# Thermodynamic topology of black holes and an invariant of spacetime

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We represent a new approach to exploring the thermodynamic topology of black holes, without introducing the nonphysical variable $\Theta \in [0, \pi]$ considered in previous studies, where black holes can exchange both energy and matter with the environment, leading to a thermal and chemical equilibrium. We construct a conserved topological tensor based on the gradient flow of the off-shell grand free energy corresponding to a two-dimensional or higher-dimensional vector field whose zeros are black hole solutions. We obtain a topological charge that is the sum of the index of all zeros. We find that black holes that share the same background geometry would have the same topological charge, hence they belong to the same kind of solutions. This point implies that the topological charge characterizing the black hole thermodynamics is also an invariant of spacetime, leading to valuable insight into the observed cosmological constant and the AdS distance conjecture.

*Introduction*—Black hole thermodynamics [1, 2] has been an active research area in modern physics because it reveals rich thermodynamic behaviors and provides clues about the quantum nature of spacetime. The Hawking-Page phase transition between the Schwarzschild anti-de Sitter (AdS) black hole and the thermal AdS spacetime [3] can provide a dual description of the confinement/deconfinement phase transition in the gauge field theory [4]. The Reissner-Nordström (RN) AdS black hole can undergo a first-order phase transition between small/large black holes, analogous to the liquid-gas phase transition in the Van der Waals fluid [5, 6]. In the extended phase space with a dynamical cosmological constant [7] whose higher-dimensional origin was proposed by using holographic braneworlds [8], various thermodynamic behaviors have been explored, including the P-V criticality [9], reentrant phase transitions [10], the heat engine [11], superfluid-like criticality [12]. In addition, microscopic structures of black holes can be probed by the thermodynamic geometry [13, 14].

The topological investigation provides a powerful approach to address universal and global properties of black hole solutions, without solving equations of motion and going into the details of solutions. Black hole solutions are identified as topological defects corresponding to zeros of a two-dimensional vector field [15–20]. A negative (positive) winding number at each zero represents a locally unstable (stable) black hole. Summing the winding number of all zeros yields a topological charge that allows us to classify the thermodynamics of black holes into three classes [15]. The analysis of the asymptotic behavior of the Hawking temperatures also unveils new classes and subclasses [21–23].

The essential point in the construction of Ref. [15] is to introduce a nonphysical variable $\Theta \in [0, \pi]$ to define a two-dimensional vector field as $\vec{\phi} = (\partial_{r_+}\mathcal{F}, -\cot\Theta\csc\Theta)$ where $\mathcal{F}$ is the off-shell Gibbs free energy [24–26] and $r_+$ is the event horizon radius. Hence, a question raised here is whether introducing the nonphysical variable $\Theta$ to construct the second component of $\vec{\phi}$ is the manifestation of a fact that certain physical processes are missing. In order to answer this question, it is important to note that black holes considered in Ref. [15] are assumed to be placed inside a cavity which is in contact with the environment (thermal bath) via the heat exchange. In fact, black holes can radiate/absorb energy and matter to/from the environment, leading to a thermal and chemical equilibrium. Therefore, the generalized thermodynamic potential should be the off-shell grand free energy [27, 28], instead of the off-shell Gibbs free energy. This insight allows us to define naturally a two-dimensional or higher-dimensional vector field as the gradient flow of the off-shell grand free energy. Its zeros correspond to extreme points of the off-shell grand free energy and thus are identified as black hole solutions because they are dominant contributions to the partition function. Based on the gradient flow of the off-shell grand free energy, we construct a conserved topological tensor in the space of thermodynamic quantities, which consists of many conserved currents flowing along various time directions corresponding to varying the ensemble parameters. As a result, we can derive a topological charge which is the sum of the index of all zeros.

