

Pinwheel Scheduling with Real Periods

Hiroshi Fujiwara

Kota Miyagi

Katsuhisa Ouchi

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Abstract

For a sequence of tasks, each with a positive integer period, the pinwheel scheduling problem involves finding a valid schedule in the sense that the schedule performs one task per day and each task is performed at least once every consecutive days of its period. It had been conjectured by Chan and Chin in 1993 that there exists a valid schedule for any sequence of tasks with density, the sum of the reciprocals of each period, at most $\frac{5}{6}$. Recently, Kawamura settled this conjecture affirmatively. In this paper we consider an extended version with real periods proposed by Kawamura, in which a valid schedule must perform each task i having a real period a_i at least l times in any consecutive $\lceil la_i \rceil$ days for all positive integer l . We show that any sequence of tasks such that the periods take three distinct real values and the density is at most $\frac{5}{6}$ admits a valid schedule. We hereby conjecture that the conjecture of Chan and Chin is true also for real periods.

1 Introduction

In the pinwheel scheduling problem [HMR⁺89], we are given a sequence of tasks, each with a positive integer period, and want to find a schedule that performs one task per day such that each task is performed at least once every consecutive days of its period. Let $\mathbb{N}$ and $\mathbb{Z}$ be the set of all positive integers and the set of all integers, respectively, and let $[k] = \{i \mid i \in \mathbb{N}, 1 \leq i \leq k\}$. Formally, an integer-valued instance of the pinwheel schedule problem is a sequence $A = (a_i)_{i \in [k]} \in \mathbb{N}^k$, where a_i is the *period* of task i . The goal is to find a *schedule* $S : \mathbb{Z} \rightarrow [k]$, which indicates that task $S(t)$ is performed on day t , such that the following condition is satisfied for each task $i \in [k]$:

for each $m \in \mathbb{Z}$, there exists a day $t \in [m, m + a_i) \cap \mathbb{Z}$ such that $S(t) = i$.

Such a schedule is said to be *valid* for instance A . In particular, when a schedule S is periodic, that is, there exists $p \in \mathbb{N}$ such that $S(t) = S(t+p)$ holds for all $t \in \mathbb{Z}$, we represent the schedule by a sequence of $|S(0)S(1) \cdots S(p-1)|$. We say that an instance is *schedulable* if there exists a valid schedule for the instance. For example, instance $(2, 4, 4)$ is schedulable, since schedule $|1213|$ is valid for it.

The *density* of instance $A = (a_i)_{i \in [k]}$ is defined as $D(A) = \sum_{i \in [k]} \frac{1}{a_i}$. Intuitively, the density of an instance means how many tasks a valid schedule for the instance has to perform on average per day. Obviously, $D(A) \leq 1$ is necessary for instance A to admit a valid schedule. (See that the density of instance $(2, 4, 4)$ above is 1.) However, this is not sufficient. Chan and Chin conjectured as follows:

Conjecture 1 ([CC93]). *If an integer-valued instance A satisfies $D(A) \leq \frac{5}{6}$, then A is schedulable.*

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This conjecture is best possible, since for any a_3 , instance $(2, 3, a_3)$, whose density is $\frac{5}{6} + \frac{1}{a_3} > \frac{5}{6}$, is not schedulable. After much work by many researchers such as [LL97, FL02, GSW22], Kawamura finally settled this conjecture in the affirmative.

Theorem 1 ([Kaw24, Theorem 1]). *If an integer-valued instance A satisfies $D(A) \leq \frac{5}{6}$, then A is schedulable.*

In his paper [Kaw24], Kawamura proposed an extended version of the pinwheel scheduling problem that allows *real* values at least 1 for periods of tasks, replacing the validity condition for each task $i \in [k]$ by:

for each $l \in \mathbb{N}$ and $m \in \mathbb{Z}$, there exist at least l values of $t \in [m, m + \lceil la_i \rceil) \cap \mathbb{Z}$ such that $S(t) = i$.

In other words, this condition is that for each $l \in \mathbb{N}$, task i is performed at least l times in any consecutive $\lceil la_i \rceil$ days. Please see that when a_i is an integer, the condition is equivalent to that of the original integer-valued problem. For example, a task with period $\frac{7}{2}$ has to be performed at least once every 4 days and at least twice every 7 days. Schedule $|1112112|$ is valid for instance $(2, \frac{7}{2})$.

It should be remarked that this is not a simple relaxation of the problem. More specifically, even if one identifies n tasks each with period a in a valid schedule, the identified task may not be performed according to a period of $\frac{a}{n}$. For example, schedule $|1111213|$ is valid for instance $(2, 7, 7)$. Identify tasks 2 and 3. Then, the identified task fails to meet the validity condition for a period of $\frac{7}{2}$; the task is not performed once every 4 days, though it is performed twice every 7 days.

Nevertheless, we conjecture that Theorem 1 holds true also for the real-valued version:

Conjecture 2. *If a real-valued instance A satisfies $D(A) \leq \frac{5}{6}$, then A is schedulable.*

This is also best possible for the same reason as the integer-valued problem. Our main theorem, Theorem 2, is a partial solution to Conjecture 2, which strengthens the plausibility of the conjecture.

Theorem 2. *If a real-valued instance A consists of three distinct values and satisfies $D(A) \leq \frac{5}{6}$, then A is schedulable.*

1.1 Previous Results

The integer-valued pinwheel scheduling problem. The next theorem is one of the results aimed at solving Conjecture 1, which can be regarded as an integer-valued version of our main theorem.

