	–
5	8	$\frac{1}{5}$	$\frac{4}{5}$	$\begin{pmatrix} q^{-\frac{1}{3}}(1+134q+1920q^2+\dots) \\ q^{-\frac{2}{15}}(1+190q+2832q^2+\dots) \\ q^{\frac{7}{15}}(57+1159q+10526q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 134 & 47500 & 1250 \\ 1 & 2 & 1 \\ 57 & 3249 & -376 \end{pmatrix}$	3^*
6	16	$\frac{4}{5}$	$\frac{6}{5}$	$\begin{pmatrix} q^{-\frac{2}{3}}(1+232q+31076q^2+\dots) \\ 5q^{-\frac{2}{15}}(9+2792q+101678q^2+\dots) \\ 5q^{\frac{8}{15}}(154+13264q+\dots) \end{pmatrix}$	$\begin{pmatrix} 232 & 1100 & 150 \\ 45 & -140 & 12 \\ 770 & 1452 & -332 \end{pmatrix}$	0
7	24	$\frac{6}{5}$	$\frac{9}{5}$	$\begin{pmatrix} q^{-1}(1+30q+87786q^2+\dots) \\ 25q^{\frac{1}{5}}(11+10212q+\dots) \\ 25q^{\frac{4}{5}}(1102+166953q+\dots) \end{pmatrix}$	$\begin{pmatrix} 30 & 30 & 10 \\ 275 & -216 & 11 \\ 27550 & 4959 & -54 \end{pmatrix}$	0
8	32	$\frac{9}{5}$	$\frac{11}{5}$	$\begin{pmatrix} q^{-\frac{4}{3}}(1+3q+62500q^2+\dots) \\ 625q^{\frac{7}{15}}(19+17100q+\dots) \\ 625q^{\frac{13}{15}}(434+135997q+\dots) \end{pmatrix}$	$\begin{pmatrix} 3 & -1 & 1 \\ 11875 & -433 & 57 \\ 271250 & 682 & 190 \end{pmatrix}$	0
9	$\frac{48}{7}$	$\frac{1}{7}$	$\frac{5}{7}$	$\begin{pmatrix} q^{-\frac{2}{7}}(1+78q+784q^2+\dots) \\ q^{-\frac{1}{7}}(1+133q+1618q^2+\dots) \\ q^{\frac{3}{7}}(55+890q+6720q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 78 & 45954 & 1702 \\ 1 & 3 & 1 \\ 55 & 2925 & -321 \end{pmatrix}$	3^*
10	$\frac{64}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\begin{pmatrix} q^{-\frac{8}{21}}(1+136q+2417q^2+\dots) \\ q^{-\frac{2}{21}}(3+632q+10787q^2+\dots) \\ q^{\frac{10}{21}}(117+2952q+32220q^2+\dots) \end{pmatrix}$	$\begin{pmatrix} 136 & 22990 & 627 \\ 3 & -2 & 2 \\ 117 & 3510 & -374 \end{pmatrix}$	2
11	$\frac{104}{7}$	$\frac{5}{7}$	$\frac{8}{7}$	$\begin{pmatrix} q^{-\frac{13}{21}}(1+188q+17260q^2+\dots) \\ q^{\frac{2}{21}}(44+13002q+424040q^2\dots) \\ q^{\frac{11}{21}}(725+52316q+\dots) \end{pmatrix}$	$\begin{pmatrix} 188 & 1564 & 138 \\ 44 & -84 & 11 \\ 725 & 1972 & -344 \end{pmatrix}$	0
12	$\frac{120}{7}$	$\frac{6}{7}$	$\frac{9}{7}$	$\begin{pmatrix} q^{-\frac{5}{7}}(1+156q+28926q^2+\dots) \\ q^{\frac{1}{7}}(78+28692q+1194804q^2+\dots) \\ q^{\frac{4}{21}}(2108+200787q+\dots) \end{pmatrix}$	$\begin{pmatrix} 156 & 475 & 76 \\ 78 & -152 & 13 \\ 2108 & 2108 & -244 \end{pmatrix}$	0
13	$\frac{160}{7}$	$\frac{8}{7}$	$\frac{12}{7}$	$\begin{pmatrix} q^{-\frac{20}{21}}(1+40q+60440q^2+\dots) \\ q^{\frac{4}{21}}(285+227848q+\dots) \\ q^{\frac{16}{21}}(27170+3857360q+\dots) \end{pmatrix}$	$\begin{pmatrix} 40 & 39 & 9 \\ 285 & -206 & 10 \\ 27170 & 5070 & -74 \end{pmatrix}$	0

14	$\frac{176}{7}$	$\frac{9}{7}$	$\frac{13}{7}$	$\begin{pmatrix} q^{-\frac{22}{21}}(1+14q+66512q^2+\dots) \\ q^{\frac{5}{21}}(782+718267q+\dots) \\ q^{\frac{17}{21}}(50922+8656740q+\dots) \end{pmatrix}$	$\begin{pmatrix} 14 & 11 & 5 \\ 782 & -217 & 17 \\ 50922 & 4797 & -37 \end{pmatrix}$	1
15	$\frac{216}{7}$	$\frac{12}{7}$	$\frac{15}{7}$	$\begin{pmatrix} q^{-\frac{9}{7}}(1+3q+52254q^2+\dots) \\ q^{\frac{3}{7}}(11495+10341870q+\dots) \\ q^{\frac{6}{7}}(260623+74348634q+\dots) \end{pmatrix}$	$\begin{pmatrix} 3 & -1 & 1 \\ 11495 & -376 & 55 \\ 260623 & 1247 & 133 \end{pmatrix}$	0

Table 7: The list of (3,3) admissible solutions. Column 3 gives the q-series along with degeneracies that follow from the duality with $(3,0)$ theories. The last column indicates the minimum value of b for which we obtain $(3,9)$ admissible solutions. In four examples, we indicate the absence of $(3,9)$ admissible solutions by a dash. For two cases, we obtain other $(3,3)$ solutions when $b = b_{\text{min}}$ and $(3,9)$ solutions when $b > b_{\text{min}}$. These are indicated by adding a * to the quoted value of b_{min}.

S. No.	c	h_1	h_2	b_i	h_1	h_2
1	$\frac{104}{7}$	$\frac{5}{7}$	$\frac{8}{7}$	$b_2 \leq 2$	$\frac{5}{7}$	$\frac{1}{7}$
2	$\frac{120}{7}$	$\frac{6}{7}$	$\frac{9}{7}$	$b_2 \leq 8$	$\frac{6}{7}$	$\frac{2}{7}$
3	$\frac{160}{7}$	$\frac{8}{7}$	$\frac{12}{7}$	$b_1 \leq 1$	$\frac{1}{7}$	$\frac{12}{7}$
				$b_2 \leq 367$	$\frac{8}{7}$	$\frac{5}{7}$
4	$\frac{176}{7}$	$\frac{9}{7}$	$\frac{13}{7}$	$b_1 \leq 3$	$\frac{2}{7}$	$\frac{13}{7}$
				$b_2 \leq 1376$	$\frac{9}{7}$	$\frac{6}{7}$
5	$\frac{216}{7}$	$\frac{12}{7}$	$\frac{15}{7}$	$b_1 \leq 3$	$\frac{12}{7}$	$\frac{15}{7}$
				$b_2 > 0$	$\frac{12}{7}$	$\frac{8}{7}$

Table 8: $(3,9)$ admissible solutions of type U_i using $(3,3)$ solutions .

S. No.	c	h_1	h_2	c_i	b	c	h_1	h_2
1.	$\frac{48}{7}$	$\frac{1}{7}$	$\frac{5}{7}$	$c_2 = 3 - b$	$0 \leq b < 3^*$	$\frac{216}{7}$	$\frac{15}{7}$	$\frac{5}{7}$
2.	$\frac{64}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$c_2 = 2$	0	$\frac{232}{7}$	$\frac{16}{7}$	$\frac{6}{7}$

Table 9: $(3,9)$ admissible solutions of type $W_{1,i}$ from the $(3,3)$ models.

S. No.	c	h_1	h_2	b_1	b_2	c	h_1	h_2
1.	$\frac{48}{7}$	$\frac{1}{7}$	$\frac{5}{7}$	$b_1 \geq 0$	$b_2 = 1 + 3b_1$	$\frac{384}{7}$	$\frac{22}{7}$	$\frac{19}{7}$
2.	$\frac{64}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$b_1 = 1 + 3x$	$b_2 = 3 + 4x$ $(x \in \mathbb{Z}_{\geq 0})$	$\frac{400}{7}$	$\frac{23}{7}$	$\frac{20}{7}$

Table 10: $(3, 9)$ admissible solutions of type V_0 using $(3, 3)$ solutions.

5 Discussion of results

5.1 GHM duality for VVMFs

Let $\mathbb{X}$ be a known RCFT with n characters and $\widetilde{\mathbb{X}}$ be its GHM dual [18]. One then has

$$\widetilde{\mathbb{X}}^T \cdot M \cdot \mathbb{X} = J(\tau) + b \ , \tag{5.1}$$

where b is a constant and M is the diagonal matrix of multiplicities, $M = \text{Diag}(m_0 = 1, m_1, \ldots, m_{n-1}$. Let Ξ (resp. $\widetilde{\Xi}$) denote the matrix of the basis obtained using the methods of Bantay and Gannon starting from $\mathbb{X}$ (resp. $\widetilde{\mathbb{X}}$). Experimentally using the examples of the $B_{r,1}$ RCFTs and their GHM duals we observe that

$$\widetilde{\Xi}(\tau)^T \cdot M \cdot \Xi(\tau) = J(\tau)\ M + B \ , \tag{5.2}$$

where B is a constant matrix. This formula determines the $\mathcal{Y}$-matrix for $\widetilde{\mathbb{X}}$ given the $\mathcal{Y}$-matrix for $\mathbb{X}$.

