Dynamic Hypersequents for Public Announcement Logic

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Abstract

Dynamic Epistemic Logic extends classical epistemic logic by modeling not only static knowledge but also its evolution through information updates. Among its various systems, Public Announcement Logic (PAL) provides one of the simplest and most studied frameworks for representing epistemic change, see [4]. While the semantics of PAL is well understood as transformation of Kripke models, the proof theory so far developed fails to represent this dynamism in purely syntactical terms. The aim of this paper is to repair this lack. In particular, building on the hypersequent calculus for S5 introduced in [8], we extend it with a mechanism that models the transition between epistemic models induced by public announcements. We call these structures *dynamic hypersequents*. Using dynamic hypersequents, we construct a calculus for PAL and we show that it enjoys several desirable properties: admissibility of all structural rules (including contraction), invertibility of logical rules, as well as syntactic cut-elimination.

Key words: Public announcement logic, proof theory, hypersequents, cut-elimination.

1 Introduction

Dynamic Epistemic Logic (DEL) is a branch of logic that emerged in the late 1980s. It extends classical epistemic logic by modeling not only static states of knowledge but also the processes by which knowledge is revised. Today, the term DEL serves as an umbrella for a variety of extensions of epistemic logic equipped with dynamic operators. Among the simplest and historically most influential is the public announcement operator, which characterises one of the earliest and most studied forms of DEL: Public Announcement Logic (PAL).

From both a Hilbert-style and a Kripke-semantic perspective, the different systems of dynamic epistemic logic display strong and distinctive features. Their most striking originality, however, lies in their semantics. Unlike in classical epistemic logic, which is concerned with static sets of possible worlds, dynamic epistemic logic focuses on transformations of these structures. After an announcement, for example, one typically moves from one model to another by altering the set of worlds or the accessibility relations between them.

The proof-theoretic side of DEL, and in particular PAL, presents a rather different picture. Several attempts have been made to develop effective proof systems for PAL. The first such attempt was the tableau method introduced in [2, 3], where results on soundness, completeness, and complexity were established. Later, sequent calculi were proposed. In [7] and [13], two distinct labelled Gentzen-style calculi are developed. In the former, the dynamics of announcements are represented by relational atoms, while in the latter, rules are introduced that simulate reduction axioms – reducing dynamic PAL-formulas into equivalent static ones. The first system admits a cut-elimination theorem, yielding a subformula property (modulo labels), whereas the second achieves only a weaker form of the subformula property.

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More recently, in [6], a non-labelled sequent calculus was proposed, again using reduction-style rules akin to those in [13]. In this calculus, only a pseudo-analytic cut theorem has been established.

The main objective of this paper is to develop a sequent calculus for PAL that captures, by purely structural means, the dynamic character of dynamic epistemic logic, and of PAL in particular. More precisely, our approach builds on hypersequents (e.g. see [1], [5], or [11]), and in particular on the hypersequent calculus for S5 introduced in [8], [9]. In this calculus, each sequent corresponds to a world in a model, and the collection of sequents represents the set of worlds in an S5 model, namely an S5 model. We extend this framework by introducing a mechanism that allows one to move from a hypersequent – representing a given model – to a new hypersequent, which corresponds to the updated model obtained through a public announcement. We refer to these new objects as *dynamic hypersequents*. They provide, to the best of our knowledge, the first attempt to syntactically reflect the dynamic character of dynamic epistemic logic at the proof-theoretical level.

Within this framework, we construct a dynamic hypersequent calculus for PAL and establish several key results: the admissibility of all structural rules (contraction rules included), the invertibility of logical rules, syntactic cut-elimination, and the full subformula property. As far as we know, none of the calculi so far proposed for PAL satisfy all these features at the same time.

The paper is structured in the following way. In *Section* 2 we will introduce the basic notions of PAL, whilst in *Section* 3 we will introduce dynamic hypersequents as well as the calculus for PAL. *Section* 4 will be dedicated to the proof of the admissibility of the structural rules, and invertitbility of logical rules, and in *Section* 5 we will prove that our calculus is sound and complete with respect to PAL. Finally *Section* 6 will serve to prove syntactically the cut-elimination theorem. In *Section* 7 we will draw some conclusions and sketch several paths of future research.

2 Public Announcement Logic

This section provides a brief overview of the language, syntax, and semantics of PAL. Although PAL is usually presented with a set of agents $\mathcal{A}$ and correspondingly as many epistemic modalities K_a as agents $a \in \mathcal{A}$, in this work, for the sake of simplicity, we consider the single-agent version of PAL. The extension of our result to the multi-agent case is left for future research but sounds natural in light of existing results for S5 with multiple agents [10].

Definition 2.1 (Language $\mathcal{L}_{PAL}$)**.** *The language of public announcement logic $\mathcal{L}_{PAL}$ is composed of a countable set of atomic sentences, p, q, etc., the classical connectives $\neg$ and $\wedge$, the modal operator $\Box$, and the public announcement operator $[\cdot]$. Formulas of $\mathcal{L}_{PAL}$ are inductively defined as follows:*

$$A \ ::= \ p \mid \neg A \mid (A \wedge A) \mid \Box A \mid [A]A$$

We follow the standard rules for omission of the parentheses. The classical connectives $\vee, \rightarrow, \leftrightarrow$, as well as the dual modalities $\Diamond$ and $\langle\cdot\rangle$ are defined as usual.

Definition 2.2 (**PAL**)**.** *Based on the language $\mathcal{L}_{PAL}$, we introduce the Hilbert-style axiomatic system **PAL**, which is composed of the axioms and rules of inference in Table 1. As standard, we will write $\vdash_{\boldsymbol{PAL}} A$, for the formula A is derivable in the Hilbert system **PAL**.*

We now present the semantics for PAL by first introducing the notion of standard epistemic model, and consequently the notions of satisfaction of a formula in a model and that of updated model.

Definition 2.3 (Epistemic model)**.** *An* epistemic model *is a tuple $M = (W, \sim, V)$ where W is a set of possible states, $\sim: W \rightarrow \mathcal{P}(W^2)$ is an accessibility relation and $V : P \rightarrow \mathcal{P}(W)$ is a valuation. For a model $M = (W, \sim, V)$ and a world $w \in W$, we call (M, w) a* pointed epistemic model*, or simply a* pointed model.

All instantiations of propositional tautologies	
Axioms of S5:	
$\Box(A \to B) \to (\Box A \to \Box B)$	distributivity of $\Box$ (axiom K)
$\Box A \to A$	truth (axiom T)
$\Box A \to \Box\Box A$	positive introspection (axiom 4)
$\neg\Box A \to \Box\neg\Box A$	negative introspection (axiom 5)
Reduction axioms:	
$[A]p \leftrightarrow (A \to p)$	atomic permanence
$[A]\neg B \leftrightarrow (A \to \neg[A]B)$	announcement and negation
$[A](B \wedge C) \leftrightarrow ([A]B \wedge [A]C)$	announcement and conjunction
$[A]\Box B \leftrightarrow (A \to \Box[A]B)$	announcement and knowledge
$[A][B]C \leftrightarrow [A \wedge [A]B]C$	announcement composition
Inference rules:	
From A and $A \to B$, infer B	modus ponens (MP)
From A, infer $\Box A$	necessitation rule (Nec)

Table 1: The axiomatisation **PAL**

In this definition, $\sim$ is an equivalence relation, *i.e.* it is reflexive, transitive, and symmetric.

Definition 2.4 (Semantics of PAL)**.** *Given an epistemic model $M = (W, \sim, V)$, the semantic relation $\vDash$ and the updated model M_A are simultaneously defined by induction as follows:*

$$
\begin{array}{lll}
M, w \vDash p & \textit{iff} & w \in V(p) \\
M, w \vDash \neg A & \textit{iff} & M, w \nvDash A \\
M, w \vDash A \wedge B & \textit{iff} & M, w \vDash A \textit{ and } M, w \vDash B \\
M, w \vDash \Box A & \textit{iff} & M, v \vDash A \textit{ for all } v \in W \textit{ such that } w \sim v \\
M, w \vDash [A]B & \textit{iff} & \textit{if } M, w \vDash A \textit{ then } M_A, w \vDash B
\end{array}
$$

After announcement of A, the updated model $M_A = (W_A, \sim_A, V_A)$ is defined as follows:

$$
\begin{aligned}
W_A &= \{w \in W : M, w \vDash A\} \\
\sim_A &= \sim \cap\, W_A \\
V_A(p) &= V(p) \cap W_A
\end{aligned}
$$

A formula A is valid*, denoted $\vDash_{PAL} A$ if, and only if, for all models $M = (W, \sim, V)$ and all worlds $w \in W$, $M, w \vDash A$. The set of all validities is denoted PAL.*

Informally, to define the updated model M_A we retain all and only the worlds in which A is true and leave the accessibility relation between remaining worlds and valuations at those worlds unchanged. In this sense, the effect of the announcement of A is to restrict the epistemic model to all and only the states where A does hold. Therefore, announcements can be seen as epistemic state *updaters*.

For our purpose, we need to introduce a notation to denote a list of announcements. We do that in the following definition.

Definition 2.5 (List of announcement)**.** *We define a list of announcement α in a public announcement formula $[\alpha]B$ by induction in the following way:*

- $[\epsilon]B := B$
- $[\alpha \cdot A]B := [\alpha][A]B$

By consequence we have

$$M_\alpha, w \models [A]B \qquad \textit{iff} \qquad \textit{if } M_\alpha, w \models A \textit{ then } M_{\alpha\cdot A}, w \models B$$

Theorem 2.6 (Soundness and Completeness of **PAL**)**.** *For all formulas $A \in \mathcal{L}_{PAL}$,*

$$\vdash_{\boldsymbol{PAL}} A \quad \textit{if, and only if,} \quad \models_{PAL} A$$

Proof. The proof can be found in [4]. □

3 Dynamic hypersequents for PAL

In this section, we introduce *dynamic hypersequents*, which extend classical hypersequents by incorporating a dynamic mechanism that enables transitions from one hypersequent to another. We introduce them step-by-step, starting from the standard notion of sequent.

Definition 3.1 (Sequent)**.** *A* sequent *is a syntactic object of the form $M \Rightarrow N$, where M and N are finite multisets of formulas. We denote sequents by capital greek letters $\Gamma, \Delta, \ldots$. By consequence, occurrences of specific formulas in sequents, e.g. $A, M \Rightarrow N$ or $M \Rightarrow N, A$, will be denoted by A, Γ and Γ, A, respectively. Finally, when $\Gamma = M \Rightarrow N$ and $\Delta = P \Rightarrow Q$, the sequent $M, P \Rightarrow N, Q$ is denoted $\Gamma\Delta$.*

Definition 3.2 (Dynamic Sequent)**.** Dynamic sequents *are inductively defined as follows:*

- *if Γ is a sequent, then Γ is a dynamic sequent;*
- *if $\Gamma_1, \cdots, \Gamma_n$ are sequents and $\alpha_1, \cdots, \alpha_n$ are (possibly empty) finite sequences of announcements, then $/\!\!/_{\alpha_1} \Gamma_1 /\!\!/ \cdots /\!\!/_{\alpha_n} \Gamma_n$ is a dynamic sequent.*

Intuitively, a dynamic sequent represents a world w in some model $M = (W, \sim, V)$, together with its copies in the successively updated models M_α that w belongs to. For instance $M \Rightarrow N /\!\!/_A M' \Rightarrow N'$, represents the world w in the initial model M as well as in the updated model M_A.

Note that in a dynamic sequent $X = /\!\!/_\epsilon \Gamma /\!\!/_{\alpha_1} \Gamma_1 /\!\!/ \cdots /\!\!/_{\alpha_n} \Gamma_n$, we call $/\!\!/_\epsilon \Gamma$, the *initial sequent*, and $/\!\!/_\alpha \Gamma$ the α-projection of X, or simply the α-*sequent*. Instead of $/\!\!/_\epsilon \Gamma$, we simply write Γ.

Definition 3.3 (Dynamic Hypersequent)**.** *Let $X_1, \cdots, X_n$ be dynamic sequents. Then, $G := X_1 \mid \cdots \mid X_n$ is a* dynamic hypersequent *(*DHS *for short).*

Intuitively, a dynamic hypersequent is the syntactic representation of a model $M = (W, \sim, V)$ and its successive updates M_α. For a better understanding, we can think of the set of updated models as a two-dimension table, where each row represents a world and its copies (if any) in the successive updated models, and each column represents an updated model, with the different worlds that belong to it. Dynamic hypersequents are a linear representation of such a table. As an example, the following table represents a model M with three worlds w, u, v, and three updated models obtained from M by announcing respectively A, A and then B, and C.[1]

M	M_A	$M_{A\cdot B}$	M_C
w	w'		w''
u			u'
v	v'	v''	v'''

[1] In this table, an empty cell indicates that the corresponding world does not belong to the updated model.