Theorem 3 ([LL97, Theorem 4]). *If an integer-valued instance A consists of three distinct values and satisfies $D(A) \leq \frac{5}{6}$, then A is schedulable.*

We will see below that it seems hopeless to extend Theorem 3 directly to the real-valued problem. (Conversely, our proof of Theorem 2 can be seen as a simpler proof of Theorem 3.) The proof of Theorem 3 in the paper [LL97] constructs a valid schedule for a given integer-valued instance

$$\underbrace{(c_1, \dots, c_1)}_{q_1}, \underbrace{(c_2, \dots, c_2)}_{q_2}, \underbrace{(c_3, \dots, c_3)}_{q_3}.$$

In the schedule, for each $i = 1, 2$, and 3 , identify all the tasks with period c_i and rename them as task i . If the resulting schedule were valid for instance $(\frac{c_1}{q_1}, \frac{c_2}{q_2}, \frac{c_3}{q_3})$, this would give a solution

to the problem with rational periods. However, this attempt fails. We have a counterexample: instance

$$(\underbrace{13, \dots, 13}_5, \underbrace{14, \dots, 14}_5, \underbrace{78, \dots, 78}_5).$$

Apply this to the construction method and identify all the 5 tasks with period 14 in the resulting schedule. It will be found that the identified task is not performed at a period of $\frac{14}{5}$.

Gąsieniec et al. partially solved Conjecture 1 for the case of 12 or fewer tasks [GSW22]. The authors provided a *Pareto surface* of the whole set of such instances, where a Pareto surface C for a set of pinwheel scheduling instances I is an inclusion minimal set of pairs of an instance with minimum periods and a schedule that is valid for the instance such that, for every instance A in I , C includes a valid schedule for A . We will also consider this point of view in Sections 2 and 3.

We introduce a stronger conjecture recently made by Kawamura:

Conjecture 3 ([Kaw25, Conjecture 1]). *If an integer-valued instance $A = (a_i)_{i \in [k]}$ satisfies $D(A) \leq 1 - \frac{a_1 - 1}{a_1(a_1 + 1)}$, then A is schedulable.*

The real-valued pinwheel scheduling problem. Kawamura [Kaw24] extended the pinwheel scheduling problem to real periods as one of the techniques for proving Theorem 1. The following theorem is a partial solution to Conjecture 2.

Theorem 4 ([Kaw24, Theorem 2]). *If a real-valued instance A contains at most two periods and satisfies $D(A) \leq 1$, then A is schedulable.*

Theorem 5 below can be seen as a partial solution to Conjecture 2 for the case where $a_1 > 103$, since the density threshold in the statement exceeds $\frac{5}{6}$ in this case.

Theorem 5 ([KKK25, Theorem 4]). *If a real-valued instance $A = (a_i)_{i \in [k]}$ satisfies $D(A) \leq 1 - \frac{1 + \ln 2}{1 + \sqrt{a_1}} - \frac{3}{2a_1}$, then A is schedulable.*

Other related problems. The pinwheel scheduling problem can be viewed as a task packing problem. Its dual problem, the pinwheel covering problem, has recently been studied [KS20, KKK25]. The bamboo garden trimming problem [GJK⁺24, Kaw24] is an optimization problem closely related to the pinwheel scheduling problem. Refer to the survey [Kaw25] for further variants and related problems.

2 A Warm-Up: Case of Two Distinct Real Values

To demonstrate the idea of the proof of our main theorem, Theorem 2, for a simpler case, we first attempt to prove a weaker theorem for the case of two distinct real values, Theorem 6. We mention that this can be obtained as a corollary of Theorem 4 and Lemma 2 below.

Theorem 6. *If a real-valued instance A consists of two distinct values and satisfies $D(A) \leq \frac{5}{6}$, then A is schedulable.*

The following two lemmas play an important role. All schedulability throughout this paper, except for that of some individual instances, is derived from these lemmas. Please see in Lemma 1 that the same schedule remains valid for the new instance.

Lemma 1 ([Kaw24, Lemma 3 (1)]). *If a schedule S is valid for a real-valued instance $(a_i)_{i \in [k]}$, then S is valid also for the instance obtained by replacing some period a_j ($j \in [k]$) by a real $b \geq a_j$.*

Lemma 2 ([Kaw24, Lemma 3 (2)]). *If a real-valued instance $(a_i)_{i \in [k]}$ is schedulable, so is the instance obtained by replacing some period a_j ($j \in [k]$) by q copies of qa_j for any $q \in \mathbb{N}$.*

Lemma 3 below states that any real-valued instance with two periods and density exactly $\frac{5}{6}$ is schedulable. Instead of such an instance $(a_1, \frac{1}{\frac{5}{6}-a_1})$ itself, we focus on its projection a_1 . Note that the condition $\frac{6}{5} < a_1 \leq \frac{12}{5}$ in the statement ensures, without loss of generality, that the periods are ordered in non-decreasing order. Lemmas 1, 2, and 3 immediately imply Theorem 6.

Lemma 3. *Every real-valued instance $A = (a_1, \frac{1}{\frac{5}{6}-a_1})$ with $\frac{6}{5} < a_1 < \frac{12}{5}$ satisfies at least one of the following conditions and the schedule S defined there is valid for instance A :*

(i) *If $\frac{6}{5} < a_1 \leq \frac{3}{2}$, then,*

$$S(t) = \begin{cases} 1, & t \equiv 0, 1, 2, 3, 4; \\ 2, & t \equiv 5 \pmod{6}, \end{cases} \quad (1)$$

which is represented as |111112|.

(ii) *If $\frac{3}{2} \leq a_1 \leq 2$, then,*

$$S(t) = \begin{cases} 1, & t \equiv 0, 1; \\ 2, & t \equiv 2 \pmod{3}, \end{cases} \quad (2)$$

which is represented as |112|.

(iii) *If $2 \leq a_1 \leq 3$, then,*

$$S(t) = \begin{cases} 1, & t \equiv 0; \\ 2, & t \equiv 1 \pmod{2}, \end{cases} \quad (3)$$

which is represented as |12|.

It does not matter that the ranges of a_1 in the cases (i), (ii), and (iii) are not exclusive and even include values out of $\frac{6}{5} < a_1 \leq \frac{12}{5}$. The former fact means that there may exist multiple valid schedules for a single instance.

Now we are going to provide some lemmas for the proof of Lemma 3. We have found by hand that the three instances each with density 1 in the following Lemma 4 are schedulable and each period is minimum possible.

Lemma 4. *The schedules defined by (1), (2), and (3) are valid for real-valued instances $(\frac{6}{5}, 6)$, $(\frac{3}{2}, 3)$, and $(2, 2)$, respectively.*

Proof. It is immediate to see that each of the schedules satisfies the condition for the tasks having an integer period. We will demonstrate that S in (1) satisfies the condition for task 1 with period $\frac{6}{5}$ in instance $(\frac{6}{5}, 6)$.

