

CLASSIFICATIONS OF DIMONOIDS WITH AT MOST THREE ELEMENTS

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ABSTRACT. In this paper, we present complete classifications, up to isomorphism, of all two-element dimonoids, all commutative three-element dimonoids, and all abelian three-element dimonoids. We show that, up to isomorphism, there exist exactly 8 two-element dimonoids, of which 3 are commutative. Among these, 4 are abelian, and the remaining nonabelian dimonoids form 2 pairs of dual dimonoids. Furthermore, there are exactly 5 pairwise nonisomorphic trivial dimonoids of order 2. For dimonoids of order 3, we prove that there are precisely 14 pairwise nonisomorphic commutative dimonoids, including 12 trivial dimonoids and a single pair of nonabelian nontrivial dual dimonoids. We also establish that, up to isomorphism, there are 17 abelian dimonoids of order 3, consisting of 12 trivial commutative dimonoids and 5 noncommutative nontrivial ones. In addition, we demonstrate the existence of at least 26 pairwise nonisomorphic nonabelian noncommutative dimonoids of order 3. Among them, there are exactly 6 pairs of trivial dual dimonoids and at least 7 pairs of nontrivial dual dimonoids.

INTRODUCTION

The notions of a dialgebra and a dimonoid were introduced by J.-L. Loday [14]. A *dimonoid* is an algebraic structure $(D, \dashv, \vdash)$ consisting of a set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \quad (D_1)$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \quad (D_2)$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z). \quad (D_3)$$

Each semigroup $(D, \dashv)$ can naturally be regarded as a dimonoid $(D, \dashv, \dashv)$, referred to as the *trivial* dimonoid and denoted simply by D . In this way, dimonoids generalize semigroups. Dimonoids also admit linear analogues known as dialgebras. A *dialgebra* is a vector space over a field equipped with two bilinear associative binary operations satisfying the axioms of a dimonoid. Consequently, many results concerning dimonoids have direct applications in dialgebra theory [1, 3, 14, 15, 19]. In recent years, dimonoids have become standard tools in the study of various structures, particularly in the theory of Leibniz algebras. Notably, T. Pirashvili [16] introduced the concept of a duplex, a generalization of dimonoids, and constructed the free duplex. The properties of free dimonoids were employed in [14] to characterize free dialgebras and to study their cohomologies. In [13], the notion of a dimonoid was used to define and investigate one-sided dirings. Furthermore, dimonoids are closely related to restrictive bisemigroups [18] and doppelsemigroups [6, 7, 8, 10, 11, 33].

One of the earliest foundational results on dimonoids is due to Loday [14], who provided a description of the absolutely free dimonoid generated by a given set. A wide range of classes of dimonoids have been systematically investigated by Anatolii Zhuchok and Yurii Zhuchok. In [25], the independence of the dimonoid axioms was established. Commutative, free commutative, and free abelian dimonoids were studied in [20], [21], and [39], respectively. The structure of dibands of subdimonoids and semilattice decompositions of dimonoids was explored in [22, 24]. Free rectangular dimonoids, as well as free normal and free (lr, rr) -dibands, were constructed in [23], [27], and [28], respectively. Free abelian dibands and some of their properties were studied in [41, 42]. The least semilattice congruence on free dimonoids was described in [26]. Free products of dimonoids and relatively free dimonoids were the focus of several works, including [30, 31, 34, 36]. Moreover, the free left n -nilpotent and free left n -dinilpotent dimonoids were constructed in [29, 32]. Representations of ordered dimonoids via binary relations were examined in [37]. Significant contributions to the theory of endomorphisms and automorphisms in the context of dimonoids were made by Y. Zhuchok in [38, 40, 43].

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In [9], we studied the properties of dual dimonoids and, within the class of noncommutative dimonoids, constructed various examples of abelian, nonabelian, and rectangular dimonoids. The algebraic structure of these dimonoids was examined in detail, including computations of their automorphism groups. In the present work, we build upon these results, as well as those obtained in [6, 7], to provide a complete classification, up to isomorphism, of all two-element dimonoids, all commutative three-element dimonoids, and all abelian three-element dimonoids.

1. PRELIMINARIES ON SEMIGROUPS

An element e of a semigroup $(S, *)$ is called a *left identity* (resp. a *right identity*) in S if $e * a = a$ (resp. $a * e = a$) for any $a \in S$. An element e is called an *identity* if e is a left identity and a right identity.

Let $(S, *)$ be a semigroup and $e \notin S$. The binary operation $*$ defined on S can be extended to $S \cup \{e\}$ putting $e * s = s * e = s$ for all $s \in S \cup \{e\}$. The notation $(S, *)^{+1}$ denotes a monoid $(S \cup \{e\}, *)$ obtained from $(S, *)$ by adjoining the extra identity e (regardless of whether $(S, *)$ is or is not a monoid).

Let $(M, *)$ be a monoid with identity e and $M^{\tilde{1}} = M \cup \{\tilde{1}\}$, where $\tilde{1} \notin M$. The binary operation $*$ defined on M can be extended to $M^{\tilde{1}}$ putting $\tilde{1} * m = m * \tilde{1} = m$ for all $m \in M$ and $\tilde{1} * \tilde{1} = e$. The notation $(M, *)^{\tilde{1}}$ denotes the semigroup obtained from $(M, *)$ by adjoining an extra element $\tilde{1}$. Note that $(M, *)^{\tilde{1}}$ is not a monoid.

An element e of a semigroup $(S, *)$ is called an *idempotent* if $e * e = e$. The semigroup is a *band*, if all its elements are idempotents. Commutative bands are called *semilattices*. By L_n we denote the *linear semilattice* $\{0, 1, \dots, n-1\}$ of order n , endowed with the operation of minimum.

