RATIONALITY PROPERTIES OF COMPLEX REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

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Abstract. For a reductive group G over a non-archimedean local field, we compare smooth representations over $\mathbb{C}$ with smooth representations over $\overline{\mathbb{Q}}$. For example, we show that tensoring with $\mathbb{C}$ over $\overline{\mathbb{Q}}$ preserves irreducibility of representations. We also show that an elliptic G-representation (in the sense of Arthur) can be realized over $\overline{\mathbb{Q}}$ if and only if its central character takes values in $\overline{\mathbb{Q}}$. That applies in particular to all essentially square-integrable G-representations.

We also study the action of the automorphism group of $\mathbb{C}/\mathbb{Q}$ on complex G-representations. We prove that essentially square-integrable representations and elliptic representations are stable under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

Contents

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1. Introduction

The paper [FaSc] develops the theory of smooth representations of a reductive group G over a non-archimedean local field F on $\overline{\mathbb{Q}}_\ell$-vector spaces. Choosing an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$, one obtains results about complex representations of G. It is natural to ask about independence of the choice of such an isomorphism. Since any two isomorphisms between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ differ by a composition with an element of the automorphism group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, this question is equivalent to understanding of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on the category of smooth complex representations of G. In this paper we show that a large part of the familiar structure of this category is invariant under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

This can be considered to be a counterpart of the recent result of Scholze [Scho] asserting that the Fargues—Scholze correspondence can be done motivically and thus is independent of the ℓ in $\overline{\mathbb{Q}}_\ell$.

1.1. Notations.

(1) $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.
(2) F is a non-archimedean local field whose residue field has characteristic p and cardinality q_F.
(3) G is the group of F-points of connected reductive F-group $\mathcal{G}$ and $Z(G)$ is the centre of G.
(4) A representation of G on a complex vector space will be identified with a representation of the group algebra $\mathbb{C}G$.
(5) $\mathrm{Irr}(\mathbb{C}G)$ is the set of isomorphism classes of smooth complex irreducible representations of G.
(6) For $\pi \in \mathrm{Irr}(\mathbb{C}G)$ we denote by $cc(\pi) : Z(G) \to \mathbb{C}^\times$ the character such that $\pi(z) = cc(\pi)(z)Id, z \in Z(G)$.
(7) $\mathrm{Irr}'(\mathbb{C}G) = \{\pi \in \mathrm{Irr}(\mathbb{C}G) \mid \mathrm{Im}(cc(\pi)) \subset \overline{\mathbb{Q}}^\times\}$.
(8) By default, our representations will have complex coefficients. So if it says G or $\mathbb{C}G$, then the coefficients are in $\mathbb{C}$, while $\overline{\mathbb{Q}}G$ means that the coefficients of the representations must be in $\overline{\mathbb{Q}}$.
(9) $\mathcal{R}_G$ is the category of smooth finite length complex representations of the group G. The variation $\mathcal{R}'_G$ is given by restricting to representations all whose irreducible subquotients lie in $\mathrm{Irr}'(\mathbb{C}G)$. When G is semisimple, $\mathcal{R}'_G$ coincides with $\mathcal{R}_G$.
(10) $\mathrm{Irr}(\overline{\mathbb{Q}}G)$ is the set of equivalence classes of smooth irreducible $\overline{\mathbb{Q}}$-representations of G.
(11) $\mathcal{R}_{\overline{\mathbb{Q}}G}$ is the category of smooth finite length $\overline{\mathbb{Q}}$-representations of G.
(12) $K(\mathcal{R}_G), K(\mathcal{R}'_G)$ and $K(\mathcal{R}_{\overline{\mathbb{Q}}G})$ are the K-groups of the categories $\mathcal{R}_G, \mathcal{R}'_G$ and $\mathcal{R}_{\overline{\mathbb{Q}}G}$.
(13) $\mathcal{K}_G := K(\mathcal{R}_G) \otimes_\mathbb{Z} \overline{\mathbb{Q}}$, $\mathcal{K}'_G := K(\mathcal{R}'_G) \otimes_\mathbb{Z} \overline{\mathbb{Q}}$, $\mathcal{K}_{\overline{\mathbb{Q}}G} := K(\mathcal{R}_{\overline{\mathbb{Q}}G}) \otimes_\mathbb{Z} \overline{\mathbb{Q}}$.
(14) $\mathcal{K}_{G,\mathrm{temp}}$ is the subspace of $\mathcal{K}_G$ spanned by the tempered G-representations. The definition of tempered [Wal, §III.2] applies just as

well to $\overline{\mathbb{Q}}G$-representations, so we can define $\mathcal{K}_{\overline{\mathbb{Q}}G,\text{temp}} \subset \mathcal{K}_{\overline{\mathbb{Q}}G}$ and $\mathcal{K}'_{G,\text{temp}} \subset \mathcal{K}'_G$ analogously.

(15) J. Arthur (see [Art, Herb]) defined a $\overline{\mathbb{Q}}$-linear subspace $\mathcal{K}_{G,\text{temp}}(ar) \subset \mathcal{K}_{G,\text{temp}}$ of elliptic tempered characters.

(16) Using only representations from $\mathcal{R}'_G$ in (15) gives $\mathcal{K}'_{G,\text{temp}}(ar)$.

(17) J. Arthur also defined a notion of elliptic tempered virtual G-representations. These span $\mathcal{K}_{G,\text{temp}}(ar)$.

1.2. **Results.**

The first general goal of this paper is a comparison of $\mathbb{C}G$-representations with $\overline{\mathbb{Q}}G$-representations. We start with results valid for all irreducible representations, which describe the relation between $\mathrm{Irr}(\overline{\mathbb{Q}}G)$ and $\mathrm{Irr}(\mathbb{C}G)$.

Theorem 1.1. (see Proposition 3.5 and Theorem 3.10)
For any irreducible $\overline{\mathbb{Q}}$-representation V of the group G the complex representation $V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is irreducible.

The functor $\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ provides a bijection from $\mathrm{Irr}(\overline{\mathbb{Q}}G)$ to the subset of $\mathrm{Irr}(\mathbb{C}G)$ which is fixed by $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$.

It follows from Theorem 1.1 that $\mathcal{K}_{\overline{\mathbb{Q}}G}$ embeds naturally in $\mathcal{K}_G$.

The group $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$ acts on (π, V) by its action on the matrix coefficients of π with respect to an arbitrary basis of V. It is clear that every irreducible $\overline{\mathbb{Q}}G$-representation is fixed by $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$, and Theorem 1.1 shows that the converse is also true. We point out that both parts of Theorem 1.1 are specific for algebraically closed fields, they would fail for other coefficient fields like $\mathbb{Q}$.

Of course not every irreducible $\mathbb{C}G$-representation comes from a $\overline{\mathbb{Q}}G$-representation. An obvious necessary condition is that its central character takes values in $\overline{\mathbb{Q}}^\times$. That is accounted for by considering $\mathcal{K}'_G$ instead of $\mathcal{K}_G$. But this condition is not sufficient, since there are representations parabolically induced from representations σ of proper Levi subgroups L such that the $Z(L)$-character of σ does not take values in $\overline{\mathbb{Q}}^\times$. Looking for a description of a subspace of $\mathcal{K}_G$ coming from $\overline{\mathbb{Q}}$-representations we restrict our attention to the part of the representation theory of G that is "orthogonal" to the parabolically induced representations.

Arthur [Art] has shown that the subgroup of $\mathcal{K}_{G,\text{temp}}$ spanned by tempered representations induced from proper Levi subgroups of G admits a complement $\mathcal{K}_{G,\text{temp}}(ar)$ of tempered elliptic representations. These constitute the discrete, non-induced part of $\mathcal{K}_{G,\text{temp}}$.

We define $\mathcal{K}_{\overline{\mathbb{Q}}G,\text{temp}}(ar) \subset \mathcal{K}_{\overline{\mathbb{Q}}G}$ as the $\overline{\mathbb{Q}}$-linear subspace spanned by tempered elliptic representations obtained from $\mathbb{C}G$-representations defined over $\overline{\mathbb{Q}}$. In Definitions (14)–(17) above the temperedness condition can be omitted, which leads to natural "nontempered" analogues of these spaces. In particular there are spaces of elliptic (virtual) representations $\mathcal{K}_G(ar) \subset \mathcal{K}_G$

and $\mathcal{K}_{\overline{\mathbb{Q}}G}(ar) \subset \mathcal{K}_{\overline{\mathbb{Q}}G}$. Again these subspaces form complements to the subspaces spanned by representations induced from proper Levi subgroups. See Paragraph 5.2 for details. For all these spaces we can impose the additional condition that the central characters of the underlying representations must take values in $\overline{\mathbb{Q}}^\times$, which will be indicated by $'$.

Theorem 1.2. (see Corollary 5.10)

(1) *Every complex elliptic representation in $\mathcal{R}'_G$ can be realized over $\overline{\mathbb{Q}}$. In particular every essentially square-integrable $\mathbb{C}G$-representation whose central character takes values in $\overline{\mathbb{Q}}^\times$ can be realized over $\overline{\mathbb{Q}}$.*
(2) *$\mathcal{K}'_{G,\mathrm{temp}}(ar)$ equals $\mathcal{K}_{\overline{\mathbb{Q}}G,\mathrm{temp}}(ar)$ and $\mathcal{K}'_G(ar)$ equals $\mathcal{K}_{\overline{\mathbb{Q}}G}(ar)$.*

For semisimple p-adic groups Theorem 1.2 says, in various ways, that the elliptic part of $\mathcal{K}_G$ or $\mathcal{K}_{G,\mathrm{temp}}$ is the same for representations over $\mathbb{C}$ and over $\overline{\mathbb{Q}}$. For reductive p-adic groups, this holds true because we restricted to $\mathcal{R}'_G$.

The second general goal of the paper is to show that well-known classes of $\mathbb{C}G$-representations are stable under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. It is clear from the definitions that the classes of irreducible representations and of cuspidal representations are stable under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Clozel showed that the same holds for square-integrable representations, when $\mathrm{char}(F) = 0$. We generalize his result to elliptic (virtual) $\mathbb{C}G$-representations, without restrictions on the characteristic of F.

Theorem 1.3. (see Theorem 4.6 and Corollary 5.13)

(1) *The set of essentially square-integrable $\mathbb{C}G$-representations is stable under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.*
(2) *The set of elliptic $\mathbb{C}G$-representations is stable under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.*
(3) *The spaces $\mathcal{K}_G(ar)$ and $\mathcal{K}'_G(ar)$ are stable under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.*

We made the classes of representations in Theorem 1.3 into sets by considering the objects up to the isomorphism.

1.3. **Brief outline of the paper.**

Section 2 is preparatory, and it recalls known material. In Section 3 the basic properties of $\overline{\mathbb{Q}}G$-representations are studied and Theorem 1.1 is proven.

Section 4 has an algebraic part, in which Theorem 1.2.(1) for essentially square-integrable representations is shown. In the analytic part of Section 4, we investigate formal degrees of square-integrable representations to prove that $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ permutes such representations.

Section 5 generalizes our results about essentially square-integrable representations and finishes the proofs of Theorems 1.2 and 1.3.

2. Cuspidal representations

We denote by $\mathcal{H}_G$ the Hecke algebra of compactly supported locally constant complex valued measures on G where the product is the convolution. A smooth complex representation (π, V) of G defines a representation of the algebra $\mathcal{H}_G$ on V, which we also denote by π.

The category $\mathcal{R}_G$ is naturally equivalent to the full subcategory of the category of finite length nondegenerate $\mathcal{H}_G$-modules. We go freely from the language of complex G-representations to the language of $\mathcal{H}_G$-modules.

Definition 2.1. Let $K \subset G$ be an open compact subgroup.

(1) $\mathcal{H}_{G,K} \subset \mathcal{H}_G$ is the subalgebra of K-bi-invariant measures.
(2) For a representation (π, V) of G we denote by $V^K \subset V$ the subspace of K-invariant vectors, which is a $\mathcal{H}_{G,K}$-module.

Lemma 2.2. [Ren, Théorème III.1.5.(i)]
A representation (π, V) of $\mathbb{C}G$ is irreducible if and only if $\mathcal{H}_{G,K}$-modules V^K are irreducible or zero for all open compact subgroups $K \subset G$.

We survey some abstract properties of cuspidal $\mathbb{C}G$-representations. Recall that a representation (π, V) of $\mathbb{C}G$ is cuspidal if, for every proper parabolic subgroup $P = LU_P$ of G, the space of U_P-coinvariants in V is zero. Equivalently, V is spanned by $\{v - \pi(u)v : v \in V, u \in U_P\}$.

The group of field automorphisms $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ act naturally on complex representations of G. Namely, for every complex representation (π, V) of G and $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we denote by (π^γ, V^γ) the complex representation of G, where the vector space V^γ is defined as the base change $\mathbb{C} \otimes_{\gamma,\mathbb{C}} V$ with respect to the isomorphism $\gamma : \mathbb{C} \to \mathbb{C}$ and G acts on V^γ by formula $g(a \otimes v) = a \otimes gv$. This means that, if $\pi(g)_{ij}$ is the matrix of $\pi(g)$ with respect to some (possibly infinite) basis of V, then the matrix of $\pi^\gamma(g)$ with respect to the corresponding basis of V^γ is $\gamma(\pi(g)_{ij})$.