We denote by $P_i(y, l)$ the proposition that for each $m \in \mathbb{Z}$, there exist at least l values of $t \in [m, m + y) \cap \mathbb{Z}$ such that $S(t) = i$, which means that schedule S performs task i at least l times in any consecutive y days. The schedulability of task 1 in instance $(\frac{6}{5}, 6)$ is represented as $P_1(\lceil l \cdot \frac{6}{5} \rceil, l)$ for all $l \in \mathbb{N}$. We are going to prove this.

For each $l = 1, 2, \dots, 5$, we have $\lceil l \cdot \frac{6}{5} \rceil = l + 1$. According to (1), $S(t) = 1$ if $t \equiv 0, 1, 2, 3, 4 \pmod{6}$. It is seen that there are l days when task 1 is performed every consecutive $l + 1$ days. That is, $P_1(\lceil l \cdot \frac{6}{5} \rceil, l)$ is true.

For $l \geq 6$, let q be the quotient and r be the remainder when l is divided by 5. We have $\lceil l \cdot \frac{6}{5} \rceil = \lceil (5q + r) \cdot \frac{6}{5} \rceil = 6q + \lceil r \cdot \frac{6}{5} \rceil$. Using what we have already shown, we obtain that: Task 1

is done at least r times in any consecutive $\lceil r \cdot \frac{6}{5} \rceil$ days by $P_1(\lceil z \cdot \frac{6}{5} \rceil, z)$ for $z = 1, 2, 3, 4$, while it is done at least $5q$ times in any consecutive $6q$ days by $P_1(\lceil 5 \cdot \frac{6}{5} \rceil, 5)$. Therefore, task 1 is done at least $5q + r = l$ times in any consecutive $6q + \lceil r \cdot \frac{6}{5} \rceil$ days, which implies $P_1(\lceil l \cdot \frac{6}{5} \rceil, l)$. $\square$

Using the following helper lemma, we can further derive a set of schedulable instances from those found to be schedulable by Lemma 4. Note that the resulting set includes instances whose periods are not in non-decreasing order. The lemma follows immediately from Lemma 1.

Lemma 5. *If a schedule S is valid for real-valued instance (b_1, b_2) , then S is valid also for real-valued instance $(a_1, \frac{1}{\frac{5}{6} - \frac{1}{a_1}})$ with $a_1 \geq b_1$ and $\frac{1}{\frac{5}{6} - \frac{1}{a_1}} \geq b_2$.*

Proof of Lemma 3. Applying Lemma 5, the schedulability of instance $(\frac{6}{5}, 6)$ by Lemma 4 leads us to the fact that the schedule defined by (1) is valid for instance $(a_1, \frac{1}{\frac{5}{6} - \frac{1}{a_1}})$ if $a_1 \geq \frac{6}{5}$ and $\frac{1}{\frac{5}{6} - \frac{1}{a_1}} \geq 6$, equivalently,

$$\frac{6}{5} < a_1 \leq \frac{3}{2}. \quad (4)$$

Similarly, we derive that the schedule defined by (2) is valid for instance $(a_1, \frac{1}{\frac{5}{6} - \frac{1}{a_1}})$ if

$$\frac{3}{2} \leq a_1 \leq 2, \quad (5)$$

and that the schedule defined by (3) is valid for instance $(a_1, \frac{1}{\frac{5}{6} - \frac{1}{a_1}})$ if

$$2 \leq a_1 \leq 3. \quad (6)$$

It is obvious that a_1 such that $\frac{6}{5} < a_1 < \frac{12}{5}$ satisfies at least one of (4), (5), or (6). $\square$

We focus on real-valued instances with two periods. The proof of Theorem 6 based on Lemma 3 tells us that for any real-valued instance with two periods and density less than or equal to $\frac{5}{6}$, the schedule defined by either (1), (2), or (3) is valid. It is seen that each schedule is instance-insensitive in the sense that the same schedule remains valid as long as either (4), (5), or (6) holds for the instance. In the context of the paper [GSW22], the result can be summarized as follows:

Corollary 1. *$\{((\frac{6}{5}, 6), |111112|), ((\frac{3}{2}, 3), |112|), ((2, 2), |12|)\}$ is a Pareto surface of the set of all real-valued instances with two periods and density less than or equal to $\frac{5}{6}$.*

Compare with the fact that $\{((2, 2), |12|)\}$ is a Pareto surface of the set of all *integer-valued* instances with two periods and density less than or equal to $\frac{5}{6}$ [GSW22]. That is, adding two instance-schedule pairs to this yields a Pareto surface for the real-valued instances.

3 Proof of Theorem 2

Theorem 2 is proved in essentially the same way as Theorem 6. The following Lemma 6 is the heart of the argument. It states that any real-valued instance with three periods and density exactly $\frac{5}{6}$ is schedulable. We deal with a pair (a_1, a_2) as a projection of such an instance. Note that the conditions $a_1 < a_2$ and $\frac{5}{6} \leq \frac{1}{a_1} + \frac{2}{a_2}$ in the statement guarantee, without loss of generality, that the periods are ordered in non-decreasing order. Lemmas 1, 2, and 6 immediately imply Theorem 2.

Lemma 6. Every real-valued instance $A = (a_1, a_2, \frac{1}{\frac{5}{6} - \frac{1}{a_1} - \frac{1}{a_2}})$ with $a_1 < a_2$, $\frac{5}{6} < \frac{1}{a_1} + \frac{2}{a_2}$, and $\frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$ satisfies at least one of the following conditions and the schedule S defined there is valid for instance A :

(I) If $\frac{1}{a_1} + \frac{2}{a_2} \leq 1$, then, letting $a'_2 = \frac{2}{1 - \frac{1}{a_1}}$,

$$S(t) = \begin{cases} 1, & t \in \{\lceil ja_1 \rceil - 1 \mid j \in \mathbb{Z}\}; \\ 2, & t \in \{\lfloor ja'_2 \rfloor \mid j \in \mathbb{Z}\}; \\ 3, & t \in \{\lfloor (j + \frac{1}{2})a'_2 \rfloor \mid j \in \mathbb{Z}\}. \end{cases} \quad (7)$$

(II) If $a_1 \geq \frac{3}{2}$, $a_2 > 5$, and $\frac{5}{6} - \frac{1}{9} \leq \frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$ then

$$S(t) = \begin{cases} 1, & t \equiv 0, 1, 3, 4, 6, 7; \\ 2, & t \equiv 2, 5; \\ 3, & t \equiv 8 \pmod{9}, \end{cases} \quad (8)$$

which is represented as |112112113|.