The following dynamic hypersequent stands for the syntactic representation of the table above, where sequents Γ, Δ and Λ represent worlds w, u and v, respectively.

$$\Gamma \mathbin{/\!\!/}_A \Gamma' \mathbin{/\!\!/}_C \Gamma'' \;\mid\; \Delta \mathbin{/\!\!/}_C \Delta' \;\mid\; \Lambda \mathbin{/\!\!/}_A \Lambda' \mathbin{/\!\!/}_{A\cdot B} \Lambda'' \mathbin{/\!\!/}_C \Lambda'''$$

In a hypersequent G, if we only consider initial sequents, then we get the syntactic representation of the initial model M, called *initial hypersequent.* Similarly, if we only consider α-sequents, we obtain the α projection of G, that is a representation of the updated model M_α: this is called *α-hypersequent.*

Definition 3.4 (Interpretation)**.** *The interpretation τ of a sequent $M \Rightarrow N$ is the standard one, namely: $(M \Rightarrow N)^\tau := \bigwedge M \to \bigvee N$. We extend it to dynamic sequents as follows:*

$$\mathbin{/\!\!/}_{\alpha_1}\Gamma_1 \mathbin{/\!\!/} \cdots \mathbin{/\!\!/}_{\alpha_n} \Gamma_n := [\alpha_1]\Gamma_1^\tau \vee \cdots \vee [\alpha_n]\Gamma_n^\tau$$

where, for all $1 \leq i \leq n$, $\alpha_i = A_{i_1} \cdot \cdots \cdot A_{i_{k_i}}$ and $[\alpha_i]B = [A_{i_1}] \cdots [A_{i_{k_i}}]B$, for $k_i \in \mathbb{N}$.

Finally, we extend the interpretation function τ to dynamic hypersequents as follows:

$$(X_1 \;\mid\; \cdots \;\mid\; X_n)^\tau := \Box X_1^\tau \vee \cdots \vee \Box X_n^\tau$$

We now have all ingredients to introduce the calculus $\mathbf{DHS}_{PAL}$, which is composed by the following axioms and inference rules.

Axioms:

$G \mid X \mathbin{/\!\!/}_\alpha p, M \Rightarrow N, p$

Propositional rules:

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A}{G \mid X \mathbin{/\!\!/}_\alpha \neg A, M \Rightarrow N}\ (\mathrm{L}\neg) \qquad \frac{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, \neg A}\ (\mathrm{R}\neg)$$

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha A, B, M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_\alpha A \wedge B, M \Rightarrow N}\ (\mathrm{L}\wedge) \qquad \frac{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A \quad G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, B}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A \wedge B}\ (\mathrm{R}\wedge)$$

Modal rules:

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha \Box A, A, M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_\alpha \Box A, M \Rightarrow N}\ (\mathrm{L}\Box_1) \qquad \frac{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \;\mid\; \mathbin{/\!\!/}_\alpha \Rightarrow A}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, \Box A}\ (\mathrm{R}\Box)$$

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha \Box A, M \Rightarrow N \mid Y \mathbin{/\!\!/}_\alpha A, P \Rightarrow Q}{G \mid X \mathbin{/\!\!/}_\alpha \Box A, M \Rightarrow N \mid Y \mathbin{/\!\!/}_\alpha P \Rightarrow Q}\ (\mathrm{L}\Box_2)$$

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha \Gamma, B_1 \;\mid\; \overline{Y} \quad G \mid X \mathbin{/\!\!/}_\alpha \Gamma \mathbin{/\!\!/}_{\alpha\cdot B_1} \Rightarrow B_2 \;\mid\; \overline{Y} \quad \cdots \quad G \mid X \mathbin{/\!\!/}_\alpha \Gamma \mathbin{/\!\!/}_{\alpha\cdot\beta} A \Rightarrow \;\mid\; \overline{Y}}{G \mid X \mathbin{/\!\!/}_\alpha \Gamma \;\mid\; Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box A, \Delta}\ (\mathrm{L}\Box_3)$$

where $\beta = B_1 \cdot B_2 \cdot \cdots \cdot B_n$ and $\overline{Y} = Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box A, \Delta$

Announcement rules:

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N' \qquad G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} B, M' \Rightarrow N'}{G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N'} \ (\mathrm{L}[\cdot])$$

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', B}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, [A]B \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N'} \ (\mathrm{R}[\cdot])$$

Dynamic rules:

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha p, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} p, M' \Rightarrow N'}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} p, M' \Rightarrow N'} \ (\mathrm{Lat}) \qquad \frac{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, p \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', p}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', p} \ (\mathrm{Rat})$$

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot A} M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_{\alpha\cdot A} M \Rightarrow N} \ (\mathrm{New}) \qquad \frac{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N'}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N'} \ (\mathrm{Recall})$$

In both the announcement rules (L[·]) and (R[·]), if the dynamic sequent $/\!\!/_{\alpha\cdot A}\ M' \Rightarrow N'$ does not occur in the conclusion, we add it to their premises – in particular, the right-hand premise in the rule (L[·]). In what follows, $(\mathrm{L}[\cdot])_1$ and $(\mathrm{R}[\cdot])_1$ will denote instances of the rules where the dynamic sequent $/\!\!/_{\alpha\cdot A}\ M' \Rightarrow N'$ occurs in the conclusion; $(\mathrm{L}[\cdot])_2$ $(\mathrm{R}[\cdot])_2$ will denote instances of the rules where the dynamic sequent $/\!\!/_{\alpha\cdot A} M' \Rightarrow N'$ does not occur in the conclusion.

We now turn to a more informal discussion of the rules of the calculus. The propositional rules are the standard ones from classical logic, adapted to the dynamic hypersequent framework. The modal rules – $(\mathrm{L}\Box_1)$, $(\mathrm{L}\Box_2)$, and $(\mathrm{R}\Box)$ – are essentially those introduced in [8], though once again reformulated to fit within the dynamic hypersequent setting.

The announcement rules become intuitive when viewed through the Kripke semantics satisfability relation of public announcement formulas. Consider the rules read bottom-up. For (R[·]), let $M = (W, \sim, V)$ be a model and w be a world of that model represented by the dynamic sequent X. If $[A]B$ is false at (M_α, w), then A must be true at w (so that it can indeed be announced): this is displayed via the sequent $/\!\!/_\alpha\ A, M \Rightarrow N$. After this announcement, however, B is false at w in the further updated model $M_{\alpha\cdot A}$, which is displayed by the sequent $/\!\!/_{\alpha\cdot A}\ M' \Rightarrow N', B$.

Similarly, for (L[·]): if $[A]B$ is true at w in M_α, then one of the following two options must hold: either A is false at (M_α, w), in which case the announcement cannot be made, or else A is true at (M_α, w), so that $w \in M_{\alpha\cdot A}$, and B holds at $(M_{\alpha\cdot A}, w)$. The former case is represented in the left-premise by $/\!\!/_\alpha\ M \Rightarrow N, A$ while the latter is displayed in the right-premise by $/\!\!/_{\alpha\cdot A}\ B, M' \Rightarrow N'$.

The dynamic rules capture important structural properties of models and their updates. Rules (Lat) and (Rat) express atomic permanence: read bottom-up, (Lat) for example states that if p holds at w in the updated model $M_{\alpha\cdot A}$ — represented by $/\!\!/_{\alpha\cdot A}\ p, M' \Rightarrow N'$ —, then p must also hold at w in the previous model M_α — which corresponds to $/\!\!/_\alpha\ p, M \Rightarrow N$. Rule (New) ensures that if $w \in M_{\alpha\cdot A}$, then w must already belong to the previous model M_α: hence from $/\!\!/_{\alpha\cdot A}\ M \Rightarrow N$ we get $/\!\!/_\alpha \Rightarrow$. Finally, (Recall) requires that if $w \in M_{\alpha\cdot A}$ — displayed by $/\!\!/_{\alpha\cdot A}\ M' \Rightarrow N'$ —, then A must be true at (M_α, w), which corresponds to $/\!\!/_\alpha\ A, M \Rightarrow N$.

Finally, rule $(\mathrm{L}\Box_3)$ should be seen as a dynamic modal rule, since it displays the relationship between knowledge and announcements. It can be understood as follows: for $\Box A$ to be true at some world v in some updated model $M_{\alpha\cdot\beta}$, every world w in $M_{\alpha\cdot\beta}$ must satisfy A. Hence, for any other world $w \in M_\alpha$, either $w \notin M_{\alpha\cdot\beta}$, or A is true at w in $M_{\alpha\cdot\beta}$. The latter case is represented by the rightmost premise of the rule. The former case, on the other hand, is represented by all the remaining premises: indeed, for $w \in M_\alpha$ not to belong to $M_{\alpha\cdot\beta}$, one of the following must hold (where $\beta = B_1 \cdots B_n$): B_1 is false at (w, M_α), so w does not belong to $M_{\alpha\cdot B_1}$, or $w \in M_{\alpha\cdot B_1}$ but B_2 is false at $(w, M_{\alpha\cdot B_2})$, etc. This is why (L$\Box_3$) has $n+1$ premises, when the sequence β is composed of n announcements.

As an example, consider the following instance of the rule, where $\beta = B$ and G is empty:

$$\frac{X \mathbin{/\!\!/}_\alpha \Gamma, B \mid Y \mathbin{/\!\!/}_{\alpha\cdot B} \Box A, \Delta \qquad X \mathbin{/\!\!/}_\alpha \Gamma \mathbin{/\!\!/}_{\alpha\cdot B} A \Rightarrow \mid Y \mathbin{/\!\!/}_{\alpha\cdot B} \Box A, \Delta}{X \mathbin{/\!\!/}_\alpha \Gamma \mid Y \mathbin{/\!\!/}_{\alpha\cdot B} \Box A, \Delta} \ (\mathrm{L}\Box_3)$$

Let $M = (W, \sim, V)$ be a model where $W = \{w, v\}$. Worlds w, v are represented by $X \mathbin{/\!\!/}_\alpha \Gamma$ and $Y \mathbin{/\!\!/}_{\alpha\cdot B} \Box A, \Delta$ respectively. Then, we have $w \in W_\alpha$ and $v \in W_{\alpha\cdot B}$ (which implies $v \in W_\alpha$). For $M_{\alpha\cdot B}, v \vDash \Box A$ to hold, either $w \notin W_{\alpha\cdot B}$ or $M_{\alpha\cdot B}, w \vDash A$. The first case arises when $M_\alpha, w \nvDash B$, which corresponds, in the first premise, to the sequent $\mathbin{/\!\!/}_\alpha \Gamma, B$. The second case is captured, in the second premise, by the sequent $\mathbin{/\!\!/}_{\alpha\cdot B} A \Rightarrow$.

Definition 3.5. Derivations *in* $\boldsymbol{DHS}_{PAL}$ *are defined in the standard way and are denoted by* d, d'*, etc.. The* height *of a derivation* d*, denoted* $h(d)$*, is also inductively defined in the standard way (see [12]). As usual, we write* $\vdash_{\boldsymbol{DHS}_{PAL}} G$ *to denote that the dynamic hypersequent* G *is derivable in the calculus* $\boldsymbol{DHS}_{PAL}$.

4 Admissibility of the structural rules and invertibility of the logical rules

In this section, we show that the structural rules of weakening, contraction and merge are hp-admissible, and that all logical rules are hp-invertible. We also introduce new dynamic structural rules and we prove their hp-admissibility.

Lemma 4.1. *All dynamic hypersequents of the form* $G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N, A$ *are derivable.*

Proof. By standard induction on the complexity of A. $\Box$

Lemma 4.2. *The rule of* internal weakening *is hp-admissible.*

$$\frac{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_\alpha P, M \Rightarrow N, Q} \ (IW)$$

Proof. The proof proceeds by straightforward induction on the height of the derivation of the premise. $\Box$

Lemma 4.3. *The rule of* external weakening *is hp-admissible.*

$$\frac{G}{G \mid X} \ (EW)$$

Proof. The proof proceeds by induction on the height of the derivation of the premise. If G is an axiom, so is $G \mid X$. If G is not an axiom, we distinguish cases on the last applied rule $\mathcal{R}$: in each of the cases, we apply the inductive hypothesis to the premise(s) of $\mathcal{R}$ and then use $\mathcal{R}$ again. We only show one example, whilst the others can be treated similarly.

If the last applied rule is (R$[\cdot]$)$_2$, we proceed as follows:

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid Y \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B\end{array}}{G \mid Y \mathbin{/\!\!/}_\alpha \Gamma, [A]B}\,(\mathrm{R}[\cdot])_2 \quad \overset{(IH)_2}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\, d'\\ G \mid Y \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B \mid X\end{array}}{G \mid Y \mathbin{/\!\!/}_\alpha \Gamma, [A]B \mid X}\,(\mathrm{R}[\cdot])_2$$

□

Lemma 4.4. *The rule of* dynamic weakening *is hp-admissible.*

$$\frac{G \mid X}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N}\,(DW)$$

Proof. The proof proceeds by induction on the height of the derivation of the premise and is similar to the one of the above lemma. □

Lemma 4.5. *All logical rules are hp-invertible.*

Proof. For each rule, the proof proceeds by induction on the height of a derivation of the conclusion.