5.2 RCFTs for $(3, 6\ell)$ admissible solutions

We have found a large number of admissible solutions using the quasi-characters given by the methods of Bantay and Gannon. While, our focus was largely on $(3, 6)$ theories, we obtained larger values of Wronskian index as well. Two questions arise from our work.

1. Which among these admissible solutions correspond to RCFTs?
2. Do all $(3, 6)$ admissible solutions arise as the admissible solutions that we obtain starting from $(3, 0)$ theories? In other words, are there any $(3, 6)$ admissible solutions that do not arise from $(3, 0)$ theories. The analogous statement for the two-character case has been proven [32].

We answer the first question in the sequel.

A simple class of $(3, 6)$ and $(3, 12)$ RCFTs

Let $\mathbb{X}$ denote the VVMF associated with a $(3, 0)$ RCFT. Then, $\mathbb{X}^{[1]} := J^{1/3}\, \mathbb{X}$ is the VVMF associated with the $(3, 6)$ obtained by tensoring the RCFT with the single character $E_{8,1}$

RCFT. Similarly, $\mathbb{X}^{[2]} := J^{2/3}\mathbb{X}$ is the VVMF associated with the $(3,12)$ obtained by tensoring the RCFT with the single character $E_{8,1}^{\otimes 2}$ RCFT or the $D_{16,1}$ single character RCFT. These RCFT's share the same S-matrix as the original RCFT but the T-matrix is multiplied by $\omega = \exp(-2\pi i/3)$ for the RCFT with character $\mathbb{X}^{[1]}$ and ω^2 for the RCFT with character $\mathbb{X}^{[2]}$.

For the $B_{r,1}$ theories, starting from a B_{r+8} theory, we obtain the $E_{8,1} \otimes B_{r,1}$ theory as one of the admissible theories of type $U_{1,2}$ when $b_2 = 2^r$ (see table).

Starting from the GHM dual of the $B_{r,1}$ theory, $\widetilde{B_{r,1}}$, with $c = \frac{47}{2} - r$ for $r \in [0,8]$, we obtain the $E_{8,1} \otimes \widetilde{B_{8-r,1}}$ theory as one of the admissible theories of type $U_{1,2}$. For $r \in [0,23]$, we also recover the the $(3,0)$ $B_{r,1}$ theory as the type $U_{1,1}$ admissible solution of the $\widetilde{B_{23-r,1}}$ RCFT. The two RCFTs are connected by a set of admissible $(3,6)$ solutions.

Similarly, for $r \in [1,15]$ and $r \neq 0 \mod 4$, we obtain the $(3,0)$ $D_{r,1}$ RCFT as the end-point of a series of $(3,6)$ admissible solutions of type $U_{1,1}$ starting from the $\widetilde{D_{24-r,1}}$ RCFT. We also obtain the $(3,6)$ RCFT $E_{8,1} \otimes D_r$ as a type $U_{1,2}$ admissible solution starting from the $D_{r+8,1}$ RCFT. A similar statement holds for the GHM dual of the D_r RCFTs.

Characters $\mathbb{X}^{[2]}$ arise as $(3,12)$ RCFTs of type U_2. For instance, when $\mathbb{X}$ is the character associated with $B_{16,1}$, $\mathbb{X}^{[2]}$ is obtained as a type U_2 admissible solution of the $\widetilde{B_{7,1}}$ RCFT when $b_1 = b_2 = 1$.

Three character RCFT's with 3 or 4 primaries

When is an admissible character associated with an RCFT? The admissible characters that we have constructed all come with known S and T matrices. These are the modular data. Further, the Verlinde coefficients are the same as the original RCFT, they are guaranteed to be positive. The additional data is in the form the fusion matrices F and braiding matrices B [33].

In the examples that we have considered, the B_r theories correspond to RCFT's with 3 primaries. Further, the D_r theories ($r \neq 0 \mod 4$) correspond to RCFT's with 4 primaries. Rayhaun has classified all RCFT's with 3 or 4 primaries and central charge $c < 24$. He obtains all of them as admissible solutions of type U [22, see Appendices E3,E4]. This is consistent with our results.

$(3,6)$ RCFTs from generalized cosets

Let $(c, h_1, h_2, \ldots, h_{n-1})$ be the data of a known RCFT with n characters and Wronskian index ℓ. Let $(\tilde{c}, \tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_{n-1})$ be the data of the generalized coset dual RCFT with

Wronskian index $\tilde{\ell}$. Then, one has the relations

$$\begin{aligned} c + \tilde{c} &= N \ , \\ h_i + \tilde{h}_i &= m_i \quad \text{with } m_i \geq 2 \ , \\ \ell + \tilde{\ell} &= n^2 + (2N-1)n - 6(m_1 + \ldots m_{n-1}) \ , \end{aligned}$$

where $N = 0 \mod 8$ is the central charge of a single-character RCFT. We will be interested in the coset duals of $(3,0)$ RCFTs which have Wronskian index $\tilde{\ell} = 6$. Following Mukhi, Poddar, Singh, we look for cases where $m_1 = m_2 = 2$. We then obtain $(3,6)$ theories when $N = 32$, i.e., these theories are generalized cosets of the single character CFTs with $c = 32$.

In unpublished work [34], Das et al. constructed $(3,6)$ RCFT's as a generalized coset of the 32 dimensional complete Kervaire lattices by one of the $(3,0)$ theories appearing from affine Kac-Moody Lie algebras. These RCFTs appear in our constructions as type W and type U admissible characters for particular values of the parameters. Tables 11 and 12 provide the relevant details of how these RCFTs are obtained in our construction of $(3,6)$ admissible solutions.

$(3,0)$ CFT	No.CFTs	$(3,6)$ CFT	b values that give an RCFT
$A_{1,1}^2 \sim D_{2,1}$ $c=2$	47	$W_0(D_{6,1})$	$90, 106, 122, 138^3, 154^2, 170^2, 186^4, 202, 218^4, 234, 250^3, 266^3, 282^2, 314^4, 330, 346^2, 362, 378^2, 442, 474^2, 506, 510, 570$
$A_{3,1} \sim D_{3,1}$ $c=3$	15	$W_0(D_{5,1})$	$129, 161, 201, 225^2, 273, 317, 321, 337, 353^2, 377, 497, 521$
$A_{2,1}^2$	9	$W_0(A_{2,1}^2, c=4)$	$112, 166, 184, 220, 316, 328^2, 352, 448$
$A_{4,1}$	4	$W_0(A_{4,1}, c=4)$	$168, 288, 368, 552$
$D_{5,1}$	6	$W_0(D_{3,1})$	$195, 291, 371, 419, 451, 531$
$D_{7,1}$	3	$W_0(D_{1,1})$	$181, 277, 469$

Table 11: $(3,6)$ RCFTs that are obtained from type W_0 admissible characters of $(3,0)$ theories. A superscript indicates the number of distinct RCFTs for that value of b.

6 Concluding Remarks

This paper largely focused on constructing quasi-characters associated with $(3,0)$ and $(3,3)$ RCFTs. The focus was to construct $(3,6)$ and $(3,9)$ admissible solutions using these quasi-characters. A nice geometric feature emerges – all admissible characters arise as integral points that lie on the boundary and interior of a polytope. This observation is not restricted

(3,0) CFT	No. CFTs	(3,6) CFT	b_i values that give an RCFT
$D_{9,1}$	4	$U_{1,2}(\widetilde{D_{1,1}})$	$b_2 = 64, 72, 100, 120$
$D_{10,1}$	11	$U_{1,2}(\widetilde{D_{2,1}})$	$b_2 = 8, 12, 16, 20^2, 24, 32^2, 40, 56^2$
$D_{14,1}$	2	$U_{1,2}(\widetilde{D_{6,1}})$	$b_2 = 1^2$
$E_{6,1}^2$	3	$U_{1,2}(A_{2,1}^2, c = 20)$	$b_2 = 9, 15, 27$
$E_{7,1}^2$	6	$U_{1,1}(\widetilde{D_{6,1}})$	$b_1 = 4, 6, 8, 12, 20, 36$

Table 12: (3,6) RCFTs that are obtained from type $U_{1,i}$ admissible characters of (3,0) theories. A superscript indicates the number of distinct RCFTs for that value of b.

to the three-characters examples considered in this paper but is valid for higher numbers of characters as well.

A slight generalization of our construction is to permit rational points in the polytope. The rational number corresponds to a denominator that is fixed to a particular value. This will require rescaling of the non-identity characters to regain non-negative integrality of the q-series. This will also modify the S-matrix. We will illustrate this with a particular example. Consider the (3,0) RCFT with $c = 20$, $(h_1, h_2) = (\frac{4}{3}, \frac{5}{3})$, $(m_1, m_2) = (4, 4)$. In this example, consider the admissible characters of type U_2.

$$U_2 = \begin{pmatrix} \frac{1}{q^{5/6}} + 4\,(27b_1 + 3b_2 + 20)\,q^{1/6} + (9936b_1 + 132b_2 + 46790)\,q^{7/6} + \cdots \\ \frac{b_1}{q^{1/2}} + 3\,(52b_1 - 6b_2 + 405)\,q^{\frac{1}{2}} + 18\,(419b_1 - 6b_2 + 11340)\,q^{3/2} + \cdots \\ \frac{b_2}{q^{1/6}} + (-1458b_1 + 16b_2 + 8748)\,q^{5/6} + (-40824b_1 + 98b_2 + 776385)\,q^{11/6} + \cdots \end{pmatrix} \tag{6.1}$$

Focusing on the identity character, we see that choosing $108b_1 \in \mathbb{Z}$ and $12b_2 \in \mathbb{Z}$ does not change the integrality of the identity character to the orders that we checked. However, if the degeneracies associated with row 2 and row 3 are scaled by n_1 and n_2 to regain integrality, then the multiplicities will be rescaled to $\frac{m_1}{n_1^2}$ and $\frac{m_2}{n_2^2}$, so that the torus partition function continues to be modular invariant. Thus, we can choose $2b_1 \in \mathbb{Z}$ and $2b_2 \in \mathbb{Z}$. Interestingly, it would change the $U_{1,2}$ upper bound to $b_2 \leq \frac{135}{2}$. Consider $U_{1,2}$ at $b_2 = \frac{135}{2}$. We then multiply row 2 and row 3 by 2 to regain integrality. This gives rise to admissible solutions but the S-matrix is modified with $(m_1, m_2) = (1, 1)$. This S-matrix does not lead to good fusion rules and thus, we do not obtain sensible solution for fractional values of b_i. For $b_1 = 0$ and $b_2 = \frac{135}{2}$ we get a (3,0) admissible solution with $c = 20$, $(h_1, h_2) = (\frac{7}{3}, \frac{2}{3})$. Consistent with our observation, this was already shown to not be an RCFT [10]. So it appears that fractional values are not permitted in the examples that we considered.