A semigroup $(S, *)$ is called *monogenic* if it is generated by some element $a \in S$ in the sense that $S = \{a^n\}_{n \in \mathbb{N}}$. If a monogenic semigroup is infinite then it is isomorphic to the additive semigroup $\mathbb{N}$ of positive integer numbers. A finite monogenic semigroup $S = \langle a \rangle$ also has simple structure, see [12]. There are positive integer numbers r and m called the *index* and the *period* of S such that

- $S = \{a, a^2, \dots, a^{r+m-1}\}$ and $r + m - 1 = |S|$;
- $a^{r+m} = a^r$;
- $C_m := \{a^r, a^{r+1}, \dots, a^{r+m-1}\}$ is a cyclic and maximal subgroup of S with the identity $e = a^n \in C_m$ and generator a^{n+1} , where $n \in (m \cdot \mathbb{N}) \cap \{r, \dots, r + m - 1\}$.

We denote by $M_{r,m}$ a finite monogenic semigroup of index r and period m .

An element z of a semigroup S is called a *left zero* (resp. a *right zero*) in S if $z * a = z$ (resp. $a * z = z$) for any $a \in S$. An element 0 is called a *zero* if 0 is a left zero and a right zero.

Let $(S, *)$ be a semigroup and $0 \notin S$. The binary operation $*$ defined on S can be extended to $S \cup \{0\}$ putting $0 * s = s * 0 = 0$ for all $s \in S \cup \{0\}$. The notation $(S, *)^{+0}$ denotes a semigroup $(S \cup \{0\}, *)$ obtained from $(S, *)$ by adjoining the extra zero 0 (regardless of whether $(S, *)$ has or has not a zero).

A semigroup $(S, *)$ is called a *null semigroup* if there exists an element $0 \in S$ such that $x * y = 0$ for all $x, y \in S$. In this case 0 is a zero of S . All null semigroups on the same set are isomorphic. By O_S we denote a null semigroup on a set S . If S is finite of cardinality $|S| = n$, then instead of O_S we use O_n .

Let S be a nonempty set, $0 \in S$ and $A \subset S \setminus \{0\}$. Define the binary operation $*$ on S in the following way:

$$x * y = \begin{cases} x, & \text{if } y = x \in A \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that a set S endowed with the operation $*$ is a commutative semigroup with zero 0 , and we denote this semigroup by O_S^A . If $A = S \setminus \{0\}$, then O_S^A is a semilattice. In the case when A is an empty set, O_S^A coincides with a null semigroup with zero 0 . The semigroups O_S^A and O_T^B are isomorphic if and only if $|S| = |T|$ and $|A| = |B|$. If S is a finite set of cardinality $|S| = n$ and $|A| = m$, then we use O_n^m instead of O_S^A .

If $(S, *)$ is a semigroup, then the semigroup $(S, *^d)$ with operation $x *^d y = y * x$ is called *dual* to $(S, *)$, denoted $(S, *)^d$. It follows that $(S, *)^d = (S, *)$ if and only if $(S, *)$ is a commutative semigroup.

A semigroup $(S, *)$ is said to be a *left* (resp. *right*) *zero semigroup* if $a * b = a$ (resp. $a * b = b$) for any $a, b \in S$. By LO_S and RO_S we denote a left zero semigroup and a right zero semigroup on a set S ,

respectively. It is easy to see that the semigroups LO_S and RO_S are dual. If S is finite of cardinality $|S| = n$, then instead of LO_S and RO_S we use LO_n and RO_n , respectively.

Let S be a nonempty set, $A \subset S$ and $0 \notin S$. Define the binary operation $*$ on $S^0 = S \cup \{0\}$ in the following way:

$$x * y = \begin{cases} x, & \text{if } y \in A \\ 0, & \text{if } y \in S^0 \setminus A. \end{cases}$$

It is easy to check that a set S^0 endowed with the operation $*$ is a semigroup with zero 0, and we denote this semigroup by $LO_{A \leftarrow S}^0$. If $A = S$, then $LO_{A \leftarrow S}^0$ coincides with LO_S^{+0} . In the case when A is an emptyset, $LO_{A \leftarrow S}^0$ coincides with a null semigroup O_{S^0} with zero 0. The semigroups $LO_{A \leftarrow S}^0$ and $LO_{B \leftarrow T}^0$ are isomorphic if and only if $|S| = |T|$ and $|A| = |B|$. If S is a finite set of cardinality $|S| = n$ and $|A| = m$, then we use $LO_{m \leftarrow n}^0$ instead of $LO_{A \leftarrow S}^0$.

By $RO_{A \leftarrow S}^0$ we denote a dual semigroup of $LO_{A \leftarrow S}^0$.

Let a and c be different elements of a set S . Define the associative binary operation $\dashv_c^a$ on S in the following way:

$$x \dashv_c^a y = \begin{cases} a, & \text{if } x = y = a \\ c, & \text{if } x = a \text{ and } y \neq a \\ x, & \text{if } x \neq a. \end{cases}$$

If $|S| \geq 3$, then $(S, \dashv_c^a)$ is a noncommutative band in which all elements $z \neq a$ are left zeros.

It is not difficult to check that for any different $b, d \in S$, the semigroups $(S, \dashv_c^a)$ and $(S, \dashv_d^b)$ are isomorphic. We denote this semigroup by LOB_S . If S is a finite set of cardinality $|S| = n$, then we use LOB_n instead of LOB_S .

By ROB_S we denote a dual semigroup of LOB_S .