– Propositional rules: we only show invertibility of (L∧), because the method is the same for the others. If $G \mid X \mathbin{/\!\!/}_\alpha A \wedge B, M \Rightarrow N$ is an axiom, then so is $G \mid X \mathbin{/\!\!/}_\alpha A, B, M \Rightarrow N$. Otherwise, we consider the last occurrence of a rule $\mathcal{R}$ in the derivation. If $\mathcal{R}$ has $A \wedge B$ as its principal formula, then it must be an occurrence of (L∧) and the premise is precisely $G \mid X \mathbin{/\!\!/}_\alpha A, B, M \Rightarrow N$. If $A \wedge B$ is not principal in $\mathcal{R}$, we apply the inductive hypothesis to the premise(s) of $\mathcal{R}$ and then use $\mathcal{R}$ again.

– Modal rules: (L$\Box_1$) and (L$\Box_2$) are hp-invertible by hp-admissibility of (IW) and (L$\Box_3$) is hp-invertible by hp-admissibility of (IW) and (DW). We now dwell on the hp-invertibility of (R□). If $G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, \Box A$ is an axiom, then so is $G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mid \mathbin{/\!\!/}_\alpha \Rightarrow A$. Otherwise, we consider the last occurrence of a rule $\mathcal{R}$ in the derivation. If the principal formula of $\mathcal{R}$ is $\Box A$, then $\mathcal{R}$ is the rule (R□) and its premise is precisely what we want, namely $G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mid \mathbin{/\!\!/}_\alpha \Rightarrow A$. If $\Box A$ is not principal in $\mathcal{R}$, we apply the inductive hypothesis to the premise(s) of the rule and then use $\mathcal{R}$ again. We show one example where $\mathcal{R}$ is an occurrence of (L$\Box_3$).

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid X \mathbin{/\!\!/}_\alpha \Gamma, \Box A, B_1 \mid Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d_n\\ G \mid X \mathbin{/\!\!/}_\alpha \Gamma, \Box A \mathbin{/\!\!/}_{\alpha\cdot\beta} C \Rightarrow \mid Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha \Gamma, \Box A \mid Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta}\,(\mathrm{L}\Box_3)$$

$\overset{(IH)}{\rightsquigarrow}$

$$\dfrac{\begin{array}{c}\vdots\, d_1'\\ G \mid X \mathbin{/\!\!/}_\alpha \Gamma, B_1 \mid \mathbin{/\!\!/}_\alpha \Rightarrow A \mid Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d_n'\\ G \mid X \mathbin{/\!\!/}_\alpha \Gamma \mathbin{/\!\!/}_{\alpha\cdot\beta} C \Rightarrow \mid \mathbin{/\!\!/}_\alpha \Rightarrow A \mid Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha \Gamma \mid \mathbin{/\!\!/}_\alpha \Rightarrow A \mid Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta}\,(\mathrm{L}\Box_3)$$

– Announcement rules: we need to consider both versions of each rule. We first consider $(\mathrm{R}[\cdot])_1$, whilst $(\mathrm{L}[\cdot])_1$ is treated likewise. If $G \mid \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', B$ is an axiom, then so is $G \mid \mathbin{/\!\!/}_\alpha M \Rightarrow N, [A]B \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N'$. Otherwise, we consider the last applied rule $\mathcal{R}$. If $\mathcal{R}$ has $[A]B$ as its principal formula, then it must be $(\mathrm{R}[\cdot])_1$ and the premise is precisely $G \mid \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', B$. If $[A]B$ is not principal, we apply the inductive hypothesis on the premise(s) and then $\mathcal{R}$ again. We show one case with (L$\Box_3$), whilst others are treated

[2] From now on, by the symbol $\overset{(IH)}{\rightsquigarrow}$, we will denote the fact that the premise(s) of the inference on the right is obtained by inductive hypothesis on the premise(s) of the inference on the left.

analogously.

$$
\dfrac{\begin{array}{c}\vdots\, d_1\\ \Gamma, [A]B \,/\!\!/_{A}\, \Gamma', B_1 \;\mid\; Y \,/\!\!/_{A\cdot\beta}\, \Box C, \Delta\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d_k\\ \Gamma, [A]B \,/\!\!/_{A}\, \Gamma' \,/\!\!/_{A\cdot\beta}\, C \Rightarrow \;\mid\; Y \,/\!\!/_{A\cdot\beta}\, \Box C, \Delta\end{array}}{\Gamma, [A]B \,/\!\!/_{A}\, \Gamma' \;\mid\; Y \,/\!\!/_{A\cdot\beta}\, \Box C, \Delta}\ \mathrm{L}(\Box_3)
$$

$\overset{(IH)}{\rightsquigarrow}$

$$
\dfrac{\begin{array}{c}\vdots\, d'_1\\ A, \Gamma \,/\!\!/_{A}\, \Gamma', B, B_1 \;\mid\; Y \,/\!\!/_{A\cdot\beta}\, \Box C, \Delta\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d'_k\\ A, \Gamma \,/\!\!/_{A}\, \Gamma', B \,/\!\!/_{A\cdot\beta}\, C \Rightarrow \;\mid\; Y \,/\!\!/_{A\cdot\beta}\, \Box C, \Delta\end{array}}{A, \Gamma \,/\!\!/_{A}\, \Gamma', B \;\mid\; Y \,/\!\!/_{A\cdot\beta}\, \Box C, \Delta}\ \mathrm{L}(\Box_3)
$$

We now turn to $(\mathrm{R}[\cdot])_2$, while $(\mathrm{L}[\cdot])_2$ is treated likewise. The proof proceeds similarly, except when the dynamic sequent $/\!\!/_{\alpha\cdot A}\, \Gamma'$ appears in the premise(s) of $\mathcal{R}$. In such cases, we use hp-invertibility of $(\mathrm{R}[\cdot])_1$ on the premise(s) displaying $/\!\!/_{\alpha\cdot A}\, \Gamma'$ instead of the inductive hypothesis. Afterwards, we might need to apply a different version of rule $\mathcal{R}$, *i.e.* $(\mathrm{L}[\cdot])_1$ (resp. $(\mathrm{R}[\cdot])_1$) instead of $(\mathrm{L}[\cdot])_2$ (resp. $(\mathrm{R}[\cdot])_2$), and $(\mathrm{L}\Box_2)$ instead of $(\mathrm{L}\Box_3)$. We provide some examples, whilst other cases can be treated similarly.

If $\mathcal{R}$ is an occurrence of $(\mathrm{L}[\cdot])_2$, we proceed as follows.

$$
\dfrac{\begin{array}{c}\vdots\, d_1\\ G \;\mid\; X \,/\!\!/_{\alpha}\, \Gamma, [A]B, A\end{array} \qquad \begin{array}{c}\vdots\, d_2\\ G \;\mid\; X \,/\!\!/_{\alpha}\, \Gamma, [A]B \,/\!\!/_{\alpha\cdot A}\, C \Rightarrow\end{array}}{G \;\mid\; X \,/\!\!/_{\alpha}\, [A]C, \Gamma, [A]B}\ (\mathrm{L}[\cdot])_2
$$

Here we apply the inductive hypothesis on the left-hand premise, to obtain $G \;\mid\; X /\!\!/_{\alpha}\, A, \Gamma, A /\!\!/_{\alpha\cdot A} \Rightarrow B$ while we use hp-invertibility of $(\mathrm{R}[\cdot])_1$ on the right-hand premise to get $G \;\mid\; X \,/\!\!/_{\alpha}\, A, \Gamma \,/\!\!/_{\alpha\cdot A}\, C \Rightarrow B$. We then apply $(\mathrm{L}[\cdot])_1$ to obtain the desired conclusion:

$$
\dfrac{\begin{array}{c}\vdots\, d'_1\\ G \;\mid\; X \,/\!\!/_{\alpha}\, A, \Gamma, A /\!\!/_{\alpha\cdot A} \Rightarrow B\end{array} \qquad \begin{array}{c}\vdots\, d'_2\\ G \;\mid\; X \,/\!\!/_{\alpha}\, A, \Gamma \,/\!\!/_{\alpha\cdot A}\, C \Rightarrow B\end{array}}{G \;\mid\; X \,/\!\!/_{\alpha}\, A, [A]C, \Gamma /\!\!/_{\alpha\cdot A} \Rightarrow B}\ (\mathrm{L}[\cdot])_1
$$

When $\mathcal{R}$ is an occurrence of $(\mathrm{L}\Box_3)$ with two premises as below

$$
\dfrac{\begin{array}{c}\vdots\, d_1\\ M \Rightarrow N, [A]B, A \;\mid\; Y \,/\!\!/_{A}\, \Box C, \Delta\end{array} \qquad \begin{array}{c}\vdots\, d_2\\ M \Rightarrow N, [A]B \,/\!\!/_{A}\, C \Rightarrow \;\mid\; Y \,/\!\!/_{A}\, \Box C, \Delta\end{array}}{M \Rightarrow N, [A]B \;\mid\; Y \,/\!\!/_{A}\, \Box C, \Delta}\ \mathrm{L}(\Box_3)
$$

we consider the right-hand premise only: we apply hp-invertibility of $(\mathrm{R}[\cdot])_1$ on it and then use $(\mathrm{L}\Box_2)$ instead of $(\mathrm{L}\Box_3)$, to get:

$$
\dfrac{\begin{array}{c}\vdots\, d'_2\\ A, M \Rightarrow N \,/\!\!/_{A}\, C \Rightarrow B \;\mid\; Y \,/\!\!/_{A}\, \Box C, \Delta\end{array}}{A, M \Rightarrow N /\!\!/_{A} \Rightarrow B \;\mid\; Y \,/\!\!/_{A}\, \Box C, \Delta}\ \mathrm{L}(\Box_2)
$$

Eventually, when $\mathcal{R}$ is an occurrence of (New) where $/\!\!/_{\alpha\cdot A}\, \Gamma'$ appears in the premise, hp-invertibility of $(\mathrm{R}[\cdot])_1$ directly provides the conclusion, with no need for further application of (New), as shown below.

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid X \sslash_\alpha \Gamma, [A]B \sslash_{\alpha\cdot A} \Rightarrow \sslash_{\alpha\cdot A\cdot C} \Gamma'\end{array}}{G \mid X \sslash_\alpha \Gamma, [A]B \sslash_{\alpha\cdot A\cdot C} \Gamma'}\ \text{(New)} \quad \overset{(R[\cdot])_1{}^3}{\dashrightarrow} \quad \begin{array}{c}\vdots\, d'\\ G \mid X \sslash_\alpha A, \Gamma \sslash_{\alpha\cdot A} \Rightarrow B \sslash_{\alpha\cdot A\cdot C} \Gamma'\end{array}$$

– Dynamic rules: (Lat) and (Rat), as well as (Recall) are hp-invertible by hp-admissibility of (IW). (New) is hp-invertible by hp-admissibility of (DW).

□

From now on, for any given rule $\mathcal{R}$, we denote by $\mathcal{R}^\star$ the inverted rule, that is derivable, by Lemma 4.5.

As is standard in a hypersequent framework, there is a merge rule that can be shown to be hp-admissible. To introduce it and establish its admissibility, we first need the following definition.

Definition 4.6. *Let $X = \sslash_{\alpha_1} \Gamma_1 \sslash \cdots \sslash_{\alpha_n} \Gamma_n$ be a dynamic sequent. The labels of X are all the sequences of announcements that label some sequent in X,* i.e. *$labels(X) = \{\alpha_1, \cdots, \alpha_n\}$.*

Definition 4.7. *Let X and Y be two DHSs, then $X \otimes Y$ is inductively defined as follows:*

- *if $X = \sslash_\alpha M \Rightarrow N$ and $Y = \sslash_\alpha P \Rightarrow Q$ then $X \otimes Y := \sslash_\alpha M, P \Rightarrow N, Q$;*
- *if $X = X' \sslash_{\alpha\cdot A} M \Rightarrow N$ and $Y = Y' \sslash_{\alpha\cdot A} P \Rightarrow Q$ then $X \otimes Y := X' \otimes Y' \sslash_{\alpha\cdot A} M, P \Rightarrow N, Q$;*
- *if $X = X' \sslash_{\alpha\cdot A} M \Rightarrow N$ and $Y = Y' \sslash_{\alpha\cdot B} P \Rightarrow Q$, where $\alpha \cdot A \notin labels(Y')$ and $\alpha \cdot B \notin labels(X')$, then $X \otimes Y := X' \otimes Y' \sslash_{\alpha\cdot A} M \Rightarrow N \sslash_{\alpha\cdot B} P \Rightarrow Q$.*

Lemma 4.8. *The rule of* Merge *is hp-admissible.*

$$\dfrac{G \mid X \sslash_\alpha M \Rightarrow N \mid Y \sslash_\alpha P \Rightarrow Q}{G \mid X \otimes Y \sslash_\alpha M, P \Rightarrow N, Q}\ \text{(Merge)}$$

Proof. The proof proceeds by induction on the height of the derivation of the premise. If $G \mid X \sslash_\alpha M \Rightarrow N \mid Y \sslash_\alpha P \Rightarrow Q$ is an axiom, so is $G \mid X \otimes Y \sslash_\alpha M, P \Rightarrow N, Q$. Otherwise, we consider the last applied rule $\mathcal{R}$: we apply the inductive hypothesis to the premise(s) of $\mathcal{R}$ and then use $\mathcal{R}$ again. We only show two paradigmatic examples.