(III) If $a_1 \geq \frac{11}{7}$, $a_2 > 4$, and $\frac{5}{6} - \frac{1}{11} \leq \frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$ then

$$S(t) = \begin{cases} 1, & t \equiv 0, 1, 3, 4, 6, 7, 9; \\ 2, & t \equiv 2, 5, 8; \\ 3, & t \equiv 10 \pmod{11}, \end{cases} \quad (9)$$

which is represented as |11211211213|.

(IV) If $a_1 \geq \frac{12}{7}$, $a_2 > 3$, and $\frac{5}{6} - \frac{1}{12} \leq \frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$ then,

$$S(t) = \begin{cases} 1, & t \equiv 0, 2, 3, 5, 7, 8, 10; \\ 2, & t \equiv 1, 4, 6, 9; \\ 3, & t \equiv 11 \pmod{12}, \end{cases} \quad (10)$$

which is represented as |121121211213|.

(V) If $a_1 > 2$, $a_2 \geq 3$, and $\frac{5}{6} - \frac{1}{6} \leq \frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$ then

$$S(t) = \begin{cases} 1, & t \equiv 0, 2, 3; \\ 2, & t \equiv 1, 4; \\ 3, & t \equiv 5 \pmod{6}, \end{cases} \quad (11)$$

which is represented as |121123|.

(VI) If $a_1 > 2$, $a_2 \geq \frac{12}{5}$, and $\frac{5}{6} - \frac{1}{11} \leq \frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$, then

$$S(t) = \begin{cases} 1, & t \equiv 0, 2, 4, 5, 7, 9; \\ 2, & t \equiv 1, 3, 6, 8, 10; \\ 3, & t \equiv 11 \pmod{12}, \end{cases} \quad (12)$$

which is represented as |121211212123|.

(VII) If $a_1 \geq \frac{12}{5}$, $a_2 \geq \frac{12}{5}$, and $\frac{5}{6} - \frac{1}{6} \leq \frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$ then

$$S(t) = \begin{cases} 1, & t \equiv 0, 2, 4, 7, 9; \\ 2, & t \equiv 1, 3, 6, 8, 10; \\ 3, & t \equiv 5, 11 \pmod{12}, \end{cases} \quad (13)$$

which is represented as |121213212123|.

Lemma 6 splits the whole set of real-valued instances with three periods and density exactly $\frac{5}{6}$ into seven cases (I) through (VII). In particular, for the analysis of cases (II) through (VII), we discovered six real-valued instances of Lemma 8 for which the covering of Lemma 10 succeeds by trial and error while checking schedulability by an integer programming solver.

We discuss some lemmas for the proof of Lemma 6. We begin with instances that satisfy the condition of case (I) of Lemma 6, that is $\frac{1}{a_1} + \frac{2}{a_2} \leq 1$. We here insist on a valid schedule in an explicit form of $S(t) = \dots$. Otherwise, the schedulability itself can easily be shown by Lemmas 1 and 2, and the schedulability of instance $(a_1, \frac{a_2}{2})$ implied by Theorem 4.

Lemma 7. *The schedule S defined by (7) is valid for real-valued instance $(a_1, a_2, \frac{1}{\frac{5}{6} - \frac{1}{a_1} - \frac{1}{a_2}})$ for any a_1 and a_2 such that $a_1 < a_2$, $\frac{5}{6} < \frac{1}{a_1} + \frac{2}{a_2} \leq 1$, and $\frac{1}{a_1} + \frac{1}{a_2} < \frac{5}{6}$.*

Proof. First, we confirm that S performs a single task for every day. Since $a'_2 = \frac{2}{1 - \frac{1}{a_1}} > 2$, the set of days when task 2 is performed $\{\lfloor ka'_2 \rfloor \mid k \in \mathbb{Z}\}$ and the set of days when task 3 is performed $\{\lfloor (k + \frac{1}{2})a'_2 \rfloor \mid k \in \mathbb{Z}\}$ have no intersection. Their union is $\{\lfloor k \cdot \frac{a'_2}{2} \rfloor \mid k \in \mathbb{Z}\}$. (See that S for tasks 2 and 3 is constructed by treating them as a single task with period $\frac{a'_2}{2}$ and then assigning tasks 2 and 3 in turn.) Moreover, for a_1 and a'_2 such that $\frac{1}{a_1} + \frac{2}{a'_2} = 1$, $\{\lfloor k \cdot \frac{a'_2}{2} \rfloor \mid k \in \mathbb{Z}\}$ and the set of days when task 1 is performed $\{\lfloor ja_1 \rfloor - 1 \mid j \in \mathbb{Z}\}$ are known to be a partition of $\mathbb{Z}$ [Gol76]. (When a_1 and a'_2 are irrational numbers, this partition is called Rayleigh's or Beatty's theorem.)

Next, we show that S is valid for instance (a_1, a'_2, a'_2) , which is sufficient to prove the lemma due to $a_2 \geq a'_2$, $\frac{1}{\frac{5}{6} - \frac{1}{a_1} - \frac{1}{a_2}} \geq a'_2$, and Lemma 1.