Theorem 2.3. (1) *If $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and π is a cuspidal representation of G, then the representation π^γ is cuspidal.*
(2) *An irreducible cuspidal representation of G is unitary if and only if its central character is unitary.*
(3) *For every representation V of G there exists a unique decomposition $V = V^{cusp} \oplus V^{nc}$ such that V^{cusp} is a cuspidal representation and every irreducible subquotient of V^{nc} is not cuspidal.*
(4) *There exists a unique decomposition $\mathcal{H}_G = \mathcal{H}_G^{cusp} \oplus \mathcal{H}_G^{nc}$ of the algebra $\mathcal{H}_G$ as a sum of two-sided ideals such that $\mathcal{H}_G^{cusp} V = V^{cusp}$ and $\mathcal{H}_G^{nc} V = V^{nc}$.*
(5) *A representation V of G is cuspidal if and only if $\mathcal{H}_G^{nc} V = 0$.*

Proof. Part (1) is clear by definition; for part (2) see [Ren, §IV.3.2], while for part (3) see [Ren, §VI.3.5]. For part (5) note that a representation V of G is cuspidal if and only if $V^{nc} = 0$, so the assertion follows from part

(4). Although part (4) is a formal consequence of part (3), we include the argument for completeness. Let

$$\mathcal{H}_G = \mathcal{H}_G^{cusp} \oplus \mathcal{H}_G^{nc} \tag{2.1}$$

be the decomposition of part (3) for the G-representation $\mathcal{H}_G$ acting by left multiplication. Since the decomposition of part (3) is automatically functorial in V, we deduce that both left ideals $\mathcal{H}_G^{cusp}, \mathcal{H}_G^{nc} \subset \mathcal{H}_G$ are stable under right multiplication by any $h \in \mathcal{H}_G$, hence are two-sided ideals. We claim that for every representation V of G we have inclusions

$$\mathcal{H}_G^{cusp} V \subseteq V^{cusp} \quad \text{and} \quad \mathcal{H}_G^{nc} V \subseteq V^{nc}. \tag{2.2}$$

We will show the first inclusion, while the second one is similar. It suffices to show that for every $v \in V$ we have an inclusion $\mathcal{H}_G^{cusp} v \subseteq V^{cusp}$. Since the map $a_v : \mathcal{H}_G \to V : h \mapsto h(v)$ is G-equivariant, the assertion follows from the fact that every homomorphism $V_1 \to V_2$ of G-representations induces a map $V_1^{cusp} \to V_2^{cusp}$.

Next, using identities $\mathcal{H}_G^{cusp} V + \mathcal{H}_G^{nc} V = \mathcal{H}_G V = V$ and $V^{cusp} \oplus V^{nc} = V$, the inclusions in (2.2) have to be equalities. Finally, the uniqueness of the decomposition in part (4) follows from the equalities $\mathcal{H}_G^{cusp} V = V^{cusp}$ and $\mathcal{H}_G^{nc} V = V^{nc}$ for $V = \mathcal{H}_G$. $\square$

Definition 2.4.
1. Let $G^1 \subset G$ be the open normal subgroup generated by all compact subgroups of G.
2. Let $X_{\mathrm{nr}}(G)$ be the group of unramified characters of G.
3. If $L \subset G$ is a Levi subgroup and σ a cuspidal representation of L we denote by $\mathrm{Irr}(\mathbb{C}G)_{L,\sigma} \subset \mathrm{Irr}(\mathbb{C}G)$ the subset of irreducible representations of G which are subquotients of representations $I_P^G(\sigma \otimes \chi)$ for a $\chi \in X_{\mathrm{nr}}(L)$, where I_P^G denotes normalized parabolic induction.
4. Two cuspidal representations σ of L and σ' of L' are inertially equivalent if $\mathrm{Irr}(\mathbb{C}G)_{L,\sigma} = \mathrm{Irr}(\mathbb{C}G)_{L',\sigma'}$.
5. $\mathfrak{B}_G$ is the set of inertial equivalence classes of pairs (L,σ).
6. For a compact open subgroup $K \subset G$, we set $\mathcal{H}_{G,K}^{cusp} := \mathcal{H}_G^{cusp} \cap \mathcal{H}_{G,K}$.

The following result is due to Bernstein, see [Ren, §VI.7].

Theorem 2.5. $\mathrm{Irr}(\mathbb{C}G)$ *is the disjoint union of subsets* $\mathrm{Irr}(\mathbb{C}G)_{\mathfrak{s}}, \mathfrak{s} \in \mathfrak{B}_G$.

We need a few more relative notions. Let $\theta : Z(G) \to \mathbb{C}^\times$ be a smooth character. Define $\mathcal{H}_G(\theta)$ as the quotient of $\mathcal{H}_G$ by the ideal generated by the elements

$$h_{\theta,z} : g \mapsto h(gz) - \theta(z)h(g) \qquad h \in \mathcal{H}_G, z \in Z(G).$$

The category of nondegenerate $\mathcal{H}_G(\theta)$-modules is naturally equivalent to the category of smooth $\mathbb{C}G$-representations with central character θ.

Moreover, since the subalgebras $\mathcal{H}_G^{cusp}$ and $\mathcal{H}_{G,K}$ are $Z(G)$-invariant, we can form quotient algebras $\mathcal{H}_G^{cusp}(\theta)$, $\mathcal{H}_{G,K}(\theta)$, and $\mathcal{H}_{G,K}^{cusp}(\theta)$.

Note that $\mathcal{H}_{G^1}$ is naturally a subalgebra of $\mathcal{H}_G$, so we can form subalgebras $\mathcal{H}_{G^1}^{cusp} := \mathcal{H}_G^{cusp} \cap \mathcal{H}_{G^1}$ and $\mathcal{H}_{G^1,K}^{cusp} := \mathcal{H}_G^{cusp} \cap \mathcal{H}_{G^1,K}$.

Definition 2.6. We say that an open compact subgroup $K \subset G$ admits an Iwahori decomposition if it satisfies the properties of [Ren, Théoréme V.5.2]. Such subgroups form a basis of open neighborhoods of the identity in G.

The following result is also due to J. Bernstein, we provide a proof because we could not find a reference formulated precisely in the form we need.

Theorem 2.7. *Let $K \subset G$ be a compact open subgroup admitting an Iwahori decomposition.*

(1) *If the centre $Z(G)$ is compact, then the algebra $\mathcal{H}^{cusp}_{G,K}$ is semisimple and thus is isomorphic to a finite product of matrix algebras.*
(2) *The algebra $\mathcal{H}^{cusp}_{G^1,K}$ is semisimple.*
(3) *For any character $\theta : Z(G) \to \mathbb{C}^\times$, the algebra $\mathcal{H}^{cusp}_{G,K}(\theta)$ is semisimple.*

Proof. We only prove part (3), while the other parts are similar. First we have the following lemma.

Lemma 2.8. *In the situation of Theorem 2.7.(3), the functor $V \mapsto V^K$ induces an equivalence between the category $\mathrm{Rep}(G,K)^{cusp}_\theta$ of cuspidal representations of G with central character θ which are generated by K-fixed vectors, and the category of $\mathcal{H}^{cusp}_{G,K}(\theta)$-modules. Moreover, the inverse functor $W \mapsto \mathcal{H}_G \otimes_{\mathcal{H}_{G,K}} W$.*

Proof. Notice that the decomposition $\mathcal{H}_G = \mathcal{H}^{cusp}_G \oplus \mathcal{H}^{nc}_G$ as a direct sum of two-sided ideals from Theorem 2.3.(4) induces a decomposition $\mathcal{H}_{G,K} = \mathcal{H}^{cusp}_{G,K} \oplus \mathcal{H}^{nc}_{G,K}$. Thus, the category of $\mathcal{H}^{cusp}_{G,K}$-modules naturally identifies with the category of $\mathcal{H}_{G,K}$-modules W such that $\mathcal{H}^{nc}_{G,K} W = 0$.

By [Ren, Proposition VI.10.6], the functor $V \mapsto V^K$ induces an equivalence between the category $\mathrm{Rep}(G,K)$ of representations of G generated by K-fixed vectors and the category of $\mathcal{H}_{G,K}$-modules, with inverse functor

$$W \mapsto \mathcal{H}_G \otimes_{\mathcal{H}_{G,K}} W.$$

Using Theorem 2.3.(5) we have to show that for $V \in \mathrm{Rep}(G,K)$ we have $\mathcal{H}^{nc}_G V = 0$ if and only if $\mathcal{H}^{nc}_{G,K} V^K = 0$, which is straightforward. This yields an equivalence of categories between the category $\mathrm{Rep}(G,K)^{cusp}$ and the category of $\mathcal{H}^{cusp}_{G,K}$-modules. Restrict that to modules which admit the $Z(G)$-character θ. $\square$

We return to the proof of the theorem. By Lemma 2.8, it suffices to show that the category $\mathrm{Rep}(G,K)^{cusp}_\theta$ is semisimple, that is, every object of V of $\mathrm{Rep}(G,K)^{cusp}_\theta$ is semisimple. Since every such object has an irreducible subquotient, it suffices to show that every irreducible object of $\mathrm{Rep}(G,K)^{cusp}_\theta$ is projective. But this follows from [Ren, Proposition VI.3.6]. $\square$

We also take a look at cuspidal $\overline{\mathbb{Q}}G$-representations.

Definition 2.9. We set $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G} := \mathcal{H}^{cusp}_G \cap \mathcal{H}_{\overline{\mathbb{Q}}G}$, $\mathcal{H}^{nc}_{\overline{\mathbb{Q}}G} := \mathcal{H}^{nc}_G \cap \mathcal{H}_{\overline{\mathbb{Q}}G}$, and $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K} := \mathcal{H}^{cusp}_{G,K} \cap \mathcal{H}_{\overline{\mathbb{Q}}G}$. For every character $\theta : Z(G) \to \overline{\mathbb{Q}}^\times$, we set $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}(\theta) := \mathcal{H}^{cusp}_{K,G}(\theta) \cap \mathcal{H}_{\overline{\mathbb{Q}}G}$.

Lemma 2.10. (1) *The $\overline{\mathbb{Q}}$-algebras $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G}$, $\mathcal{H}^{nc}_{\overline{\mathbb{Q}}G}$, $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}$ and $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}(\theta)$ are the $\overline{\mathbb{Q}}$-forms of the $\mathbb{C}$-algebras $\mathcal{H}_G$, $\mathcal{H}^{nc}_{\overline{\mathbb{Q}}G}$, $\mathcal{H}^{cusp}_{G,K}$ and $\mathcal{H}^{cusp}_{G,K}(\theta)$, respectively.*
(2) *$\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}(\theta)$ is a semisimple $\overline{\mathbb{Q}}$-algebra.*

Proof. (1) Notice that the $\mathbb{C}$-vector space $\mathcal{H}_G$ is defined over $\overline{\mathbb{Q}}$, with $\overline{\mathbb{Q}}$-form $\mathcal{H}_{\overline{\mathbb{Q}}G}$. Therefore it suffices to show that $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G}$ spans $\mathcal{H}^{cusp}_G$ over $\mathbb{C}$, and similarly for the other algebras.

By Theorem 2.3.(1), the decomposition $\mathcal{H}_G = \mathcal{H}^{cusp}_G \oplus \mathcal{H}^{nc}_G$ is stable under the action of $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$, thus the projection $\mathcal{H}_G \to \mathcal{H}^{cusp}_G$ is defined over $\overline{\mathbb{Q}}$, thus it induces a surjection $\mathcal{H}_{\overline{\mathbb{Q}}G} \to \mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G}$, implying the assertion for $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G}$. The proof of the assertion for $\mathcal{H}^{nc}_{\overline{\mathbb{Q}}G}$ is similar. Next, the assertion $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}$ follows from the fact that the averaging maps $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G} \to \mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}$ and $\mathcal{H}^{cusp}_G \to \mathcal{H}^{cusp}_{G,K}$ are surjective. Finally, the assertion for $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}(\theta)$ follows from that for $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}$.
(2) This is a consequence of part (1) and Theorem 2.7.(3). ☐

3. Irreducible G-representations over $\overline{\mathbb{Q}}$

Let $\mathrm{Rep}(\mathbb{C}G)$ be the category of smooth G-representations on $\mathbb{C}$-vector spaces, and define $\mathrm{Rep}(\overline{\mathbb{Q}}G)$ analogously. The irreducible objects (up to isomorphism) in here form sets $\mathrm{Irr}(\mathbb{C}G)$ and $\mathrm{Irr}(\overline{\mathbb{Q}}G)$. Since $\overline{\mathbb{Q}}$ is algebraically closed and of characteristic zero, most of the abstract representation theory of G works the same over $\overline{\mathbb{Q}}$ as over $\mathbb{C}$. For instance, Lemma 2.2 applies just as well over $\overline{\mathbb{Q}}$.

3.1. Schur's lemma and admissibility.

Schur's lemma does not apply automatically over $\overline{\mathbb{Q}}$, because the cardinality of $\overline{\mathbb{Q}}$ is not larger than that of G/K for an open compact subgroup K of G. We will show that nevertheless Schur's lemma holds in this setting.

Lemma 3.1. *Every irreducible admissible representation V of $\overline{\mathbb{Q}}G$ or $\overline{\mathbb{Q}}G^1$ has endomorphism algebra $\overline{\mathbb{Q}}$ and therefore admits a central character.*

Proof. The argument applies to any totally disconnected locally compact group G.

Let V be such a representation, and let $K \subset G$ be a compact open subgroup. By assumption $\dim_{\overline{\mathbb{Q}}} V^K$ is finite. If it is nonzero, then by Lemma 2.2 V^K is an irreducible representation of $\mathcal{H}_{\overline{\mathbb{Q}}G,K}$. Since V^K is finite dimensional, it follows from Schur's lemma that $\mathrm{End}_{\mathcal{H}_{K,\overline{\mathbb{Q}}G}}(V^K) = \overline{\mathbb{Q}}$.