The black hole solutions that have the same asymptotic geometry possess the same topological charge. This implies that the topological charge obtained by black hole thermodynamics provides an invariant of spacetime, which is used to classify spacetime solutions into universal classes. We find that Minkowski-flat/dS spacetimes are the same class, and the AdS spacetime belongs to a different class. Interestingly, this result has indications of the observed cosmological constant [29] and the AdS distance conjecture [30, 31].

*The simple case*—Let us consider the charged black hole which is characterized by the entropy $S$ (or equivalently its event horizon radius $r_+$) and the electric charge $Q$. When the black hole can exchange both energy and matter with the environment due to its radiation and absorption, leading to a thermal and chemical equilibrium, the thermodynamic system is described by the off-shell grand free energy given by [27, 28]

$$\hat{\mathcal{F}} = M - \frac{S}{\tau_1} - \frac{Q}{\tau_2}, \tag{1}$$

where $M$ is the ADM mass of the black hole, and $\tau_{1,2}$ are the inverse temperature and the inverse chemical potential of the ensemble which are adjustable. Additionally, the first law reads

$$dM = TdS + \mu dQ, \tag{2}$$

where the Hawking temperature $T$ and the chemical potential $\mu$ are determined by

$$T = \left(\frac{\partial M}{\partial S}\right)_Q, \quad \mu = \left(\frac{\partial M}{\partial Q}\right)_S. \tag{3}$$

We introduce a vector field $\vec{\phi}$ on a two-dimensional space formed by $(S, Q)$ as

$$\vec{\phi} = \nabla\hat{\mathcal{F}} = \left(\partial_S\hat{\mathcal{F}}, \partial_Q\hat{\mathcal{F}}\right) \equiv (\phi^S, \phi^Q). \tag{4}$$

The vector field $\vec{\phi}$ determines the gradient flow of the off-shell grand free energy $\hat{\mathcal{F}}$ or the rate of the change of $\hat{\mathcal{F}}$ with respect to the entropy $S$ and the electric charge $Q$. The vanishing of $\vec{\phi}$ corresponds to the extreme points of $\hat{\mathcal{F}}$, which contribute dominantly to the partition function. On the other hand, the zeros of the vector field $\vec{\phi}$, leading to $\tau_1 = 1/T$ and $\tau_2 = 1/\mu$, correspond to the black hole solutions.

Including both the ensemble variables $\tau_{1,2}$ and the thermodynamic quantities $(S, Q)$, we construct a four-dimensional space corresponding to the evolution of the topological defects (zeros of $\vec{\phi}$), where $\tau_{1,2}$ play the role of time parameters. In this space, we define a topological tensor as follows

$$j^{\mu\nu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho\lambda}\epsilon_{ab}\partial_\rho n^a\partial_\lambda n^b, \tag{5}$$

where $\epsilon^{\mu\nu\rho\lambda}$ and $\epsilon_{ab}$ are the totally antisymmetric Levi-Civita tensors in four and two dimensions, respectively, $\partial_\rho \equiv (\partial_{\tau_1}, \partial_{\tau_2}, \partial_S, \partial_Q)$, and $n^a = \phi^a/||\vec{\phi}||$ is a normalized vector field. We can decompose the topological tensor $j^{\mu\nu}$ into three components as $(j^{\tau_1\mu}, j^{\tau_2\mu}, j^{SQ})$, where $j^{\tau_1\mu}$ is the current flowing along the time direction $\tau_2$ when adjusting the chemical potential of the ensemble, whereas, the current $j^{\tau_2\mu}$ flows along the time direction $\tau_1$ when varying the ensemble temperature.