Using a duality due to Bantay and Gannon, we were able to obtain the S-matrices and the fusion matrices for the 15 known (3,3) admissible characters. This enables us to rule out 8 of the 15 arising from the MLDE method. The remaining 7 all have the same

S-matrix and have three primaries. This rank-three S-matrix appears in the classification of Modular Tensor Categories by Rowell et al. [35, see Theorem 3.2].

A natural extension of our work is carry out similar constructions for theories with four and five characters. For four character theories, the Wronskian index is even. The Bantay-Gannon dual of $(4,0)$ theories have Wronskian index 4 and $(4,2)$ theories get mapped to $(4,2)$ theories. It appears that it might suffice to obtain all $(4,0)$ and $(4,2)$ theories and use our method of quasi-characters and Bantay-Gannon duality to obtain theories with higher values of Wronskian index. This is being currently studied [36].

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A Modular background

In this appendix we define some modular forms of $PSL(2,\mathbb{Z})$ that are used in this paper. This is to provide a quick description to set up notations. For details, refer to the introduction to the topic by Zagier [37, 38].

A modular form of weight w is a holomorphic function on the Poincaré upper half plane $\mathbb{H}$ such that

$$\phi\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w\ \phi(\tau) \tag{A.1}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{Z})$. The invariance under $\tau \to \tau+1$ implies that one has the expansion

$$\phi(\tau) = \sum_{n=-\infty}^{\infty} a_n q^n\ .$$

If the modular form is bounded as $\tau \to i\infty$ (or $q \to 0$), one has $a_n = 0$ for $n < 0$. Such modular forms are called *holomorphic* modular forms. A modular form is called weakly holomorphic if $a_n = 0$ for some $n < -N$ where N is a positive integer.

We provide explicit formulae for the Eisenstein series E_2, E_4, and E_6, the cusp form of weight 12, and Klein-J invariant. We define $q = \exp(2\pi i\tau)$.

$$
\begin{aligned}
E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty}\frac{n\,q^n}{(1-q^n)} = 1 - 24q - 72q^2 - 96q^3 + \ldots \\
E_4(\tau) &= 1 + 240\sum_{n=1}^{\infty}\frac{n^3\,q^n}{(1-q^n)} = 1 + 240q + 2160q^2 + 6720q^3 + \ldots \\
E_6(\tau) &= 1 - 504\sum_{n=1}^{\infty}\frac{n^5\,q^n}{(1-q^n)} = 1 - 504q - 16632q^2 - 122976q^3 + \ldots \\
\Delta(\tau) &= \frac{E_4^3 - E_6^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \ldots \\
J(\tau) &= \frac{E_4^3}{\Delta} = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots
\end{aligned}
$$

Further, only the J-invariant is a weakly holomorphic modular function while the others are holomorphic modular forms.
The Serre-Ramanujan covariant derivative, acting on a modular form of weight w, is defined by

$$
\mathcal{D}_w := q\frac{d}{dq} - \frac{w}{12}E_2. \tag{A.2}
$$

It maps a modular form of weight w to one of weight $(w+2)$. The operators ∇_i defined in the main text do not change the weight.

B Admissible $(3,6)$ Admissible Solutions obtained from $(3,0)$ theories

This appendix provides a complete listing of $(3,6)$ admissible solutions of type U, W and V that we obtain starting from all $(3,0)$ examples that we have considered in this paper.

B.1 $(3,6)$ admissible solutions of type U

S. No.	(3,0) c	(3,0) h_1	(3,0) h_2	b_i	(3,6) h_1	(3,6) h_2
1.	$\frac{17}{2}$	$\frac{1}{2}$	$\frac{17}{16}$	$b_2 \leq 1$	$\frac{1}{2}$	$\frac{1}{16}$
2.	$\frac{19}{2}$	$\frac{1}{2}$	$\frac{19}{16}$	$b_2 \leq 2$	$\frac{1}{2}$	$\frac{3}{16}$
3.	$\frac{21}{2}$	$\frac{1}{2}$	$\frac{21}{16}$	$b_2 \leq 5$	$\frac{1}{2}$	$\frac{5}{16}$
4.	$\frac{23}{2}$	$\frac{1}{2}$	$\frac{23}{16}$	$b_2 \leq 11$	$\frac{1}{2}$	$\frac{7}{16}$
5.	$\frac{25}{2}$	$\frac{3}{2}$	$\frac{9}{16}$	$b_1 < 25^*$	$\frac{1}{2}$	$\frac{9}{16}$
6.	$\frac{27}{2}$	$\frac{3}{2}$	$\frac{11}{16}$	$b_1 < 27^*$	$\frac{1}{2}$	$\frac{11}{16}$
7.	$\frac{29}{2}$	$\frac{3}{2}$	$\frac{13}{16}$	$b_1 < 29^*$	$\frac{1}{2}$	$\frac{13}{16}$
8.	$\frac{31}{2}$	$\frac{3}{2}$	$\frac{15}{16}$	$b_1 < 31^*$	$\frac{1}{2}$	$\frac{15}{16}$
9.	$\frac{33}{2}$	$\frac{3}{2}$	$\frac{17}{16}$	$b_1 < 33^*$	$\frac{1}{2}$	$\frac{17}{16}$
				$b_2 \leq 1$	$\frac{3}{2}$	$\frac{1}{16}$
10.	$\frac{35}{2}$	$\frac{3}{2}$	$\frac{19}{16}$	$b_1 < 35^*$	$\frac{1}{2}$	$\frac{19}{16}$
				$b_2 \leq 2$	$\frac{3}{2}$	$\frac{3}{16}$
11.	$\frac{37}{2}$	$\frac{3}{2}$	$\frac{21}{16}$	$b_1 < 37^*$	$\frac{1}{2}$	$\frac{21}{16}$
				$b_2 \leq 6$	$\frac{3}{2}$	$\frac{5}{16}$
12.	$\frac{39}{2}$	$\frac{3}{2}$	$\frac{23}{16}$	$b_1 < 39^*$	$\frac{1}{2}$	$\frac{23}{16}$
				$b_2 \leq 17$	$\frac{3}{2}$	$\frac{7}{16}$
13.	$\frac{41}{2}$	$\frac{3}{2}$	$\frac{25}{16}$	$b_1 < 41^*$	$\frac{1}{2}$	$\frac{25}{16}$
				$b_2 \leq 45$	$\frac{3}{2}$	$\frac{9}{16}$
14.	$\frac{43}{2}$	$\frac{3}{2}$	$\frac{27}{16}$	$b_1 < 43^*$	$\frac{1}{2}$	$\frac{27}{16}$
				$b_2 \leq 125$	$\frac{3}{2}$	$\frac{11}{16}$
15.	$\frac{45}{2}$	$\frac{3}{2}$	$\frac{29}{16}$	$b_1 < 45^*$	$\frac{1}{2}$	$\frac{29}{16}$
				$b_2 \leq 398$	$\frac{3}{2}$	$\frac{13}{16}$
16.	$\frac{47}{2}$	$\frac{3}{2}$	$\frac{31}{16}$	$b_1 < 47^*$	$\frac{1}{2}$	$\frac{31}{16}$
				$b_2 \leq 2185$	$\frac{3}{2}$	$\frac{15}{16}$

Table 13: $(3,6)$ admissible solutions of type $U_{1,i}$ using $(3,0)$ $B_{r,1}$ theories and their GHM duals. An asterisk indicates that we obtain another $(3,0)$ admissible solution on equality.