Let S be a nonempty set, A be a nonempty subset of S , and $a \in A$. Define the associative binary operation $*$ on S in the following way:

$$x * y = \begin{cases} x, & \text{if } x \in A \\ a, & \text{if } x \notin A. \end{cases}$$

We denote the semigroup $(S, *)$ by $LO_{A \leftarrow S}$. It follows that all elements $z \in A$ are left zeros of $LO_{A \leftarrow S}$. If $A = \{a\}$, then $LO_{A \leftarrow S}$ coincides with a null semigroup O_S with zero a . If $A = S$, then $LO_{A \leftarrow S}$ coincides with a left zero semigroup LO_S . The semigroups $LO_{A \leftarrow S}$ and $LO_{B \leftarrow T}$ are isomorphic if and only if $|S| = |T|$ and $|A| = |B|$. If S is a finite set of cardinality $|S| = n$ and $|A| = m$, then we use $LO_{m \leftarrow n}$ instead of $LO_{A \leftarrow S}$.

By $RO_{A \leftarrow S}$ we denote a dual semigroup of $LO_{A \leftarrow S}$.

Following the algebraic tradition, we take for a model of the class of cyclic groups of order n the multiplicative group $C_n = \{z \in \mathbb{C} : z^n = 1\}$ of n -th roots of 1. For a set X by S_X we denote the group of all bijections of X .

2. SOME DEFINITIONS AND BASIC PROPERTIES OF DIMONOIDS

In this section, we recall several useful results on dimonoids and establish auxiliary propositions that will be frequently used in the subsequent investigations.

An element $0 \in D$ is called a *zero of a dimonoid* $(D, \dashv, \vdash)$ [29] if 0 is a zero of $(D, \dashv)$ and a zero of $(D, \vdash)$. Let $(D, \dashv, \vdash)$ be a dimonoid and $0 \notin D$. The binary operations defined on D can be extended to $D \cup \{0\}$ putting $0 \dashv d = d \dashv 0 = 0 = 0 \vdash d = d \vdash 0$ for all $d \in D \cup \{0\}$. The notation $(D, \dashv, \vdash)^{+0}$ denotes a dimonoid $D \cup \{0\}$ obtained from D by adjoining the extra zero 0.

A dimonoid $(D, \dashv, \vdash)$ is called *abelian* [39] if $x \dashv y = y \vdash x$ for all $x, y \in D$.

Let $(D, \dashv, \vdash)$ be a dimonoid. Define new operations $\dashv^d$ and $\vdash^d$ on D by

$$x \dashv^d y = y \vdash x \quad \text{and} \quad x \vdash^d y = y \dashv x.$$

It is immediate to check that $(D, \dashv^d, \vdash^d)$ is a new dimonoid, called the *dual dimonoid of* $(D, \dashv, \vdash)$ [14], which we denote by $(D, \dashv, \vdash)^d$. It follows that the unary duality operation is involutive in the sense

that $((D, \dashv, \vdash)^d)^d = (D, \dashv, \vdash)$. In fact, $(D, \dashv, \vdash)^d$ is a dimonoid if and only if $(D, \dashv, \vdash)$ is a dimonoid. As usual, a dimonoid $(D, \dashv, \vdash)$ is said to be *self-dual* if $(D, \dashv, \vdash)^d = (D, \dashv, \vdash)$. As established in [9], a dimonoid $(D, \dashv, \vdash)$ is abelian if and only if it is self-dual, which in turn holds if and only if the semigroups $(D, \vdash)$ and $(D, \dashv)$ are dual to each other. Consequently, nonabelian dimonoids are divided into the pairs of dual dimonoids.

A dimonoid $(D, \dashv, \vdash)$ is called *commutative* [20] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative.

Since commutative semigroups $(D, \dashv)$ and $(D, \vdash)$ are dual if and only if their operations coincide, all commutative nontrivial dimonoids are nonabelian. On the other hand, it is clear to see that all commutative trivial dimonoids are abelian and all noncommutative trivial dimonoids are nonabelian. A left zero and a right zero dimonoid $(D, \dashv, \vdash)$ with operations $x \dashv y = x$ and $x \vdash y = y$ [14] is an example of a nontrivial abelian noncommutative dimonoid. In the section 4 we give examples of commutative nonabelian dimonoids, see also [20].

The axioms (D_1) and (D_3) of a dimonoid imply the following proposition.

Proposition 2.1. *Let $(D, \dashv, \vdash)$ be a dimonoid. If a semigroup $(D, \dashv)$ contains a left identity or a semigroup $(D, \vdash)$ contains a right identity, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide.*

Proposition 2.2. *Let $(D, \dashv, \vdash)$ be a dimonoid. If $z \in D$ is a left zero of a semigroup $(D, \vdash)$, then z is a left zero of a semigroup $(D, \dashv)$ as well.*

Proof. Taking into account that for any $a \in D$ the following equalities hold

$$z \dashv a = (z \vdash a) \dashv a = z \vdash (a \dashv a) = z,$$

we conclude that z is a left zero of a semigroup $(D, \dashv)$. $\square$

Dually, we prove the following proposition.

Proposition 2.3. *Let $(D, \dashv, \vdash)$ be a dimonoid. If $z \in D$ is a right zero of a semigroup $(D, \dashv)$, then z is a right zero of a semigroup $(D, \vdash)$ as well.*

Proof. Since for any $a \in D$ the following equalities hold

$$a \vdash z = a \vdash (a \dashv z) = (a \vdash a) \dashv z = z,$$

we conclude that z is a right zero of a semigroup $(D, \vdash)$. $\square$

Corollary 2.4. *Let $(D, \dashv, \vdash)$ be a commutative dimonoid. An element $z \in D$ is a zero of a semigroup $(D, \dashv)$ if and only if z is a zero of a semigroup $(D, \vdash)$.*

Propositions 2.2 and 2.3 imply the following corollary.