Because the topological tensor $j^{\mu\nu}$ satisfies the continuity equation $\partial_\nu j^{\mu\nu} = 0$, it is identically conserved. A corresponding topological charge is

$$\begin{aligned} Q_t &= \int_\Sigma j^{\tau_1\tau_2}d^2x = \frac{1}{2\pi}\int_\Sigma \Delta_\phi\left(\ln||\vec{\phi}||\right)J_2(\phi|x)\,d^2x, \\ &= \int_\Sigma \delta^2(\phi)J_2(\phi|x)\,d^2x, \end{aligned} \tag{6}$$

where $J_2(\phi|x) = \frac{1}{2}\epsilon^{\tau_1\tau_2\rho\lambda}\epsilon_{ab}\partial_\rho\phi^a\partial_\lambda\phi^b$ is the Jacobian of the transformation $x = (S, Q) \longrightarrow \phi = (\phi^S, \phi^Q)$. Note that, we have used the relation $\Delta_\phi\left(\ln||\vec{\phi}||\right) = 2\pi\delta^2(\phi)$ to obtain the second line. The topological charge $Q_t$ satisfies $\partial_{\tau_1}Q_t = 0$ and $\partial_{\tau_2}Q_t = 0$, meaning that the topological charge $Q_t$ is independent on the values of the inverse temperature and the inverse chemical potential of the ensemble. Therefore, the topological charge $Q_t$ characterizes the intrinsically thermodynamic properties of the black hole.

For the zeros $\{x_i\}$ of the vector field $\vec{\phi}$ which are nondegenerate, i.e. $J_2(\phi|x)|_{x_i} \neq 0$,[1] we can express the delta function $\delta^2(\phi)$ as follows [33]

$$\delta^2(\phi) = \sum_{i=1}^{K}\frac{1}{|J_2(\phi|x)|_{x_i}|}\delta^2(x - x_i), \tag{7}$$

where $K$ is the number of the isolated zeros of the vector field $\vec{\phi}$. By substituting this relation into Eq. (6), we find

$$Q_t = \sum_{i=1}^{K}\text{index}_{x_i}(\vec{\phi}), \tag{8}$$

where $\text{index}_{x_i}(\vec{\phi}) = \text{sign}\left(J_2(\phi|x)|_{x_i}\right) = \pm 1$ refers to the index of the vector field $\vec{\phi}$ at the zero point $x_i$, which is in two dimensions the winding number, which describes the behavior of the gradient flow of the off-shell grand free energy around its extreme point $x_i$.

We can show that the Jacobian $J_2(\phi|x)|_{x_i}$ is expressed in terms of the heat capacity at a fixed chemical potential $C_\mu = T\left(\partial S/\partial T\right)_\mu$ as follows

$$J_2(\phi|x)|_{x_i} = \partial_S T\partial_Q\mu - (\partial_Q T)^2 = \frac{T\partial_Q\mu}{C_\mu}. \tag{9}$$

This relation means that the sign of the Jacobian $J_2(\phi|x)|_{x_i}$ is the same as that of the heat capacity $C_\mu$ corresponding to the zero $x_i$. As a result, the locally stable black holes ($C_\mu > 0$) would have $\text{index}_{x_i}(\vec{\phi}) = +1$. On the contrary, $\text{index}_{x_i}(\vec{\phi}) = -1$ describes the locally unstable black holes ($C_\mu < 0$).

For the RN black hole which is the charged black hole in the Minkowski-flat background, its vector field possesses a unique zero determined by $\sqrt{S} = \tau_1(\tau_2^2 - 1)/(4\sqrt{\pi}\tau_2^2)$ and $Q = \sqrt{S/\pi}/\tau_2$, denoted by $\text{ZP}_1$ as depicted in the left panel of Fig. 1. This zero has $\text{index}_{x_i}(\vec{\phi}) = -1$ because the heat capacity at the fixed chemical potential $C_\mu = -2S$ is always negative. As

[1] The Jacobian $J_2(\phi|x)|_{x_i}$ vanishes at second-order phase transition points. In this case, the bifurcation theory can be used to study the evolution of the topological currents [32].