According to (7), for any $l \in \mathbb{N}$, the maximum time span in which task 1 is performed exactly $(l - 1)$ times is from day $\lceil ma_1 \rceil$ to day $(\lceil (m + l)a_1 \rceil - 2)$ for some integer m , and its length is $(\lceil (m + l)a_1 \rceil - \lceil ma_1 \rceil - 1)$ days. (i) If la_1 is an integer, we evaluate $\lceil (m + l)a_1 \rceil - \lceil ma_1 \rceil - 1 = ma_1 + \lceil la_1 \rceil - ma_1 - 1 = la_1 - 1 = \lceil la_1 \rceil - 1$. (ii) Otherwise, we get $\lceil (m + l)a_1 \rceil - \lceil ma_1 \rceil - 1 \leq (\lceil ma_1 \rceil + \lceil la_1 \rceil) - \lceil ma_1 \rceil - 1 = \lceil la_1 \rceil - 1$. In any case, we observe that task 1 is done at least l times in any consecutive $\lceil la_1 \rceil$ days.

We finally check the validity of tasks 2 and 3 together. Set $d = 0$ for task 2 and $\frac{1}{2}$ for task 3. For any $l \in \mathbb{N}$, the maximum time span in which the task is performed exactly $(l - 1)$ times is from day $(\lfloor (m + d)a'_2 \rfloor + 1)$ to day $(\lfloor (m + d + l)a'_2 \rfloor - 1)$ for some integer m , and its length is $(\lfloor (m + d + l)a'_2 \rfloor - \lfloor (m + d)a'_2 \rfloor - 1)$ days. (i) If la'_2 is an integer, we obtain $\lfloor (m + d + l)a'_2 \rfloor - \lfloor (m + d)a'_2 \rfloor - 1 = \lfloor (m + d)a'_2 \rfloor + la'_2 - \lfloor (m + d)a'_2 \rfloor - 1 = la'_2 - 1 = \lceil la'_2 \rceil - 1$. (ii) Otherwise, we get $\lfloor (m + d + l)a'_2 \rfloor - \lfloor (m + d)a'_2 \rfloor - 1 \leq (\lfloor (m + d)a'_2 \rfloor + \lceil la'_2 \rceil + 1) - \lfloor (m + d)a'_2 \rfloor - 1 = \lceil la'_2 \rceil = \lceil la'_2 \rceil - 1$. In any case, we see that the task is done at least l times in any consecutive $\lceil la'_2 \rceil$ days. $\square$

The following lemma ensures the schedulability of the six real-valued instances. Each of the periods, except those involving ε , is minimum possible. Substituting zero for ε makes the instance non-schedulable.

Lemma 8. *The schedules defined by (8), (9), (10), (11), and (12) are valid for real-valued instances $(\frac{3}{2}, 5 + \varepsilon, 9)$, $(\frac{11}{7}, 4 + \varepsilon, 11)$, $(\frac{12}{7}, 3 + \varepsilon, 12)$, $(2 + \varepsilon, 3, 6)$, and $(2 + \varepsilon, \frac{12}{5}, 12)$ for any $\varepsilon > 0$, respectively. The schedule defined by (13) is valid for $(\frac{12}{5}, \frac{12}{5}, 6)$.*

Proof. It is easy to check for the tasks with integer periods. For the tasks with rational periods, we can verify the validity as the same as in the proof of Lemma 4. We therefore focus on the tasks with periods that involve ε . We will demonstrate that for any $\varepsilon > 0$, S in (8) satisfies the condition for task 2 with period $5 + \varepsilon$ in instance $(\frac{3}{2}, 5 + \varepsilon, 9)$.

Fix $\varepsilon > 0$ arbitrarily. As in the proof of Lemma 4, we denote by $P_i(y, l)$ the proposition that for each $m \in \mathbb{Z}$, there exist at least l values of $t \in [m, m + y) \cap \mathbb{Z}$ such that $S(t) = i$, which means that schedule S performs task i at least once every consecutive y days. Clearly, for $y \leq y'$ and $l \geq l'$, $P_i(y, l) \Rightarrow P_i(y', l')$ holds true. In addition, if $P_i(y, l)$ and $P_i(y', l')$ hold true, then so does $P_i(y + y', l + l')$.

The schedulability of task 2 in instance $(\frac{3}{2}, 5 + \varepsilon, 9)$ is equivalent to $P_2(\lceil l \cdot (5 + \varepsilon) \rceil, l)$ for all $l \in \mathbb{N}$. We are going to prove this. We know from (8) that S is itself has a period of 9 and $S(t) = 2$ for $t \equiv 2, 5 \pmod{9}$, which implies that $P_2(9, 2)$ is true. See that $P_2(9, 2)$ indicates that task 2 is performed approximately on average once every $\frac{9}{2} (< 5 + \varepsilon)$ days. Roughly speaking, $P_2(\lceil l \cdot (5 + \varepsilon) \rceil, l)$ holds true for large l and therefore we only need to check this for small l . We will often use the following inequality, $\lceil l \cdot (5 + \varepsilon) \rceil = 5l + \lceil l\varepsilon \rceil \geq 5l + 1$ for any integer l .