Corollary 2.5. *Let $(D, \dashv, \vdash)$ be a dimonoid. If $(D, \dashv)$ is a right zero semigroup or $(D, \vdash)$ is a left zero semigroup, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide.*

A bijective map $\psi : D_1 \rightarrow D_2$ is called an *isomorphism* from a dimonoid $(D_1, \dashv_1, \vdash_1)$ to a dimonoid $(D_2, \dashv_2, \vdash_2)$ if

$$\psi(a \dashv_1 b) = \psi(a) \dashv_2 \psi(b) \quad \text{and} \quad \psi(a \vdash_1 b) = \psi(a) \vdash_2 \psi(b)$$

for all $a, b \in D_1$.

If there exists an isomorphism from a dimonoid $(D_1, \dashv_1, \vdash_1)$ to a dimonoid $(D_2, \dashv_2, \vdash_2)$, then $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are said to be *isomorphic*, denoted $(D_1, \dashv_1, \vdash_1) \cong (D_2, \dashv_2, \vdash_2)$. An isomorphism $\psi : D \rightarrow D$ is called an *automorphism* of a dimonoid $(D, \dashv, \vdash)$. By $\text{Aut}(D, \dashv, \vdash)$ we denote the automorphism group of a dimonoid $(D, \dashv, \vdash)$. It follows that $\text{Aut}((D, \dashv, \vdash)^{+0}) = \text{Aut}(D, \dashv, \vdash)$.

For a dimonoid $(D, \dashv, \vdash)$, if $\mathbb{S}$ and $\mathbb{T}$ denote the semigroups $(D, \dashv)$ and $(D, \vdash)$, respectively, then $\mathbb{S} \mathbin{\mathcal{J}} \mathbb{T}$ stands for the dimonoid $(D, \dashv, \vdash)$.

Proposition 2.6. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be dimonoids such that $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are left zero semigroups. Dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if semigroups $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$ are isomorphic.*

Proof. It is immediate to observe that if dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic, then semigroups $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$ are also isomorphic. Conversely, let $\psi : D_1 \rightarrow D_2$ be an isomorphism from a semigroup $(D_1, \vdash_1)$ to a semigroup $(D_2, \vdash_2)$. Then, necessarily, $|D_1| = |D_2|$. Taking into account that any bijective map is an isomorphism from a left zero semigroup $(D_1, \dashv_1)$ to a left zero semigroup $(D_2, \dashv_2)$, it follows that ψ is also an isomorphism from a left zero semigroup $(D_1, \dashv_1)$ to a left zero semigroup $(D_2, \dashv_2)$. Therefore, ψ is an isomorphism from a dimonoid $(D_1, \dashv_1, \vdash_1)$ to a dimonoid $(D_2, \dashv_2, \vdash_2)$. $\square$

Dually, one can prove the following proposition.

Proposition 2.7. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be dimonoids such that $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$ are right zero semigroups. Dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic.*

Proposition 2.8. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be abelian dimonoids. Dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic.*

Proof. It is immediate to show that if dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic, then semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic as well. Conversely, let $\psi : D_1 \rightarrow D_2$ be an isomorphism from a semigroup $(D_1, \dashv_1)$ to a semigroup $(D_2, \dashv_2)$. Since $\psi(x \dashv_1 y) = \psi(y \dashv_1 x) = \psi(y) \dashv_2 \psi(x) = \psi(x) \dashv_2 \psi(y)$ for all $x, y \in D_1$, it follows that $\psi : D_1 \rightarrow D_2$ is an isomorphism from a dimonoid $(D_1, \dashv_1, \vdash_1)$ to a dimonoid $(D_2, \dashv_2, \vdash_2)$. $\square$

3. TWO-ELEMENT DIMONOIDS AND THEIR AUTOMORPHISM GROUPS

In this section we describe, up to isomorphism, all two-element dimonoids and their automorphism groups.

Theorem 3.1. *Up to isomorphism, there exist 8 two-element dimonoids among which 3 dimonoids are commutative. Also, up to isomorphism, there are 4 abelian dimonoids of order 2, and nonabelian dimonoids are divided into 2 pairs of dual dimonoids. There exist exactly 5 pairwise nonisomorphic two-element trivial dimonoids.*

Proof. It is well-known that there are exactly five pairwise nonisomorphic semigroups having two elements: the multiplicative cyclic group $C_2 = \{-1, 1\}$, the linear semilattice $L_2 = \{0, 1\}$ with min-operation, the null semigroup $O_2 = \{0, 1\}$ with zero 0, the left zero semigroup LO_2 with operation $ab = a$, and the right zero semigroup RO_2 with operation $ab = b$.

In the sequel, we divide our investigation into cases. In the case of a semigroup $(S, *)$ we shall find all pairwise nonisomorphic dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv)$ is isomorphic to $(S, *)$.

Cases C_2 and L_2 . According to Proposition 2.1, if a semigroup $(D, \dashv)$ possesses a left identity or a semigroup $(D, \vdash)$ possesses a right identity, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide. Therefore, up to isomorphism, there exist a unique dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong C_2$ or $(D, \vdash) \cong C_2$, and this dimonoid is the trivial dimonoid C_2 . Similarly, L_2 is a unique dimonoid in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong L_2$ or $(D, \vdash) \cong L_2$. The trivial dimonoids C_2 and L_2 are commutative and abelian.