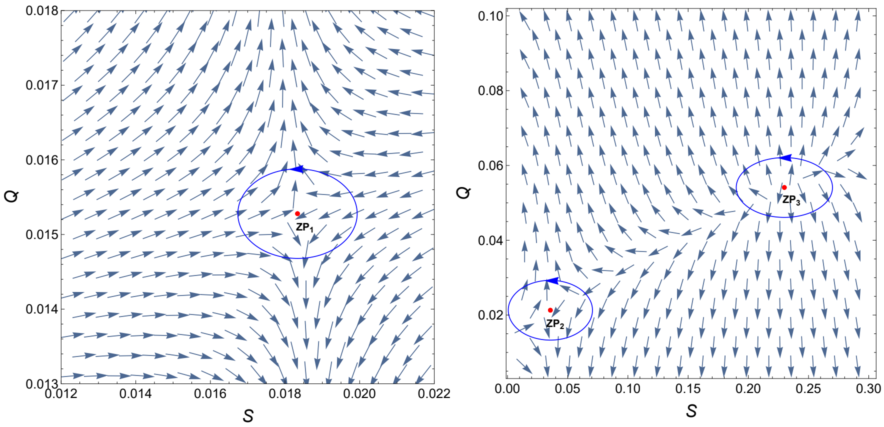


FIG. 1: The left and right panels represent the unit vector field $\vec{\phi}/||\vec{\phi}||$ for the RN and RN-AdS black holes, respectively, with $\tau_1 = 1$, $\tau_2 = 5$, and $l = 0.3$. Red dots refer to the zeros of $\vec{\phi}$. The blue closed curves denote the curves $\mathcal{C}_i$, enclosing solely each zero, through which we compute $\mathrm{index}_{x_i}(\vec{\phi}) = \frac{1}{2\pi}\int_{\mathcal{C}_i} \epsilon_{ab} n^a dn^b$.

a result, the RN black hole has the topological charge $Q_t = -1$.

For the RN-AdS black hole which is the charged black hole in the AdS background, if $\tau_2 > 1$ ($\mu < 1$), its vector field has two distinguished zeros, which are determined by

$$\sqrt{S} = \frac{2l^2\pi^{3/2}}{3\tau_1}\left[1 \pm \sqrt{1 - \frac{3\tau_1^2(\tau_2^2-1)}{4l^2\pi^2\tau_2^2}}\right], \tag{10}$$

$$Q = \frac{1}{\tau_2}\sqrt{\frac{S}{\pi}}, \tag{11}$$

where the solutions with the signs "$-$" and "$+$" are denoted by ZP$_2$ and ZP$_3$ as shown in the right panel of Fig. 1. With $C_\mu = -2S\left[1 - 6S/\left(3S - \pi l^2(1-\mu^2)\right)\right]$, we can indicate that the ZP$_2$ and ZP$_3$ zeros have $\mathrm{index}_{x_i}(\vec{\phi}) = -1$ and $+1$, corresponding to the locally unstable and stable black hole solutions. This means that the topological charge of the RN-AdS black hole in the case of $\tau_2 > 1$ is given by $Q_t = -1 + 1 = 0$. However, there exists only one zero in the case of $\tau_2 \leq 1$ ($\mu \geq 1$), which is the solution with the sign "+" given in Eq. (10), corresponding to $\mathrm{index}_{x_i}(\vec{\phi}) = +1$. Hence, in this case, the topological charge of the RN-AdS black hole is $Q_t = 1$. We can realize this change of the topological charge of the RN-AdS black hole through observing the behavior of the Hawking temperature, given shown in Fig. 2. We see that the temperature of the RN-AdS black hole behaves discontinuously when changing from the region $\tau_2 \leq 1$ to the one $\tau_2 > 1$. With the combined regions, we find the topological charge of the RN-AdS black hole as $Q_t \in \{0, 1\}$.