S. No.	(3,0) c	h_1	h_2	b_i	(3,6) h_1	h_2
1.	9	$\frac{1}{2}$	$\frac{9}{8}$	$b_2 \leq 1$	$\frac{1}{2}$	$\frac{1}{8}$
2.	11	$\frac{1}{2}$	$\frac{11}{8}$	$b_2 \leq 5$	$\frac{1}{2}$	$\frac{3}{8}$
3.	13	$\frac{3}{2}$	$\frac{5}{8}$	$b_1 < 26^*$	$\frac{1}{2}$	$\frac{5}{8}$
4.	15	$\frac{3}{2}$	$\frac{7}{8}$	$b_1 < 30^*$	$\frac{1}{2}$	$\frac{7}{8}$
5.	17	$\frac{3}{2}$	$\frac{9}{8}$	$b_1 < 34^*$	$\frac{1}{2}$	$\frac{9}{8}$
				$b_2 \leq 1$	$\frac{3}{2}$	$\frac{1}{8}$
6.	19	$\frac{3}{2}$	$\frac{11}{8}$	$b_1 < 38^*$	$\frac{1}{2}$	$\frac{11}{8}$
				$b_2 \leq 7$	$\frac{3}{2}$	$\frac{3}{8}$
7.	21	$\frac{3}{2}$	$\frac{13}{8}$	$b_1 < 42^*$	$\frac{1}{2}$	$\frac{13}{8}$
				$b_2 \leq 53$	$\frac{3}{2}$	$\frac{5}{8}$
8.	23	$\frac{3}{2}$	$\frac{15}{8}$	$b_1 < 46^*$	$\frac{1}{2}$	$\frac{15}{8}$
				$b_2 < 575^*$	$\frac{3}{2}$	$\frac{7}{8}$
9.	10	$\frac{1}{2}$	$\frac{5}{4}$	$b_2 \leq 2$	$\frac{1}{2}$	$\frac{1}{4}$
10.	14	$\frac{3}{2}$	$\frac{3}{4}$	$b_1 < 28^*$	$\frac{1}{2}$	$\frac{3}{4}$
11.	18	$\frac{3}{2}$	$\frac{5}{4}$	$b_1 < 36^*$	$\frac{1}{2}$	$\frac{5}{4}$
				$b_2 \leq 3$	$\frac{3}{2}$	$\frac{1}{4}$
12.	22	$\frac{3}{2}$	$\frac{7}{4}$	$b_1 < 44^*$	$\frac{1}{2}$	$\frac{7}{4}$
				$b_2 < 154^*$	$\frac{3}{2}$	$\frac{3}{4}$

Table 14: $(3,6)$ admissible solutions of type $U_{1,i}$ using $(3,0)$ $D_{r,1}$ theories and their GHM duals. An asterisk indicates that we obtain another $(3,0)$ admissible solution on equality.

S. No.	(3,0) c	h_1	h_2	b_i	(3,6) h_1	h_2
1.	$\frac{60}{7}$	$\frac{3}{7}$	$\frac{8}{7}$	$b_2 \leq 1$	$\frac{3}{7}$	$\frac{1}{7}$
2.	$\frac{68}{7}$	$\frac{3}{7}$	$\frac{9}{7}$	$b_2 \leq 3$	$\frac{3}{7}$	$\frac{2}{7}$
3.	$\frac{100}{7}$	$\frac{5}{7}$	$\frac{11}{7}$	$b_2 \leq 55$	$\frac{5}{7}$	$\frac{4}{7}$
4.	$\frac{108}{7}$	$\frac{6}{7}$	$\frac{11}{7}$	$b_2 < 39^*$	$\frac{6}{7}$	$\frac{4}{7}$
5.	$\frac{116}{7}$	$\frac{8}{7}$	$\frac{10}{7}$	$b_1 \leq 1$	$\frac{1}{7}$	$\frac{10}{7}$
				$b_2 \leq 16$	$\frac{8}{7}$	$\frac{3}{7}$

6.	$\frac{124}{7}$	$\frac{9}{7}$	$\frac{10}{7}$	$b_1 \leq 4$	$\frac{2}{7}$	$\frac{10}{7}$
				$b_2 \leq 27$	$\frac{9}{7}$	$\frac{3}{7}$
5.	$\frac{156}{7}$	$\frac{11}{7}$	$\frac{12}{7}$	$b_1 \leq 95$	$\frac{4}{7}$	$\frac{12}{7}$
				$b_2 < 130^*$	$\frac{11}{7}$	$\frac{5}{7}$
6.	$\frac{164}{7}$	$\frac{11}{7}$	$\frac{13}{7}$	$b_1 \leq 65$	$\frac{4}{7}$	$\frac{13}{7}$
				$b_2 \leq 436$	$\frac{11}{7}$	$\frac{6}{7}$
7.	$\frac{236}{7}$	$\frac{16}{7}$	$\frac{17}{7}$	$b_1 = 17x$	$\frac{9}{7}$	$\frac{17}{7}$
	$x \in \mathbb{Z}_{>0}$			$b_2 = 17x$	$\frac{16}{7}$	$\frac{10}{7}$

Table 15: $(3,6)$ admissible solutions using $(3,0)$ solutions of type $U_{1,i}$ from type LY_2 theories.

S. No.	(3,0) c	h_1	h_2	b_i	(3,6) h_1	h_2
1.	$\frac{44}{5}$	$\frac{2}{5}$	$\frac{6}{5}$	$b_2 \leq 1$	$\frac{2}{5}$	$\frac{1}{5}$
2.	$\frac{52}{5}$	$\frac{3}{5}$	$\frac{6}{5}$	$b_2 \leq 3$	$\frac{3}{5}$	$\frac{1}{5}$
3.	$\frac{68}{5}$	$\frac{4}{5}$	$\frac{7}{5}$	$b_2 \leq 11$	$\frac{4}{5}$	$\frac{2}{5}$
4.	$\frac{76}{5}$	$\frac{4}{5}$	$\frac{8}{5}$	$b_2 < 57^*$	$\frac{4}{5}$	$\frac{3}{5}$
5.	$\frac{84}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$b_1 \leq 1$	$\frac{1}{5}$	$\frac{7}{5}$
				$b_2 \leq 17$	$\frac{6}{5}$	$\frac{2}{5}$
6.	$\frac{92}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$b_1 \leq 4$	$\frac{1}{5}$	$\frac{8}{5}$
				$b_2 \leq 59$	$\frac{6}{5}$	$\frac{3}{5}$
5.	$\frac{108}{5}$	$\frac{7}{5}$	$\frac{9}{5}$	$b_1 \leq 16$	$\frac{2}{5}$	$\frac{9}{5}$
				$b_2 < 459^*$	$\frac{7}{5}$	$\frac{4}{5}$
6.	$\frac{116}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$b_1 \leq 100$	$\frac{3}{5}$	$\frac{9}{5}$
				$b_2 \leq 165$	$\frac{8}{5}$	$\frac{4}{5}$
7.	$\frac{164}{5}$	$\frac{12}{5}$	$\frac{11}{5}$	$b_1 = 11x$	$\frac{7}{5}$	$\frac{11}{5}$
	$x \in \mathbb{Z}_{>0}$			$b_2 = 11x$	$\frac{12}{5}$	$\frac{6}{5}$

Table 16: $(3,6)$ admissible solutions of type $U_{1,i}$ using $(3,0)$ theories of type $(LY_1)^{\otimes 2}$.

S. No.	(3,0) c	h_1	h_2	b_i	(3,6) h_1	h_2
1.	12	$\frac{2}{3}$	$\frac{4}{3}$	$b_2 \leq 4$	$\frac{2}{3}$	$\frac{1}{3}$
2.	20	$\frac{4}{3}$	$\frac{5}{3}$	$b_1 < 6^*$	$\frac{1}{3}$	$\frac{5}{3}$
				$b_2 < 67$	$\frac{4}{3}$	$\frac{2}{3}$
3.	20	$\frac{7}{5}$	$\frac{8}{5}$	$b_1 \leq 12$	$\frac{2}{5}$	$\frac{8}{5}$
				$b_2 \leq 51$	$\frac{7}{5}$	$\frac{3}{5}$

Table 17: $(3,6)$ admissible solutions of type $U_{1,i}$ using $(3,0)$ theories of type $A_{2,1}^{\otimes 2}$.

B.2 $(3,6)$ admissible solutions of type W

S. No.	(3,0) Sol. c	h_1	h_2	c_i	b	$(3,6)$ Adm. c	h_1	h_2
1.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}$	$2 \leq c_1 < 49^*$	$b = 25c_1 - 49$	$\frac{49}{2}$	$\frac{1}{2}$	$\frac{33}{16}$
				$c_2 = 1$	$b = 274$	$\frac{49}{2}$	$\frac{5}{2}$	$\frac{1}{16}$
2.	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{16}$	$4 \leq c_1 < 51^*$	$b = 3(9c_1 - 34)$	$\frac{51}{2}$	$\frac{1}{2}$	$\frac{35}{16}$
3.	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{5}{16}$	$6 \leq c_1 < 53^*$	$b = 29c_1 - 159$	$\frac{53}{2}$	$\frac{1}{2}$	$\frac{37}{16}$
				$c_2 = 4$	$b = 268$	$\frac{53}{2}$	$\frac{5}{2}$	$\frac{5}{16}$
4.	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{7}{16}$	$8 \leq c_1 < 55^*$	$b = 31c_1 - 220$	$\frac{55}{2}$	$\frac{1}{2}$	$\frac{39}{16}$
	$x \in [0,2]$			$c_2 = 8 + 7x$	$b = 263 + 512x$	$\frac{55}{2}$	$\frac{5}{2}$	$\frac{7}{16}$
5.	$\frac{9}{2}$	$\frac{1}{2}$	$\frac{9}{16}$	$9 \leq c_1 < 57^*$	$b = 3(11c_1 - 95)$	$\frac{57}{2}$	$\frac{1}{2}$	$\frac{41}{16}$
	$x \in [0,6]$			$c_2 = 10 + 9x$	$b = 86 + 256x$	$\frac{57}{2}$	$\frac{5}{2}$	$\frac{9}{16}$
6.	$\frac{11}{2}$	$\frac{1}{2}$	$\frac{11}{16}$	$11 \leq c_1 < 59^*$	$b = 35c_1 - 354$	$\frac{59}{2}$	$\frac{1}{2}$	$\frac{43}{16}$
	$x \in [0,16]$			$c_2 = 21 + 11x$	$b = 121 + 128x$	$\frac{59}{2}$	$\frac{5}{2}$	$\frac{11}{16}$
7.	$\frac{13}{2}$	$\frac{1}{2}$	$\frac{13}{16}$	$12 \leq c_1 < 61^*$	$b = 37c_1 - 427$	$\frac{61}{2}$	$\frac{1}{2}$	$\frac{45}{16}$
	$x \in [0,55]$			$c_2 = 25 + 13x$	$b = 48 + 64x$	$\frac{61}{2}$	$\frac{5}{2}$	$\frac{13}{16}$
8.	$\frac{15}{2}$	$\frac{1}{2}$	$\frac{15}{16}$	$13 \leq c_1 < 63^*$	$b = 3(13c_1 - 168)$	$\frac{63}{2}$	$\frac{1}{2}$	$\frac{47}{16}$
	$x \in [0,316]$			$c_2 = 33 + 15x$	$b = 27 + 32x$	$\frac{63}{2}$	$\frac{5}{2}$	$\frac{15}{16}$
9.	$\frac{17}{2}$	$\frac{1}{2}$	$\frac{17}{16}$	$15 \leq c_1 < 65^*$	$b = 41c_1 - 585$	$\frac{65}{2}$	$\frac{1}{2}$	$\frac{49}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 35 + 17x$	$b = 10 + 16x$	$\frac{65}{2}$	$\frac{5}{2}$	$\frac{17}{16}$
10.	$\frac{19}{2}$	$\frac{1}{2}$	$\frac{19}{16}$	$16 \leq c_1 < 67^*$	$b = 43c_1 - 670$	$\frac{67}{2}$	$\frac{1}{2}$	$\frac{51}{16}$