If $\mathcal{R}$ is an occurrence of (R□):

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid X \sslash_\alpha \Gamma \mid Y \sslash_\alpha \Delta \mid \sslash_\alpha \Rightarrow A\end{array}}{G \mid X \sslash_\alpha \Gamma, \Box A \mid Y \sslash_\alpha \Delta}\ (\mathrm{R}\Box) \quad \overset{(IH)}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\, d'\\ G \mid X \otimes Y \sslash_\alpha \Gamma\Delta \mid \sslash_\alpha \Rightarrow A\end{array}}{G \mid X \otimes Y \sslash_\alpha \Gamma\Delta, \Box A}\ (\mathrm{R}\Box)$$

If $\mathcal{R}$ is an occurrence of $(\mathrm{L}\Box_3)$ of the following shape,

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid X \sslash_\alpha \Gamma, B_1 \mid Y \sslash_\alpha \Delta \sslash_{\alpha\cdot\beta} \Box A, \Delta'\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d_n\\ G \mid X \sslash_\alpha \Gamma \sslash_{\alpha\cdot\beta} A \Rightarrow \mid Y \sslash_\alpha \Delta \sslash_{\alpha\cdot\beta} \Box A, \Delta'\end{array}}{G \mid X \sslash_\alpha \Gamma \mid Y \sslash_\alpha \Delta \sslash_{\alpha\cdot\beta} \Box A, \Delta'}\ (\mathrm{L}\Box_3)$$

we simply apply the inductive hypothesis to the rightmost premise, and conclude with $(\mathrm{L}\Box_1)$:

[3] We write $\overset{\mathcal{R}}{\dashrightarrow}$ in an informal proof to indicate that we use the hp-invertibility of rule $\mathcal{R}$.

$$\dfrac{\begin{array}{c}\vdots\, d'_n\\ G \mid X \otimes Y \mathbin{/\!\!/}_{\alpha} \Gamma\Delta \mathbin{/\!\!/}_{\alpha\cdot\beta} A, \Box A, \Delta'\end{array}}{G \mid X \otimes Y \mathbin{/\!\!/}_{\alpha} \Gamma\Delta \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box A, \Delta'}\ (\mathrm{L}\Box_1)$$

$\Box$

Lemma 4.9. *The rules of* contraction *are hp-admissible.*

$$\dfrac{G \mid X \mathbin{/\!\!/}_{\alpha} A, A, M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N}\ (LC) \qquad\qquad \dfrac{G \mid X \mathbin{/\!\!/}_{\alpha} M \Rightarrow N, A, A}{G \mid X \mathbin{/\!\!/}_{\alpha} M \Rightarrow N, A}\ (RC)$$

Proof. We simultaneously prove the hp-admissibility of (LC) and (RC) by induction on the height of the derivation of the premise of each rule. If $G \mid X \mathbin{/\!\!/}_{\alpha} A, A, M \Rightarrow N$ and $G \mid X \mathbin{/\!\!/}_{\alpha} M \Rightarrow N, A, A$ are axioms, then so are $G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N$ and $G \mid X \mathbin{/\!\!/}_{\alpha} M \Rightarrow N, A$. Else, we distinguish cases on the last applied rule $\mathcal{R}$. If neither of the occurrences of A is principal, we apply the inductive hypothesis to the premise(s) of $\mathcal{R}$, and then use $\mathcal{R}$ again. If one of the occurrence of A is principal, we start from the premise(s) of $\mathcal{R}$ and use the hp-invertibility of the rules before applying the inductive hypothesis; then we use $\mathcal{R}$ again. We only show one example, while the others can be treated similarly.

If the last applied rule is $(\mathrm{R}[\cdot])_2$, we first use hp-invertibility of $(\mathrm{R}[\cdot])_1$, then we apply the inductive hypothesis twice, and eventually we use $(\mathrm{R}[\cdot])_2$ again:

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid X \mathbin{/\!\!/}_{\alpha} A, \Gamma[A]B \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B\end{array}}{G \mid X \mathbin{/\!\!/}_{\alpha} \Gamma, [A]B, [A]B}\ (\mathrm{R}[\cdot]) \quad \overset{(R[\cdot])_1}{\dashrightarrow} \quad G \mid X \mathbin{/\!\!/}_{\alpha} A, A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B, B$$

$$\overset{(IH)}{\rightsquigarrow} \quad G \mid X \mathbin{/\!\!/}_{\alpha} A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B, B$$

$$\overset{(IH)}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\, d'_1\\ G \mid X \mathbin{/\!\!/}_{\alpha} A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B\end{array}}{G \mid X \mathbin{/\!\!/}_{\alpha} \Gamma, [A]B}\ (\mathrm{R}[\cdot])$$

$\Box$

We end this section by proving the admissibility of the rule New^A, which will be crucial to prove the admissibility of the Cut-rule.

Lemma 4.10. *The rule* New^A *is admissible.*

$$\dfrac{G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow}{G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N}\ (New^A)$$

Proof. We proceed by induction on the height of the derivation of the premise. If $G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow$ is an axiom then so is $G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N$. If $G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow$ is of the form $G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot A\cdot B} M' \Rightarrow N'$ - namely the dynamic sequent X contains a sequent of the form $\mathbin{/\!\!/}_{\alpha\cdot A\cdot B} M' \Rightarrow N'$ - we simply apply (New) to obtain the desired conclusion:

$$\dfrac{G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot A\cdot B} M' \Rightarrow N'}{G \mid X \mathbin{/\!\!/}_{\alpha} A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A\cdot B} M' \Rightarrow N'}\ (\mathrm{New})$$

Finally, if $G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow$ is neither an axiom nor of the form indicated above, we consider the last applied rule $\mathcal{R}$. If $\mathcal{R}$ is an application of a right-rule, we apply the inductive hypothesis to the premise(s) of $\mathcal{R}$ and then use $\mathcal{R}$ again, as in the example below, where $\mathcal{R}$ is an occurrence of (R$\Box$).

$$\dfrac{\begin{array}{c}\vdots\, d\\ G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow \mid\ \mathbin{/\!\!/}_\alpha \Rightarrow B\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N, \Box B \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow}\,(\mathrm{R}\Box) \quad \overset{(IH)}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\, d'\\ G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mid\ \mathbin{/\!\!/}_\alpha \Rightarrow B\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N, \Box B}\,(\mathrm{R}\Box)$$

If $\mathcal{R}$ is an application of a left-rule that does not involve neither A nor the dynamic sequent $\mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow$, we reason as before. Here is an example with (L$[\cdot]$):

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma, B \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow\end{array} \quad \begin{array}{c}\vdots\, d_2\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot B} C \Rightarrow\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha [B]C, A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow}\,(\mathrm{L}[\cdot])$$

$$\overset{(IH)}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\, d_1'\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma, B\end{array} \quad \begin{array}{c}\vdots\, d_2'\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot B} C \Rightarrow\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha [B]C, A, \Gamma}\,(\mathrm{L}[\cdot])$$

If $\mathcal{R}$ is an application of a left-rule that does involve either A or the dynamic sequent $\mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow$, then we obtain the desired result using hp-admissibility of weakening and contraction, applying the inductive hypothesis or directly using $\mathcal{R}$ again. We only show the most significant cases.

– $\mathcal{R}$ is an application of (L$\Box_3$) involving $\mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow$, with active dynamic sequent $\mathbin{/\!\!/}_{\alpha\cdot A\cdot\beta} \Box C, \Delta$ where $\beta = B_1 \cdots B_n$, as below

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B_1 \mid \overline{Y}\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d_{n+1}\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot A\cdot\beta} C \Rightarrow \mid \overline{Y}\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow \mid Y \mathbin{/\!\!/}_{\alpha\cdot A\cdot\beta} \Box C, \Delta}\,(\mathrm{L}\Box_3)$$

we apply the inductive hypothesis on all the premises except the leftmost, and use $G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma, A \mid Y \mathbin{/\!\!/}_{\alpha\cdot A\cdot\beta} \Box C, \Delta$ (that is derivable by Lemma 4.1) as a first premise to apply (L$\Box_3$) with $n+2$ premises:

$$\dfrac{G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma, A \mid \overline{Y} \quad \begin{array}{c}\vdots\, d_1\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A} \Rightarrow B_1 \mid \overline{Y}\end{array} \quad \cdots \quad \begin{array}{c}\vdots\, d_{n+1}'\\ G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mathbin{/\!\!/}_{\alpha\cdot A\cdot\beta} C \Rightarrow \mid \overline{Y}\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha A, \Gamma \mid Y \mathbin{/\!\!/}_{\alpha\cdot A\cdot\beta} \Box C, \Delta}\,(\mathrm{L}\Box_3)$$

– $\mathcal{R}$ is an application of (L$[\cdot]$) where A is the principal formula:

$$\dfrac{\begin{array}{c}\vdots\, d_1\\ G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A \mathbin{/\!\!/}_{[A]B} \Rightarrow\end{array} \quad \begin{array}{c}\vdots\, d_2\\ G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N \mathbin{/\!\!/}_{[A]B} \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot A} B \Rightarrow\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N \mathbin{/\!\!/}_{[A]B} \Rightarrow}\,(\mathrm{L}[\cdot])$$

We first use the hp-admissibility of (IW) to get

$$\begin{array}{cc} \vdots\, d_1' & \vdots\, d_2' \\ G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N, A \mathbin{/\!\!/}_{[A]B} \Rightarrow & G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N \mathbin{/\!\!/}_{[A]B} \Rightarrow \mathbin{/\!\!/}_{\alpha\cdot A} B \Rightarrow \end{array}$$

then we apply the inductive hypothesis, use (L[·]) and then hp-admissibility of (LC) to conclude:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, d_1'' \\ G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N, A\end{array} \quad \begin{array}{c}\vdots\, d_2'' \\ G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} B \Rightarrow\end{array}}{G \mid X \mathbin{/\!\!/}_\alpha [A]B, [A]B, M \Rightarrow N}\ (\mathrm{L}[\cdot])}{G \mid X \mathbin{/\!\!/}_\alpha [A]B, M \Rightarrow N}\ (\mathrm{LC})$$

□

5 Soundness and Completeness

In this section, we show that $\mathbf{DHS}_{PAL}$ is sound and complete.

Theorem 5.1 (Soundness)**.** *For all formulas $A \in \mathcal{L}_{PAL}$, if $\vdash_{\mathbf{DHS}_{PAL}} \Rightarrow A$ then $\models_{PAL} A$.*

Proof. We need to show that axioms are valid and that all rules are correct, *i.e.* that they preserve validity. As for the axioms, this is straightforward. As for the rules, for the sake of clarity, we only present in detail the case of the rule $(\mathrm{R}[\cdot])_1$. The other rules can be treated similarly.

For $(\mathrm{R}[\cdot])_1$, let $\alpha = A_1 \cdots A_n$ and suppose $\models (G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', B)^\tau$ but $\not\models (G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, [A]B \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N')^\tau$. Then, there are a model $M = (W, \sim, V)$ and a world $w \in W$ such that

$$M, w \not\models G^\tau \vee \Box(X^\tau \vee [\alpha](\bigwedge M \to \bigvee N \vee [A]B) \vee [\alpha][A](\bigwedge M' \to \bigvee N')$$

so there is $u \in W$ such that $w \sim u$ and:

$$\begin{array}{ll} (1) & M, u \not\models X^\tau \\ (2) & M, u \not\models [\alpha](\bigwedge M \to \bigvee N \vee [A]B) \\ (3) & M, u \not\models [\alpha][A](\bigwedge M' \to \bigvee N') \end{array}$$

From (3), we get $M, u \models A_1, M_{A_1}, u \models A_2, \cdots, M_\alpha, u \models A$ and $M_{\alpha\cdot A} \not\models \bigwedge M' \to \bigvee N'$. From (2) we get $M_\alpha, u \models \bigwedge M$ and $M_\alpha, u \not\models [A]B$. From that and $M_\alpha, u \models A$, we conclude that $M_{\alpha\cdot A}, u \not\models B$.

However, since $\models (G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N', B)^\tau$ and $M, w \not\models G^\tau$, necessarily

$$M, w \models \Box(X^\tau \vee [\alpha](A \wedge \bigwedge M \to \bigvee N) \vee [\alpha][A](\bigwedge M' \to \bigvee N' \vee B)).$$

Now, because $M, u \not\models X^\tau$, either $M, u \models [\alpha](A \wedge \bigwedge M \to \bigvee N)$ or $M, u \models [\alpha][A](\bigwedge M' \to \bigvee N' \vee B)$. In the first case, we get a contradiction because $M_\alpha, u \models \bigwedge M$ and $M_\alpha, u \models A$ but $M_\alpha, u \not\models \bigvee N$. In the other case, we get a contradiction from $M_{\alpha\cdot A}, u \not\models \bigwedge M' \to \bigvee N'$ and $M_{\alpha\cdot A}, u \not\models B$ which implies $M_{\alpha\cdot A} \not\models \bigwedge M' \to \bigvee N' \vee B$, hence $M, u \not\models [\alpha][A](\bigwedge M' \Rightarrow \bigvee N' \vee B)$.

Therefore $\models (G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, [A]B \mathbin{/\!\!/}_{\alpha\cdot A} M' \Rightarrow N')^\tau$.