*Generalization*—We generalize the topological study of the black hole thermodynamics, represented above, to black holes which are characterized by the entropy $S$ and other quantities denoted by $\{\tilde{Q}_2, \tilde{Q}_3, ..., \tilde{Q}_N\}$. The corresponding first law reads

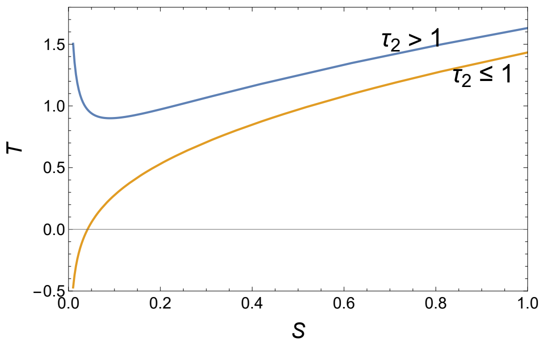


FIG. 2: The behavior of the RN-AdS black hole temperature in the entropy $S$ for two regions of the time parameter $\tau_2$ with the AdS radius $l$ kept fixed.

$$dM = TdS + \sum_{m=2}^{N} \mu_m d\tilde{Q}_m, \tag{12}$$

where the AMD mass $M$ is a function of the entropy $S$ and the quantities $\{\tilde{Q}_2, \tilde{Q}_3, ..., \tilde{Q}_N\}$, and $\mu_m$ is the conjugate to $\tilde{Q}_m$ which is called the generalized chemical potential. The Hawking temperature and the generalized chemical potentials are determined by

$$T = \left(\frac{\partial M}{\partial S}\right)_{\tilde{Q}_2,..,\tilde{Q}_N}, \tag{13}$$

$$\mu_m = \left(\frac{\partial M}{\partial \tilde{Q}_m}\right)_{S,\tilde{Q}_2,..,\tilde{Q}_{m-1},\tilde{Q}_{m+1},...,\tilde{Q}_N}. \tag{14}$$

When the black hole is in contact with the environment where it can exchange both energy and matter, the black hole thermodynamics is described by the following off-shell grand free energy

$$\hat{\mathcal{F}} = M - \frac{S}{\tau_1} - \sum_{m=2}^{N} \frac{\tilde{Q}_m}{\tau_m}, \tag{15}$$

where $\{\tau_m\}$ with $m = 2, ..., N$ are the inverse generalized chemical potentials of the ensemble.

The gradient flow of the off-shell grand free energy in the $N$-dimensional space specified by $x = (S, \tilde{Q}_2, ..., \tilde{Q}_N)$ is determined by the vector field $\vec{\phi} = \nabla\hat{\mathcal{F}}$ given by

$$\vec{\phi} = \left(\partial_S\hat{\mathcal{F}}, \partial_{\tilde{Q}_2}\hat{\mathcal{F}}, ..., \partial_{\tilde{Q}_N}\hat{\mathcal{F}}\right) \equiv \left(\phi^S, \phi^{\tilde{Q}_2}, ..., \phi^{\tilde{Q}_N}\right). \tag{16}$$

The zeros of this vector field are determined by $T = 1/\tau_1$ and $\mu_m = 1/\tau_m$, with $m = 2, ..., N$, corresponding to black hole solutions. We introduce a topological tensor with $N$ indices, which is totally antisymmetric, as follows

$$j^{\nu_1...\nu_N} = \frac{1}{\omega_{N-1}}\epsilon^{\nu_1...\nu_N\rho_1...\rho_N}\epsilon_{a_1...a_N}\partial_{\rho_1}n^{a_1}...\partial_{\rho_N}n^{a_N}, \tag{17}$$

where $\omega_{N-1} = 2\pi^{N/2}/\Gamma(\frac{N}{2})$ is the surface area of the $(N-1)$-dimensional unit sphere. This topological tensor consists of $N$ currents which flow along the time directions $\tau_1$, $\tau_2$,...,$\tau_N$ corresponding to the change of the temperature and the generalized chemical potentials of the ensemble. We can check that $j^{\nu_1\cdots\nu_N}$ is identically conserved

$$\partial_{\nu_k}j^{\nu_1...\nu_k...\nu_N} = 0, \quad \text{with} \quad k = 1, ..., N. \tag{18}$$