For every endomorphism $f \in \mathrm{End}_{\overline{\mathbb{Q}}G}(V)$, its restriction $f|_{V^K}$ is an element $\mathrm{End}_{\mathcal{H}_{\overline{\mathbb{Q}}G,K}}(V^K)$. By what we proved above, f acts on each V^K by a scalar. All these scalars coincide because the spaces V^K overlap. Hence f acts by a scalar on the whole of $V = \bigcup_K V^K$. □

In view of Lemma 3.1, we want to prove that every irreducible $\overline{\mathbb{Q}}G$-representation is admissible. We start by checking that for a simpler group.

Lemma 3.2. *Let $r \in \mathbb{N}$. Every irreducible $\overline{\mathbb{Q}}\mathbb{Z}^r$-representation has dimension one.*

Proof. An irreducible $\overline{\mathbb{Q}}\mathbb{Z}^r$ representation is the same as a simple module of the group algebra $A = \overline{\mathbb{Q}}[\mathbb{Z}^r]$, that is, a quotient $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset A$. Since A is a finitely generated $\overline{\mathbb{Q}}$-algebra, the assertion follows from the Nullstellensatz. □

For reductive p-adic groups, admissibility of irreducible representations is by no means obvious. Like for complex representations, we derive it from properties of compact representations.

Let $Z(G)^1$ be the unique maximal compact subgroup of $Z(G)$. Recall that the group $G^1 Z(G)/G^1 \cong Z(G)/Z(G)^1$ is free abelian and finitely generated. We fix a subset $\{x_1, \ldots, x_d\} \subset Z(G)$ whose images in $Z(G)/Z(G)^1$ is $\mathbb{Z}$-basis. We define

$$\mathbb{Z}^d_G := \text{subgroup of } Z(G) \text{ generated by } \{x_1, \ldots, x_d\}. \tag{3.1}$$

This is a group isomorphic to $\mathbb{Z}^d$ and

$$G^1 Z(G) = G^1 \times \mathbb{Z}^d_G. \tag{3.2}$$

Proposition 3.3. *Every irreducible cuspidal representation of $\overline{\mathbb{Q}}G^1$ or $\overline{\mathbb{Q}}G$ is admissible.*

Proof. First we consider an irreducible cuspidal representation π of $\overline{\mathbb{Q}}G^1$. By [Ren, Théorème VI.2.1] it is compact, and then [Ren, Proposition IV.1.3] says that π is admissible.
Now we consider an irreducible cuspidal representation (π, V) of $\overline{\mathbb{Q}}(G^1 Z(G))$. Since the restriction $V|_{G^1}$ is cuspidal as well, [Ren, Théorème VI.2.1] says that $V|_{G^1}$ is compact and hence [Ren, Corollaire IV.1.6] says that $V|_{G^1}$ is semisimple.

Let V' be an irreducible $\overline{\mathbb{Q}}G^1$-subrepresentation of $V|_{G^1}$. Since V is irreducible, we conclude that $V|_{G^1}$ is a sum $\sum_{z \in Z(G)} z(V')$. Now we can write $V|_{G^1} = W \otimes_{\overline{\mathbb{Q}}} V'$ with $W = \mathrm{Hom}_{\overline{\mathbb{Q}}G^1}(V', V)$.

The irreducibility of V and (3.2) imply that W is an irreducible $\overline{\mathbb{Q}}\mathbb{Z}^d_G$-representation. By Lemma 3.2 we conclude that $\dim_{\overline{\mathbb{Q}}} W = 1$, so V is already irreducible as a representation of $\overline{\mathbb{Q}}G^1$. As observed at the beginning of the proof, V is admissible as G^1-representation. Every compact open subgroup of $G^1 Z(G)$ is contained in G^1, so V is admissible as a $G^1 Z(G)$-representation.

Finally, we consider an irreducible cuspidal representation σ of $\overline{\mathbb{Q}}G$. As $[G : G^1 Z(G)]$ is finite, the restriction of σ to $G^1 Z(G)$ is a finite direct sum of irreducible cuspidal representations π of $\overline{\mathbb{Q}}(G^1 Z(G))$, and each π is admissible by the above. Hence σ is admissible as well. $\square$

The next result will be used many times in this paper, sometimes implicitly.

Theorem 3.4. *Every irreducible representation of $\overline{\mathbb{Q}}G$ or $\overline{\mathbb{Q}}G^1$ is admissible and has a central character.*

Proof. It suffices to prove the admissibility, then Lemma 3.1 guarantees the existence of a central character.

Let $\pi \in \mathrm{Irr}(\overline{\mathbb{Q}}G)$. By [Ren, Corollaire VI.2.1], π is isomorphic to a subrepresentation of some $I_P^G(\sigma)$, where $P = LU$ is a parabolic subgroup of G and σ is an irreducible cuspidal $\overline{\mathbb{Q}}L$-representation. By Proposition 3.3 and [Ren, Lemme III.2.3], $I_P^G(\sigma)$ is admissible. Hence π is also admissible.

Consider $(\rho, W) \in \mathrm{Irr}(\overline{\mathbb{Q}}G^1)$. By [Tad, §2] there exists a $(\pi, V) \in \mathrm{Irr}(\overline{\mathbb{Q}}G)$ whose restriction to G^1 contains ρ. Then $\dim W^K \leq \dim V^K$, so by the above ρ is admissible. $\square$

3.2. Comparison with G-representations over $\mathbb{C}$.

We have the functor

$$\otimes_{\overline{\mathbb{Q}}}\mathbb{C} : \mathrm{Rep}(\overline{\mathbb{Q}}G) \to \mathrm{Rep}(\mathbb{C}G). \tag{3.3}$$

Proposition 3.5. *Let $(\pi, V) \in \mathrm{Irr}(\overline{\mathbb{Q}}G)$.*

(1) *The $\mathbb{C}G$-representation $(\pi, V \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$ is irreducible.*
(2) *Suppose that $(\pi', V') \in \mathrm{Irr}(\overline{\mathbb{Q}}G)$ and that $(\pi, V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) \cong (\pi', V' \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$. Then $\pi \cong \pi'$.*

Proof. (1) For any compact open subgroup $K \subset G$, let $e_K \in \mathcal{H}_G$ be the corresponding idempotent. Then

$$(V \otimes_{\overline{\mathbb{Q}}} \mathbb{C})^K = \pi(e_K)(V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) = (\pi(e_K)V) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = V^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}. \tag{3.4}$$

By Lemma 2.2 (over $\overline{\mathbb{Q}}$) the $\mathcal{H}_{\overline{\mathbb{Q}}G,K}$-module V^K is either zero or irreducible. Assume the latter. By Theorem 3.4 every finite length $\overline{\mathbb{Q}}G$-representation is admissible, so $\dim_{\overline{\mathbb{Q}}} V^K$ is finite. By Burnside's theorem the image of $\mathcal{H}_{\overline{\mathbb{Q}}G,K}$ in $\mathrm{End}_{\overline{\mathbb{Q}}}(V^K)$ by π is a simple $\overline{\mathbb{Q}}$-algebra with V^K as irreducible module. As $\overline{\mathbb{Q}}$ is algebraically closed in fact $\pi(\mathcal{H}_{\overline{\mathbb{Q}}G,K}) = \mathrm{End}_{\overline{\mathbb{Q}}}(V^K)$. It follows that

$$\pi(\mathcal{H}_{G,K}) = \pi(\mathcal{H}_{\overline{\mathbb{Q}}G,K} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) = \pi(\mathcal{H}_{\overline{\mathbb{Q}}G,K}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = \mathrm{End}_{\mathbb{C}}(V^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}).$$

With (3.4) we conclude that $(V \otimes_{\overline{\mathbb{Q}}} \mathbb{C})^K$ is an irreducible $\mathcal{H}_{G,K}$-module, whenever it is nonzero. Lemma 2.2 says that $(\pi, V \otimes_{\overline{\mathbb{Q}}} \mathbb{C})$ is irreducible.
(2) Let K be as above. By (3.4), the $\mathcal{H}_{G,K}$-modules $V^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ and $V'^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$

are isomorphic. These are finite dimensional modules, so their traces are the same. For any $f \in \mathcal{H}_{\overline{\mathbb{Q}}G,K}$ we have

$$\mathrm{tr}(f, V^K) = \mathrm{tr}(f, V^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) = \mathrm{tr}(f, V'^K \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) = \mathrm{tr}(f, V'^K).$$

Therefore V^K and V'^K are isomorphic $\mathcal{H}_{\overline{\mathbb{Q}}G,K}$-modules. Now [Ren, Théorème III.1.5.(ii)] says that π and π' are isomorphic. □

Proposition 3.5.(1) can be generalized to finite length representations, in the following sense:

Lemma 3.6. *Let $(\pi, V) \in \mathrm{Rep}(\mathbb{C}G)$ have finite length, and suppose that π can be realized over $\overline{\mathbb{Q}}$. Then every irreducible constituent of π can be realized as an irreducible $\overline{\mathbb{Q}}G$-representation.*

Proof. By assumption, there exists a representation $(\pi_{\overline{\mathbb{Q}}}, V_{\overline{\mathbb{Q}}}) \in \mathrm{Rep}(\overline{\mathbb{Q}}G)$ with $(\pi_{\mathbb{C}}, V_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) \cong (\pi, V)$. The functor $\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ is exact and maps non-zero representations to non-zero representations, and $\pi_{\mathbb{C}}$ is Noetherian and Artinian. We conclude that $\pi_{\overline{\mathbb{Q}}}$ is Noetherian and Artinian, thus of finite length. Let

$$0 \subset \tau_1 \subset \tau_2 \subset \cdots \subset \tau_M = \pi_{\overline{\mathbb{Q}}} \tag{3.5}$$

be a composition series in $\mathrm{Rep}(\overline{\mathbb{Q}}G)$. By Proposition 3.5.(1) all the $\mathbb{C}G$-representations $(\tau_i/\tau_{i-1}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ are irreducible. Thus applying $\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ to (3.5) yields a composition series of π. Hence every irreducible constituent of π is isomorphic to $(\tau_i/\tau_{i-1}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ for some i. □

We note that $\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ sends cuspidal representations to cuspidal representations. In fact, for cuspidal representations there is only little difference between working over $\overline{\mathbb{Q}}$ and over $\mathbb{C}$:

Proposition 3.7. (1) *Every cuspidal $V_1 \in \mathrm{Irr}(\mathbb{C}G^1)$ can be realized over $\overline{\mathbb{Q}}$.*
(2) *Let $(\pi, V) \in \mathrm{Irr}(\mathbb{C}G)$ be cuspidal. Then π can be realized over $\overline{\mathbb{Q}}$ if and only if the central character $cc(\pi) : Z(G) \to \mathbb{C}^\times$ takes values in $\overline{\mathbb{Q}}^\times$.*

Proof. We only prove part (2), because part (1) is similar and easier.

$\Rightarrow$ Every irreducible $\overline{\mathbb{Q}}G$-representation has a central character with values in $\overline{\mathbb{Q}}^\times$. Hence any $\mathbb{C}G$-representation which can be realized over $\overline{\mathbb{Q}}$ has the same property.

$\Leftarrow$ Let $\theta : Z(G) \to \overline{\mathbb{Q}}^\times$ be the central character of π. Let $K \subset G$ be a compact open subgroup admitting an Iwahori decomposition such that V^K is nonzero. By Lemma 2.8 we have an isomorphism

$$V \cong \mathcal{H}_G \otimes_{\mathcal{H}_{G,K}} V^K.$$

Since $\mathbb{C}$-algebras $\mathcal{H}_G$ and $\mathcal{H}_{G,K}$ are obtained by extension of scalars from $\overline{\mathbb{Q}}$-algebras $\mathcal{H}_{\overline{\mathbb{Q}}G}$ and $\mathcal{H}_{\overline{\mathbb{Q}}G,K}$, respectively, it suffices to show that the $\mathcal{H}_{G,K}$-module V^K is obtained by extension of scalars from a $\mathcal{H}_{\overline{\mathbb{Q}}G,K}$-module.

Using Lemma 2.8 again, we see that V^K corresponds to a simple $\mathcal{H}^{cusp}_{G,K}(\theta)$-module. Since $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}(\theta)$ is a $\overline{\mathbb{Q}}$-form of $\mathcal{H}^{cusp}_{G,K}(\theta)$ (by Lemma 2.10.(1)), it suffices to show that V^K is obtained extension of scalars from a simple $\mathcal{H}^{cusp}_{\overline{\mathbb{Q}}G,K}(\theta)$-module.

By Theorem 2.7.(3), $\mathcal{H}^{cusp}_{G,K}(\theta)$ is a semisimple algebra, therefore every simple $\mathcal{H}^{cusp}_{G,K}(\theta)$-module satisfies this property. $\square$

The following simple observation will be important later.

Lemma 3.8. *For every $\pi \in \mathrm{Irr}(\mathbb{C}G)$ there exists $\chi \in X_{\mathrm{nr}}(G)$ such that the image of central character $cc(\chi \otimes \pi)$ has finite image, thus takes values in $\{z \in \overline{\mathbb{Q}}^\times : |z| = 1\}$.*

Proof. Since $Z(G)^1$ is compact and $\mathrm{cc}(\pi)$ is smooth, the image $\mathrm{cc}(\pi)(Z(G)^1)$ is a finite subgroup of $\mathbb{C}^\times$. Let $\mathbb{Z}^d_G$ be as in (3.1) and define a character $\chi' : Z(G)/Z(G)^1 \to \mathbb{C}^\times$ by $\chi'(x) = \mathrm{cc}(\pi)(x^{-1})$ for $x \in \mathbb{Z}^d_G$. Then the map $\mathrm{cc}(\pi) \otimes \chi' : Z(G) \to \mathbb{C}^\times$ sends $\mathbb{Z}^d_G$ to 1, so has finite image.