Case LO_2 . If $(D, \vdash) \cong RO_2$, then we obtain the abelian noncommutative dimonoid $LO_2 \frown RO_2$. According to Proposition 2.8, $LO_2 \frown RO_2$ is a unique dimonoid in the class of abelian dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong LO_2$ and $(D, \vdash) \cong RO_2$. It follows that $\text{Aut}(LO_2 \frown RO_2) = \text{Aut}(LO_2) = S_2 \cong C_2$.

In the case $(D, \vdash) \cong O_2$, we obtain the noncommutative nonabelian dimonoid $LO_2 \frown O_2$. By Proposition 2.6, $LO_2 \frown O_2$ is a unique dimonoid in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong LO_2$ and $(D, \vdash) \cong O_2$. It follows that $\text{Aut}(LO_2 \frown O_2) = \text{Aut}(O_2) = C_1$.

In the remaining case, we obtain the trivial noncommutative nonabelian dimonoid LO_2 . By Proposition 2.6, LO_2 is a unique dimonoid in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong (D, \vdash) \cong LO_2$.

Case RO_2 . According to Proposition 2.5, if $(D, \dashv)$ is a right zero semigroup, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide. Consequently, the trivial dimonoid RO_2 is a unique dimonoid in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong RO_2$. This dimonoid is noncommutative and nonabelian, and it is dual to the dimonoid LO_2 .

Case O_2 . If $(D, \vdash)$ is a right zero semigroup, then we obtain the noncommutative nonabelian dimonoid $O_2 \frown RO_2$, which is unique in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong O_2$ and $(D, \vdash) \cong RO_2$, in

accordance with Proposition 2.7. This dimonoid is dual to the dimonoid $LO_2 \mathbin{\dot{\smile}} O_2$, and $\text{Aut}(O_2 \mathbin{\dot{\smile}} RO_2) = C_1$.

According to Proposition 2.5, if $(D, \vdash)$ is a left zero semigroup, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide. Therefore, there does not exist a dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong O_2$ and $(D, \vdash) \cong LO_2$.

In the final case, we obtain the trivial commutative abelian dimonoid O_2 . It follows from Corollary 2.4 that O_2 is a unique dimonoid in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong (D, \vdash) \cong O_2$. $\square$

In the following table we present, up to isomorphism, all two-element dimonoids and their automorphism groups.

D	C_2	L_2	O_2	LO_2	RO_2	$LO_2 \mathbin{\dot{\smile}} RO_2$	$LO_2 \mathbin{\dot{\smile}} O_2$	$O_2 \mathbin{\dot{\smile}} RO_2$
$\text{Aut}(D)$	C_1	C_1	C_1	C_2	C_2	C_2	C_1	C_1

TABLE 1. Two-element dimonoids and their automorphism groups

4. THREE-ELEMENT DIMONIDS AND THEIR AUTOMORPHISM GROUPS

In the remaining part of this paper, we focus on describing, up to isomorphism, all three-element dimonoids.

Among the 19683 possible binary operations on a three-element set S , precisely 113 are associative. In other words, there exist exactly 113 distinct three-element semigroups. However, many of these semigroups are isomorphic, and as a result, there are essentially only 24 pairwise nonisomorphic semigroups of order 3, see [2, 4, 5].

Among these 24 pairwise nonisomorphic semigroups of order 3, there are 12 commutative semigroups. The remaining 12 pairwise nonisomorphic noncommutative semigroups are partitioned into pairs of dual semigroups. Moreover, the automorphism groups of dual semigroups coincide.

List of all pairwise nonisomorphic semigroups of order 3 and their automorphism groups are presented in Table 2 and Table 3 taken from [5].

S	C_3	O_3	$M_{2,2}$	C_2^{+1}	$C_2^{\bar{1}}$	$M_{3,1}$	O_2^{+1}	O_2^{+0}	L_3	C_2^{+0}	O_3^2	O_3^1
$\text{Aut}(S)$	C_2	C_2	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_2	C_1

TABLE 2. Commutative semigroups of order 3 and their automorphism groups

S	LO_3, RO_3	LO_2^{+0}, RO_2^{+0}	$LO_{1\leftarrow 2}^{\sim 0}, RO_{1\leftarrow 2}^{\sim 0}$	LO_2^{+1}, RO_2^{+1}	LOB_3, ROB_3	$LO_{2\leftarrow 3}, RO_{2\leftarrow 3}$
$\text{Aut}(S)$	S_3	C_2	C_1	C_2	C_1	C_2

TABLE 3. Noncommutative three-element semigroups and their automorphism groups

4.1. Commutative three-element dimonoids. The classification of three-element commutative dimonoids will be based on our results concerning the classification of three-element doppelsemigroups from [6].

A *doppelsemigroup* is an algebraic structure $(D, \dashv, \vdash)$ consisting of a nonempty set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the axiom (D_2) and the following axiom:

$$(x \dashv y) \vdash z = x \dashv (y \vdash z) \quad (D_4).$$

For a doppelsemigroup $(D, \dashv, \vdash)$, if $\mathbb{S}$ and $\mathbb{T}$ denote the semigroups $(D, \dashv)$ and $(D, \vdash)$, respectively, then $\mathbb{S} \bowtie \mathbb{T}$ stands for the doppelsemigroup $(D, \dashv, \vdash)$.

In [6], the problem of classifying all doppelsemigroups with at most three elements up to isomorphism was completely solved. According to Proposition 1 from [33], every commutative dimonoid is a doppelsemigroup. Therefore, in order to describe all three-element commutative dimonoids up to

isomorphism, it suffices to select those dimonoids among the commutative pairwise nonisomorphic doppelsemigroups of order 3.