This implies a topological charge $Q_t$ given by

$$\begin{aligned} Q_t &= \int_{\mathcal{V}} j^{\tau_1\cdots\tau_N}d^Nx \\ &= -\frac{1}{\omega_{N-1}}\int_{\mathcal{V}}\Delta_\phi\left(\frac{1}{||\vec{\phi}||^{N-2}}\right)J_N(\phi|x)\,d^Nx, \end{aligned} \tag{19}$$

where $J_N(\phi|x)$ is the Jacobian of the transformation $x = (S, \tilde{Q}_2, ..., \tilde{Q}_N) \longrightarrow \phi = (\phi^S, \phi^{\tilde{Q}_2}, ..., \phi^{\tilde{Q}_N})$, given by

$$\begin{aligned} J_N(\phi|x) &= \frac{1}{N!}\epsilon^{\tau_1...\tau_N\nu_1...\nu_N}\epsilon_{a_1...a_N}\partial_{\nu_1}\phi^{a_1}...\partial_{\nu_N}\phi^{a_N} \\ &= \begin{vmatrix} \partial_{\tilde{Q}_2}\mu_2 & \ldots & \partial_{\tilde{Q}_2}\mu_N \\ \vdots & \ldots & \vdots \\ \partial_{\tilde{Q}_N}\mu_2 & \ldots & \partial_{\tilde{Q}_N}\mu_N \end{vmatrix}\frac{T}{C_{\mu_2,...,\mu_N}}, \end{aligned} \tag{20}$$

where $|...|$ denotes the determinant of the $(N-1)\times(N-1)$ matrix $\partial_{\tilde{Q}_m}\mu_n$ and $C_{\mu_2,...,\mu_N} = T\left(\partial S/\partial T\right)_{\mu_2,...,\mu_N}$ is the heat capacity at the generalized chemical potentials kept fixed. This topological charge $Q_t$ satisfies

$$\frac{\partial Q_t}{\partial\tau_k} = 0 \quad \text{with} \quad k = 1, ..., N. \tag{21}$$

This means that $Q_t$ is independent of the temperature and the generalized chemical potentials of the ensemble, hence it refers to the intrinsic characteristic of the black hole.

By substituting the following relations

$$\Delta_\phi\left(\frac{1}{||\vec{\phi}||^{N-2}}\right) = -\omega_{N-1}\delta^N(\phi), \tag{22}$$

$$\delta^N(\phi) = \sum_{i=1}^{K}\frac{\delta^N(x - x_i)}{|J_N(\phi|x)|_{x_i}|}, \tag{23}$$

into the second line of Eq. (19), we compute the topological charge as

$$Q_t = \sum_{i=1}^{K}\text{index}_{x_i}(\vec{\phi}) = \sum_{i=1}^{K}\text{sign}\left(J_N(\phi|x)|_{x_i}\right). \tag{24}$$

If the determinant given in Eq. (20) is positive, then $\text{index}_{x_i}(\vec{\phi})$ is determined by the sign of the heat capacity $C_{\mu_2,...,\mu_N}$. This means that the black hole corresponding to the zero $x_i$ would be locally stable if $\text{index}_{x_i}(\vec{\phi}) = +1$, whereas $\text{index}_{x_i}(\vec{\phi}) = -1$ indicates the locally unstable black hole. On the contrary, if the determinant is negative, the $\text{index}_{x_i}(\vec{\phi}) = +1$ and $-1$ correspond to the black holes which are locally unstable and stable, respectively.