The group $Z(G)/Z(G)^1$ embeds in G/G^1 with finite index. As $\mathbb{C}^\times$ is divisible, we can extend χ' to a $\mathbb{C}^\times$-valued character χ of G/G^1, or equivalently an unramified character of G. Then $\mathrm{cc}(\pi \otimes \chi) = \mathrm{cc}(\pi) \otimes \chi'$ has a finite image. The last assertion follows from the observation that every finite subgroup of $\mathbb{C}^\times$ is contained in $\{z \in \overline{\mathbb{Q}}^\times : |z| = 1\}$. $\square$

For an arbitrary irreducible $\mathbb{C}G$-representation π the cuspidal support $\mathrm{Sc}(\pi)$ consists of a Levi subgroup $L \subset G$ and a cuspidal $\mathbb{C}L$-representation σ, such that π is a constituent of the normalized parabolic induction of σ (with respect to any parabolic subgroup with Levi factor L). This determines (L, σ) uniquely up to G-conjugacy. We will also express that by saying that (L, σ) represents $\mathrm{Sc}(\pi)$.

The question whether or not π can be defined over $\overline{\mathbb{Q}}$ can be reduced to the cuspidal case:

Proposition 3.9. *Let $(\pi, V) \in \mathrm{Irr}(\mathbb{C}G)$. Then π can be realized over $\overline{\mathbb{Q}}$ if and only if its cuspidal support can be realized over $\overline{\mathbb{Q}}$.*

Proof. $\Rightarrow$ For a parabolic subgroup $P = LU$ of G, the (normalized) Jacquet functor $J^G_P : \mathrm{Rep}(\mathbb{C}G) \to \mathrm{Rep}(\mathbb{C}L)$ is defined over $\overline{\mathbb{Q}}$. Hence $J^G_P(\pi)$ can be realized over $\overline{\mathbb{Q}}$. It has finite length [Ren, §VI.6.4], so by Lemma 3.6 all its irreducible constituents can be realized over $\overline{\mathbb{Q}}$. By construction [Ren, §VI.7.1], $\mathrm{Sc}(\pi)$ can be represented by any of the constituents of $J^G_P(\pi)$, for a suitable P. Hence $\mathrm{Sc}(\pi)$ can be realized over $\overline{\mathbb{Q}}$.
$\Leftarrow$ The (normalized) parabolic induction functor $I^G_P : \mathrm{Rep}(\mathbb{C}L) \to \mathrm{Rep}(\mathbb{C}G)$ is defined over $\overline{\mathbb{Q}}$. By definition, π is an irreducible subquotient of $I^G_P(\sigma)$,

where (L, σ) represents $\mathrm{Sc}(\pi)$. By assumption σ can be realized over $\overline{\mathbb{Q}}$, and hence $I_P^G(\sigma)$ can be realized over $\overline{\mathbb{Q}}$. It has a finite length [Ren, Lemme VI.6.2], so by Lemma 3.6 all its irreducible constituents can be realized over $\overline{\mathbb{Q}}$. By Lemma 3.6, π can be realized over $\overline{\mathbb{Q}}$. □

Theorem 3.10. *The functor* $\otimes_{\overline{\mathbb{Q}}}\mathbb{C}$ *provides a bijection*

$$\mathrm{Irr}(\overline{\mathbb{Q}}G) \longrightarrow \mathrm{Irr}(\mathbb{C}G)^{\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})}.$$

Proof. For $(\pi, V) \in \mathrm{Irr}(\overline{\mathbb{Q}}G)$ and $\gamma \in \mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$ we have

$$(V \otimes_{\overline{\mathbb{Q}}} \mathbb{C})^\gamma = V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}^\gamma \cong V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

That and Proposition 3.5.(1) show that

$$(\pi, V \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) \in \mathrm{Irr}(\mathbb{C}G)^{\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})}. \tag{3.6}$$

Proposition 3.5.(2) ensures that the map from the statement is injective.

Conversely, consider any $\pi \in \mathrm{Irr}(\mathbb{C}G)^{\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})}$. We represent $\mathrm{Sc}(\pi)$ by (L, ω) for a parabolic subgroup $P = LU$. Since I_P^G is $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$-equivariant, $\mathrm{Sc}(\pi)$ can be represented by (L, ω^γ) for any $\gamma \in \mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$. On the other hand, $\mathrm{Sc}(\pi)$ is unique up to G-conjugation. If we fix L, that means that $\mathrm{Sc}(\pi)$ can be represented only by $(L, n{\cdot}\omega)$ for some $n \in N_G(L)$. As $N_G(L)/L$ is finite, there are only finitely many such pairs $(L, n \cdot \omega)$. It follows that the orbit $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})\omega \subset \mathrm{Irr}(\mathbb{C}L)$ is finite. In particular the set of central characters $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})\mathrm{cc}(\omega) \subset \mathrm{Irr}(\mathbb{C}Z(L))$ is finite.

For any $z \in \mathbb{C} \setminus \overline{\mathbb{Q}}$, the orbit $\mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})z \subset \mathbb{C}$ is infinite, because z is not algebraic over $\overline{\mathbb{Q}}$. We deduce that $\mathrm{cc}(\omega)$ takes values in $\overline{\mathbb{Q}}^\times$. By Proposition 3.7 ω can be realized over $\overline{\mathbb{Q}}$, and then Proposition 3.9 says that π can be realized over $\overline{\mathbb{Q}}$ as well. □

4. Essentially square-integrable representations

Definition 4.1.
(1) A complex G-representation (π, V) is unitary if there exists a G-invariant positive definite Hermitian form $(\,,)$ on V. If (π, V) is irreducible then such a form is unique up to multiplication by a positive scalar.
(2) For a unitary irreducible representation $(\pi, V, (\,,))$ of G and $v \in V$ we define a matrix coefficient $f_v(g) := (\pi(g)v, v)$. Then f_v and $|f_v|$ are functions on G, and $|f_v|$ can also be regarded as a function on $G/Z(G)$.
(3) An irreducible representation (π, V) of G is square-integrable (modulo centre) if it is unitary and $|f_v| \in L^2(G/Z(G))$ for some non-zero $v \in V$. Then $|f_v| \in L^2(G/Z(G))$ for all $v \in V$.
(4) An irreducible representation (π, V) of G is square-integrable (also called discrete series) if $Z(G)$ is compact and $f_v \in L^2(G)$ for all $v \in V$.

(5) An irreducible G-representation (π, V) is essentially square-integrable if its restriction to G_{der} is a direct sum of square-integrable representations.

4.1. Some general results.

Let $\mathcal{G}_{\text{der}}$ be the derived group of $\mathcal{G}$ and write $G_{\text{der}} = \mathcal{G}_{\text{der}}(F)$. Recall from [Sil4] that the restriction of an irreducible $\mathbb{C}G$-representation to G_{der} is always a finite direct sum of irreducible representations. The following lemma is well-known, we include a proof because we could not find a reference.

Lemma 4.2. *Let π be an irreducible $\mathbb{C}G$-representation. Then π is essentially square-integrable if and only if $\pi \otimes \chi$ is square-integrable (modulo centre) for some $\chi \in X_{\text{nr}}(G)$.*

Proof. $\Leftarrow$ See [Tad, Lemma 2.1 and Proposition 2.7].
$\Rightarrow$ The absolute value of the central character of π extends uniquely to an unramified character of G with values in $\mathbb{R}_{>0}$, say ν_G. Then $\pi \otimes \nu_G^{-1}$ has unitary central character and its restriction to G_{der} is the same as that of π, so a direct sum of square-integrable representations π_1. Every matrix coefficient f of π from a vector in π_1 is square-integrable on G_{der}. Hence $|f|$ is square-integrable on $G_{\text{der}} Z(G)/Z(G)$. As $G/G_{\text{der}} Z(G)$ is compact, it follows that $|f|$ is also square-integrable on $G/Z(G)$. Hence $\pi \otimes \nu_G^{-1}$ is square-integrable modulo centre and π is essentially square-integrable. $\square$

For every parabolic subgroup $P = LU \subset G$, the normalized parabolic induction functor I_P^G gives a homomorphism $K(\mathcal{R}_L) \to K(\mathcal{R}_G)$. (It only depends on L, not on choice of P when L is given.) Let $K_{\text{ind}}(\mathcal{R}_G)$ be the subgroup generated by the sets $I_P^G(K(\mathcal{R}_L))$, where L runs over all proper Levi subgroups of G.

Lemma 4.3. *Let $\pi \in \text{Irr}(\mathbb{C}G)$ be essentially square-integrable. Then the class of π in $K(\mathcal{R}_G)$ does not belong to $K_{\text{ind}}(\mathcal{R}_G)$.*

Proof. By assumption there exists a $\chi \in X_{\text{nr}}(G)$ such that $\sigma := \pi \otimes \chi$ is square-integrable modulo centre, and in particular tempered. The subgroup $K_{\text{ind}}(\mathcal{R}_G)$ is stable under tensoring by unramified characters of G, so it suffices to prove that σ does not lie in $K_{\text{ind}}(\mathcal{R}_G)$.

Recall that an element $g \in G$ is called regular elliptic if $Z_G(g)/Z(G)$ is an anisotropic torus. This is equivalent to requiring that g does not belong to any proper parabolic subgroup $P = LU$ of G. For any finite length $\mathbb{C}L$-representation ρ, the Harish-Chandra character $\chi_{I_P^G(\rho)}$ is supported on the set of elements of G that are conjugate to an element of P. Therefore $\chi_{I_P^G(\rho)}(g) = 0$ for any regular elliptic $g \in G$. This also follows from [KaVa, Proposition 4.1]. Thus the character of any virtual representation in $K_{\text{ind}}(\mathcal{R}_G)$ vanishes at any regular elliptic element g.

According to [Art, Proposition 2.1.c], the square-integrable modulo centre representation σ is elliptic, in the sense that $\chi_\sigma(g) \neq 0$ for some regular elliptic element $g \in G$. Therefore σ does not belong to $K_{\text{ind}}(\mathcal{R}_G)$. $\square$

The set of irreducible square-integrable $\mathbb{C}G$-representations is countable (but empty when $Z(G)$ is not compact). To get from there to finite sets of representations, one may involve compact open subgroups.

Theorem 4.4. *Fix a compact open subgroup $K \subset G$.*

1. *The set of essentially square-integrable $\mathbb{C}G$-representations (π, V) with $V^K \neq 0$ (considered up to isomorphism) forms a union of finitely many $X_{\mathrm{nr}}(G)$-orbits.*
2. *There are only finitely many non-isomorphic irreducible square-integrable $\mathbb{C}G^1$-representations (ρ, W) with $W^K \neq 0$.*

Proof. (1) By [Ren, Théorème VI.10.6] there are only finitely many Bernstein components in $\mathrm{Irr}(\mathbb{C}G)$ containing representations with nonzero K-fixed vectors. Therefore we may restrict to essentially square-integrable G-representations in one Bernstein component $\mathrm{Irr}(\mathbb{C}G)_{\mathfrak{s}}$ (with the notation from Theorem 2.5).

In Lemma 4.3 we saw that no essentially square-integrable $\mathbb{C}G$-representation π lies in $K_{\mathrm{ind}}(\mathcal{R}_G)$. In the terminology of [BDK], π is discrete. According to [BDK, Proposition 3.1], the discrete irreducible $\mathbb{C}G$-representations in one Bernstein component form a finite union of $X_{\mathrm{nr}}(G)$-orbits. That remains true if we impose the additional condition of essential square-integrability.

(2) Let (ρ, W) be such a $\mathbb{C}G^1$-representation. Let $\mathbb{Z}_G^d \subset Z(G)$ be as in (3.1). By (3.2) we can extend (ρ, W) to a $G^1 Z(G)$-representation ρ_Z by defining it to be trivial on $\mathbb{Z}_G^d$. Since ρ is square-integrable, ρ_Z is square-integrable modulo centre.

As $[G : G^1 Z(G)]$ is finite, the $\mathbb{C}G$-representation $\mathrm{ind}_{G^1 Z(G)}^G \rho_Z$ has finite length, and all its irreducible constituents are square-integrable modulo centre. By Frobenius reciprocity ρ is a constituent of $\mathrm{Res}^G_{G^1} \omega$, for an irreducible constituent ω of $\mathrm{ind}_{G^1 Z(G)}^G \rho_Z$ with nonzero K-fixed vectors. From part (1) we know that there are only finitely many possibilities for ω, up to tensoring by unramified characters and up to isomorphism. By [Sil3], $\mathrm{Res}^G_{G^1} \omega$ is a finite direct sum of irreducible representations. As $\mathrm{Res}^G_{G^1}(\omega \otimes \chi) = \mathrm{Res}^G_{G^1} \omega$ for $\chi \in X_{\mathrm{nr}}(G)$, there are only finitely many possible ρ (up to isomorphism). $\square$

The next result generalizes Proposition 3.7.