The following Table 4 of all pairwise nonisomorphic nontrivial commutative three-element doppelsemigroups and their automorphism groups is taken from [6].

D	$C_3 \bowtie C_3^{-1}$	$O_3 \bowtie M_{3,1}$	$O_3 \bowtie O_2^{+1}$	$O_3 \bowtie O_2^{+0}$	$O_3 \bowtie L_3$	$O_3 \bowtie C_2^{+0}$
$\text{Aut}(D)$	C_1	C_1	C_1	C_1	C_1	C_1
D	$O_3 \bowtie O_3^2$	$O_3 \bowtie O_3^1$	$M_{2,2} \bowtie C_2^{+1}$	$M_{2,2} \bowtie C_2^{\bar{1}}$	$C_2^{+1} \bowtie C_2^{\bar{1}}$	$C_2^{+1} \bowtie M_{2,2}$
$\text{Aut}(D)$	C_2	C_1	C_1	C_1	C_1	C_1
D	$C_2^{\bar{1}} \bowtie M_{2,2}$	$C_2^{\bar{1}} \bowtie C_2^{+1}$	$M_{3,1} \bowtie O_2^{+1}$	$M_{3,1} \bowtie O_3$	$O_2^{+1} \bowtie M_{3,1}$	$O_2^{+1} \bowtie O_3$
$\text{Aut}(D)$	C_1	C_1	C_1	C_1	C_1	C_1
D	$(O_2 \bowtie L_2)^{+0}$	$O_2^{+0} \bowtie O_3$	$L_3 \bowtie O_3$	$(L_2 \bowtie O_2)^{+0}$	$(C_2 \bowtie C_2^{-1})^{+0}$	$C_2^{+0} \bowtie O_3$
$\text{Aut}(D)$	C_1	C_1	C_1	C_1	C_1	C_1
D	$O_3^2 \bowtie O_3^1$	$O_3^2 \bowtie O_3$	$O_3^a \bowtie O_3^b$	$O_3^1 \bowtie O_3^2$	$O_3^1 \bowtie O_3$	
$\text{Aut}(D)$	C_1	C_2	C_1	C_1	C_1	

TABLE 4. Three-element nontrivial commutative doppelsemigroups and their automorphism groups

We begin by establishing several auxiliary propositions.

Proposition 4.1. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup such that $(D, \dashv)$ is a null semigroup with zero 0. A doppelsemigroup $(D, \dashv, \vdash)$ is a dimonoid if and only if $D \vdash D \vdash D = \{0\}$.*

Proof. Taking into account that for a doppelsemigroup $(D, \dashv, \vdash)$ an element $0 \in D$ is a zero of a semigroup $(D, \dashv)$ if and only if 0 is a zero of a semigroup $(D, \vdash)$, see [6], we conclude that the axioms (D_1) and (D_2) of a dimonoid hold:

$$(x \dashv y) \dashv z = 0 = x \dashv (y \vdash z), \quad (x \vdash y) \dashv z = 0 = x \vdash 0 = x \vdash (y \dashv z).$$

Since $(x \dashv y) \vdash z = 0 \vdash z = 0$ for any $x, y, z \in D$, we conclude that the axiom (D_3) of a dimonoid holds if and only if $x \vdash (y \vdash z) = 0$ for any $x, y, z \in D$, that is $D \vdash D \vdash D = \{0\}$. $\square$

Dually, one can prove the following proposition.

Proposition 4.2. *Let $(D, \dashv, \vdash)$ be a doppelsemigroup such that $(D, \vdash)$ is a null semigroup with zero 0. A doppelsemigroup $(D, \dashv, \vdash)$ is a dimonoid if and only if $D \dashv D \dashv D = \{0\}$.*

The following theorem provides a complete classification of all pairwise nonisomorphic commutative dimonoids of order 3.

Theorem 4.3. *Up to isomorphism, there exist 14 three-element commutative dimonoids among which 12 trivial dimonoids and a pair of nontrivial nonabelian dual dimonoids.*

Proof. Since a trivial dimonoid $(D, \dashv, \dashv)$ is commutative if and only if a semigroup $(D, \dashv)$ is commutative, we obtain that, up to isomorphism, there exist 12 trivial commutative dimonoids, see Table 2.

Our further investigation is carried out by distinguishing several cases.

Case 1. Consider the doppelsemigroups $O_3 \bowtie M_{3,1}$ and $M_{3,1} \bowtie O_3$. Recall that $M_{3,1} = \{a, a^2, a^3 \mid a^4 = a^3\}$ is a monogenic semigroup of index 3 and period 1 with zero a^3 . Since $M_{3,1} * M_{3,1} * M_{3,1} = \{a^3\}$, we conclude that $O_3 \bowtie M_{3,1}$ and $M_{3,1} \bowtie O_3$ are (nonabelian dual) dimonoids according to Propositions 4.1 and 4.2. These dimonoids are examples of commutative nonabelian dimonoids.