	$x \in \mathbb{Z}_{>0}$			$c_2 = 37 + 19x$	$b = 5 + 8x$	$\frac{67}{2}$	$\frac{5}{2}$	$\frac{19}{16}$
11.	$\frac{21}{2}$	$\frac{1}{2}$	$\frac{21}{16}$	$17 \leq c_1 < 69^*$	$b = 3(15c_1 - 253)$	$\frac{69}{2}$	$\frac{1}{2}$	$\frac{53}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 23 + 21x$	$b = 4x$	$\frac{69}{2}$	$\frac{5}{2}$	$\frac{21}{16}$
12.	$\frac{23}{2}$	$\frac{1}{2}$	$\frac{23}{16}$	$19 \leq c_1 < 71^*$	$b = 47c_1 - 852$	$\frac{71}{2}$	$\frac{1}{2}$	$\frac{55}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 47 + 23x$	$b = 1 + 2x$	$\frac{71}{2}$	$\frac{5}{2}$	$\frac{23}{16}$
13.	$\frac{25}{2}$	$\frac{3}{2}$	$\frac{9}{16}$	$c_1 = 146 + 25b$	$b \geq 0$	$\frac{73}{2}$	$\frac{3}{2}$	$\frac{41}{16}$
14.	$\frac{27}{2}$	$\frac{3}{2}$	$\frac{11}{16}$	$c_1 = 325 + 27b$	$b \geq 0$	$\frac{75}{2}$	$\frac{3}{2}$	$\frac{43}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 87$	$b = 736$	$\frac{77}{2}$	$\frac{7}{2}$	$\frac{11}{16}$
15.	$\frac{29}{2}$	$\frac{3}{2}$	$\frac{13}{16}$	$c_1 = 616 + 29b$	$b \geq 0$	$\frac{77}{2}$	$\frac{3}{2}$	$\frac{45}{16}$
16.	$\frac{31}{2}$	$\frac{3}{2}$	$\frac{15}{16}$	$c_1 = 1027 + 31b$	$b \geq 0$	$\frac{79}{2}$	$\frac{3}{2}$	$\frac{47}{16}$
	$x = 0, 1$			$c_2 = 930 + 3875x$	$b = 1754 + 8704x$	$\frac{79}{2}$	$\frac{7}{2}$	$\frac{15}{16}$
17.	$\frac{33}{2}$	$\frac{3}{2}$	$\frac{17}{16}$	$c_1 = 1566 + 33b$	$b \geq 0$	$\frac{81}{2}$	$\frac{3}{2}$	$\frac{49}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 2024 + 4301x$	$b = 1545 + 3840x$	$\frac{81}{2}$	$\frac{7}{2}$	$\frac{17}{16}$
18.	$\frac{35}{2}$	$\frac{3}{2}$	$\frac{19}{16}$	$c_1 = 2241 + 35b$	$b \geq 0$	$\frac{83}{2}$	$\frac{3}{2}$	$\frac{51}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 588 + 4655x$	$b = 4 + 1664x$	$\frac{83}{2}$	$\frac{7}{2}$	$\frac{19}{16}$
19.	$\frac{37}{2}$	$\frac{3}{2}$	$\frac{21}{16}$	$c_1 = 3060 + 37b$	$b \geq 0$	$\frac{85}{2}$	$\frac{3}{2}$	$\frac{53}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 2109 + 4921x$	$b = 139 + 704x$	$\frac{85}{2}$	$\frac{7}{2}$	$\frac{21}{16}$
20.	$\frac{39}{2}$	$\frac{3}{2}$	$\frac{23}{16}$	$c_1 = 4031 + 39b$	$b \geq 0$	$\frac{87}{2}$	$\frac{3}{2}$	$\frac{55}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 6188 + 1105x$	$b = 222 + 288x$	$\frac{87}{2}$	$\frac{7}{2}$	$\frac{23}{16}$
21.	$\frac{41}{2}$	$\frac{3}{2}$	$\frac{25}{16}$	$c_1 = 5162 + 41b$	$b \geq 0$	$\frac{89}{2}$	$\frac{3}{2}$	$\frac{57}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 5986 + 5125x$	$b = 29 + 112x$	$\frac{89}{2}$	$\frac{7}{2}$	$\frac{25}{16}$
22.	$\frac{43}{2}$	$\frac{3}{2}$	$\frac{27}{16}$	$c_1 = 6461 + 43b$	$b \geq 0$	$\frac{91}{2}$	$\frac{3}{2}$	$\frac{59}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 11180 + 5031x$	$b = 8 + 40x$	$\frac{91}{2}$	$\frac{7}{2}$	$\frac{27}{16}$
23.	$\frac{45}{2}$	$\frac{3}{2}$	$\frac{29}{16}$	$c_1 = 7936 + 45b$	$b \geq 0$	$\frac{93}{2}$	$\frac{3}{2}$	$\frac{61}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 27027 + 1595x$	$b = 3 + 4x$	$\frac{93}{2}$	$\frac{7}{2}$	$\frac{21}{16}$
24.	$\frac{47}{2}$	$\frac{3}{2}$	$\frac{31}{16}$	$c_1 = 9595 + 47b$	$b \geq 0$	$\frac{95}{2}$	$\frac{3}{2}$	$\frac{63}{16}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 115197 + 4371x$	$b = 2x$	$\frac{95}{2}$	$\frac{7}{2}$	$\frac{23}{16}$

Table 18: $(3, 6)$ admissible solutions of type $W_{1,i}$ using $(3, 0)$ solutions of type $B_{r,1}$ and their GHM duals

S. No.	(3,0) Sol. c	h_1	h_2	c_i	b	(3,6) Adm. c	h_1	h_2
1.	1	$\frac{1}{2}$	$\frac{1}{8}$	$3 \leq c_1 < 50^*$	$b = 26c_1 - 75$	25	$\frac{1}{2}$	$\frac{17}{8}$
				$c_2 = 1$	$b = 273$	25	$\frac{5}{2}$	$\frac{1}{8}$
2.	3	$\frac{1}{2}$	$\frac{3}{8}$	$7 \leq c_1 < 54^*$	$b = 3(10c_1 - 63)$	27	$\frac{1}{2}$	$\frac{19}{8}$
	$x \in [0, 2]$			$c_2 = 3 + 3x$	$b = 95 + 512x$	27	$\frac{5}{2}$	$\frac{3}{8}$
3.	5	$\frac{1}{2}$	$\frac{5}{8}$	$10 \leq c_1 < 58^*$	$b = 34c_1 - 319$	29	$\frac{1}{2}$	$\frac{21}{8}$
	$x \in [0, 14]$			$c_2 = 11 + 5x$	$b = 125 + 128x$	29	$\frac{5}{2}$	$\frac{5}{8}$
4.	7	$\frac{1}{2}$	$\frac{7}{8}$	$13 \leq c_1 < 62^*$	$b = 38c_1 - 465$	31	$\frac{1}{2}$	$\frac{23}{8}$
	$x \in [0, 163]$			$c_2 = 15 + 7x$	$b = 11 + 32x$	31	$\frac{5}{2}$	$\frac{7}{8}$
5.	9	$\frac{1}{2}$	$\frac{9}{8}$	$15 \leq c_1 < 66^*$	$b = 3(14c_1 - 209)$	33	$\frac{1}{2}$	$\frac{25}{8}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 19 + 9x$	$b = 1 + 8x$	33	$\frac{5}{2}$	$\frac{9}{8}$
6.	11	$\frac{1}{2}$	$\frac{11}{8}$	$18 \leq c_1 < 70^*$	$b = 23(2c_1 - 35)$	35	$\frac{1}{2}$	$\frac{27}{8}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 23 + 11x$	$b = 1 + 2x$	35	$\frac{5}{2}$	$\frac{11}{8}$
7.	13	$\frac{3}{2}$	$\frac{5}{8}$	$c_1 = 222 + 26b$	$b \geq 0$	37	$\frac{3}{2}$	$\frac{21}{8}$
8.	15	$\frac{3}{2}$	$\frac{7}{8}$	$c_1 = 806 + 30b$	$b \geq 0$	39	$\frac{3}{2}$	$\frac{23}{8}$
	$x \in [0, 3]$			$c_2 = 78 + 455x$	$b = 15 + 2304x$	39	$\frac{7}{2}$	$\frac{7}{8}$
9.	17	$\frac{3}{2}$	$\frac{9}{8}$	$c_1 = 1886 + 34b$	$b \geq 0$	41	$\frac{3}{2}$	$\frac{25}{8}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 748 + 561x$	$b = 365 + 448x$	41	$\frac{7}{2}$	$\frac{9}{8}$
10.	19	$\frac{3}{2}$	$\frac{11}{8}$	$c_1 = 3526 + 38b$	$b \geq 0$	43	$\frac{3}{2}$	$\frac{27}{8}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 1596 + 627x$	$b = 59 + 80x$	43	$\frac{7}{2}$	$\frac{11}{8}$
11.	21	$\frac{3}{2}$	$\frac{13}{8}$	$c_1 = 5790 + 42b$	$b \geq 0$	45	$\frac{3}{2}$	$\frac{29}{8}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 5292 + 637x$	$9 + 12x$	45	$\frac{7}{2}$	$\frac{13}{8}$
12.	23	$\frac{3}{2}$	$\frac{15}{8}$	$c_1 = 8742 + 46b$	$b \geq 0$	47	$\frac{3}{2}$	$\frac{31}{8}$
				$c_2 = 33511 + 575b$	$b \geq 0$	47	$\frac{7}{2}$	$\frac{15}{8}$