Theorem 4.5. *Let $\pi \in \mathrm{Irr}(\mathbb{C}G)$ be essentially square-integrable.*

1. *π can be realized over $\overline{\mathbb{Q}}$ if and only if $cc(\pi)$ takes values in $\overline{\mathbb{Q}}^\times$.*
2. *The $X_{\mathrm{nr}}(G)$-orbit of π contains a $\pi' \in \mathrm{Irr}(\mathbb{C}G)$ such that the image of $cc(\pi')$ is finite subgroup of $\{z \in \overline{\mathbb{Q}}^\times : |z| = 1\}$. Such a π' can be realized over $\overline{\mathbb{Q}}$ and is unitary and square-integrable (modulo centre).*

Proof. (1) The condition on the central character is clearly necessary.

Let $(\pi, V) \in \mathrm{Irr}(\mathbb{C}G)$ with $cc(\pi) : Z(G) \to \overline{\mathbb{Q}}^\times$. By Proposition 3.7 and Lemma 3.8, there exists a Levi subgroup $L \subset G$, a cuspidal $\overline{\mathbb{Q}}L$-representation σ and an unramified character $\chi \in X_{\mathrm{nr}}(L)$ such that π is an irreducible subquotient of $I_P^G(\sigma \otimes \chi)$. By Proposition 3.9, it suffices to show that the character χ takes values in $\overline{\mathbb{Q}}^\times$.

For every $\gamma \in \mathrm{Gal}(\mathbb{C}/\overline{\mathbb{Q}})$ the conjugate π^γ is an irreducible subquotient of $I_P^G(\sigma \otimes \chi^\gamma)$. Arguing as in Theorem 4.4.(1) all conjugates π^γ lie in a finite number of $X_{\mathrm{nr}}(G)$-orbits. Indeed, π is discrete in the sense of [BDK], so π^γ is discrete as well and the assertion follows from [BDK, Proposition 3.1].

With the uniqueness of the cuspidal support (up to G-conjugation), it follows that all conjugates χ^γ of χ belong to a finite union of $X_{\mathrm{nr}}(G)$-orbits. Therefore the set of conjugates $\chi^\gamma|_{L \cap G^1}$ is finite, and thus the restriction $\chi|_{L\cap G^1}$ has values in $\overline{\mathbb{Q}}^\times$.

By assumption, the central character $cc(\pi) = cc(\sigma)|_{Z(G)} \cdot \chi|_{Z(G)}$ takes values in $\overline{\mathbb{Q}}^\times$. Since σ is defined over $\overline{\mathbb{Q}}$, the central character $cc(\sigma)|_{Z(G)}$ has values in $\overline{\mathbb{Q}}^\times$.

Thus, the restriction $\chi|_{Z(G)}$ and hence also $\chi|_{L\cap G^1 Z(G)}$ takes values in $\overline{\mathbb{Q}}^\times$. Since $[G : G^1 Z(G)] < \infty$, we conclude that χ has values in $\overline{\mathbb{Q}}^\times$.
(2) By Lemma 3.8, there exists a $\chi \in X_{\mathrm{nr}}(G)$ such that the image of the central character of $\pi' := \pi \otimes \chi$ is a finite subgroup of $\{z \in \overline{\mathbb{Q}}^\times : |z| = 1\}$. By part (1), π' can be realized over $\overline{\mathbb{Q}}$. Further, since $\mathrm{cc}(\pi')$ is unitary, we conclude that π' is unitary and square-integrable (modulo centre). $\square$

4.2. **Clozel's result about the Galois action.**

L. Clozel has shown that the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathrm{Irr}(\mathbb{C}G)$ preserves the discrete series. We thank him for sharing his unpublished proof with us. In this paragraph we work his argument out and generalize it.

Theorem 4.6. *Let (π, V) be an essentially square-integrable $\mathbb{C}G$-representation and let $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then the representation (π^γ, V) is essentially square-integrable.*

The proof of Theorem 4.6 will occupy the remainder of this paragraph. An important tool to study square-integrable representations is the Plancherel measure, which exists for any locally compact unimodular group of type I.

We fix a Haar measure $\mu_{\mathbb{Q}}$ on G such that $\mu_{\mathbb{Q}}(K) \in \mathbb{Q}_{>0}$ for every compact open subgroup K of G. That provides an identification between $\mathcal{H}_G$ and the space of locally constant compact supported $\mathbb{C}$-valued functions on G.

The Plancherel measure μ_{Pl} is usually defined via unitary G-representations which are not smooth. We recall from [Sol1, (95)] that there is a canonical bijection between the set of irreducible unitary G-representations on Hilbert spaces and the set of irreducible smooth unitary G-representations (both considered up to isomorphism). In one direction the functor is completion, in the opposite direction the functor is taking smooth vectors. Therefore we may replace unitary G-representations by smooth unitary G-representations for μ_{Pl}. Further, we may extend the Plancherel measure to $\mathrm{Irr}(\mathbb{C}G)$ by defining it to be supported on the unitary representations in $\mathrm{Irr}(\mathbb{C}G)$. That leads to:

Theorem 4.7. [Dix, §18.8]

(1) *There exists a unique measure μ_{Pl} on $\mathrm{Irr}(\mathbb{C}G)$ which is supported on unitary representations and satisfies*

$$f(1) = \int_{\mathrm{Irr}(\mathbb{C}G)} \mathrm{tr}\, \pi(f\mu_{\mathbb{Q}}) \, \mathrm{d}\mu_{Pl}(\pi) \qquad \text{for all } f\mu_{\mathbb{Q}} \in \mathcal{H}_G.$$

(2) *Assume that $Z(G)$ is compact. An irreducible $\mathbb{C}G$-representation π is square-integrable if and only if $\mu_{Pl}(\{\pi\}) > 0$.*

We focus first on the cases of Theorem 4.6 where G is semisimple and the underlying field F is p-adic (so of characteristic zero). By [BoHa, Remark 1.13] there exists a number field $\mathbb{E}$ such that all infinite places of $\mathbb{E}$ are real and $\mathbb{E}_v \cong F$ for a finite place v. Let $\mathcal{V}_{\mathbb{E}}$ be the set of places of $\mathbb{E}$ and let $\mathcal{V}_{\mathbb{E}}^{\infty}$ be the subset of infinite places. Recall that any semisimple real group has a compact form. By [BoHa, Theorem B], there exists an $\mathbb{E}$-group $\mathcal{G}$ such that $\mathcal{G}(\mathbb{E}_v) \cong \mathcal{G}(F) \cong G$ and such that $\mathcal{G}(\mathbb{E}_w)$ is compact for every $w \in \mathcal{V}_{\mathbb{E}}^{\infty}$.

Theorem 4.8. [BoHa, Theorem A and page 60]
Let $\Gamma \subset \mathcal{G}(\mathbb{E})$ be an S-arithmetic subgroup, where $S = \mathcal{V}_{\mathbb{E}}^{\infty} \cup \{v\}$. Then Γ is a discrete cocompact subgroup of the semisimple group $\mathcal{G}(\mathbb{E}_v)$ over the p-adic field $\mathbb{E}_v \cong F$.

By varying Γ in Theorem 4.8, we can construct a sequence $(\Gamma_n)_{n=0}^{\infty}$ such that:

- each Γ_n is a discrete cocompact subgroup of $\mathcal{G}(\mathbb{E}_v) \cong G$,
- $\Gamma_{n+1} \subset \Gamma_n$ and $\bigcap_n \Gamma_n = \{1\}$,
- each Γ_n is normal in Γ_0.

For example if $\mathcal{G}$ is defined over the ring of integers $\mathfrak{o}_{\mathbb{E}}$, we can take

$$\Gamma_0 = \mathcal{G}(\mathfrak{o}_{\mathbb{E},S}) \ \subset \ \bigcap\nolimits_{w \in S} \mathcal{G}(\mathbb{E}_w) \cap \bigcap\nolimits_{w \in \mathcal{V}_{\mathbb{E}} \setminus S} \mathcal{G}(\mathfrak{o}_{\mathbb{E}_w}),$$

and define Γ_n by putting more congruence conditions on the places in $\mathcal{V}_{\mathbb{E}} \setminus S$. For each $n \in \mathbb{Z}_{\geq 0}$ we have a unitary representation of G on $L^2(\Gamma_n \backslash G)$, by right translations. The following theorem is a reformulation of a result of Sauvageot [Sau].

Theorem 4.9. *Take Γ_n as above.*

(1) *For any irreducible unitary G-representation $\tilde{\pi}$ on a Hilbert space:*

$$\mu_{Pl}(\{\tilde{\pi}\}) = \lim_{n \to \infty} \mathrm{vol}(\Gamma_n \backslash G)^{-1} \dim \mathrm{Hom}_{\mathbb{C}G}(\tilde{\pi}, L^2(\Gamma_n \backslash G)).$$

(2) *For any $\pi \in \mathrm{Irr}(\mathbb{C}G)$:*

$$\mu_{Pl}(\{\pi\}) = \lim_{n \to \infty} \mathrm{vol}(\Gamma_n \backslash G)^{-1} \dim \mathrm{Hom}_{\mathbb{C}G}(\pi, C^{\infty}(\Gamma_n \backslash G)).$$

Proof. (1) See [Sau, Introduction].
(2) It follows from the remarks before Theorem 4.7 that in part (1) we may replace both $\tilde{\pi}$ and $L^2(\Gamma_n \backslash G)$ by their smooth parts. That does not change the dimension of the Hom-space because $\tilde{\pi}$ is irreducible. The smooth

part of $L^2(\Gamma_n \backslash G)$ is $C^\infty(\Gamma_n \backslash G)$ and the smooth part of $\tilde{\pi}$ is an irreducible unitary representation in $\mathrm{Rep}(\mathbb{C}G)$ [Sol1, (95)]. Moreover any irreducible unitary smooth G-representation can be obtained in this way, which proves (2) when π is unitary.

When π is not unitary, $\mu_{Pl}(\{\pi\}) = 0$ by definition. $\mathrm{Hom}_{\mathbb{C}G}(\pi, C^\infty(\Gamma_n \backslash G))$ is also zero because all subrepresentations of $C^\infty(\Gamma_n \backslash G)$ are unitary. $\square$

The next result is the crucial part of Clozel's argument.

Proposition 4.10. *Let π be a discrete series representation of a semisimple group G over a p-adic field, and let $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then the representation π^γ is also a discrete series and has the same formal degree as π.*

Proof. Note that the $\mathbb{C}$-vector space $C^\infty(\Gamma_n \backslash G)$ has a natural $\mathbb{Q}$-structure, and there is an isomorphism of $\mathbb{C}G$-representations

$$\begin{array}{ccc} C^\infty(\Gamma_n \backslash G) & \to & C^\infty(\Gamma_n \backslash G)^\gamma \\ f & \mapsto & \gamma^{-1} \circ f \end{array} . \tag{4.1}$$

Hence the representation of G on $C^\infty(\Gamma_n \backslash G)$ is stable under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. By Theorem 4.9.(2)

$$\begin{aligned} \mu_{Pl}(\{\pi^\gamma\}) &= \lim_{n\to\infty} \mathrm{vol}(\Gamma_n \backslash G)^{-1} \dim \mathrm{Hom}_{\mathbb{C}G}(\pi^\gamma, C^\infty(\Gamma_n \backslash G)) \\ &= \lim_{n\to\infty} \mathrm{vol}(\Gamma_n \backslash G)^{-1} \dim \mathrm{Hom}_{\mathbb{C}G}(\pi, C^\infty(\Gamma_n \backslash G)) = \mu_{Pl}(\{\pi\}). \end{aligned}$$

Now Theorem 4.7 and the square-integrability of π imply that π^γ is square-integrable. $\square$

Unfortunately, Theorem 4.9 cannot be applied to semisimple groups G' over local function fields, because it is not known whether G' has discrete cocompact subgroups. To establish Theorem 4.6 for such groups, we use the method of close local fields from [Del, Gan, Kaz]. Let F' be a local field of positive characteristic and write $G' = \mathcal{G}'(F')$. Let $G'_{x',0}$ be the parahoric subgroup of G' associated to a special vertex x' of the Bruhat–Tits building of G'. For $m \in \mathbb{Z}_{>0}$, let $G'_{x',m}$ be the Moy–Prasad subgroup of level m. We will choose m so large that the representations which we consider have nonzero $G'_{x',m}$-fixed vectors.

Let $\mathrm{Rep}(\mathbb{C}G', G'_{x',m})$ be the category of $\mathbb{C}G'$-modules which are generated by their $G'_{x',m}$-fixed vectors. Recall from [Ren, Proposition VI.9.4] that there is an equivalence of categories

$$\begin{array}{ccc} \mathrm{Rep}(\mathbb{C}G', G'_{x',m}) & \to & \mathrm{Rep}(\mathcal{H}_{G',G'_{x',m}}) \\ V & \mapsto & V^{G'_{x',m}} \end{array} . \tag{4.2}$$

We recall that a non-archimedean local field F is said to be l-close to F' if there is a ring isomorphism $\mathfrak{o}_{F'}/\mathfrak{p}_{F'}^l \cong \mathfrak{o}_F/\mathfrak{p}_F^l$. Given F' and $l \in \mathbb{Z}_{\geq 0}$, there always exists an l-close p-adic field F [Del].

Theorem 4.11. [Gan, Theorem 4.1]
Suppose that F' and F are l-close, where l is large enough compared to m. Then there exist

- *a semisimple F-group $\mathcal{G}$,*
- *a special vertex x of the Bruhat–Tits building of $G = \mathcal{G}(F)$,*
- *a bijection $G'_{x',m} \backslash G' / G'_{x',m} \to G_{x,m} \backslash G / G_{x,m}$,*

which induce an $\mathbb{C}$-algebra isomorphism $\mathcal{H}_{G',G'_{x'm}} \cong \mathcal{H}_{G,G_{x,m}}$.