□

We now move to the completeness of the calculus. In order to prove that $\mathbf{DHS}_{PAL}$ is complete with respect to PAL, we need to show that all axioms of PAL are derivable, as well as that the inference rules of PAL are admissible. This can be standardly carried out except for the case of the axiom of announcement composition, which requires us to show a derivability lemma (see Lemma 5.3). On the other hand, in order to prove this derivability lemma, we first need an enhanced notion of complexity of formulas, *i.e.* a notion of complexity of formulas that is relative to the sequents formulas belong to in the context of a dynamic hypersequent.[4] We thus also introduce the notion of *DHS-complexity* in Definition 5.2. Finally, note that many other calculi for PAL (*e.g.* [7],[3]) require some preliminary lemmas in order to show a completeness result; hence our procedure is standard in the context of proof theory for public announcement logic.

Definition 5.2 (DHS-Complexity)**.** *The DHS-complexity of a formula A* relatively to a sequence of announcements *α, $c(A,\alpha)$, is defined as:*

$$c(A,\alpha) := c(A) + c(\alpha),$$

where $c(A)$ is inductively defined as follows:

$$\begin{aligned} c(p) &:= 1 \quad \textit{for } p \in P \\ c(\neg A) &:= c(A)+1 & \qquad c(\Box A) &:= c(A)+1 \\ c(A \wedge B) &:= c(A)+c(B)+1 & \qquad c([A]B) &:= c(A)+c(B)+1 \end{aligned}$$

whilst $c(\alpha)$ is inductively defined as follows:

$$c(\epsilon) := 0 \qquad\qquad c(\alpha \cdot A) := c(\alpha) + c(A).$$

Lemma 5.3. *Let A, B, C be formulas of $\mathcal{L}_{PAL}$, and α, β be finite sequences of announcements. Then, the following dynamic hypersequents are derivable:*

$$\begin{array}{ll} (1) & G \mid X \mathbin{/\!\!/}_{\alpha\cdot A\cdot B\cdot\beta}\ C, M \Rightarrow N \mathbin{/\!\!/}_{\alpha\cdot A\wedge[A]B\cdot\beta}\ M' \Rightarrow N', C \\ (2) & G \mid X \mathbin{/\!\!/}_{\alpha\cdot A\cdot B\cdot\beta}\ M \Rightarrow N, C \mathbin{/\!\!/}_{\alpha\cdot A\wedge[A]B\cdot\beta}\ C, M' \Rightarrow N' \end{array}$$

Proof. We simultaneously prove that (1) and (2) are derivable by induction on the sum of the DHS-complexity of each occurrence of C, denoted $\overline{c}(C) := c(C, \alpha\cdot A\cdot B\cdot\beta) + c(C, \alpha\cdot A\wedge[A]B\cdot\beta)$.

Without loss of generality, we prove the result for empty G, X and α. In the following, $\beta = B_1 \cdot B_2 \cdots B_{n-1}$ and we write φ for $A \wedge [A]B$. We distinguish the following cases, where we only present the details for (1), whilst (2) is treated similarly.

• Case $C = p$. To derive (1), we (bottom-up) apply $2 \times n$ times (New) to go from $/\!\!/_{\varphi\cdot\beta}\ \Gamma$ to $/\!\!/_{\epsilon}\ \Gamma'$ and from $/\!\!/_{A\cdot B\cdot\beta}\ \Delta$ to $/\!\!/_{A}\ \Delta'$, then we apply $n+1$ times (Rat) and n times (Lat) to obtain an axiom with $p \Rightarrow p$. We write $\mathcal{R}^*$ when we apply several times rule $\mathcal{R}$.

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p /\!\!/_{A} \ \Rightarrow p /\!\!/_{A\cdot B} \ \Rightarrow p /\!\!/ \cdots /\!\!/_{A\cdot B\cdot\beta}\, M \Rightarrow N, p /\!\!/_{\varphi}\ \ p \Rightarrow /\!\!/ \cdots /\!\!/_{\varphi\cdot\beta}\ \ p, M' \Rightarrow N'}{\Rightarrow p /\!\!/_{A} \ \Rightarrow p /\!\!/_{A\cdot B} \ \Rightarrow p /\!\!/ \cdots /\!\!/_{A\cdot B\cdot\beta}\, M \Rightarrow N, p /\!\!/_{\varphi}\ \ \Rightarrow /\!\!/ \cdots /\!\!/_{\varphi\cdot\beta}\ \ p, M' \Rightarrow N'}\ (\mathrm{Lat})^*\ (n \text{ times})}{\Rightarrow /\!\!/_{A} \ \Rightarrow /\!\!/_{A\cdot B} \ \Rightarrow /\!\!/ \cdots /\!\!/_{A\cdot B\cdot\beta}\, M \Rightarrow N, p /\!\!/_{\varphi}\ \ \Rightarrow /\!\!/ \cdots /\!\!/_{\varphi\cdot\beta}\ \ p, M' \Rightarrow N'}\ (\mathrm{Rat})^*\ (n+1 \text{ times})}{/\!\!/_{A\cdot B\cdot\beta} M \Rightarrow N, p /\!\!/_{\varphi\cdot\beta}\ \ p, M' \Rightarrow N'}\ (\mathrm{New})^*\ (2n \text{ times})$$

• Case $C = \neg C_1$. Since $\overline{c}(C_1) < \overline{c}(\neg C_1)$, we apply the inductive hypothesis. Then, we simply apply (L$\neg$) and (R$\neg$). This is straightforward.

[4]This complexity measure will also be used later to prove Cut-admissibility 6.

• Case $C = C_1 \wedge C_2$. We can apply the inductive hypothesis because $\overline{c}(C_i) < \overline{c}(C_1 \wedge C_2)$ for $i \in \{1, 2\}$. Then, we apply admissibility of (IW) and use (L$\wedge$) and (R$\wedge$). This is straightforward.

• Case $C = \Box C_1$. Note that $\overline{c}(C_1) < \overline{c}(\Box C_1)$ and, for all $\beta' \cdot B_i \subseteq \beta$, $c(B_i, A \cdot B \cdot \beta') + c(B_i, A \wedge [A]B \cdot \beta') < \overline{c}(\Box C_1)$. This allows us to use the inductive hypothesis to obtain the following DHSs:

$$
\begin{array}{rl}
(1) & \sslash_{A \cdot B} \Rightarrow B_1 \sslash_{\varphi} B_1 \Rightarrow \\
(2) & \sslash_{A \cdot B \cdot B_1} \Rightarrow B_2 \sslash_{\varphi \cdot B_1} B_2 \Rightarrow \\
\vdots & \\
(n-1) & \sslash_{A \cdot B \cdots B_{n-2}} \Rightarrow B_{n-1} \sslash_{\varphi \cdots B_{n-2}} B_{n-1} \Rightarrow \\
(n) & \sslash_{A \cdot B \cdot \beta} C_1 \Rightarrow \sslash_{\varphi \cdot \beta} \Rightarrow C_1
\end{array}
$$

By admissibility of (IW), (EW) and (DW), from each (i) we get the following (i'):

$$
\begin{array}{ll}
(1)' & \sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid A \Rightarrow \sslash_{A} B \Rightarrow \sslash_{A \cdot B} \Rightarrow B_1 \sslash_{\varphi} B_1 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1 \\
(2)' & \sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid A \Rightarrow \sslash_{A} B \Rightarrow \sslash_{A \cdot B \cdot B_1} \Rightarrow B_2 \sslash_{\varphi} B_1 \Rightarrow \sslash_{\varphi \cdot B_1} B_2 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1 \\
\vdots & \\
(n)' & \sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid A \Rightarrow \sslash_{A} B \Rightarrow \sslash_{A \cdot B \cdot \beta} C_1 \Rightarrow \sslash_{\varphi} B_1 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1
\end{array}
$$

To apply (L$\Box_3$) with $\sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma'$, we need $n + 1$ premises, since the sequence of announcements here is $B \cdot B_1 \cdots B_{n-1}$. Hence, we also need the following DHS, that is derivable by Lemma 4.1:

$$
(0)' \quad \sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid A \Rightarrow \sslash_{A} B \Rightarrow B \sslash_{\varphi} B_1 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1
$$

We now apply (L$\Box_3$) to $(0)'$–$(n)'$ and continue the derivation as follows, where $X := \sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma'$:

$$
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X \mid A \Rightarrow A \sslash \cdots
\qquad
\dfrac{(0)' \quad (1)' \quad \cdots \quad (n)'}{X \mid A \Rightarrow \sslash_{A} B \Rightarrow \sslash_{\varphi} B_1 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1}\,(\mathrm{L}\Box_3)
}{\sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid A, [A]B \Rightarrow \sslash_{\varphi} B_1 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1}\,(\mathrm{L}[\cdot])
}{\sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid A \wedge [A]B \Rightarrow \sslash_{\varphi} B_1 \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1}\,(\mathrm{L}\wedge)
}{\sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid \Rightarrow \sslash_{\varphi} \Rightarrow \sslash \cdots \sslash_{\varphi \cdot \beta} \Rightarrow C_1}\,(\mathrm{Recall})^* \ (n \text{ times})
}{\sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma' \mid \sslash_{\varphi \cdot \beta} \Rightarrow C_1}\,(\mathrm{New})^* \ (n \text{ times})
}{\sslash_{A \cdot B \cdot \beta} \Box C_1, \Gamma \sslash_{\varphi \cdot \beta} \Gamma', \Box C_1}\,(\mathrm{R}\Box)
$$

• Case $C = [C_1]C_2$. We use the inductive hypothesis and admissibility of (DW). Note that the inductive hypothesis applies because $\overline{c}(C_1) < \overline{c}([C_1]C_2)$ and $c(C_2, A \cdot B \cdot \beta \cdot C_1) + c(C_2, A \wedge [A]B \cdot \beta \cdot C_1) < \overline{c}([C_1]C_2)$.

$$
\dfrac{
\dfrac{
\dfrac{\sslash_{A \cdot B \cdot \beta} \Gamma, C_1 \sslash_{\varphi \cdot \beta} C_1, \Gamma'}{\sslash_{A \cdot B \cdot \beta} \Gamma, C_1 \sslash_{\varphi \cdot \beta} C_1, \Gamma' \sslash_{\varphi \cdot \beta \cdot C_1} \Rightarrow C_2}\,(\mathrm{DW})
\qquad
\dfrac{\sslash_{A \cdot B \cdot \beta \cdot C_1} C_2 \Rightarrow \sslash_{\varphi \cdot \beta \cdot C_1} \Rightarrow C_2}{\sslash_{A \cdot B \cdot \beta} \Gamma \sslash_{\varphi \cdot \beta} C_1, \Gamma' \sslash_{A \cdot B \cdot \beta \cdot C_1} C_2 \Rightarrow \sslash_{\varphi \cdot \beta \cdot C_1} \Rightarrow C_2}\,(\mathrm{DW})
}{\sslash_{A \cdot B \cdot \beta} [C_1]C_2, \Gamma \sslash_{\varphi \cdot \beta} C_1, \Gamma' \sslash_{\varphi \cdot \beta \cdot C_1} \Rightarrow C_2}\,(\mathrm{L}[\cdot])
}{\sslash_{A \cdot B \cdot \beta} [C_1]C_2, \Gamma \sslash_{\varphi \cdot \beta} \Gamma', [C_1]C_2}\,(\mathrm{R}[\cdot])
$$

$\Box$

Note that Lemma 5.3 can be understood as a way of identifying dynamic sequents labelled by $A \cdot B$ and those labelled by $A \wedge [A]B$ (and more generally by $\alpha \cdot A \cdot B \cdot \beta$ and by $\alpha \cdot A[A]B \cdot \beta$). This result has a semantic counterpart. Indeed, it can be proved that, for all models $M = (W, \sim, V)$ and all formulas A, B, $W_{A \cdot B} = W_{A \wedge [A]B}$ (see for instance [4]).