For $l = 1$, (8) says that there is at least one day t satisfying $t \equiv 2, 5 \pmod{9}$ in any consecutive $5l + 1 = 6$ days, thus $P_2(6, 1) \Rightarrow P_2(\lceil 1 \cdot (5 + \varepsilon) \rceil, 1)$. For $l = 2$, since $\lceil 2 \cdot (5 + \varepsilon) \rceil \geq 5 \cdot 2 + 1 = 11 > 1 \cdot 9$, $P_2(9, 2) \Rightarrow P_2(\lceil 2 \cdot (5 + \varepsilon) \rceil, 2)$ holds. For $l = 3$, it follows that $\lceil 3 \cdot (5 + \varepsilon) \rceil \geq 5 \cdot 3 + 1 = 16 = 1 \cdot 9 + 7$. Since $P_2(6, 1) \Rightarrow P_2(7, 1)$ and $P_2(9, 2)$, we have $P_2(9 + 7, 2 + 1) \Rightarrow P_2(\lceil 3 \cdot (5 + \varepsilon) \rceil, 3)$. For $l = 4$, $\lceil 4 \cdot (5 + \varepsilon) \rceil \geq 5 \cdot 4 + 1 = 21 > 2 \cdot 9$ holds. Using $P_2(9, 2)$, we derive $P_2(2 \cdot 9, 2 \cdot 2) \Rightarrow P_2(\lceil 4 \cdot (5 + \varepsilon) \rceil, 4)$. For $l = 5$, we have $\lceil 5 \cdot (5 + \varepsilon) \rceil \geq 5 \cdot 5 + 1 = 26 = 2 \cdot 9 + 8$. From $P_2(6, 1) \Rightarrow P_2(8, 1)$ and $P_2(9, 2)$, we get $P_2(2 \cdot 9 + 8, 2 \cdot 2 + 1) \Rightarrow P_2(\lceil 5 \cdot (5 + \varepsilon) \rceil, 5)$. For $l = 6$, $\lceil 6 \cdot (5 + \varepsilon) \rceil \geq 5 \cdot 6 + 1 = 31 > 3 \cdot 9$ holds. We have $P_2(3 \cdot 9, 3 \cdot 2) \Rightarrow P_2(\lceil 6 \cdot (5 + \varepsilon) \rceil, 6)$ by $P_2(9, 2)$. Finally, for $l \geq 7$, we have $\lceil l \cdot (5 + \varepsilon) \rceil \geq 5l + 1 \geq 9 \cdot \lfloor \frac{5l+1}{9} \rfloor$. It also holds true that $2 \cdot \lfloor \frac{5l+1}{9} \rfloor \geq l$, since $2 \cdot \lfloor \frac{5l+1}{9} \rfloor - l > 2 \cdot (\frac{5l+1}{9} - 1) - l = \frac{l-7}{9} \geq 0$. Applying $P_2(9, 2)$, we conclude $P_2(9 \cdot \lfloor \frac{5l+1}{9} \rfloor, 2 \cdot \lfloor \frac{5l+1}{9} \rfloor) \Rightarrow P_2(\lceil l \cdot (5 + \varepsilon) \rceil, l)$. $\square$

Lemma 9 below is a helper lemma, which is the counterpart of Lemma 5 in Section 2, that derives a set of schedulable real-valued instances from those found to be schedulable by Lemma 8. Note that the resulting set includes instances whose periods are not in non-decreasing order. Lemma 9 follows immediately from Lemma 1.

Lemma 9. *If a schedule S is valid for real-valued instance (b_1, b_2, b_3) , then S is valid also for real-valued instance $(a_1, a_2, \frac{1}{\frac{5}{6} - \frac{1}{a_1} - \frac{1}{a_2}})$ with $a_1 \geq b_1$, $a_2 \geq b_2$, and $\frac{1}{\frac{5}{6} - \frac{1}{a_1} - \frac{1}{a_2}} \geq b_3$.*

The conditions of cases (II) through (VII) of Lemma 6 turn out to be the conditions that are fulfilled by schedulable real-valued instances obtained by applying Lemma 9 to the instances found to be schedulable by Lemma 8.

Lemma 6 holds if the union of such real-valued instances and real-valued instances that apply to case (I), which are shown to be schedulable by Lemma 7, contains any real-valued instance with three periods and density exactly $\frac{5}{6}$. To facilitate this discussion, we map a projection of a real-valued instance $(a_1, a_2) \in \{(x, y) \mid 1 \leq x, 1 \leq y\}$ by a bijection $(a_1, a_2) \mapsto (\frac{1}{a_1}, \frac{1}{a_2}) \in \{(X, Y) \mid 0 < X \leq 1, 0 < Y \leq 1\}$. In other words, we look at *frequencies* rather than periods.

The image of the whole set of real-valued instances with three periods and density exactly $\frac{5}{6}$ is

$$J := \left\{ (X, Y) \mid X > Y, \frac{5}{6} < X + 2Y, X + Y < \frac{5}{6} \right\}.$$

(Refer to the beginning of the statement of Lemma 6 for the condition.) The images of the

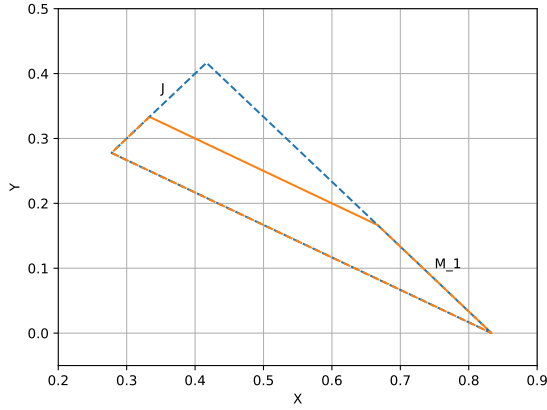


Figure 1: J and M_1 . The dotted lines are excluded.

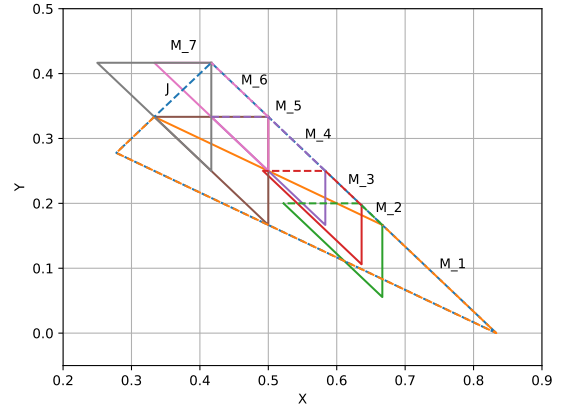


Figure 2: J and $M_1, M_2, \dots, M_7$. The dotted lines are excluded.