Theorem 4.11 also holds for reductive groups, but we do not need that here. From (4.2) (for G' and for G) and Theorem 4.11, we obtain an equivalence of categories

$$\mathrm{Rep}(\mathbb{C}G', G'_{x',m}) \cong \mathrm{Rep}(\mathbb{C}G, G_{x,m}). \tag{4.3}$$

Lemma 4.12. *The equivalence of categories (4.3):*

1. *commutes with the action of* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$*,*
2. *preserves square-integrability of irreducible representations.*

Proof. (1) This holds because the functors in (4.2) and the algebra isomorphism in Theorem 4.11 commute with the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.
(2) For general linear groups over division algebras, this is [Bad, Théorème 2.17.b]. That proof can be adapted to the setting of [Gan], but we prefer a different argument. We regard $\mathcal{H}_{G',G'_{x',m}}$ as the algebra of compactly supported functions on $G'_{x',m} \backslash G' / G'_{x',m}$, where $G'_{x',m}$ has volume 1. We make it into a unital Hilbert algebra with *-operation $f^*(g) = \overline{f(g^{-1})}$ and trace $f \mapsto f(1)$. Then it has a C^*-completion and a Plancherel measure [BHK, §3.2]. Now Theorem 4.11 provides an isomorphism of Hilbert algebras $\mathcal{H}_{G',G'_{x'm}} \cong \mathcal{H}_{G,G_{x,m}}$, so it preserves the Plancherel measures. Furthermore, by [BHK, Theorem 3.5] the functor (4.2) preserves Plancherel measures, up to positive real factor that depends on the normalization of the Haar measure on G'. The same holds with G instead of G'. It follows that the equivalence of categories (4.3) preserves Plancherel measures, up to a positive real factor. That and Theorem 4.7 entail that (4.3) preserves square-integrability of irreducible representations. $\square$

We are ready to finish the proof of Theorem 4.6, first for semisimple groups and then in general.

Proof of Theorem 4.6
Let G' be as in Theorem 4.11 and let $\pi' \in \mathrm{Irr}(\mathbb{C}G')$ be square-integrable. We choose $m \in \mathbb{Z}_{>0}$ such that π' has nonzero $G'_{x',m}$-fixed vectors. Let G be the semisimple group from Theorem 4.11, over a p-adic field F, and let $\pi \in \mathrm{Irr}(\mathbb{C}G)$ be the image of π' under (4.3). By Lemma 4.12.(2), π is square-integrable. For $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, Lemma 4.12.(1) says that the image of π'^γ under (4.3) is π^γ. By Proposition 4.10, π^γ is square-integrable. Now Lemma 4.12.(2) says that π'^γ is square-integrable.

This and Proposition 4.10 establish Theorem 4.6 for all semisimple groups over non-archimedean local fields.

Consider any reductive group G over a non-archimedean local field, and an essentially square-integrable $\pi \in \mathrm{Irr}(\mathbb{C}G)$. Thus the restriction of π to the derived group G_{der} is a direct sum of irreducible representations π_1, each of which is square-integrable. By Theorem 4.6 for semisimple groups, π_1^γ is square-integrable. Thus π^γ is an irreducible G-representation whose restriction to G_{der} is a direct sum of square-integrable representations π_1^γ. By definition π^γ is essentially square-integrable. $\Box$

5. Elliptic representations

The goal of this section is to generalize the results from Section 4 to elliptic representations. Before we come to their definition, we recall some necessary background.

5.1. **Intertwining operators and μ-functions.**

First we recall the definition of Harish-Chandra's intertwining operators. Consider two parabolic subgroups $P = LU_P$ and $P' = LU_{P'}$ with a common Levi factor L. Let (σ, V_σ) be an irreducible $\mathbb{C}L$-representation. All the representations $I_P^G(\sigma \otimes \chi)$ with $\chi \in X_{\mathrm{nr}}(L)$ can be realized on the same vector space, namely $\mathrm{ind}_{P\cap K_0}^{K_0} V_\sigma$ for a good maximal compact subgroup K_0 of G. This makes it possible to speak of objects on $I_P^G(\sigma \otimes \chi)$ that vary regularly or rationally as functions of $\chi \in X_{\mathrm{nr}}(L)$. Consider the intertwining operators

$$(5.1)\qquad \begin{array}{llll} J_{P'|P}(\sigma\otimes\chi): & I_P^G(\sigma\otimes\chi) & \to & I_{P'}^G(\sigma\otimes\chi) \\ & f & \mapsto & [g \mapsto \int_{U_P \cap U_{P'} \backslash U_{P'}} f(ug)\,\mathrm{d}u] \end{array}.$$

This is well-defined as a family of G-homomorphisms depending rationally on $\chi \in X_{\mathrm{nr}}(L)$ [Wal, Théorème IV.1.1]. There is an alternative algebraic construction of (5.1) in [Wal, proof of Théorème IV.1.1], which also works for representations over $\overline{\mathbb{Q}}$.

We denote the maximal F-split torus in $Z(L)$ by A_L. Let $\Phi(G, A_L)$ be the set of reduced roots of (G, A_L) and denote the subset of roots appearing in the Lie algebra of P by $\Phi(G, A_L)^+$. For $\alpha \in \Phi(G, A_L)^+$, let U_α (resp. $U_{-\alpha}$) be the root subgroup of G for all positive (resp. negative) multiples of α. Let L_α be the Levi subgroup of G generated by $L \cup U_\alpha \cup U_{-\alpha}$. Then L is a maximal proper Levi subgroup of L_α, while LU_α and $LU_{-\alpha}$ are opposite parabolic subgroups with Levi factor L. Consider the composition

(5.2)
$$J_{LU_\alpha|LU_{-\alpha}}(\sigma\otimes\chi) J_{LU_{-\alpha}|LU_\alpha}(\sigma\otimes\chi) : I_{LU_\alpha}^{L_\alpha}(\sigma\otimes\chi) \to I_{LU_\alpha}^{L_\alpha}(\sigma\otimes\chi) \quad \chi \in X_{\mathrm{nr}}(L).$$

This operator depends rationally on χ, and for generic χ the representation $I_{LU_\alpha}^{L_\alpha}(\sigma \otimes \chi)$ is irreducible [Sau, Théorème 3.2]. Therefore (5.2) is a scalar operator [Wal, §IV.3], say

$$(5.3)\qquad j_{L_\alpha,L}(\sigma\otimes\chi)\mathrm{id} \quad \text{with} \quad j_{L_\alpha,L} : X_{\mathrm{nr}}(L)\sigma \to \mathbb{C}\cup\{\infty\}.$$

By definition [Wal, §V.2] Harish-Chandra's μ-function in this corank one setting is

$$\mu_{L_\alpha,L}(\sigma\otimes\chi) = \gamma(L_\alpha|L)^2 j_{L_\alpha,L}(\sigma\otimes\chi)^{-1}, \tag{5.4}$$

where the constant $\gamma(L_\alpha|L) \in \mathbb{Q}_{>0}$ is defined in [Wal, p. 241].

5.2. R-groups and elliptic representations.

This paragraph is based on constructions and results of J. Arthur presented in [Art]. We note that Arthur considered only tempered representations reductive groups defined over local fields of characteristic zero. But all the harmonic analysis which is necessary for these results has been developed for tempered and nontempered representations of reductive groups over nonarchimedean local fields, so we may include those as well.

Let $L \subset G$ be a Levi subgroup, and let the group $N_G(L)$ act on $\mathrm{Irr}(\mathbb{C}L)$, by $(n\cdot\pi)(l) = \pi(n^{-1}ln)$. This descends to an action of

$$W_L := N_G(L)/L$$

on $\mathrm{Irr}(\mathbb{C}L)$ and on $\mathrm{Irr}(\overline{\mathbb{Q}}L)$, which sends $X_{\mathrm{nr}}(L)$ to $X_{\mathrm{nr}}(L)$. We denote the stabilizer of $\pi\in\mathrm{Irr}(\mathbb{C}L)$ by $W_{L,\pi}\subset W_L$.

Let $\delta\in\mathrm{Irr}(\mathbb{C}L)$ be an essentially square-integrable. Consider the set of reduced roots α of (G,A_L) such that Harish-Chandra's function $\mu_{L_\alpha,L}$ (see [Wal, §V.2] or the previous paragraph) has a zero at δ. These roots form a finite integral root system, say Φ_δ. The group $W_{L,\delta}$ acts on Φ_δ and contains the Weyl group $W(\Phi_\delta)$ as a normal subgroup. Let Φ_δ^+ be the positive system of roots appearing in the Lie algebra of P. The R-group R_δ is defined as the stabilizer of Φ_δ^+ in $W_{L,\delta}$. Since $W(\Phi_\delta)$ acts simply transitively on the collection of positive systems in Φ_δ, we have a decomposition

$$W_{L,\delta} = W(\Phi_\delta)\rtimes R_\delta. \tag{5.5}$$

Let $P = LU_P$ be a parabolic subgroup of G and consider the G-representation $I_P^G(\delta)$ obtained by normalized parabolic induction. Every $w\in W_{L,\delta}$ gives rise to an invertible intertwining operator $J_\delta(w)$ in $\mathrm{End}_G(I_P^G(\delta))$ [ABPS, Lemma 1.3], unique up to scalars. It relates to the intertwining operators from (5.1) by a normalization procedure. By results of Knapp–Stein [Sil1], and by [ABPS, Lemma 1.5] in the non-tempered cases, $J_\delta(w)$ is a scalar multiple of the identity if and only if $w\in W(\Phi_\delta)$. Therefore it suffices to consider the intertwining operators $J_\delta(r)$ with $r\in R_\delta$. These operators span a twisted group algebra $\mathbb{C}[R_\delta,\natural_\delta]$, for some 2-cocycle $R_\delta\times R_\delta\to\mathbb{C}^\times$. In other words, J_δ yields a projective representation of R_δ on $I_P^G(\delta)$. By [ABPS, Theorem 1.6] there is a decomposition of $\mathbb{C}[R_\delta,\natural_\delta]\times\mathbb{C}G$-modules

$$I_P^G(\delta) = \bigoplus\nolimits_{\kappa\in\mathrm{Irr}(\mathbb{C}[R_\delta,\natural_\delta])}\kappa\otimes I_P^G(\delta)_\kappa. \tag{5.6}$$

For $r\in R_\delta$ we have the G-invariant distribution

$$\mathcal{H}_G\to\mathbb{C} : f\mapsto \mathrm{tr}\big(J_\delta(r)I_P^G(\delta)(f)\big).$$

This is the trace-distribution of the virtual G-representation

$$I_P^G(\delta)_r = \sum_{\kappa \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta])} \mathrm{tr}(\kappa(r)) I_P^G(\delta)_\kappa. \tag{5.7}$$

We note that $I_P^G(\delta)_r$ is uniquely determined up to scalars (because $J_\delta(r)$ is) and that it depends only on r up to R_δ-conjugacy.

Let $X_*(Z(L))$ and $X_*(Z(G))$ be the groups of F-rational cocharacters of $Z(\mathcal{L})$ and $Z(\mathcal{G})$. We say that

$$w \in W_L \quad \text{is elliptic if} \quad X_*(Z(L))^w = X_*(Z(G)). \tag{5.8}$$

Equivalently, an element of W_L is elliptic if it fixes only finitely many points of $X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(G)$. We let $W_{L,\mathrm{ell}}$ be the set of elliptic elements of W_L. Notice that it is not a group because $e \in W_L$ is not elliptic (unless $L = G$, but in that case everything in this section is trivial). We write

$$W_{L,\delta,\mathrm{ell}} := W_{L,\delta} \cap W_{L,\mathrm{ell}} \quad \text{and} \quad R_{\delta,\mathrm{ell}} = R_\delta \cap W_{L,\mathrm{ell}}.$$

Definition 5.1. The virtual representations $I_P^G(\delta)_r$ with $r \in R_{\delta,\mathrm{ell}}$ are called elliptic G-representations. If δ is square-integrable modulo centre, then $I_P^G(\delta)_r$ is called an elliptic tempered representation.

Although there is in general no canonical normalization of the $J_\delta(r)$, we can be a bit more precise than above.

Lemma 5.2. (1) *We can normalize the $J_\delta(r)$ such that each $J_\delta(r)$ has finite order and all values of $\natural_\delta$ are roots of unity.*

(2) *For any normalization as in part (1), $J_\delta(r)$ is uniquely determined up to roots of unity, and in particular up to elements of $\overline{\mathbb{Q}}^\times$.*

(3) *If $\delta = \delta_{\overline{\mathbb{Q}}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ for a $\overline{\mathbb{Q}}L$-representation $\delta_{\overline{\mathbb{Q}}}$, then we can choose all the $J_\delta(r)$ in $\mathrm{End}_{\overline{\mathbb{Q}}G}(I_P^G(\delta_{\overline{\mathbb{Q}}}))$.*

Proof. (1) Pick a finite central extension $\tilde{R}_\delta$ of R_δ such that the projective representation $r \mapsto J_\delta(r)$ lifts to a linear representation $\tilde{J}_\delta$ of $\tilde{R}_\delta$ on $I_P^G(\delta)$. Notice that the restriction of $\tilde{J}_\delta$ to $\ker(\tilde{R}_\delta \to R_\delta)$ is a character, say ζ_δ, times the identity.