Theorem 5.4 (Completeness)**.** *For all formulas $A \in \mathcal{L}_{PAL}$, if $\vDash_{PAL} A$ then $\vdash_{\boldsymbol{DHS}_{PAL}} \Rightarrow A$.*

Proof. Since **PAL** is complete w.r.t. the standard semantics of PAL, it is sufficient to demonstrate that all axioms of **PAL** are derivable in $\mathbf{DHS}_{PAL}$ and that the rules of modus ponens and necessitation are admissible. Modus ponens can be shown to be admissible using the cut-rule (see Section 6); whilst the necessitation rule can be shown to be admissible using external weakening. Derivations for the axioms of propositional logic and modal logic **S5** are obtained by direct (bottom-up) proof-search, see [8] for details. Derivations for the reduction axioms are also obtained by (bottom-up) proof search (except for the announcement composition axiom that we will consider separately). As an illustrative example, consider the axiom of announcement and knowledge, which we prove to be a theorem by means of the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \mid A \Rightarrow \sslash_A \Rightarrow B \qquad \dfrac{A \Rightarrow \sslash_A \Box B \Rightarrow \mid A \Rightarrow \sslash_A B \Rightarrow B}{A \Rightarrow \sslash_A \Box B \Rightarrow \mid A \sslash_A \Rightarrow B}\,(\mathrm{L}\Box_2)}{A, [A]\Box B \Rightarrow \mid A \Rightarrow \sslash_A \Rightarrow B}\,(\mathrm{L}[\cdot])}{A, [A]\Box B \Rightarrow \mid \Rightarrow [A]B}\,(\mathrm{R}[\cdot])}{A, [A]\Box B \Rightarrow \Box [A]B}\,(\mathrm{R}\Box)}{[A]\Box B \Rightarrow A \to \Box [A]B}\,(\mathrm{R}\to)}{\Rightarrow [A]\Box B \to (A \to \Box [A]B)}\,(\mathrm{R}\to)$$

and

$$\dfrac{\dfrac{A \Rightarrow A \sslash_A \Rightarrow \Box B \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \mid A \Rightarrow A \sslash_A \Rightarrow B \qquad A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \mid A \Rightarrow \sslash_A B \Rightarrow B}{A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \mid A, [A]B \Rightarrow \sslash_A \Rightarrow B}\,(\mathrm{L}[\cdot])}{A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \mid A \Rightarrow \sslash_A \Rightarrow B}\,(\mathrm{L}\Box_2)}{A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \mid \Rightarrow \sslash_A \Rightarrow B}\,(\mathrm{Recall})}{A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \mid \sslash_A \Rightarrow B}\,(\mathrm{New})}{A, \Box[A]B \Rightarrow \sslash_A \Rightarrow \Box B}\,(\mathrm{R}\Box)}{\dfrac{A, A \to \Box[A]B \Rightarrow \sslash_A \Rightarrow \Box B}{\dfrac{A \to \Box[A]B \Rightarrow [A]\Box B}{\Rightarrow (A \to \Box[A]B) \to [A]\Box B}\,(\mathrm{R}\to)}\,(\mathrm{R}[\cdot])}\,(\mathrm{L}\to)$$

We now move to the axiom of announcements composition $[A][B]C \leftrightarrow [A \wedge [A]B]C$. Providing a direct proof for that axiom schema requires Lemma 5.3 that demonstrates derivability of dynamic hypersequents $\sslash_{A \cdot B} \Rightarrow C \sslash_{A \wedge [A]B} C \Rightarrow$ and $\sslash_{A \cdot B} C \Rightarrow \sslash_{A \wedge [A]B} \Rightarrow C$, for all formulas C. We only show the derivation for one direction, whilst the other can be obtained in a similar way.

$$
\dfrac{\dfrac{A \Rightarrow A \mathbin{/\!/}_A B \Rightarrow \mathbin{/\!/}_{A\cdot B} \Rightarrow C \quad \dfrac{A, A \Rightarrow \mathbin{/\!/}_A B \Rightarrow B \mathbin{/\!/}_{A\cdot B} \Rightarrow C}{A \Rightarrow [A]B \mathbin{/\!/}_A B \Rightarrow \mathbin{/\!/}_{A\cdot B} \Rightarrow C}\,(\mathrm{R}[\cdot])}{A \Rightarrow A \wedge [A]B \mathbin{/\!/}_A B \Rightarrow \mathbin{/\!/}_{A\cdot B} \Rightarrow C}\,(\mathrm{R}\wedge) \quad \begin{array}{c} \vdots \text{ Lemma 5.3} \\ A \Rightarrow \mathbin{/\!/}_A B \Rightarrow \mathbin{/\!/}_{A\cdot B} \Rightarrow C \mathbin{/\!/}_{A\wedge[A]B} C \Rightarrow \end{array}}{\dfrac{A, [A \wedge [A]B]C \Rightarrow \mathbin{/\!/}_A B \Rightarrow \mathbin{/\!/}_{A\cdot B} \Rightarrow C}{\dfrac{A, [A \wedge [A]B]C \Rightarrow \mathbin{/\!/}_A \Rightarrow [B]C}{\dfrac{[A \wedge [A]B]C \Rightarrow [A][B]C}{\Rightarrow [A \wedge [A]B]C \to [A][B]C}\,(\mathrm{R}\to)}\,(\mathrm{R}[\cdot])}\,(\mathrm{R}[\cdot])}\,(\mathrm{L}[\cdot])
$$

□

6 Cut Admissibility

In this section we demonstrate that the cut-rule is admissible in the calculus $\mathbf{DHS}_{PAL}$[5]. We first prove the following lemma.

Lemma 6.1. *Propositional rules, modal rules, announcement rules, as well as (New), (Recall) and (Lat) permute down with respect to n applications of (Rat). Furthermore, any occurrence of (Rat) also permutes down with respect to n applications of (Rat) when the atom p that is removed in its conclusion is not active in either of these n applications of (Rat).*

Proof. Let us first consider the permutation with 1-premise rules, that is straightforward. We show the permutability of n consecutive applications of (Rat) with the rule (R[·]), while other cases can be treated similarly. In the following, (Rat)* denotes n applications of (Rat).

$$
\begin{array}{c}
\dfrac{M \Rightarrow N, p \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha A, M' \Rightarrow N', p \mathbin{/\!/}_{\alpha\cdot A} \Rightarrow B}{M \Rightarrow N, p \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha M' \Rightarrow N', p, [A]B}\,(\mathrm{R}[\cdot]) \\
\vdots\ (\mathrm{Rat})^* \\
M \Rightarrow N \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha M \Rightarrow N', p, [A]B \\[1em]
\downarrow \\[1em]
M \Rightarrow N, p \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha A, M' \Rightarrow N', p \mathbin{/\!/}_{\alpha\cdot A} \Rightarrow B \\
\vdots\ (\mathrm{Rat})^* \\
\dfrac{M \Rightarrow N \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha A, M' \Rightarrow N', p \mathbin{/\!/}_{\alpha\cdot A} \Rightarrow B}{M \Rightarrow N \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha M' \Rightarrow N', p, [A]B}\,(\mathrm{R}[\cdot])
\end{array}
$$

Let us now consider the permutation with n-premises rules (for $n \geq 2$). As a way of paradigmatic example, we show that rule $(\mathrm{L}\Box_3)$ permutes down with n applications of (Rat). Without loss of generality, let us assume the first premise has been obtained by n applications of (Rat). We then proceed as follows (where $\beta = B_1 \cdots B_{k-1}$):

$$
\dfrac{\begin{array}{c} \Gamma, p \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha \Gamma', p, B_1 \mid Y \mathbin{/\!/}_{\alpha\cdot\beta} \Box C, \Delta \\ \vdots\ (\mathrm{Rat})^* \\ \Gamma \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha \Gamma', p, B_1 \mid Y \mathbin{/\!/}_{\alpha\cdot\beta} \Box C, \Delta \end{array} \quad \cdots \quad \begin{array}{c} \vdots\ d \\ \Gamma \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha \Gamma', p \mathbin{/\!/}_{\alpha\cdot\beta} C \Rightarrow \mid Y \mathbin{/\!/}_{\alpha\cdot\beta} \Box C, \Delta \end{array}}{\Gamma \mathbin{/\!/} \cdots \mathbin{/\!/}_\alpha \Gamma', p \mid Y \mathbin{/\!/}_{\alpha\cdot\beta} \Box C, \Delta}\,(\mathrm{L}\Box_3)
$$

Here, we use hp-admissibility of weakening on all the premises but the leftmost one to obtain the required premises and apply $(\mathrm{L}\Box_3)$ before applying (Rat) n times.

[5]Hereafter, for clarity and succinctness, we omit explicit notations of derivations of premises $\vdots\ (d)$.

$$\dfrac{\Gamma, p \mathbin{/\!\!/} \cdots \mathbin{/\!\!/}_\alpha \Gamma', p, B_1 \ \mid\ Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta \qquad \cdots \qquad \begin{array}{c} \vdots\, d' \\ \Gamma, p \mathbin{/\!\!/} \cdots \mathbin{/\!\!/}_\alpha \Gamma', p \mathbin{/\!\!/}_{\alpha\cdot\beta} C \Rightarrow\ \mid\ Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta \end{array}}{\dfrac{\Gamma, p \mathbin{/\!\!/} \cdots \mathbin{/\!\!/}_\alpha \Gamma', p \ \mid\ Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta}{\begin{array}{c} \vdots\, (\mathrm{Rat})^* \\ \Gamma \mathbin{/\!\!/} \cdots \mathbin{/\!\!/}_\alpha \Gamma', p \ \mid\ Y \mathbin{/\!\!/}_{\alpha\cdot\beta} \Box C, \Delta \end{array}}} \ (\mathrm{L}\Box_3)$$

□

Lemma 6.2. *Propositional rules, modal rules, announcement rules, as well as (New), (Recall) and (Rat) permute down with respect to n applications of (Lat).*

Furthermore, any occurrence of (Lat) also permutes down with respect to n applications of (Lat) when the atom p that is removed in its conclusion is not active in either of these n applications of (Lat).

Proof. The proof is similar to that for (Rat). □

Theorem 6.3. *The rule of* Cut *is admissible.*

$$\dfrac{\begin{array}{c} \vdots\, d_1 \\ G \ \mid\ X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A \end{array} \qquad \begin{array}{c} \vdots\, d_2 \\ H \ \mid\ Y \mathbin{/\!\!/}_\alpha A, P \Rightarrow Q \end{array}}{G \ \mid\ H \ \mid\ X \otimes Y \mathbin{/\!\!/}_\alpha M, P \Rightarrow N, Q} \ \textit{(Cut)}$$

Proof. The proof proceeds by main induction on the DHS-complexity of the cut-formula (see Definition 5.2), and secondary induction on the sum of the heights of the derivations d_1 and d_2 of each premise. We distinguish cases on the last rule applied on the left premise.[6]

– If $G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A$ is an axiom, then either the conclusion is an axiom as well, or the cut can be replaced by several applications of the rules (IW), (EW) and (DW) on $H \mid Y \mathbin{/\!\!/}_\alpha p, P \Rightarrow Q$ so to obtain the conclusion.

– If $G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A$ has been obtained by a rule $\mathcal{R}$ where the formula A is not principal, then we apply the inductive hypothesis to the premise(s) of $\mathcal{R}$ and then $\mathcal{R}$ again.[7]

– If $G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, A$ has been obtained by a rule $\mathcal{R}$ where the formula A is principal, then we distinguish the following subcases. (i) The rule $\mathcal{R}$ is a propositional rule, (ii) the rule $\mathcal{R}$ is a modal rule, (iii) the rule $\mathcal{R}$ is an announcement rule, (iv) the rule $\mathcal{R}$ is a dynamic rule.

(i) As an example, we consider the case where $\mathcal{R}$ is the rule (R¬), we have:

$$\dfrac{\dfrac{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N}{G \mid X \mathbin{/\!\!/}_\alpha M \Rightarrow N, \neg A}\ (\mathrm{R}\neg) \qquad H \mid Y \mathbin{/\!\!/}_\alpha \neg A, P \Rightarrow Q}{G \mid H \mid X \otimes Y \mathbin{/\!\!/}_\alpha M, P \Rightarrow N, Q} \ (\mathrm{Cut})$$

Here, we consider the right premise of the Cut. We first apply hp-invertibility of (L¬) to obtain $H \mid Y \mathbin{/\!\!/}_\alpha P \Rightarrow Q, A$ and then cut A, where this application of (Cut) is admissible by induction on the DHS-complexity of A:

$$\dfrac{G \mid X \mathbin{/\!\!/}_\alpha A, M \Rightarrow N \qquad H \mid Y \mathbin{/\!\!/}_\alpha P \Rightarrow Q, A}{G \mid H \mid X \otimes Y \mathbin{/\!\!/}_\alpha M, P \Rightarrow N, Q} \ (\mathrm{Cut})$$

[6]Note that in this proof, we denote by $\mathcal{R}^*$ any finitely many applications of rule $\mathcal{R}$.

[7]Note that, in some cases, we might need to apply another version of the rule, *e.g.* (L$\Box_2$) instead of (L$\Box_3$) or (R[·])$_1$ instead of (R[·])$_2$, or no rule at all if $\mathcal{R}$ is (New).