whole set of real-valued instances that apply to cases (I) through (VII) are, respectively,

$$\begin{aligned}
M_1 &:= \left\{ (X, Y) \mid X > Y, \frac{5}{6} < X + 2Y \leq 1, X + Y < \frac{5}{6} \right\}, \\
M_2 &:= \left\{ (X, Y) \mid X \leq \frac{2}{3}, Y < \frac{1}{5}, \frac{5}{6} - \frac{1}{9} \leq X + Y < \frac{5}{6} \right\}, \\
M_3 &:= \left\{ (X, Y) \mid X \leq \frac{7}{11}, Y < \frac{1}{4}, \frac{5}{6} - \frac{1}{11} \leq X + Y < \frac{5}{6} \right\}, \\
M_4 &:= \left\{ (X, Y) \mid X \leq \frac{7}{12}, Y < \frac{1}{3}, \frac{5}{6} - \frac{1}{12} \leq X + Y < \frac{5}{6} \right\}, \\
M_5 &:= \left\{ (X, Y) \mid X < \frac{1}{2}, Y \leq \frac{1}{3}, \frac{5}{6} - \frac{1}{6} \leq X + Y < \frac{5}{6} \right\}, \\
M_6 &:= \left\{ (X, Y) \mid X < \frac{1}{2}, Y \leq \frac{5}{12}, \frac{5}{6} - \frac{1}{12} \leq X + Y < \frac{5}{6} \right\}, \text{ and} \\
M_7 &:= \left\{ (X, Y) \mid X \leq \frac{5}{12}, Y \leq \frac{5}{12}, \frac{5}{6} - \frac{1}{6} \leq X + Y < \frac{5}{6} \right\}.
\end{aligned}$$

See Figures 1 and 2.

Lemma 10 may be visually clear from the figures.

Lemma 10. *It holds that $J \subset \bigcup_{i=1}^7 M_i$.*

Proof. First, we obtain $J \setminus M_1 = \{(X, Y) \mid X > Y, 1 < X + 2Y, X + Y < \frac{5}{6}\}$, which is the sides and interior of the triangle $(\frac{2}{3}, \frac{1}{6}), (\frac{5}{12}, \frac{5}{12}), (\frac{1}{3}, \frac{1}{3})$, excluding the two sides $(\frac{2}{3}, \frac{1}{6}), (\frac{5}{12}, \frac{5}{12})$ and $(\frac{2}{3}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{3})$ (see Figure 1). We will show that $J \setminus M_1 \subset \bigcup_{i=2}^7 M_i$ (see Figure 2) by dividing into regions $\frac{7}{11} < X \leq 1, \frac{7}{12} < X \leq \frac{7}{11}, \frac{1}{2} \leq X \leq \frac{7}{12}, \frac{5}{12} < X < \frac{1}{2}$, and $0 < X \leq \frac{5}{12}$.

(i) $(J \setminus M_1) \cap \{(X, Y) \mid \frac{7}{11} < X \leq 1, 0 < Y \leq 1\}$ is the interior of the triangle $(\frac{2}{3}, \frac{1}{6}), (\frac{7}{11}, \frac{13}{66}), (\frac{7}{11}, \frac{2}{11})$. On the other hand, $M_2 \cap \{(X, Y) \mid \frac{7}{11} < X \leq 1, 0 < Y \leq 1\}$ is the sides and interior of the parallelogram $(\frac{2}{3}, \frac{1}{18}), (\frac{2}{3}, \frac{1}{6}), (\frac{7}{11}, \frac{13}{66}), (\frac{7}{11}, \frac{17}{198})$, excluding the side $(\frac{7}{11}, \frac{13}{66}), (\frac{7}{11}, \frac{17}{198})$. Noting that $\frac{17}{198} < \frac{2}{11} < \frac{13}{66}$ for the Y coordinates of the vertices, we obtain $(J \setminus M_1) \cap \{(X, Y) \mid \frac{7}{11} < X \leq 1, 0 < Y \leq 1\} \subset M_2 \cap \{(X, Y) \mid \frac{7}{11} < X \leq 1, 0 < Y \leq 1\}$.

(ii) $(J \setminus M_1) \cap \{(X, Y) \mid \frac{7}{12} < X \leq \frac{7}{11}, 0 < Y \leq 1\}$ is the sides and interior of the trapezoid $(\frac{7}{11}, \frac{2}{11}), (\frac{7}{11}, \frac{13}{66}), (\frac{7}{12}, \frac{1}{4}), (\frac{7}{12}, \frac{5}{24})$, excluding the three sides $(\frac{7}{11}, \frac{13}{66}), (\frac{7}{12}, \frac{1}{4}), (\frac{7}{12}, \frac{5}{24})$, and $(\frac{7}{12}, \frac{5}{24}), (\frac{7}{11}, \frac{2}{11})$. $M_3 \cap \{(X, Y) \mid \frac{7}{12} < X \leq \frac{7}{11}, 0 < Y \leq 1\}$ is the sides and interior of the parallelogram $(\frac{7}{11}, \frac{7}{66}), (\frac{7}{11}, \frac{13}{66}), (\frac{7}{12}, \frac{1}{4}), (\frac{7}{12}, \frac{7}{44})$, excluding the two sides $(\frac{7}{11}, \frac{13}{66}), (\frac{7}{12}, \frac{1}{4})$ and

$(\frac{7}{12}, \frac{1}{4}), (\frac{7}{12}, \frac{7}{44})$. Since $\frac{7}{66} < \frac{2}{11} < \frac{13}{66}$ and $\frac{7}{44} < \frac{5}{24} < \frac{1}{4}$, we have $(J \setminus M_1) \cap \{(X, Y) \mid \frac{7}{12} < X \leq \frac{7}{11}, 0 < Y < 1\} \subset M_3 \cap \{(X, Y) \mid \frac{7}{12} < X \leq \frac{7}{11}, 0 < Y \leq 1\}$.

(iii) $(J \setminus M_1) \cap \{(X, Y) \mid \frac{1}{2} \leq X \leq \frac{7}{12}, 0 < Y \leq 1\}$ is the sides and interior of the trapezoid $(\frac{7}{12}, \frac{5}{24}), (\frac{7}{12}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{4})$, excluding the two sides $(\frac{7}{12}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{4}), (\frac{7}{12}, \frac{5}{24})$. $M_4 \cap \{(X, Y) \mid \frac{1}{2} \leq X \leq \frac{7}{12}, 0 < Y \leq 1\}$ is the sides and interior of the parallelogram $(\frac{7}{12}, \frac{1}{6}), (\frac{7}{12}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{4})$, excluding the side $(\frac{7}{12}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3})$. Due to $\frac{1}{6} < \frac{5}{24} < \frac{1}{4}$, it holds that $(J \setminus M_1) \cap \{(X, Y) \mid \frac{1}{2} \leq X \leq \frac{7}{12}, 0 < Y \leq 1\} \subset M_4 \cap \{(X, Y) \mid \frac{1}{2} \leq X \leq \frac{7}{12}, 0 < Y \leq 1\}$.