Let $\tilde{r} \in \tilde{R}_\delta$ be a lift of $r \in R_\delta$ and normalize $J_\delta(r)$ as $\tilde{J}_\delta(\tilde{r})$. Now the multiplication rules in $\mathbb{C}[\tilde{R}_\delta]$ entail that $J_\delta(r)$ has finite order and that each value of $\natural_\delta$ is a value of ζ_δ (a character of a finite group).

(2) This is clear, we only need to know that there exists some normalization of $J_\delta(r)$ of finite order.

(3) The algebraic construction of the intertwining operators (5.1) from [Wal, Théorème IV.1.1] can be applied to $I_P^G(\delta_{\overline{\mathbb{Q}}})$. With that as starting point for the normalizations in [Art, §2] and [ABPS, §1], we obtain $J_\delta(r) \in \mathrm{End}_{\overline{\mathbb{Q}}G}(I_P^G(\delta_{\overline{\mathbb{Q}}}))$ for all $r \in R_\delta$. □

We pick any normalization of the $J_\delta(r)$ as in Lemma 5.2.

Lemma 5.3. *For any $r \in R_\delta$, the virtual representation $I_P^G(\delta)_r$ belongs to $\mathcal{K}_G = K(\mathcal{R}_G) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$. It is uniquely determined up to scaling by elements of $\overline{\mathbb{Q}}^\times$.*

Proof. For any $\kappa \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta])$, $\mathrm{tr}\,\kappa(r)$ is a sum of roots of unity, so belongs to $\overline{\mathbb{Q}}$. Hence $I_P^G(\delta)_r$ is an $\overline{\mathbb{Q}}$-linear combination of $\mathbb{C}G$-representations. The second claim follows from (5.7) and Lemma 5.2.(2). $\square$

In the setting of the proof of Lemma 5.2.(1), $\mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta])$ identifies with the set of irreducible representations of $\tilde{R}_\delta$ whose restriction to $\ker(\tilde{R}_\delta \to R_\delta)$ is a multiple of ζ_δ. This shows that (5.6) and Definition 5.1 agree with the setup in [Art, Herb], where our κ is called $\rho^\vee$ in [Art].

One can characterize characters of elliptic tempered G-representations also as characters of G that decrease rapidly (in a precise sense) when $g \in G$ goes to infinity [Herb, Theorem 3.1].

Definition 5.4.
(1) For every irreducible essentially square-integrable representation δ of a Levi subgroup of G we denote by $\mathcal{K}_{G,\delta}(ar) \subset \mathcal{K}_G$ the $\overline{\mathbb{Q}}$-span of the virtual representations $I_P^G(\delta)_r$ with $r \in R_{\delta,\mathrm{ell}}$.
(2) We denote by $\mathcal{K}_{G,\mathrm{temp}}(ar) \subset \mathcal{K}_{G,\mathrm{temp}}$ the span of all the subspaces $\mathcal{K}_{G,\delta}$ with δ square-integrable modulo centre. We denote by $\mathcal{K}_G(ar) \subset \mathcal{K}_G$ the $\overline{\mathbb{Q}}$-span of the subspaces $\mathcal{K}_{G,\delta}$ with δ essentially square-integrable.
(3) We set $\mathcal{K}'_G(ar) := \mathcal{K}_G(ar) \cap \mathcal{K}'_G$, $\mathcal{K}'_{G,\mathrm{temp}}(ar) := \mathcal{K}_{G,\mathrm{temp}}(ar) \cap \mathcal{K}'_G$, $\mathcal{K}_{\overline{\mathbb{Q}}G}(ar) := \mathcal{K}_G(ar) \cap \mathcal{K}_{\overline{\mathbb{Q}}G}$, $\mathcal{K}'_{G,\mathrm{temp}}(ar) := \mathcal{K}_{G,\mathrm{temp}}(ar) \cap \mathcal{K}'_G$, and $\mathcal{K}_{\overline{\mathbb{Q}}G,\mathrm{temp}}(ar) := \mathcal{K}_{G,\mathrm{temp}}(ar) \cap \mathcal{K}_{\overline{\mathbb{Q}}G}$.

The next result provides a basis of $\mathcal{K}_{G,\delta}(ar)$.

Lemma 5.5. *We say that $r \in R_\delta$ is $\natural_\delta$-good if $J_\delta(r)$ and $J_\delta(r')$ commute whenever r and $r' \in R_\delta$ commute.*

(1) *The virtual representation $I_P^G(\delta)_r$ is nonzero if and only if $r \in R_\delta$ is $\natural_\delta$-good.*
(2) *The virtual representations $I_P^G(\delta)_r$, where r runs through the conjugacy classes of $\natural_\delta$-good elements in R_δ, are linearly independent.*
(3) *When $r \in R_\delta$ is not elliptic, $I_P^G(\delta)_r$ belongs to $\mathcal{K}_{\mathrm{ind}}(\mathcal{R}_G) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}$.*

Proof. (1) The space of trace functions on $\mathbb{C}[R_\delta, \kappa_\delta]$ has a basis formed by the characteristic functions of the conjugacy classes consisting of $\natural_\delta$-good elements [Sol2, Lemma 1.1]. Combine that with (5.6) and (5.7).
(2) This follows from the same arguments as (1).
(3) When δ is square-integrable modulo centre, this is [Art, Proposition 2.1.b]. With the intertwining operators from [ABPS], the same arguments work when δ is essentially square-integrable. $\square$

With Lemma 5.5 we can provide a classification of elliptic representations. Namely, each elliptic G-representation is (up to scaling by $\overline{\mathbb{Q}}^\times$) determined by a triple (L, δ, r) where L is a Levi subgroup of G, $\delta \in \mathrm{Irr}(\mathbb{C}L)$ is essentially square integrable and $r \in R_\delta$ is elliptic and $\natural_\delta$-good. For a given elliptic G-representation, (L, δ) is unique up to G-conjugation and r is unique up to R_δ-conjugation.

The following two results highlight the importance of the space of elliptic (tempered) G-representations. For a Levi subgroup L of G we denote by $\mathcal{K}_G^L(ar) \subset \mathcal{K}_G$ the image of the space $\mathcal{K}_L(ar)$ under the map $I_P^G : \mathcal{K}_L \to \mathcal{K}_G$. This subspace depends only on the G-conjugacy class of L.

Theorem 5.6. [Art, §2]
$\mathcal{K}_{G,\mathrm{temp}}$ equals $\bigoplus_L \mathcal{K}_{G,\mathrm{temp}}^L(ar)$, where L runs through the set of conjugacy classes of Levi subgroups of G (including G itself). In particular the subspace $\mathcal{K}_{G,\mathrm{temp}}(ar)$ of $\mathcal{K}_{G,\mathrm{temp}}$ is a complement to $\overline{\mathbb{Q}}$-span of the set of tempered representations induced from proper Levi subgroups of G.

Corollary 5.7. $\mathcal{K}_G = \bigoplus_L \mathcal{K}_G^L(ar) = \mathcal{K}_G(ar) \oplus (\mathcal{K}_{\mathrm{ind}}(\mathcal{R}_G) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}})$.

Proof. This follows from Theorem 5.6 and the comparison of the geometric structures of $\mathrm{Irr}(\mathbb{C}G)$ and its subspace of tempered representations, as in [ABPS, Proposition 2.1]. $\square$

5.3. Realization of elliptic representations over $\overline{\mathbb{Q}}$.

As before, let $L \subset G$ be a Levi subgroup.

Lemma 5.8. *Let $\delta \in \mathrm{Irr}(\mathbb{C}L)$.*

1. *Assume that $W_{L,\delta,\mathrm{ell}}$ is nonempty. There exists $\chi_G \in X_{\mathrm{nr}}(G)$ such that the image of the central character of $\delta \otimes \chi_G|_L$ is finite.*
2. *The set $\{\chi \in X_{\mathrm{nr}}(L) : W_{L,\delta\otimes\chi,\mathrm{ell}}$ is nonempty$\}$ is a finite union of $X_{\mathrm{nr}}(G)$-orbits.*

Proof. (1) Arguing as in Lemma 3.8, we conclude that there exists $\chi_G \in X_{\mathrm{nr}}(G)$ such that the image of the tensor product $\mathrm{cc}(\delta)|_{Z(G)} \otimes \chi_G|_{Z(G)}$ is finite. We claim that this χ_G satisfies the required property. Indeed, replacing δ by $\delta \otimes (\chi_G|_L)$ we can assume that $w(\delta) \cong \delta$ for some $w \in W_{L,\mathrm{ell}}$ and the image $\mathrm{cc}(\delta)(Z(G))$ is finite. We want to show that the image of $\mu := \mathrm{cc}(\delta) : Z(L) \to \mathbb{C}^\times$ is finite.

Choose an uniformizer $\varpi \in F$. Then the map $\nu \mapsto \nu(\varpi)$ gives an injective map $X_*(Z(L)) \hookrightarrow Z(L)$ such that $X_*(Z(L))Z(L)^1 \subset Z(L)$ is a subgroup of finite index. Since $Z(L)^1$ is compact and μ is smooth, the image $\mu(Z(L)^1)$ is finite. Therefore it suffices to show that the image $\mu(X_*(Z(L)))$ is finite.

Since $w(\delta) \cong \delta$, we conclude that $w(\mu) = \mu$, thus μ is tvivial on the image $\mathrm{Im}(w-1) \subset X_*(Z(L))$. On the other hand, the assumption that w is elliptic implies that $X_*(Z(G)) + \mathrm{Im}(w-1) \subset X_*(Z(L))$ is a subgroup of finite index. Therefore finiteness of $\mu\big(X_*(Z(L))\big)$ follows from the finiteness of $\mu(Z(G))$.

(2) Since W_L is finite, it suffices to show that for every $w \in W_{L,\mathrm{ell}}$ the set $\{\chi \in X_{\mathrm{nr}}(L) : w \in W_{L,\delta\otimes\chi}\}$ is a finite union of $X_{\mathrm{nr}}(G)$-orbits. Replacing δ by $\delta \otimes \chi$, if necessary, we may assume that $w(\delta) \cong \delta$. Let $X_{\mathrm{nr}}(L, \delta)$ be the

collection of all $\chi \in X_{\mathrm{nr}}(L)$ such that $\chi \otimes \delta \cong \delta$, so that there is a bijection

$$X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\delta) \to X_{\mathrm{nr}}(L)\delta : \chi \mapsto \delta \otimes \chi.$$

This map is w-equivariant, so it induces a bijection between the w-fixed points on both sides. Since w is assumed to be elliptic, $(X_{\mathrm{nr}}(L)/X_{\mathrm{nr}}(L,\delta))^w$ is a finite union of $X_{\mathrm{nr}}(G)$-orbits. Hence $(X_{\mathrm{nr}}(L)\delta)^w$ is a finite union of $X_{\mathrm{nr}}(G)$-orbits as well. $\square$

We return to an essentially square-integrable $\delta \in \mathrm{Irr}(\mathbb{C}L)$. The representation $\delta \otimes \chi_G|_L$ from Lemma 5.8 is square-integrable modulo centre because its central character is unitary. The equality

$$I_P^G(\delta)_r = I_P^G(\delta \otimes \chi_G|_L)_r \otimes \chi_G^{-1} \tag{5.9}$$

shows that all elliptic G-representations can be obtained from elliptic tempered G-representations by tensoring with unramified characters.

The condition that $W_{L,\delta}$ contains elliptic elements is necessary for $R_{\delta,\mathrm{ell}}$ to be nonempty and for $\mathcal{K}_{G,\delta}$ to be nonzero, but not sufficient. Fortunately, it already gives us enough structure to derive some nice properties of representations.

Theorem 5.9. *Let $\delta \in \mathrm{Irr}(\mathbb{C}L)$ be an essentially square-integrable representation such that $R_{\delta,\mathrm{ell}}$ is nonempty.*

1. *The G-representations $I_P^G(\delta)_\kappa$ with $\kappa \in \mathrm{Irr}(\mathbb{C}[R_\delta, \natural_\delta])$ can be realized over $\overline{\mathbb{Q}}$ if and only if $cc(I_P^G(\delta)_\kappa) = cc(\delta)|_{Z(G)}$ takes values in $\overline{\mathbb{Q}}^\times$.*
2. *In the case of (1), δ can be realized over $\overline{\mathbb{Q}}$ and $\mathcal{K}_{G,\delta} = \mathcal{K}_{\overline{\mathbb{Q}}G,\delta}$.*

Proof. (1) $\Rightarrow$ Each $I_P^G(\delta)_\kappa$ can be realized over $\overline{\mathbb{Q}}$, so by Proposition 3.9 their cuspidal support $\mathrm{Sc}(I_P^G(\delta)_\kappa) = \mathrm{Sc}(\delta)$ can be realized over $\overline{\mathbb{Q}}$. Another application of Proposition 3.9 shows that δ can be realized over $\overline{\mathbb{Q}}$. Hence $\mathrm{cc}(\delta)$ takes values in $\overline{\mathbb{Q}}^\times$.
$\Leftarrow$ By Theorem 4.5.(2) there exists $\chi \in X_{\mathrm{nr}}(L)$ such that $\delta' := \delta \otimes \chi^{-1}$ can be realized over $\overline{\mathbb{Q}}$. We claim that χ takes values in $\overline{\mathbb{Q}}^\times$.