Case 2. According to Proposition 2.1 for a dimonoid $(D, \dashv, \vdash)$, if a semigroup $(D, \dashv)$ contains a left identity or a semigroup $(D, \vdash)$ contains a right identity, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide. Therefore, the doppelsemigroups $C_3 \bowtie C_3^{-1}$, $O_3 \bowtie O_2^{+1}$, $O_3 \bowtie L_3$, $O_3 \bowtie C_2^{+0}$, $M_{2,2} \bowtie C_2^{+1}$, $C_2^{+1} \bowtie C_2^{\bar{1}}$, $C_2^{+1} \bowtie M_{2,2}$, $C_2^{\bar{1}} \bowtie C_2^{+1}$, $M_{3,1} \bowtie O_2^{+1}$, $O_2^{+1} \bowtie M_{3,1}$, $O_2^{+1} \bowtie O_3$, $(O_2 \bowtie L_2)^{+0}$, $L_3 \bowtie O_3$, $(L_2 \bowtie O_2)^{+0}$, $(C_2 \bowtie C_2^{-1})^{+0}$, and $C_2^{+0} \bowtie O_3$ can not be dimonoids.

Case 3. Consider the doppelsemigroups $O_3 \bowtie O_2^{+0}$ and $O_2^{+0} \bowtie O_3$. Let 0 and z be zeros of the semigroup O_3 and O_2 , respectively. Taking into account that $O_2^{+0} * O_2^{+0} * O_2^{+0} = \{0, z\} \neq \{0\}$, we conclude according to Propositions 4.1 and 4.2 that $O_3 \bowtie O_2^{+0}$ and $O_2^{+0} \bowtie O_3$ are not dimonoids.

Case 4. Consider the doppelsemigroups $O_3 \bowtie O_3^2$ and $O_3^2 \bowtie O_3$. Recall that O_3^2 is a nonlinear semilattice isomorphic to the semigroup $\{a, b, 0\}$ with the operation $*$:

$$x * y = \begin{cases} x, & \text{if } y = x \in \{a, b\} \\ 0, & \text{otherwise.} \end{cases}$$

Since $O_3^2 * O_3^2 * O_3^2 = O_3^2 \neq \{0\}$, we conclude that $O_3 \bowtie O_3^2$ and $O_3^2 \bowtie O_3$ are not dimonoids by Propositions 4.1 and 4.2.

Case 5. Consider the doppelsemigroups $O_3 \bowtie O_3^1$ and $O_3^1 \bowtie O_3$. Recall that O_3^1 is isomorphic to the semigroup $\{a, b, 0\}$ with the operation $*$:

$$x * y = \begin{cases} x, & \text{if } y = x = a \\ 0, & \text{otherwise.} \end{cases}$$

Since $O_3^1 * O_3^1 * O_3^1 = \{0, a\} \neq \{0\}$, we conclude that $O_3 \bowtie O_3^1$ and $O_3^1 \bowtie O_3$ are not dimonoids according to Propositions 4.1 and 4.2.

Case 6. Consider the doppelsemigroups $O_3^2 \bowtie O_3^1$ and $O_3^1 \bowtie O_3^2$. According to Lemma 3 of [25] for a dimonoid $(D, \dashv, \vdash)$, if $(D, \dashv)$ is a semilattice, then the operations of a dimonoid $(D, \dashv, \vdash)$ coincide. Since O_3^2 is a semilattice, the doppelsemigroup $O_3^2 \bowtie O_3^1$ can not be a dimonoid. The doppelsemigroup $O_3^1 \bowtie O_3^2$ also cannot be a dimonoid either, because otherwise $O_3^2 \bowtie O_3^1$ would be its dual dimonoid.

Case 7. Consider the doppelsemigroups $M_{2,2} \bowtie C_2^{\bar{1}}$ and $C_2^{\bar{1}} \bowtie M_{2,2}$. Recall that $M_{2,2} = \{a, a^2, a^3 \mid a^4 = a^2\}$ with operation $\dashv$ is a monogenic semigroup of index 2 and period 2 and $C_2^{\bar{1}} = \{a^2, a^3\}^{\bar{1}}$ with operation $\vdash$ is a semigroup obtained from the cyclic group $\{a^2, a^3\}$ with identity a^2 by adjoining an element a with $a \vdash s = s \vdash a = s$ for $s \in \{a^2, a^3\}$ and $a \vdash a = a^2$. Taking into account that $(a \dashv a) \dashv a^2 = a^2 \dashv a^2 = a^4 = a^2$ and $a \dashv (a \vdash a^2) = a \dashv a^2 = a^3 \neq a^2$, we conclude that $M_{2,2} \bowtie C_2^{\bar{1}}$ is not a dimonoid. By analogy $C_2^{\bar{1}} \bowtie M_{2,2}$ is not a dimonoid.

Case 8. Consider the last doppelsemigroup $O_3^a \bowtie O_3^b$. Recall that $O_3^a \bowtie O_3^b$ is the doppelsemigroup $(\{a, b, 0\}, *_a, *_b)$, where for $t \in \{a, b\}$

$$x *_t y = \begin{cases} x, & \text{if } y = x = t \\ 0, & \text{otherwise.} \end{cases}$$

Taking into account that $(a *_a a) *_a a = a *_a a = a$ and $a *_a (a *_b a) = a *_a 0 = 0 \neq a$, we conclude that $O_3^a \bowtie O_3^b$ is not a dimonoid. $\square$

In the following Table 5 we present, up to isomorphism, all three-element commutative dimonoids and their automorphism groups.

D	C_3	O_3	$M_{2,2}$	C_2^{+1}	$C_2^{\bar{1}}$	$M_{3,1}$	O_2^{+1}	O_2^{+0}	L_3	C_2^{+0}	O_3^2	O_3^1	$M_{3,1} \dot{\cup} O_3$	$O_3 \dot{\cup} M_{3,1}$
$\text{Aut}(D)$	C_2	C_2	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_1	C_2	C_1	C_1	C_1

TABLE 5. Commutative three-element dimonoids and their automorphism groups

4.2. Abelian three-element dimonoids. The following theorem provides a complete classification of all pairwise nonisomorphic abelian dimonoids of order 3.