Based on the framework developed above, let us study the thermodynamic topology of the Kerr-Newman black hole [34, 35] with the charges $\{\tilde{Q}_2, \tilde{Q}_3\}$ corresponding to the electric charge $Q$ and the angular momentum $J$. Its off-shell grand free energy is $\hat{\mathcal{F}} = M - S/\tau_1 - Q/\tau_2 - J/\tau_3$. The corresponding vector field reads

$$\vec{\phi} = \nabla\hat{\mathcal{F}} = \left(T - \frac{1}{\tau_1}, \mu - \frac{1}{\tau_2}, \Omega - \frac{1}{\tau_3}\right), \tag{25}$$

where the Hawking temperature $T$, the chemical potential $\mu$, and the angular velocity $\Omega$ are given by

$$T = \frac{r_+}{4\pi(r_+^2 + a^2)}\left(1 - \frac{a^2}{r_+^2} - \frac{Q^2}{r_+^2}\right), \tag{26}$$

$$\mu = \frac{Qr_+}{r_+^2 + a^2}, \quad \Omega = \frac{a}{r_+^2 + a^2}, \tag{27}$$

where the parameter $a$ and the horizon radius $r_+$ are related to $S$ and $J$ as $a = 2J\sqrt{\pi S/[4J^2\pi^2 + (S + \pi Q^2)^2]}$ and $r_+ = \sqrt{S/\pi - a^2}$. The behavior of $\vec{\phi}/||\vec{\phi}||$ is depicted in Fig. 3 with the ensemble parameters given by $\tau_1 = 1$, $\tau_2 = 4$, and $\tau_3 = 5$. A zero of $\vec{\phi}$ is located at $S \approx 0.01747$, $Q \approx 0.01865$, and $J \approx 0.00004$. The black hole solution corresponding to this zero has a negative heat capacity $C_{\mu,\Omega}$ (the heat capacity at the chemical potential and angular velocity kept fixed), which indicates its thermodynamic instability. In addition, we show that this black hole solution has $\text{index}_{x_i}(\vec{\phi}) = -1$, leading to the topological charge $Q_t = -1$.

The Hawking temperature of the Kerr-Newman black hole at the chemical potential and angular velocity kept

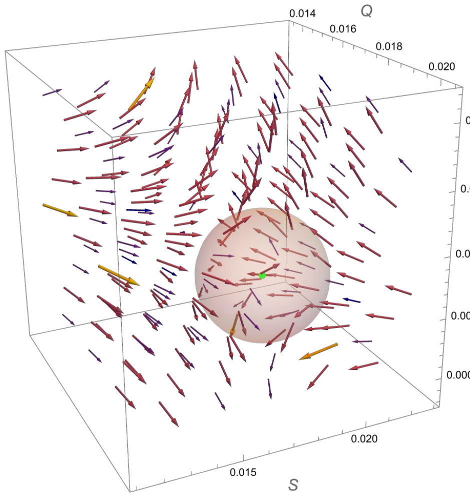


FIG. 3: The behavior of $\vec{\phi}/||\vec{\phi}||$ in the $(S, Q, J)$ space with $\tau_1 = 1$, $\tau_2 = 4$, and $\tau_3 = 5$. The green dot refers to a zero of $\vec{\phi}$. The light orange closed surface refers to the surface $\mathcal{S}_i$, enclosing the zero, by which we compute $\text{index}_{x_i}(\vec{\phi}) = \frac{1}{\omega_2}\int_{\mathcal{S}_i} \epsilon_{abc} n^a dn^b \wedge dn^c$ with $\wedge$ being the wedge product.

fixed reads

$$T = \frac{\pi^2(1-\mu^2) - S\Omega^2\left(3\pi - 2S\Omega^2\right)}{4\pi\sqrt{S}\left(\pi - S\Omega^2\right)^{3/2}}, \tag{28}$$

with $\mu^2 < 1$ and $S\Omega^2 < \pi$ for the existence of the black hole solution. We observe that the Hawking temperature behaves continuously when varying the time parameters $\tau_2 = 1/\mu$ and $\tau_3 = 1/\Omega$. This means that the topological charge $Q_t$ for the Kerr-Newman black hole is always equal to $-1$.