(ii) Consider the case where $\mathcal{R}$ is the rule (R$\Box$). We then need to look at the right-hand premise $H \mid Y \sslash_\alpha \Box A, P \Rightarrow Q$. If it is an axiom, then the conclusion is an axiom as well. If it has been obtained by a rule $\mathcal{R}'$ where $\Box A$ is not the principal formula, we use the inductive hypothesis – with induction on the sum of the heights – and then apply $\mathcal{R}'$ again. If finally $H \mid Y \sslash_\alpha \Box A, P \Rightarrow Q$ has been obtained by a rule $\mathcal{R}'$ which is either (L$\Box_1$) or (L$\Box_2$) and has $\Box A$ as principal formula, we apply the inductive hypothesis twice, to cut $\Box A$ and then A, with first induction of the sum of the heights and second on the DHS-complexity of the cut-formula. We only analyse the case of the rule (L$\Box_2$), whilst that of (L$\Box_1$) is dealt with analogously. We have the following situation:

$$\dfrac{\dfrac{G \mid X \sslash_\alpha \Gamma \mid \sslash_\alpha \Rightarrow A}{G \mid X \sslash_\alpha \Gamma, \Box A}\,(\mathrm{R}\Box) \qquad \dfrac{H \mid Y \sslash_\alpha \Box A, \Delta \mid Z \sslash_\alpha A, \Lambda}{H \mid Y \sslash_\alpha \Box A, \Delta \mid Z \sslash_\alpha \Lambda}\,(\mathrm{L}\Box_2)}{G \mid H \mid X \otimes Y \sslash_\alpha \Gamma\Delta \mid Z \sslash_\alpha \Lambda}\,(\mathrm{Cut})$$

We first apply (Cut) on the right-hand premise and on $G \mid X \sslash_\alpha \Gamma, \Box A$. This application of (Cut) is admissible by inductive hypothesis on the sum of the heights:

$$\dfrac{G \mid X \sslash_\alpha \Gamma, \Box A \qquad H \mid Y \sslash_\alpha \Box A, \Delta \mid Z \sslash_\alpha A, \Lambda}{G \mid H \mid X \otimes Y \sslash_\alpha \Gamma\Delta \mid Z \sslash_\alpha A, \Lambda}\,(\mathrm{Cut})$$

We then cut the remaining A, with an application of (Cut) that is admissible by induction on the DHS-complexity of the cut-formula. Eventually, the desired conclusion is obtained by several applications of the contraction rules as well as (Merge):

$$\dfrac{\dfrac{G \mid X \sslash_\alpha \Gamma \mid \sslash_\alpha \Rightarrow A \qquad G \mid H \mid X \otimes Y \sslash_\alpha \Gamma\Delta \mid Z \sslash_\alpha A, \Lambda}{G \mid G \mid H \mid X \otimes Y \sslash_\alpha \Gamma\Delta \mid X \sslash_\alpha \Gamma \mid Z \sslash_\alpha \Lambda}\,(\mathrm{Cut})}{G \mid H \mid X \otimes Y \sslash_\alpha \Gamma\Delta \mid Z \sslash_\alpha \Lambda}\,(\mathrm{Merge})^*{+}(\mathrm{C})^*$$

Suppose now that the rule $\mathcal{R}'$ is (L$\Box_3$) and the cut-formula is principal in (L$\Box_3$). For the sake of simplicity, we show how to proceed with an example where only three premises of the rule (L$\Box_3$) occur. The generalisation to n premises can be found in the Appendix.[8]

$$\dfrac{\dfrac{G \mid X \sslash_{B\cdot C} \Gamma \mid \sslash_{B\cdot C} \Rightarrow A}{G \mid X \sslash_{B\cdot C} \Gamma, \Box A}\,(\mathrm{R}\Box) \qquad \dfrac{\begin{matrix}\vdots\,(a)\\ \overline{H} \mid \Lambda, B\end{matrix} \qquad \begin{matrix}\vdots\,(b)\\ \overline{H} \mid \Lambda \sslash_B \Rightarrow C\end{matrix} \qquad \begin{matrix}\vdots\,(c)\\ \overline{H} \mid \Lambda \sslash_{B\cdot C} A \Rightarrow\end{matrix}}{H \mid Y \sslash_{B\cdot C} \Box A, \Delta \mid \Lambda}\,(\mathrm{L}\Box_3)}{G \mid H \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda}\,(\mathrm{Cut})$$

We first work with premise (c) in the following way:

$$\dfrac{\dfrac{\dfrac{\dfrac{G \mid X \sslash_{B\cdot C} \Gamma \mid \sslash_{B\cdot C} \Rightarrow A \qquad \dfrac{G \mid X \sslash_{B\cdot C} \Gamma, \Box A \qquad \begin{matrix}\vdots\,(c)\\ H \mid Y \sslash_{B\cdot C} \Box A, \Delta \mid \Lambda \sslash_{B\cdot C} A \Rightarrow\end{matrix}}{G \mid H \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda \sslash_{B\cdot C} A \Rightarrow}\,(\mathrm{Cut})}{G \mid G \mid H \mid X \sslash_{B\cdot C} \Gamma \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda \sslash_{B\cdot C} \Rightarrow}\,(\mathrm{Cut})}{G \mid H \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda \sslash_{B\cdot C} \Rightarrow}\,(\mathrm{Merge})^*{+}(\mathrm{C})^*}{G \mid H \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda \sslash_B C \Rightarrow \sslash_{B\cdot C} \Rightarrow}\,(\mathrm{DW})$$
$$\dfrac{G \mid H \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda \sslash_B C \Rightarrow \sslash_{B\cdot C} \Rightarrow}{G \mid H \mid X \otimes Y \sslash_{B\cdot C} \Gamma\Delta \mid \Lambda \sslash_B C \Rightarrow}\,(\mathrm{New}^A)$$

[8]Here, $\overline{H} := H \mid Y \sslash_{B\cdot C} \Box A, \Delta$.

where the first cut is admissible by induction on the sum of the heights, whilst the second by induction on DHS-complexity of the cut-formula. Then we re-use the previously obtained DHS in order to apply the same method to premise (b):

$$
\dfrac{\dfrac{\dfrac{\dfrac{G \mid H \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda \;/\!\!/_B\, C \Rightarrow \qquad \dfrac{\begin{matrix}\vdots\,(b)\\ G \mid X /\!\!/_{B\cdot C}\, \Gamma, \Box A \qquad H \mid Y /\!\!/_{B\cdot C}\, \Box A, \Delta \mid \Lambda /\!\!/_B \;\Rightarrow C\end{matrix}}{G \mid H \mid X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda /\!\!/_B \;\Rightarrow C}\;(\mathrm{Cut})}{G \mid G \mid H \mid H \mid X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda\Lambda /\!\!/_B \;\Rightarrow}\;(\mathrm{Cut})}{G \mid H \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda /\!\!/_B \;\Rightarrow}\;(\mathrm{Merge})^*{+}(\mathrm{C})^*}{G \mid H \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; B, \Lambda \;\Rightarrow /\!\!/_B \;\Rightarrow}\;(\mathrm{IW})}{G \mid H \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; B, \Lambda}\;(\mathrm{New}^A)
$$

where the first cut is admissible by induction on the sum of the heights, whilst the second by induction of DHS-complexity of the cut-formula. Finally, we reach the desired conclusion with the following derivation:

$$
\dfrac{\dfrac{G \mid H \mid X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; B, \Lambda \qquad \dfrac{\begin{matrix}\vdots\,(a)\\ G \mid X /\!\!/_{B\cdot C}\, \Gamma, \Box A \qquad H \mid Y /\!\!/_{B\cdot C}\, \Box A, \Delta \mid \Lambda, B\end{matrix}}{G \mid H \mid X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda, B}\;(\mathrm{Cut})}{G \mid G \mid H \mid H \mid X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda\Lambda}\;(\mathrm{Cut})}{G \mid H \;\mid\; X \otimes Y /\!\!/_{B\cdot C}\, \Gamma\Delta \;\mid\; \Lambda}\;(\mathrm{Merge})^*{+}(\mathrm{C})^*
$$

where the first cut is admissible by induction on the sum of the heights, whilst the second by induction of DHS-complexity of the cut-formula.

(iii) Consider the case where $\mathcal{R}$ is the rule $(\mathrm{R}[\cdot])_2$.[9] We then need to look at the right-hand premise $H \mid Y /\!\!/_\alpha [A]B, P \Rightarrow Q$. If it is an axiom, or it has been obtained by a rule $\mathcal{R}'$ where $[A]B$ is not the principal formula, then the procedure is the standard one. Consider then the case where $\mathcal{R}'$ is the rule $(\mathrm{L}[\cdot])$. We have the following situation:[10]

$$
\dfrac{\dfrac{G \mid X /\!\!/_\alpha A, \Gamma /\!\!/_{\alpha\cdot A} \Rightarrow B}{G \mid X /\!\!/_\alpha \Gamma, [A]B}\;(\mathrm{R}[\cdot])_2 \qquad \dfrac{H \mid Y /\!\!/_\alpha \Delta, A \qquad H \mid Y /\!\!/_\alpha \Delta /\!\!/_{\alpha\cdot A} B \Rightarrow}{H \mid Y /\!\!/_\alpha [A]B, \Delta}\;(\mathrm{L}[\cdot])_2}{G \mid H \mid X \otimes Y /\!\!/_\alpha \Gamma\Delta}\;(\mathrm{Cut})
$$

We proceed in the following way:

$$
\dfrac{\dfrac{H \mid Y /\!\!/_\alpha \Delta, A \qquad \dfrac{\dfrac{G \mid X /\!\!/_\alpha A, \Gamma /\!\!/_{\alpha\cdot A} \Rightarrow B \qquad H \mid Y /\!\!/_\alpha \Delta /\!\!/_{\alpha\cdot A} B \Rightarrow}{G \mid H \mid X \otimes Y /\!\!/_\alpha A, \Gamma\Delta /\!\!/_{\alpha\cdot A} \Rightarrow}\;(\mathrm{Cut})}{G \mid H \mid X \otimes Y /\!\!/_\alpha A, \Gamma\Delta}\;(\mathrm{New}^A)}{G \mid H \mid H \mid X \otimes Y \otimes Y /\!\!/_\alpha \Gamma\Delta\Delta}\;(\mathrm{Cut})}{G \mid H \mid X \otimes Y /\!\!/_\alpha \Gamma\Delta}\;(\mathrm{Merge})^*{+}\ (\mathrm{C})^*
$$

[9]The case where $\mathcal{R}$ is $(\mathrm{R}[\cdot])_1$ can be treated in an analogous, though simpler manner.

[10]We consider the case where $\mathcal{R}'$ is $(\mathrm{L}[\cdot])_2$; the case where $\mathcal{R}'$ is $(\mathrm{L}[\cdot])_1$ can be treated analogously.

where the first cut is admissible by induction on the sum of the heights, whilst the second is admissible by induction on the DHS-complexity of the cut-formula.

(iv) Consider the case where $\mathcal{R}$ is the rule (Rat) – the cases of the other dynamic rules can be treated straightforwardly. We have the following situation, where $\alpha = A_1 \cdots A_n$ and for all $1 \leq i \leq n$, $\alpha_i = A_1 \cdots A_i$:

$$\dfrac{\dfrac{\begin{matrix} \vdots\, d_1' \\ G \mid X \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1}, p \,/\!\!/_{\alpha_n} \Gamma_n, p \end{matrix}}{G \mid X \,/\!\!/_{\alpha_{n-1}} \Gamma_{n-1} \,/\!\!/_{\alpha_n} \Gamma_n, p}\,(\mathrm{Rat}) \qquad \begin{matrix} \vdots\, d_2 \\ H \mid Y \,/\!\!/_{\alpha_n} p, \Delta_n \end{matrix}}{G \mid H \mid X \otimes Y \,/\!\!/ \cdots /\!\!/_{\alpha_n} \Gamma_n \Delta_n}\,(\mathrm{Cut})$$

Here we go up derivation d_1' until we find an application of a rule $\mathcal{R}''$ different from (Rat) on that atom p. There are two possibilities:

(1) After m applications of (Rat), we indeed find a rule $\mathcal{R}''$ different from (Rat) on that atom p. Thanks to Lemma 6.1, we can permute $\mathcal{R}''$ down with the m applications of (Rat). We thus obtain a DHS that we can use, together with $H \mid Y \,/\!\!/_{\alpha_n}\ p, \Delta_n$ to cut atom p. This cut is admissible by induction on the sum of heights. We then apply rule $\mathcal{R}''$ again[11], to obtain the desired result.

(2) Alternatively, there is no such rule because $G \mid X \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1}, p \,/\!\!/_{\alpha} \Gamma_n, p$ is an axiom.

$$\dfrac{\dfrac{\begin{matrix} G \mid X' \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1}, p \,/\!\!/_{\alpha_n} \Gamma_n, p \\ \vdots\, (\mathrm{Rat})^* \\ G \mid X \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1}, p \,/\!\!/_{\alpha_n} \Gamma_n, p \end{matrix}}{G \mid X \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1} \,/\!\!/_{\alpha_n} \Gamma_n, p}\,(\mathrm{Rat}) \qquad \begin{matrix} \vdots\, d_2 \\ H \mid Y \,/\!\!/_{\alpha_{n-1}} \Delta_{n-1} \,/\!\!/_{\alpha_n} p, \Delta_n \end{matrix}}{G \mid H \mid X \otimes Y \,/\!\!/ \cdots /\!\!/_{\alpha_n} \Gamma_n \Delta_n}\,(\mathrm{Cut})$$

We distinguish three cases, that we analyse in the following:

($2a$) p is principal in Γ_n;

($2b$) p is principal in another dynamic sequent Γ_i (for $1 \leq i < n$);

($2c$) p is principal in none of them.

($2a$) We have $G \mid X' \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1}, p \,/\!\!/_{\alpha_n} p, \Gamma_n, p$ so we start from the right-hand premise $H \mid Y \,/\!\!/_{\alpha_{n-1}} \Delta_{n-1} \,/\!\!/_{\alpha_n} p, \Delta_n$ and obtain the conclusion by admissibility of (IW), (EW) and (DW).

($2b$) Let $1 \leq i < n$. The axiom we have is $G \mid X' \,/\!\!/ \cdots /\!\!/_{\alpha_i} p, \Gamma_i, p \,/\!\!/ \cdots /\!\!/_{\alpha_n} \Gamma_n, p$. Here, we need to look at the right-hand premise and distinguish the following cases:

($2bi$) If it is an axiom, where p is principal in Δ_n, we start from the left-hand premise $G \mid X \,/\!\!/ \cdots /\!\!/_{\alpha_{n-1}} \Gamma_{n-1} \,/\!\!/_{\alpha_n} \Gamma_n, p$ and obtain the conclusion by admissibility of the weakening rules.