Recall that a semigroup $(S, *)$ is called *right commutative* [28], if it satisfies the identity $s * x * y = s * y * x$ for all $s, x, y \in S$.

Theorem 4.4. *Up to isomorphism, there exist 17 three-element abelian dimonoids among which 12 commutative trivial dimonoids and 5 noncommutative nontrivial dimonoids.*

Proof. Since a trivial dimonoid $(D, \dashv, \dashv)$ is abelian if and only if a semigroup $(D, \dashv)$ is commutative, we obtain that, up to isomorphism, there exist 12 trivial abelian dimonoids, see Table 2.

Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ be a dual semigroup to $(D, \dashv)$. According to Lemma 3 of [28], an algebraic structure $(D, \dashv, \vdash)$ is an abelian dimonoid if and only if $(D, \dashv)$ is a right commutative semigroup. If an abelian dimonoid $(D, \dashv, \vdash)$ has a commutative semigroup $(D, \dashv)$, then for all $x, y \in D$, it holds that $x \vdash y = y \dashv x = x \dashv y$. In this case, both operations coincide, and the dimonoid is trivial.

Consider two abelian dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$. By Proposition 2.8, dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic.

From the previous considerations it follows that the task of describing all pairwise nonisomorphic nontrivial abelian three-element dimonoids reduces to the task of recognizing right commutative semigroups among the nontrivial noncommutative semigroups listed in Table 3.

It was proved in [9] that the semigroups LO_3 , $LO_{2\leftarrow 3}$, LOB_3 , $LO_{1\leftarrow 2}^0$, and LO_2^{+0} are right commutative.

It follows directly from the definition of a right commutative semigroup that a noncommutative semigroup containing a left identity cannot be right commutative. Therefore, the semigroups RO_3 , RO_2^{+0} , $RO_{1\leftarrow 2}^0$, LO_2^{+1} , RO_2^{+1} , and ROB_3 are not right commutative. Consider the remaining semigroup $RO_{2\leftarrow 3}$, which contains two right zeros. Denote these zeros by a and b . For any $s \in RO_{2\leftarrow 3}$, it holds that $sab = b \neq a = sba$, and therefore, the semigroup $RO_{2\leftarrow 3}$ is not right commutative.

We conclude that up to isomorphism there exist 5 abelian noncommutative nontrivial dimonoids: $LO_3 \wr RO_3$, $LO_{2\leftarrow 3} \wr RO_{2\leftarrow 3}$, $LOB_3 \wr ROB_3$, $LO_{1\leftarrow 2}^0 \wr RO_{1\leftarrow 2}^0$, and $(LO_2 \wr RO_2)^{+0} = LO_2^{+0} \wr RO_2^{+0}$. $\square$

Based on the results of [9] concerning the automorphism groups of abelian noncommutative dimonoids, Table 6 lists, up to isomorphism, all abelian noncommutative nontrivial three-element dimonoids and their corresponding automorphism groups.

D	$LO_3 \wr RO_3$	$LO_{2\leftarrow 3} \wr RO_{2\leftarrow 3}$	$LOB_3 \wr ROB_3$	$LO_{1\leftarrow 2}^0 \wr RO_{1\leftarrow 2}^0$	$(LO_2 \wr RO_2)^{+0}$
$\text{Aut}(D)$	S_3	C_2	C_1	C_1	C_2

TABLE 6. Abelian noncommutative nontrivial 3-element dimonoids and their automorphism groups

4.3. Nonabelian noncommutative three-element dimonoids. Based on the results of [9] concerning noncommutative nonabelian dimonoids and their automorphism groups and properties of dual dimonoids, Table 7 lists some pairwise nonisomorphic noncommutative nonabelian nontrivial three-element dimonoids and their automorphism groups.

D	$LO_3 \wr O_3$	$LO_{2\leftarrow 3} \wr O_3$	$LO_3 \wr RO_{2\leftarrow 3}$	$LO_3 \wr LO_{2\leftarrow 3}$	$LOB_3 \wr O_3^1$	$LO_{1\leftarrow 2}^0 \wr O_3^1$	$(LO_2 \wr O_2)^{+0}$
$\text{Aut}(D)$	C_2	C_1	C_2	C_2	C_1	C_1	C_1
D	$O_3 \wr RO_3$	$O_3 \wr RO_{2\leftarrow 3}$	$LO_{2\leftarrow 3} \wr RO_3$	$RO_{2\leftarrow 3} \wr RO_3$	$O_3^1 \wr ROB_3$	$O_3^1 \wr RO_{1\leftarrow 2}^0$	$(O_2 \wr RO_2)^{+0}$
$\text{Aut}(D)$	C_2	C_1	C_2	C_2	C_1	C_1	C_1

TABLE 7. Nonabelian noncommutative nontrivial 3-element dimonoids and their automorphism groups

Since a trivial dimonoid $(D, \dashv, \dashv)$ is nonabelian if and only if a semigroup $(D, \dashv)$ is noncommutative, we obtain that, up to isomorphism, there exist 12 trivial nonabelian noncommutative dimonoids, see Table 3. It follows that we have proved the following theorem.

Theorem 4.5. *There exist at least 26 pairwise nonisomorphic nonabelian noncommutative three-element dimonoids among which there are exactly 6 pairs of trivial dual dimonoids and at least 7 pairs of nontrivial dual dimonoids.*

Solving the following problem will provide a complete classification, up to isomorphism, of all three element dimonoids.

Problem 4.6. *Give a complete classification, up to isomorphism, of all noncommutative nonabelian nontrivial three-element dimonoids.*

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