*$Q_t$ as an invariant of spacetime*—If there are no discontinuous changes in the behavior of the thermodynamic quantities, Eq. (21) implies that the topological charge $Q_t$ is conserved even if the time parameter $\tau_m$ (the generalized chemical potential $\mu_m$) approaches infinite (zero) corresponding to $\tilde{Q}_m \to 0$. This means that two black hole solutions which have $\tilde{Q}_m \neq 0$ and $\tilde{Q}_m = 0$, respectively, should have the same topological charge $Q_t$. For example, the Kerr-Newman and RN black holes, which correspond to $\Omega \neq 0$ and $\Omega = 0$, both have $Q_t = -1$, as derived above. In this sense, black holes that have the same background spacetime would possess the same topological charge $Q_t$. With detailed calculations, we can show explicitly that the neutral, charged, and rotating black holes, which are asymptotic to the Minkowski-flat/dS geometries, have $Q_t = -1$, hence they belong to the same class of black hole solutions. Whereas, these types of black holes, which are asymptotic to the AdS geometry, have $Q_t \in \{0, 1\}$, which means that they belong to a different class.

A remarkable consequence of introducing the topological charge $Q_t$, given by Eq. (6) or (19) based on the black hole thermodynamics, is that $Q_t$ provides an invariant to classify spacetime solutions into the universal classes because $Q_t$ only depends on the background geometry of black holes. We show the classification for constant curvature spacetimes, which are maximally symmetric, in the Table. I. According to this classification, Minkowski-flat/dS spacetimes are the same class, and the AdS spacetime belongs to a distinct class. From the mathematical aspect, this is consistent with the fact that the dS spacetime can be embedded into the Minkowski-flat spacetime with a higher dimension, whereas the embedding of the AdS spacetime requires the higher-dimensional flat space including two time-like dimensions [36].

| Kind of geometries | $Q_t$ |
|---|---|
| Minkowski-flat/dS spacetimes | -1 |
| AdS spacetime | {0,1} |

TABLE I: The universal classes of constant curvature spacetimes based on the topological charge $Q_t$.

Additionally, the fact that the Minkowski-flat/dS spacetimes belong to the same kind of solutions, which is different from the AdS solution class, implies the existence of a gap between the Minkowski-flat and AdS spacetimes. On the other hand, one cannot get the Minkowski-flat spacetime from the AdS spacetime by taking the negative cosmological constant $\Lambda$ to zero, without leading to physical/mathematical inconsistencies. For instance, the AdS distance conjecture [30, 31] states that the limit of $\Lambda \to 0$ corresponds to an infinite tower of light states breaking the low-energy description of effective field theories and hence this limit belongs to the Swampland [37]. Therefore, the topological charge $Q_t$ gives valuable insight into the observed cosmological constant, which is tiny and positive [29] (corresponding to the dS geometry, but it can be approximated by the Minkowski-flat geometry), instead of being tiny and negative.

*Conclusions*—The previous approach to the thermodynamic topology of black holes [15] requires introducing a nonphysical variable $\Theta \in [0, \pi]$, which manifests the fact that certain physical processes are missing. We have represented a new topological approach to explore the thermodynamics of black holes that are in a thermal and chemical equilibrium with the environment. The gradient flow of the off-shell grand free energy corresponds to a vector field whose zeros are black hole solutions. The index of this vector field at each zero characterizes the locally stable or unstable property of the corresponding black hole. The sum of the index of all zeros yields a topological charge characterizing the intrinsically thermodynamic properties of the black hole.

An important point in our approach is that black holes which have the same background spacetime have the same topological charge, which means that they belong to the same kind of solutions and hence share the same universal thermodynamic properties. First, this is very useful in situations where black hole solutions cannot be obtained analytically due to the complexity of the equations of motion. Second, the topological charge constructed from the black hole thermodynamics provides an invariant that allows us to classify spacetime solutions into the universal classifications. We have classified constant curvature spacetimes into two distinct classes: the first class consists of the Minkowski-flat and dS spacetimes, whereas the AdS spacetime belongs to a different class. This implies the presence of a gap between the Minkowski-flat and AdS spacetimes, which provides a reason why the observed cosmological constant is tiny and positive [29], instead of being tiny and negative.

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