Table 19: $(3, 6)$ admissible solutions of type $W_{1,i}$ using $(3, 0)$ solutions of type $B_{r,1}$ (odd r) and their GHM duals

S. No.	(3,0) Sol. c	h_1	h_2	c_i	b	(3,6) Adm. c	h_1	h_2
1.	2	$\frac{1}{2}$	$\frac{1}{4}$	$5 \leq c_1 < 52^*$	$b = 2(14c_1 - 65)$	26	$\frac{1}{2}$	$\frac{9}{4}$
				$c_2 = 2, 3$	$b = 2(256c_2 - 377)$	26	$\frac{5}{2}$	$\frac{1}{4}$
2.	6	$\frac{1}{2}$	$\frac{3}{4}$	$11 \leq c_1 < 60^*$	$b = 6(6c_1 - 65)$	30	$\frac{1}{2}$	$\frac{11}{4}$
	$x \in [0, 86]$			$c_2 = 10 + 3x$	$b = 15 + 48x$	30	$\frac{5}{2}$	$\frac{3}{4}$
3.	10	$\frac{1}{2}$	$\frac{5}{4}$	$17 \leq c_1 < 68^*$	$b = 2(22c_1 - 357)$	34	$\frac{1}{2}$	$\frac{13}{4}$
	$x \in [0, \infty)$			$c_2 = 17 + 5x$	$b = 2x$	34	$\frac{5}{2}$	$\frac{5}{4}$
4.	14	$\frac{3}{2}$	$\frac{3}{4}$	$c_1 = 456 + 28b$	$b \geq 0$	38	$\frac{3}{2}$	$\frac{11}{4}$
	$x \in [0, 7]$			$c_2 = 49 + 49x$	$b = 146 + 640x$	38	$\frac{7}{2}$	$\frac{3}{4}$
5.	18	$\frac{3}{2}$	$\frac{5}{4}$	$c_1 = 2632 + 36b$	$b \geq 0$	42	$\frac{3}{2}$	$\frac{13}{4}$
	$x \in [0, \infty)$			$c_2 = 591 + 25x$	$b = 6 + 8x$	42	$\frac{7}{2}$	$\frac{5}{4}$
6.	22	$\frac{3}{2}$	$\frac{7}{4}$	$c_1 = 7176 + 44b$	$b \geq 0$	46	$\frac{3}{2}$	$\frac{15}{4}$
				$c_2 = 11132 + 154b$	$b \geq 0$	46	$\frac{7}{2}$	$\frac{7}{4}$

Table 20: $(3, 6)$ admissible solutions of type $W_{1,i}$ using $(3, 0)$ solutions of type $B_{r,1}$ (even r) and their GHM duals

S. No.	(3,0) Sol. c	h_1	h_2	c_i, b	(3,6) Adm. c	h_1	h_2
1.	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$c_1 = 1$, $b = 326$	$\frac{172}{7}$	$\frac{1}{7}$	$\frac{17}{7}$
				$3 \leq c_2 \leq 22$, $b = 55c_2 - 129$	$\frac{172}{7}$	$\frac{15}{7}$	$\frac{3}{7}$
2.	$\frac{12}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$c_1 = 2, 4$, $b = 69, 696$	$\frac{180}{7}$	$\frac{2}{7}$	$\frac{17}{7}$
3.	$\frac{44}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$c_1 = 22 + 44x$, $b = 71 + 725x$, $x \in [0, 2]$	$\frac{212}{7}$	$\frac{4}{7}$	$\frac{19}{7}$
				$c_2 = 20 + 11x$ $b = 116 + 138x$, $x \in [0, 17]$	$\frac{212}{7}$	$\frac{18}{7}$	$\frac{5}{7}$
4.	$\frac{52}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$c_1 = 33, 72$, $b = 528, 1582$	$\frac{220}{7}$	$\frac{4}{7}$	$\frac{20}{7}$
				$c_2 = 26 + 13x$, $b = 72 + 76x$, $x \in [0, 62]$	$\frac{220}{7}$	$\frac{18}{7}$	$\frac{6}{7}$
5.	$\frac{60}{7}$	$\frac{3}{7}$	$\frac{8}{7}$	$c_2 = 13 + 10x$, $b = 6 + 9x$	$\frac{228}{7}$	$\frac{17}{7}$	$\frac{8}{7}$
6.	$\frac{68}{7}$	$\frac{3}{7}$	$\frac{9}{7}$	$c_2 = 17x^*$, $b = 5x$	$\frac{236}{7}$	$\frac{17}{7}$	$\frac{9}{7}$
7.	$\frac{100}{7}$	$\frac{5}{7}$	$\frac{11}{7}$	$c_2 = 670 + 55b$, $b \geq 0$	$\frac{268}{7}$	$\frac{19}{7}$	$\frac{11}{7}$
8.	$\frac{108}{7}$	$\frac{6}{7}$	$\frac{11}{7}$	$c_1 = 195 + 351x$, $b = 853 + 2299x$, $x \in [0, 3]$	$\frac{276}{7}$	$\frac{6}{7}$	$\frac{25}{7}$

				$c_2 = 828 + 39b,\ b \geq 0$	$\frac{276}{7}$	$\frac{20}{7}$	$\frac{11}{7}$
9.	$\frac{116}{7}$	$\frac{8}{7}$	$\frac{10}{7}$	$c_2 = 1189 + 725x,\ b = 4 + 44x$	$\frac{284}{7}$	$\frac{22}{7}$	$\frac{10}{7}$
10.	$\frac{124}{7}$	$\frac{9}{7}$	$\frac{10}{7}$	$c_1 = 868 + 2108x,\ b = 41 + 475x$	$\frac{292}{7}$	$\frac{23}{7}$	$\frac{10}{7}$
11.	$\frac{156}{7}$	$\frac{11}{7}$	$\frac{12}{7}$	$c_2 = 11310 + 130b,\ b \geq 0$	$\frac{324}{7}$	$\frac{25}{7}$	$\frac{12}{7}$
12.	$\frac{164}{7}$	$\frac{11}{7}$	$\frac{13}{7}$	$c_1 = 11070 + 1107x,\ b = 4 + 17x$	$\frac{332}{7}$	$\frac{11}{7}$	$\frac{27}{7}$
	$x \in \mathbb{Z}_{>0}$			$c_1 = 30135 + 4797x,\ b = 1 + 11x$	$\frac{332}{7}$	$\frac{25}{7}$	$\frac{13}{7}$

Table 21: $(3,6)$ admissible solutions of type $W_{1,i}$ using $(3,0)$ solutions of type LY_2.

S. No.	(3,0) Sol.			$c_i,\ b$	$(3,6)$ Adm.		
	c	h_1	h_2		c	h_1	h_2
1.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$c_1 = 1,\ b = 382$	$\frac{124}{5}$	$\frac{1}{5}$	$\frac{12}{5}$
				$4 \leq c_2 \leq 22,\ b = 3(19c_2 - 62)$	$\frac{124}{5}$	$\frac{11}{5}$	$\frac{2}{5}$
2.	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$c_1 = 3, 4,\ b = 3(153c_1 - 385)$	$\frac{132}{5}$	$\frac{1}{5}$	$\frac{13}{5}$
				$c_2 = 4 + 3x, b = 8 + 50x,\ x \in [0, 30]$	$\frac{132}{5}$	$\frac{11}{5}$	$\frac{3}{5}$
3.	$\frac{28}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$c_2 = 18 + 7x,\ b = 14 + 26x,\ x \in [0, 121]$	$\frac{148}{5}$	$\frac{12}{5}$	$\frac{4}{5}$
4.	$\frac{36}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$c_1 = 20 + 9x,\ b = 36 + 154x,\ x \in [0, 14]$	$\frac{156}{5}$	$\frac{3}{5}$	$\frac{14}{5}$
5.	$\frac{44}{5}$	$\frac{2}{5}$	$\frac{6}{5}$	$c_1 = 13, 24,\ b = 378, 1480$	$\frac{164}{5}$	$\frac{2}{5}$	$\frac{16}{5}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 11x^*,\ b = 10x$	$\frac{164}{5}$	$\frac{12}{5}$	$\frac{6}{5}$
6.	$\frac{52}{5}$	$\frac{3}{5}$	$\frac{6}{5}$	$c_1 = 26 + 26x,\ b = 109 + 625x,\ x \in [0, 4]$	$\frac{172}{5}$	$\frac{3}{5}$	$\frac{16}{5}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 91 + 26x,\ b = 3 + 7x$	$\frac{172}{5}$	$\frac{13}{5}$	$\frac{6}{5}$
7.	$\frac{68}{5}$	$\frac{4}{5}$	$\frac{7}{5}$	$c_1 = 102 + 68x,\ b = 49 + 299x,\ x \in [0, 18]$	$\frac{188}{5}$	$\frac{4}{5}$	$\frac{17}{5}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 391 + 119x,\ b = 6 + 10x$	$\frac{188}{5}$	$\frac{14}{5}$	$\frac{7}{5}$
8.	$\frac{76}{5}$	$\frac{4}{5}$	$\frac{8}{5}$	$c_1 = 171, 342,\ b = 2010, 4510$	$\frac{196}{5}$	$\frac{4}{5}$	$\frac{18}{5}$
				$c_2 = 931 + 57b,\ b \geq 0$	$\frac{196}{5}$	$\frac{14}{5}$	$\frac{8}{5}$
9.	$\frac{84}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$c_1 = 253 + 33x,\ b = 16 + 25x$	$\frac{204}{5}$	$\frac{6}{5}$	$\frac{17}{5}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 1508 + 154x,\ b = 6 + 9x$	$\frac{204}{5}$	$\frac{16}{5}$	$\frac{7}{5}$
10.	$\frac{92}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$c_1 = 1127 + 299x,\ b = 28 + 68x$	$\frac{212}{5}$	$\frac{6}{5}$	$\frac{18}{5}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 3289 + 299x,\ b = 2 + 5x$	$\frac{212}{5}$	$\frac{16}{5}$	$\frac{8}{5}$
11.	$\frac{108}{5}$	$\frac{7}{5}$	$\frac{9}{5}$	$c_1 = 3944 + 833x,\ b = 32 + 50x$	$\frac{228}{5}$	$\frac{7}{5}$	$\frac{19}{5}$
				$c_2 = 25194 + 459b,\ b \geq 0$	$\frac{228}{5}$	$\frac{17}{5}$	$\frac{8}{5}$
12.	$\frac{116}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	$c_1 = 14877 + 1102x\ ,\ b = 1 + 11x$	$\frac{236}{5}$	$\frac{8}{5}$	$\frac{19}{5}$