The proof of the claim is almost identical to that of Theorem 4.5.(1). Arguing as over there, we deduce from Lemma 5.8.(2) that the restriction $\chi|_{L\cap G^1}$ has values in $\overline{\mathbb{Q}}^\times$. Next, since $cc(\delta)|_{Z(G)}$ has values in $\overline{\mathbb{Q}}^\times$ and δ' can be realized over $\overline{\mathbb{Q}}$, we deduce that $\chi|_{Z(G)}$ takes values in $\overline{\mathbb{Q}}^\times$. Hence $\chi|_{L\cap G^1 Z(G)}$ takes values in $\overline{\mathbb{Q}}^\times$, and χ as well because $[G : G^1 Z(G)] < \infty$. That proves our claim.

By our claim $\delta = \delta' \otimes \chi$ can be realized over $\overline{\mathbb{Q}}$, say as $\delta_{\overline{\mathbb{Q}}}$. By Lemma 5.2.(3), we can normalize the $J_\delta(r)$ so that they span a twisted group algebra

$$\overline{\mathbb{Q}}[R_\delta, \natural_\delta] \subset \mathrm{End}_{\overline{\mathbb{Q}}L}(I_P^G(\delta_{\overline{\mathbb{Q}}})).$$

This gives rise to a version of the decomposition (5.6) for $I_P^G(\delta_{\overline{\mathbb{Q}}})$, and in particular $I_P^G(\delta_{\overline{\mathbb{Q}}})_\kappa$ is defined for $\kappa \in \mathrm{Irr}(\overline{\mathbb{Q}}[R_\delta, \natural_\delta])$. Extension of scalars from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ recovers $I_P^G(\delta)_\kappa$.

(2) When the above equivalent conditions are fulfilled, Lemma 5.3 shows that the elliptic representations $I_P^G(\delta)_r$ with $r \in R_{\delta,\mathrm{ell}}$ are defined over $\overline{\mathbb{Q}}$. Both $\mathcal{K}_{G,\delta}$ and $\mathcal{K}_{\overline{\mathbb{Q}}G,\delta}$ are the $\overline{\mathbb{Q}}$-span of these representations. □

Theorem 5.9 shows that an elliptic $\mathbb{C}G$-representation $I_P^G(\delta)_r$ can be realized over $\overline{\mathbb{Q}}$ if and only if its central character $\mathrm{cc}(\delta)|_{Z(G)}$ takes values in $\overline{\mathbb{Q}}^\times$.

We are ready to conclude the proof of Theorem 1.2.

Corollary 5.10. *Consider complex G-representations in the category $\mathcal{R}'_G$.*

1. *Every elliptic representation in $K(\mathcal{R}'_G) \otimes_\mathbb{Z} \overline{\mathbb{Q}}$ can be realized over $\overline{\mathbb{Q}}$.*
2. *$\mathcal{K}'_{G,\mathrm{temp}}(ar)$ equals $\mathcal{K}_{\overline{\mathbb{Q}}G,\mathrm{temp}}(ar)$ and $\mathcal{K}'_G(ar)$ equals $\mathcal{K}_{\overline{\mathbb{Q}}G}(ar)$.*

Proof. (1) Since we work in $\mathcal{R}'_G$, central characters of irreducible representations take values in $\overline{\mathbb{Q}}^\times$. Apply Theorem 5.9 and the remark below it.
(2) This is a direct consequence of part (1). □

5.4. The action of Gal$(\mathbb{C}/\mathbb{Q})$.

We consider the Galois action on elliptic representations, generalizing the Galois action on essentially square-integrable representations studied in Section 4. We have to take into account that I_P^G need not commute with the action of an arbitrary $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. To this end, we consider the alternative square root of δ_P. Recall from [Sil2, §1.2.1] that

$$\delta_P(l) = \big|\det\big(\mathrm{Ad}(l) : \mathrm{Lie}(U_P) \to \mathrm{Lie}(U_P)\big)\big|_F.$$

This shows in particular that δ_P takes values in $q_F^\mathbb{Z} \subset \mathbb{Q}_{>0}$. Whenever $\delta_P(l)$ is a square in $\mathbb{Q}^\times$, it has a canonical root in $\mathbb{Q}$, namely $\sqrt{\delta_P(l)} \in \mathbb{Q}_{>0}$. When $\delta_P(l)$ is not a square in $\mathbb{Q}^\times$, it has the roots $\sqrt{\delta_P(l)} \in \mathbb{R}_{>0}$ and $\sqrt[\prime]{\delta_P(l)} \in \mathbb{R}_{<0}$, which are exchanged by $\mathrm{Gal}(\mathbb{Q}(\sqrt{q_F})/\mathbb{Q})$. This gives rise to two roots of δ_P, denoted $\sqrt{\delta_P}$ and $\sqrt[\prime]{\delta_P}$. Let v_p be the p-adic valuation on $\mathbb{Q}$, then

$$\begin{aligned} \sqrt[\prime]{\delta_P} &= \sqrt{\delta_P}\,\chi_-, \\ \chi_-(l) &= (-1)^{v_p(\delta_P(l))} \qquad l \in L. \end{aligned} \tag{5.10}$$

Notice that χ_- is a quadratic unramified character of L, which could be 1.

Lemma 5.11. (1) *$\chi_-|_L$ depends only on L, not on P.*
(2) *$\chi_-|_L$ is fixed by $N_G(L)$.*

Proof. (1) Let L_α be the Levi subgroup of G generated by $L \cup U_\alpha \cup U_{-\alpha}$, where U_α denotes the root subgroup for all positive multiples of α. Multiplication provides an isomorphism

$$\prod\nolimits_{\alpha \in \Phi(G,A_L)^+} (U_P \cap L_\alpha) \to U_P,$$

for any ordering of the set of roots $\Phi(G, A_L)^+$ [Bor, Proposition 14.4]. This implies that

$$\delta_P(l) = \prod\nolimits_{\alpha \in \Phi(G,A_L)^+} \delta_{P\cap L_\alpha}(l) \qquad l \in L.$$

Writing χ_-^α for χ_- with respect to $L_\alpha \supset P \cap L_\alpha = LU_\alpha \supset L$, we obtain

$$\chi_-|_L = \prod\nolimits_{\alpha \in \Phi(G,A_L)^+} \chi_-^\alpha|_L. \tag{5.11}$$

As $\delta_{LU_{-\alpha}} = \delta_{LU_\alpha}^{-1} = \delta_{P \cap L_\alpha}^{-1}$ and χ_-^α is quadratic, we can also interpret χ_-^α as χ_- for $L_\alpha \supset LU_{-\alpha} \supset L$. This enables us to rewrite (5.11) as

$$\chi_-|_L = \prod\nolimits_{\alpha \in \Phi(G,A_L)/\{\pm 1\}} \chi_-^\alpha|_L.$$

That expression does not depend on the choice of a parabolic subgroup with Levi factor L.
(2) For $n \in N_G(L)$ and $l \in L$ we have $\delta_P(n^{-1}ln) = \delta_{nPn^{-1}}(l)$. Hence $\chi_-(n^{-1}ln) = \chi_-(l)$, where the second χ_- is for $G \supset nPn^{-1} \supset L$. By part (1), χ_- coincides with the original χ_-. $\square$

For $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we write $\epsilon(\gamma) = 0$ if γ fixes $\sqrt{q_F}$, and $\epsilon(\gamma) = 1$ otherwise. This definition is designed so that $\sqrt{\delta_P}^{\,\gamma} = \sqrt{\delta_P} \otimes \chi_-^{\epsilon(\gamma)}$.

For $\pi \in \mathrm{Rep}(\mathbb{C}L)$ and $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we have

$$\begin{aligned} I_P^G(\pi)^\gamma &= \mathrm{ind}_P^G\big(\pi \otimes \sqrt{\delta_P}\big)^\gamma \cong \mathrm{ind}_P^G\big(\pi^\gamma \otimes \sqrt{\delta_P}^{\,\gamma}\big) \\ &= \mathrm{ind}_P^G\big(\pi^\gamma \otimes \sqrt{\delta_P} \otimes \chi_-^{\epsilon(\gamma)}\big) = I_P^G\big(\pi^\gamma \otimes \chi_-^{\epsilon(\gamma)}\big). \end{aligned} \tag{5.12}$$

Proposition 5.12. *Let $\delta \in \mathrm{Irr}(\mathbb{C}L)$ be essentially square-integrable, such that $R_{\delta,\mathrm{ell}}$ is nonempty. Let $r \in R_{\delta,\mathrm{ell}}$ and let $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then*

(1) *$R_{\delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}} = R_\delta \subset W_L$ and $R_{\delta^\gamma \otimes \chi_-^{\epsilon(\gamma)},\mathrm{ell}} = R_{\delta,\mathrm{ell}}$.*

(2) *With the identifications from part (1) and $I_P^G(\delta)^\gamma \cong I_P^G\big(\delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}\big)$ from (5.12), we can normalize the involved intertwining operators such that ${J_\delta}^\gamma = J_{\delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}}$.*

(3) *With the normalizations as in (2), $I_P^G(\delta)_r^\gamma$ is isomorphic to the elliptic representation $I_P^G\big(\delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}\big)_r$.*

Proof. We abbreviate $\delta' = \delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}$.
(1) By Theorem 4.6, $\delta^\gamma \in \mathrm{Irr}(\mathbb{C}L)$ is an essentially square-integrable representation. By Lemma 5.11.(2), $W_{L,\delta'}$ equals W_{L,δ^γ}. The group W_{L,δ^γ} equals $W_{L,\delta}$ because the actions of $N_G(L)$ and $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathrm{Irr}(L)$ commute. It follows easily from (5.12), algebraic description of intertwining operators in [Wal, proof of Théorème IV.1.1], and the definition of $j_{L_\alpha,L}$ that

$$\gamma(j_{L_\alpha,L}(\delta)) = j_{L_\alpha,L}\big(\delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}\big). \tag{5.13}$$

Therefore the set Φ_δ of roots α such that $\mu_{L_\alpha,L}(\delta) = 0$ does not change when we replace δ by δ'. Now the definitions before (5.5) show that the groups $W(\Phi_{\delta'})$ and $R_{\delta'}$ are the same for $\delta' = \delta^\gamma \otimes \chi_-^{\epsilon(\gamma)}$ and for δ. By (5.8), the same holds for the subsets of elliptic elements.

(2) For every $r \in R_\delta$, the intertwining operator $J_\delta(r) \in \mathrm{Aut}_G(I_P^G(\delta))$ (defined up to a scalar) is characterized by the fact that it is a member of an algebraic family

$$J_{\delta\otimes\chi}(r) \in \mathrm{Hom}_G\big(I_P^G(\delta\otimes\chi), I_P^G(r\cdot(\delta\otimes\chi))\big) \qquad \chi \in X_{\mathrm{nr}}(L).$$

Indeed, for generic χ, $I_P^G(\delta\otimes\chi)$ is irreducible and $J_{\delta\otimes\chi}(r)$ is unique up to a scalar. For any $\chi \in X_{\mathrm{nr}}(L)$, the operator $J_{\delta\otimes\chi}(r)^\gamma$ is an element of

$$\mathrm{Hom}_G\big(I_P^G(\delta\otimes\chi)^\gamma, I_P^G(r\cdot(\delta\otimes\chi))^\gamma\big) = \\ \mathrm{Hom}_G\big(I_P^G(\delta^\gamma\otimes\chi^\gamma\otimes\chi_-^{\epsilon(\gamma)}), I_P^G(r\cdot(\delta^\gamma\otimes\chi^\gamma)\otimes\chi_-^{\epsilon(\gamma)})\big).$$

As $r\cdot\chi_- = \chi_-$ (by Lemma 5.11.2), a family of such operators determines $J_{\delta'}(r)$. Therefore the operators $J_\delta(r)^\gamma$ and $J_{\delta'}(r)$ are equal up to a scalar. Hence we can normalize them so that they become equal.

(3) For $\kappa \in \mathrm{Irr}(\mathbb{C}[R_\delta,\natural_\delta])$, part (2) entails that $\kappa^\gamma \in \mathrm{Irr}(\mathbb{C}[R_{\delta'},\natural_{\delta'}])$ and

$$I_P^G(\delta)_\kappa^\gamma = I_P^G(\delta')_{\kappa^\gamma}. \tag{5.14}$$

Upon applying γ and using (5.14), the decomposition (5.6) becomes

$$\begin{aligned} I_P^G(\delta') \cong I_P^G(\delta)^\gamma &\cong \bigoplus\nolimits_{\kappa\in\mathrm{Irr}(\mathbb{C}[R_\delta,\natural_\delta])} \kappa^\gamma\otimes I_P^G(\delta)_\kappa^\gamma \\ &= \bigoplus\nolimits_{\kappa^\gamma\in\mathrm{Irr}(\mathbb{C}[R_{\delta'},\natural_{\delta'}])} \kappa^\gamma\otimes I_P^G(\delta')_{\kappa^\gamma}. \end{aligned} \tag{5.15}$$

It follows that (5.7) transforms into

$$I_P^G(\delta)_r^\gamma = \sum_{\kappa^\gamma\in\mathrm{Irr}(\mathbb{C}[R_{\delta'},\natural_{\delta'}])} \mathrm{tr}(\kappa^\gamma(r))\, I_P^G(\delta')_{\kappa^\gamma} = I_P^G(\delta')_r. \qquad \square$$

We are ready to conclude the proof of Theorem 1.3.

Corollary 5.13. (1) *For every elliptic G-representation π and every $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, the Galois conjugate π^γ is elliptic.*
(2) *The spaces $\mathcal{K}_G(ar)$ and $\mathcal{K}'_G(ar)$ are stable under* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

Proof. (1) This is a less precise formulation of Proposition 5.12.(3).
(2) Since these spaces are spanned by elliptic G-representations, this is a direct consequence of part (1). $\square$

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