(iv) $(J \setminus M_1) \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y \leq 1\}$ is the interior of the trapezoid $(\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3}), (\frac{5}{12}, \frac{5}{12}), (\frac{5}{12}, \frac{7}{24})$. $M_5 \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y < 1\}$ is the sides and interior of the trapezoid $(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{1}{3}), (\frac{5}{12}, \frac{1}{3}), (\frac{5}{12}, \frac{1}{4})$, excluding the two sides $(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{1}{3})$, and $(\frac{5}{12}, \frac{1}{3}), (\frac{5}{12}, \frac{1}{4})$. Besides, $M_6 \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y \leq 1\}$ is the sides and interior of the parallelogram $(\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3}), (\frac{5}{12}, \frac{5}{12}), (\frac{5}{12}, \frac{1}{3})$, excluding the two sides $(\frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{3})$, and $(\frac{5}{12}, \frac{5}{12}), (\frac{5}{12}, \frac{1}{3})$. Note that $\frac{1}{6} < \frac{1}{4}$ and $\frac{1}{4} < \frac{7}{24}$ for the Y coordinates of the vertices. The part of region $(J \setminus M_1) \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y \leq 1\}$ where $Y \leq \frac{1}{3}$ is a subset of $M_5 \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y \leq 1\}$, while the rest of the region is a subset of $M_6 \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y \leq 1\}$. Thus, $(J \setminus M_1) \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y \leq 1\} \subset (M_5 \cup M_6) \cap \{(X, Y) \mid \frac{5}{12} < X < \frac{1}{2}, 0 < Y < 1\}$.

(v) $(J \setminus M_1) \cap \{(X, Y) \mid 0 < X \leq \frac{5}{12}, 0 < Y \leq 1\}$ is the sides and interior of the triangle $(\frac{5}{12}, \frac{7}{24}), (\frac{5}{12}, \frac{5}{12}), (\frac{1}{3}, \frac{1}{3})$, excluding the two sides $(\frac{5}{12}, \frac{5}{12}), (\frac{1}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{1}{3}), (\frac{5}{12}, \frac{7}{24})$. On the other hand, $M_7 \cap \{(X, Y) \mid 0 < X \leq \frac{5}{12}, 0 < Y \leq 1\}$ is the sides and interior of the triangle $(\frac{5}{12}, \frac{1}{4}), (\frac{5}{12}, \frac{5}{12}), (\frac{1}{4}, \frac{5}{12})$. Noting that $\frac{1}{4} < \frac{7}{24}$ for the Y coordinates of the vertices and the point $(\frac{1}{3}, \frac{1}{3})$ lies on the side $(\frac{1}{4}, \frac{5}{12}), (\frac{5}{12}, \frac{1}{4})$, we obtain $(J \setminus M_1) \cap \{(X, Y) \mid 0 < X \leq \frac{5}{12}, 0 < Y < 1\} \subset M_7 \cap \{(X, Y) \mid 0 < X \leq \frac{5}{12}, 0 < Y < 1\}$. $\square$

Proof of Lemma 6. Lemmas 7, 8, and 9 imply that instances that apply to cases (I) through (VII) are schedulable and a valid schedule is given either of by (7), (8), (9), (10), (11), (12), or (13). Lemma 10 states that any real-valued instance with three periods and density less than or equal to $\frac{5}{6}$ satisfies one of the conditions of cases (I) through (VII). $\square$

We continue the analysis of real-valued instances with three periods. The proof of Theorem 2 based on Lemma 6 provides us with valid schedules for real-valued instances with three periods and density less than or equal to $\frac{5}{6}$. For an instance that applies to case (I) of Lemma 6, the schedule is determined by the individual period values of the instance. Namely, the schedule is instance-sensitive, which did not occur in the case of two periods in Section 2. (The same is true if the schedulability is derived from Lemmas 1 and 2, and Theorem 4.)

On the other hand, for an instance that applies to cases (II) through (VII), the schedule is independent of the individual period values of the instance. In particular, the last instance of Lemma 8, which corresponds to the condition of case (VII), has no ε , and the instance itself is the one with minimum periods among those schedulable. This fact can be expressed in the context of the paper [GSW22] as follows:

Corollary 2. $\{((\frac{12}{5}, \frac{12}{5}, 6), |121213212123|)\}$ is a Pareto surface of the set of all real-valued instances (a_1, a_2, a_3) such that $a_1 \geq \frac{12}{5}$, $a_2 \geq \frac{12}{5}$, $a_3 \geq 6$, and $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{5}{6}$.

Unlike Corollary 1, we conjecture that there is no Pareto surface of the set of all real-valued instances with three periods and density less than or equal to $\frac{5}{6}$. In contrast, $\{((2, 4, 4), |1213|), (3, 3, 3), |123|)\}$ is known to be a Pareto surface of the set of all *integer-valued* instances with three periods and density less than or equal to $\frac{5}{6}$ [GSW22].

Rather than a Pareto surface, let us focus on a common valid schedule for a given set of instances. As we have seen in Lemmas 6, 8, and 9, such a schedule is found for instances that apply to each of cases (II) through (VII). Then, what about the condition of case (I)? It is

seen in Figure 2 that almost half area of the trapezoid M_1 is covered by $M_2, M_3, \dots, M_7$, which means that there is a finite set of valid schedules for instances that correspond to the covered region. However, we do not know anything about the uncovered region.

4 Future Work

Towards Conjecture 2, we would like to build a framework that comprehensively handles real-valued instance sets, analogous to the Pareto surface for integer-valued instance sets [GSW22]. Although our current methodology could be extended to the case where the periods take four or five distinct values, solving the general case would be difficult.

The method of Kawamura [Kaw24] for deriving Theorem 4 seems hopeless to immediately apply. The method relies on the fact that: The subset of all integer-valued instances such that the periods and the density are less than a certain threshold is finite and therefore the schedulability can be investigated using a computer. However, a subset of such real-valued instances is in general infinite.

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