	$x \in \mathbb{Z}_{>0}$			$c_2 = 14326 + 1653x,\ b = 8 + 10x$	$\frac{236}{5}$	$\frac{18}{5}$	$\frac{9}{5}$

Table 22: $(3,6)$ admissible solutions of type $W_{1,i}$ using $(3,0)$ solutions of type $(LY_1)^{\otimes 2}$

S. No.	(3,0) Sol.			c_i	b	$(3,6)$ Adml.		
	c	h_1	h_2			c	h_1	h_2
1.	4	$\frac{1}{3}$	$\frac{2}{3}$	$3 \leq c_1 \leq 7$	$b = 2(135c_1 - 322)$	28	$\frac{1}{3}$	$\frac{8}{3}$
	$x \in \mathbb{Z}_{>0}$			$5 \leq c_1 \leq 112$	$b = (18c_2 - 77)$	28	$\frac{7}{3}$	$\frac{2}{3}$
2.	12	$\frac{2}{3}$	$\frac{4}{3}$	$21 \leq c_1 < 162^*$	$b = 24(c_1 - 21)$	36	$\frac{2}{3}$	$\frac{10}{3}$
	$x \in \mathbb{Z}_{>0}$			$c_2 = 81 + 9x$	$b = 2x$	36	$\frac{8}{3}$	$\frac{4}{3}$
3.	20	$\frac{4}{3}$	$\frac{5}{3}$	$c_2 = 4158 + 135x$	$b = 2x,\ x \in \mathbb{Z}_{>0}$	44	$\frac{10}{3}$	$\frac{5}{3}$
1.	4	$\frac{2}{5}$	$\frac{3}{5}$	$c_2 = 10 + 5x$	$b = 106 + 119x,\ x \in [0, 13]$	28	$\frac{12}{5}$	$\frac{3}{5}$
2.	20	$\frac{7}{5}$	$\frac{8}{5}$	$c_2 = 625(7 + 4x)$	$b = 49x + 6.\ x \in \mathbb{Z}_{>0}$	44	$\frac{17}{5}$	$\frac{8}{5}$

Table 23: $(3,6)$ admissible sols. of type $W_{1,i}$ using $(3,0)$ solutions of type $A_{2,1}^{\otimes 2}$ and $A_{4,1}$.

B.3 $(3,6)$ admissible solutions of type V

S. No.	(3,0) Sol.			$b_1,\ b_2,\ s_i$	$(3,6)$ Adm.		
	c	h_1	h_2		c	h_1	h_2
1.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{16}$	$b_1 \geq 0,\ b_2 = 285956 + 1176b_1,\ s_1 = 11446 + 49b_1$	$\frac{97}{2}$	$\frac{3}{2}$	$\frac{65}{16}$
				$b_1 = 64756 + 102400x,$ $b_2 = 63943315 + 96538624x,\ s_2 = 1001(76 + 119x)$	$\frac{97}{2}$	$\frac{9}{2}$	$\frac{17}{16}$
2.	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{16}$	$b_1 \geq 0,\ b_2 = 363627 + 1275b_1,\ s_1 = 13497 + 51b_1$	$\frac{99}{2}$	$\frac{3}{2}$	$\frac{67}{16}$
				$b_1 = 24087 + 26624x,\ b_2 = 26433783 + 27992064x,$ $s_2 = 629(123 + 133x)$	$\frac{99}{2}$	$\frac{9}{2}$	$\frac{19}{16}$
3.	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{5}{16}$	$b_1 \geq 0,\ b_2 = 455005 + 1378b_1,\ s_1 = 15756 + 53b_1$	$\frac{101}{2}$	$\frac{3}{2}$	$\frac{69}{16}$
				$b_1 = 53342 + 56320x,\ b_2 = 64559448 + 67351552x,$ $s_2 = 1961(234 + 245x)$	$\frac{101}{2}$	$\frac{9}{2}$	$\frac{21}{16}$
4.	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{7}{16}$	$b_1 \geq 0,\ b_2 = 561350 + 1485b_1,\ s_1 = 18231 + 55b_1$	$\frac{103}{2}$	$\frac{3}{2}$	$\frac{71}{16}$
5.	$\frac{9}{2}$	$\frac{1}{2}$	$\frac{9}{16}$	$b_1 \geq 0,\ b_2 = 683970 + 1596b_1,\ s_1 = 20930 + 57b_1$	$\frac{105}{2}$	$\frac{3}{2}$	$\frac{73}{16}$

				$b_1 = 8(139+224x)$, $b_2 = 2523377 + 3078912x$, $s_2 = 24149(4+5x)$	$\frac{105}{2}$	$\frac{9}{2}$	$\frac{25}{16}$
6.	$\frac{11}{2}$	$\frac{1}{2}$	$\frac{11}{16}$	$b_1 \geq 0$, $b_2 = 824221 + 1711b_1$, $s_1 = 23861 + 59b_1$	$\frac{107}{2}$	$\frac{3}{2}$	$\frac{75}{16}$
7.	$\frac{13}{2}$	$\frac{1}{2}$	$\frac{13}{16}$	$b_1 \geq 0$, $b_2 = 983507 + 1830b_1$, $s_1 = 27032 + 61b_1$	$\frac{109}{2}$	$\frac{3}{2}$	$\frac{77}{16}$
8.	$\frac{15}{2}$	$\frac{1}{2}$	$\frac{15}{16}$	$b_1 \geq 0$, $b_2 = 1163280 + 1953b_1$, $s_1 = 30451 + 63b_1$	$\frac{111}{2}$	$\frac{3}{2}$	$\frac{79}{16}$
				$b_1 = 29 + 160x$, $b_2 = 36(64543 + 45136x)$, $s_2 = 705(1546 + 1085x)$	$\frac{111}{2}$	$\frac{9}{2}$	$\frac{31}{16}$
9.	$\frac{17}{2}$	$\frac{1}{2}$	$\frac{17}{16}$	$b_1 \geq 0$, $b_2 = 1365040 + 2080b_1$, $s_1 = 34126 + 65b_1$	$\frac{113}{2}$	$\frac{3}{2}$	$\frac{81}{16}$
10.	$\frac{19}{2}$	$\frac{1}{2}$	$\frac{19}{16}$	$b_1 \geq 0$, $b_2 = 1590335 + 2211b_1$, $s_1 = 38065 + 67b_1$	$\frac{115}{2}$	$\frac{3}{2}$	$\frac{83}{16}$
11.	$\frac{21}{2}$	$\frac{1}{2}$	$\frac{21}{16}$	$b_1 \geq 0$, $b_2 = 1840761 + 2346b_1$, $s_1 = 42276 + 69b_1$	$\frac{117}{2}$	$\frac{3}{2}$	$\frac{85}{16}$
12.	$\frac{23}{2}$	$\frac{1}{2}$	$\frac{23}{16}$	$b_1 \geq 0$, $b_2 = 2117962 + 2485b_1$, $s_1 = 46767 + 71b_1$	$\frac{119}{2}$	$\frac{3}{2}$	$\frac{87}{16}$
13.	$\frac{31}{2}$	$\frac{3}{2}$	$\frac{15}{16}$	$b_1 = 4716 + 8704x$, $b_2 = 92434598 + 156088832x$, $s_2 = 761165(55 + 93x)$	$\frac{127}{2}$	$\frac{11}{2}$	$\frac{31}{16}$

Table 24: $(3,6)$ admissible solutions of type $V_{1,i}$ using $(3,0)$ solutions of type $B_{r,1}$.