[11] As already mentioned, in some cases, we might need to apply another version of the rule – *e.g.* (L$\Box_2$) instead of (L$\Box_3$) – or no rule at all – in case of (New).

($2bii$) If it is an axiom with a principal formula different from p in Δ_n, then the conclusion is also an axiom.

($2biii$) If it has been obtained by a rule $\mathcal{R}'$ that does not operate on p, then we consider the premise of $\mathcal{R}'$, and we apply a cut between such premise and $G \mid X \mathbin{/\!\!/} \cdots \mathbin{/\!\!/}_{\alpha_n} \Gamma_n, p$. The cut is admissible by induction on the sum of heights. We then apply $\mathcal{R}'$ again to obtain the desired conclusion.

($2biv$) If it is the result of l applications of (Lat), then, as before, we go up the derivation until we find a rule other than (Lat) on the atom p. There are two possibilities that are analogous to (1) and (2) above. Let us call them ($1'$) and ($2'$), respectively. As for ($1'$), we proceed analogously to (1). As for ($2'$), we have the situation where after l applications of (Lat), we find an axiom. We need to distinguish cases again.

($2'a$) p is principal in Δ_n, we proceed as in ($2a$);

($2'b$) p is principal in a dynamic sequent labelled by the same sequence α_i as the axiom in the left-hand premise. Then the conclusion is an axiom.

($2'c$) p is principal in a dynamic sequent with a label α_j different from α_i, *i.e.* where $i \neq j$. Without loss of generality, let us consider $i < j$. In such a case, we start from the axioms, removing the p in Γ_k (resp. Δ_k) for all $k > j$. We apply the inductive hypothesis on the sum of heights, and then apply (Lat) or (Rat) as many times as needed to obtain the conclusion. Below is an example.

$$\dfrac{\dfrac{\dfrac{\dfrac{p, \Gamma_1, p \mathbin{/\!\!/}_A \Gamma_2, p \mathbin{/\!\!/}_{A\cdot B} \Gamma_3, p \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4, p}{p, \Gamma_1 \mathbin{/\!\!/}_A \Gamma_2, p \mathbin{/\!\!/}_{A\cdot B} \Gamma_3, p \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4, p}\,(\mathrm{Rat})}{p, \Gamma_1 \mathbin{/\!\!/}_A \Gamma_2 \mathbin{/\!\!/}_{A\cdot B} \Gamma_3, p \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4, p}\,(\mathrm{Rat})}{p, \Gamma_1 \mathbin{/\!\!/}_A \Gamma_2 \mathbin{/\!\!/}_{A\cdot B} \Gamma_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4, p}\,(\mathrm{Rat}) \quad \dfrac{\dfrac{\Delta_1 \mathbin{/\!\!/}_A p, \Delta_2, p \mathbin{/\!\!/}_{A\cdot B} p, \Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} p, \Delta_4}{\Delta_1 \mathbin{/\!\!/}_A \Delta_2, p \mathbin{/\!\!/}_{A\cdot B} p, \Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} p, \Delta_4}\,(\mathrm{Lat})}{\Delta_1 \mathbin{/\!\!/}_A \Delta_2, p \mathbin{/\!\!/}_{A\cdot B} \Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} p, \Delta_4}\,(\mathrm{Lat})}{p, \Gamma_1\Delta_1 \mathbin{/\!\!/}_A \Gamma_2\Delta_2, p \mathbin{/\!\!/}_{A\cdot B} \Gamma_3\Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4\Delta_4}\,(\mathrm{Cut})$$

We proceed as explained above, where $i = 1$ and $j = 2$: we start from the axioms wherein we remove p from $\Gamma_3, \Gamma_4, \Delta_3$ and Δ_4. We then cut p in Γ_2 and Δ_2 and apply (Rat) to remove the atom p that still needs to be removed in Γ_1:

$$\dfrac{\dfrac{p, \Gamma_1, p \mathbin{/\!\!/}_A \Gamma_2, p \mathbin{/\!\!/}_{A\cdot B} \Gamma_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4 \qquad \Delta_1 \mathbin{/\!\!/}_A p, \Delta_2, p \mathbin{/\!\!/}_{A\cdot B} \Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} \Delta_4}{p, \Gamma_1\Delta_1, p \mathbin{/\!\!/}_A \Gamma_2\Delta_2, p \mathbin{/\!\!/}_{A\cdot B} \Gamma_3\Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4\Delta_4}\,(\mathrm{Cut})}{p, \Gamma_1\Delta_1 \mathbin{/\!\!/}_A \Gamma_2\Delta_2, p \mathbin{/\!\!/}_{A\cdot B} \Gamma_3\Delta_3 \mathbin{/\!\!/}_{A\cdot B\cdot C} \Gamma_4\Delta_4}\,(\mathrm{Rat})$$

The method generalises naturally: when (Rat) has been applied m times, and (Lat) l times, if $i < j$, then $m > l$. Therefore, after having applied the inductive hypothesis, we need to apply $(m - l)$ times (Rat) to remove the remaining atoms p in sequents Γ_k for all $i \leq k < j$.

($2c$) Then the conclusion is an axiom so we can just start from it.

□

Corollary 6.4. ***DHS**$_{PAL}$ enjoys the subformula property: all formulas occurring in a **DHS**$_{PAL}$-derivation ending with the dynamic hypersequent G are subformulas of one of the formulas occurring in G or of one of the formulas occurring as announcement in G.*

Proof. Straightforward from the rules of the calculus $\mathbf{DHS}_{PAL}$ and Theorem 6.3. □

7 Conclusion and future work

In this paper, we introduced a syntactic method for extending hypersequents to their dynamic counterparts. On the basis of this method, we developed a calculus for Public Announcement Logic, one of the most prominent and widely studied dynamic logics. The calculus was shown to satisfy several fundamental properties, including soundness, completeness, and cut-elimination. These results are of intrinsic interest, but they also suggest a number of directions for further research. First, it is reasonable to expect that additional metatheoretical results could be established for the calculus of PAL, such as interpolability, decidability, and the construction of countermodels. Second, the framework of dynamic hypersequents appears well suited for the systematic development of sequent calculi for a broader class of dynamic epistemic logics involving different dynamic operators.

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Appendix

Let us explain the general method to prove admissibility of (Cut) when the left-hand premise has been obtained by an application of (R$\Box$) and the right-hand premise has been obtained by an application of (L$\Box_3$) with k premises, for $\beta = B_1 \cdots B_{k-1}$. Let $(1), \cdots, (k)$ denote each premise of (L$\Box_3$), let (i) denote $G \mid X \sslash_\alpha \Gamma \mid \sslash_\alpha \Rightarrow A$, and (ii) $G \mid X \sslash_\alpha \Gamma, \Box A$. The derivation has this shape[12]:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ G \mid X \sslash_{\alpha\cdot\beta} \Gamma \mid \sslash_\alpha \Rightarrow A\end{matrix}}{G \mid X \sslash_{\alpha\cdot\beta} \Gamma, \Box A}\,(\mathrm{R}\Box) \qquad \dfrac{\begin{matrix}\vdots\,(1)\\ \overline{H} \mid Z \sslash_\alpha \Lambda, B_1\end{matrix} \quad \cdots \quad \begin{matrix}\vdots\,(k)\\ \overline{H} \mid Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot\beta} A \Rightarrow\end{matrix}}{H \mid Y \sslash_{\alpha\cdot\beta} \Box A, \Delta \ \mid\ Z \sslash_\alpha \Lambda}\,(\mathrm{L}\Box_3)}{G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \mid Z \sslash_\alpha \Lambda}\,(\mathrm{Cut})$$

We first apply k times (Cut) between (ii) and each (j) premise (for $1 \le j \le k$), where each of these applications is admissible, by induction on the sum of heights. We denote by $(I_1), \cdots (I_k)$ the DHSs obtained this way:

$$\dfrac{\begin{matrix}\vdots\,(ii)\\ G \mid X \sslash_{\alpha\cdot\beta} \Gamma, \Box A\end{matrix} \qquad \begin{matrix}\vdots\,(1)\\ H \mid Y \sslash_{\alpha\cdot\beta} \Box A, \Delta \ \mid\ Z \sslash_\alpha \Lambda, B_1\end{matrix}}{(I_1): \quad G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda, B_1}\,(\mathrm{Cut})$$

$$\vdots$$

$$\dfrac{\begin{matrix}\vdots\,(ii)\\ G \mid X \sslash_{\alpha\cdot\beta} \Gamma, \Box A\end{matrix} \qquad \begin{matrix}\vdots\,(k)\\ H \mid Y \sslash_{\alpha\cdot\beta} \Box A, \Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot\beta} A \Rightarrow\end{matrix}}{(I_k): \quad G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot\beta} A \Rightarrow}\,(\mathrm{Cut})$$

We now apply the inductive hypothesis between (i) and (I_k), with induction on the DHS-complexity of the cut-formula. We then use admissibility of (Merge), contraction rules, (DW) and (New^A) as follows, to obtain $(J_k): G \ \mid\ H \ \mid\ X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot B_1 \cdots B_{k-2}} B_{k-1} \Rightarrow$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,(i)\\ G \ \mid\ X \sslash_{\alpha\cdot\beta} \Gamma \ \mid\ \sslash_{\alpha\cdot\beta} \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots\,(I_k)\\ G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot\beta} A \Rightarrow\end{matrix}}{G \mid G \mid H \mid X \mid X \otimes Y \sslash_{\alpha\cdot\beta} \ \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot\beta} \Rightarrow}\,(\mathrm{Cut})}{G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \ \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot\beta} \Rightarrow}\,(\mathrm{Merge})^* + (\mathrm{C})^*}{G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \ \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot B_1\cdots B_{k-2}} B_{k-1} \Rightarrow \sslash_{\alpha\cdot\beta} \Rightarrow}\,(\mathrm{DW})}{(J_k): \quad G \ \mid\ H \ \mid\ X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot B_1\cdots B_{k-2}} B_{k-1} \Rightarrow}\,(\mathrm{New}^A)$$

We now take (I_{k-1}) and (J_k) and follow the same procedure: we apply the inductive hypothesis to cut B_{k-1}, and then use admissibility of the structural rules, thereby obtaining DHS $(J_{k-1}) : G \mid H \mid X \otimes Y \sslash_{\alpha\cdot\beta} \Gamma\Delta \ \mid\ Z \sslash_\alpha \Lambda \sslash_{\alpha\cdot B_1\cdots B_{k-3}} B_{k-2} \Rightarrow$. The induction is here on the DHS-complexity of the cut-formula. The derivation proceeds as follows[13]:

[12] Here $\overline{H} := H \ \mid\ Y \sslash_{\alpha\cdot\beta} \Box A, \Delta$.

[13] Here $\beta' := B_1 \cdots B_{k-2}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}\vdots\,(I_{k-1}) \\ G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda \mathbin{/\!\!/_{\alpha\cdot\beta'}} \Rightarrow B_{k-1}\end{matrix} \qquad \begin{matrix}\vdots\,(J_k) \\ \cdots \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda \mathbin{/\!\!/_{\alpha\cdot\beta'}} B_{k-1} \Rightarrow\end{matrix}}{G \mid G \mid H \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \otimes Z \mathbin{/\!\!/_{\alpha}} \Lambda\Lambda \mathbin{/\!\!/_{\alpha\cdot\beta'}} \Rightarrow}\,(\mathrm{Cut})}{G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda \mathbin{/\!\!/_{\alpha\cdot\beta'}} \Rightarrow}\,(\mathrm{Merge})^*+(\mathrm{C})^*}{G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda \mathbin{/\!\!/_{\alpha\cdot B_1\cdots B_{k-3}}} B_{k-2} \Rightarrow \mathbin{/\!\!/_{\alpha\cdot\beta'}} \Rightarrow}\,(\mathrm{DW})}{(J_{k-1}):\quad G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda \mathbin{/\!\!/_{\alpha\cdot B_1\cdots B_{k-3}}} B_{k-2} \Rightarrow}\,(\mathrm{New}^A)$$

We proceed this way until we obtain $(J_2) : G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} B_1, \Lambda$. Then, we use it together with (I_1) to apply (Cut), which application is admissible by induction on the DHS-complexity of the cut-formula. We conclude with the admissibility of the structural rules:

$$\cfrac{\cfrac{\begin{matrix}\vdots\,(I_1) \\ G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda, B_1\end{matrix} \qquad \begin{matrix}\vdots\,(J_2) \\ G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} B_1, \Lambda\end{matrix}}{G \mid G \mid H \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \otimes Z \mathbin{/\!\!/_{\alpha}} \Lambda\Lambda}\,(\mathrm{Cut})}{G \mid H \mid X \otimes Y \mathbin{/\!\!/_{\alpha\cdot\beta}} \Gamma\Delta \mid Z \mathbin{/\!\!/_{\alpha}} \Lambda}\,(\mathrm{Merge})^*+(\mathrm{C})^*$$