S. No.	(3,0) Sol.			$b_1,\ b_2,\ s_i$	$(3,6)$ Adm.		
	c	h_1	h_2		c	h_1	h_2
1.	1	$\frac{1}{2}$	$\frac{1}{8}$	$b_1 \geq 0,\ b_2 = 323155 + 1225b_1,\ s_1 = 12446 + 50b_1$	49	$\frac{3}{2}$	$\frac{33}{8}$
				$b_1 = 609 + 3584x$, $b_2 = 4(539131 + 890240x)$, $s_2 = 1615(1 + 3x)$	49	$\frac{9}{2}$	$\frac{9}{8}$
2.	2	$\frac{1}{2}$	$\frac{1}{4}$	$b_1 \geq 0,\ b_2 = 407525 + 1326b_1,\ s_1 = 14600 + 52b_1$	50	$\frac{3}{2}$	$\frac{17}{4}$
				$b_1 = 22 + 32x$, $b_2 = 917721 + 35776x$, $s_2 = 117(16 + x)$	50	$\frac{9}{2}$	$\frac{5}{4}$
3.	3	$\frac{1}{2}$	$\frac{3}{8}$	$b_1 \geq 0,\ b_2 = 506226 + 1431b_1,\ s_1 = 16966 + 54b_1$	51	$\frac{3}{2}$	$\frac{35}{8}$
				$b_1 = 751 + 3200x$, $b_2 = 27(61505 + 152704x)$, $s_2 = 2907(4 + 11x)$	51	$\frac{9}{2}$	$\frac{11}{8}$
4.	5	$\frac{1}{2}$	$\frac{5}{8}$	$b_1 \geq 0,\ b_2 = 751805 + 1653b_1,\ s_1 = 22366 + 58b_1$	53	$\frac{3}{2}$	$\frac{37}{8}$
				$b_1 = 141 + 160x$, $b_2 = 898974 + 312736x$, $s_2 = 29(1244 + 455x)$	53	$\frac{9}{2}$	$\frac{13}{8}$
5.	6	$\frac{1}{2}$	$\frac{3}{4}$	$b_1 \geq 0,\ b_2 = 901395 + 1770b_1,\ s_1 = 25416 + 60b_1$	54	$\frac{3}{2}$	$\frac{19}{4}$
				$b_1 = 2x$, $b_2 = 724275 + 5556x$, $s_2 = 68013 + 539x$	54	$\frac{9}{2}$	$\frac{7}{4}$
6.	7	$\frac{1}{2}$	$\frac{7}{8}$	$b_1 \geq 0,\ b_2 = 1070740 + 1891b_1,\ s_1 = 28710 + 62b_1$	55	$\frac{3}{2}$	$\frac{39}{8}$
7.	9	$\frac{1}{2}$	$\frac{9}{8}$	$b_1 \geq 0,\ b_2 = 1474647 + 2145b_1,\ s_1 = 36062 + 66b_1$	57	$\frac{3}{2}$	$\frac{41}{8}$
8.	10	$\frac{1}{2}$	$\frac{5}{4}$	$b_1 \geq 0,\ b_2 = 1712305 + 2278b_1,\ s_1 = 40136 + 68b_1$	58	$\frac{3}{2}$	$\frac{21}{4}$
9.	11	$\frac{1}{2}$	$\frac{11}{8}$	$b_1 \geq 0,\ b_2 = 1975910 + 2415b_1,\ s_1 = 44486 + 70b_1$	59	$\frac{3}{2}$	$\frac{43}{8}$
10.	13	$\frac{3}{2}$	$\frac{5}{8}$	$b_1 = 1137 + 2816x$, $b_2 = 6039998 + 9273088x$, $s_2 = 24679(8 + 13x)$	61	$\frac{11}{2}$	$\frac{13}{8}$

11.	14	$\frac{3}{2}$	$\frac{3}{4}$	$b_1 = 106 + 280x,\ b_2 = 3213267 + 1333040x,\ s_2 = 2299(110 + 49x)$	62	$\frac{11}{2}$	$\frac{7}{4}$

Table 25: $(3, 6)$ admissible solutions of type $V_{1,i}$ using $(3, 0)$ solutions of type $D_{r,1}$.

S. No.	(3,0) Sol.			$b_1,\ b_2,\ s_i$	(3,6) Adm.		
	c	h_1	h_2		c	h_1	h_2
1.	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$b_1 = 8500 + 14467x,\ b_2 = 9481960 + 14051712x,\ s_1 = 2494(5 + 8x)$	$\frac{340}{7}$	$\frac{8}{7}$	$\frac{31}{7}$
2.	$\frac{12}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$b_1 = 23 + 209x,\ b_2 = 36(20839 + 6479x),\ s_1 = 124(20 + 9x)$	$\frac{348}{7}$	$\frac{9}{7}$	$\frac{31}{7}$
3.	$\frac{44}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$b_1 = 532 + 551x,\ b_2 = 1897442 + 1088196x,\ s_1 = 530(236 + 143x)$	$\frac{380}{7}$	$\frac{11}{7}$	$\frac{33}{7}$
				$b_1 = 50 + 207x,\ b_2 = 890018 + 533922x,\ s_1 = 8957(8 + 5x)$	$\frac{380}{7}$	$\frac{32}{7}$	$\frac{19}{7}$
4.	$\frac{52}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$b_1 = 97 + 527x,\ b_2 = 194(6395 + 5797x),\ s_1 = 492(97 + 99x)$	$\frac{388}{7}$	$\frac{11}{7}$	$\frac{34}{7}$
5.	$\frac{60}{7}$	$\frac{3}{7}$	$\frac{8}{7}$	$b_1 = 484 + 15730x,\ b_2 = 2517810 + 30403230x,\ s_2 = 15523(2 + 29x)$	$\frac{396}{7}$	$\frac{10}{7}$	$\frac{36}{7}$

Table 26: $(3, 6)$ admissible solutions of type $V_{1,i}$ using $(3, 0)$ solutions of type LY_2 and $x \in \mathbb{Z}_{>0}$.

S. No.	(3,0) Sol.			$b_1,\ b_2,\ s_i$	(3,6) Adm.		
	c	h_1	h_2		c	h_1	h_2
1.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$b_1 = 2012 + 2500x,\ b_2 = 2917568 + 2518750x,\ s_1 = 3751(1 + x)$	$\frac{244}{5}$	$\frac{6}{5}$	$\frac{22}{5}$
				$b_1 = 15 + 19x,\ b_2 = 417449 + 21204x,\ s_2 = 7409 + 434x$	$\frac{244}{5}$	$\frac{21}{5}$	$\frac{7}{5}$
2.	$\frac{28}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$b_1 = 3788 + 4375x,\ b_2 = 6973730 + 6845000x,\ s_1 = 1887(48 + 49x)$	$\frac{268}{5}$	$\frac{7}{5}$	$\frac{24}{5}$
				$b_1 = 3 + 13x,\ b_2 = 706078 + 41366x,\ s_2 = 222(857 + 51x)$	$\frac{268}{5}$	$\frac{22}{5}$	$\frac{9}{5}$
3.	$\frac{36}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$b_1 = 6 + 49x,\ b_2 = 975090 + 107562x,\ s_1 = 57681 + 7163x$	$\frac{276}{5}$	$\frac{8}{5}$	$\frac{24}{5}$
				$b_1 = 4 + 50x,\ b_2 = 975986 + 180600x,\ s_2 = 551(142 + 27x)$	$\frac{276}{5}$	$\frac{23}{5}$	$\frac{9}{5}$
4.	$\frac{44}{5}$	$\frac{2}{5}$	$\frac{6}{5}$	$b_1 = 796 + 2204x,\ b_2 = 3264955 + 4201926x,\ s_2 = 1271(32 + 49x)$	$\frac{284}{5}$	$\frac{7}{5}$	$\frac{26}{5}$
5.	$\frac{68}{5}$	$\frac{4}{5}$	$\frac{7}{5}$	$b_1 = 139 + 391x,\ b_2 = 3543566 + 2183620x,\ s_1 = 10434(79 + 51x)$	$\frac{308}{5}$	$\frac{9}{5}$	$\frac{27}{5}$
6.	$\frac{76}{5}$	$\frac{4}{5}$	$\frac{8}{5}$	$b_1 = 2390 + 3750x,\ b_2 = 18130982 + 23030000x,\ s_1 = 188993(7 + 9x)$	$\frac{316}{5}$	$\frac{9}{5}$	$\frac{28}{5}$

Table 27: $(3, 6)$ admissible solutions of type $V_{1,i}$ using $(3, 0)$ solutions of type $(LY_1)^{\otimes 2}$ and $x \in \mathbb{Z}_{>0}$.

S. No.	(3,0) Sol. c	h_1	h_2	$b_1,\ b_2,\ s_i$	(3,6) Adml. c	h_1	h_2
1.	4	$\frac{1}{3}$	$\frac{2}{3}$	$b_1 = 6 + 20x,\ b_2 = 963517 + 26810x,\ s_1 = 147(25 + x)$	52	$\frac{4}{3}$	$\frac{14}{3}$
				$b_1 = 2x,\ b_2 = 487942 + 3896x,\ s_1 = 27144 + 225x$	52	$\frac{13}{3}$	$\frac{8}{3}$
2.	12	$\frac{2}{3}$	$\frac{4}{3}$	$b_1 \geq 0,\ b_2 = 1971810 + 3384b_1,\ s_1 = 83592 + 162b_1$	60	$\frac{5}{3}$	$\frac{16}{3}$
1.	4	$\frac{2}{5}$	$\frac{3}{5}$	$b_1 = 12896 + 15548x,\ b_2 = 18921702 + 21867664x,\ s_1 = 34375(6 + 7x)$	52	$\frac{7}{5}$	$\frac{23}{5}$

Table 28: $(3,6)$ admissible solutions of type $V_{1,i}$ using $(3,0)$ solutions of type $A_{2,1}^{\otimes 2}$ and $A_{4,1}$ and $x \in \mathbb{Z}_{>0}$.

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