DEFORMATION OF NEF ADJOINT CANONICAL LINE BUNDLES

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Abstract. Much inspired by J. A. Wiśniewski's nef-value function method, we prove that in a smooth projective family over the unit disk, if the adjoint bundle of the canonical line bundle with a relatively semiample line bundle is nef on one fiber, then it remains nef on all fibers. We further extend this result to the semiampleness of the adjoint canonical line bundles. Using these, we prove the deformation invariance of any generalized plurigenera by assuming that only one fiber admits the semiample canonical line bundle and improve the first author–Xiao-Lei Liu's recent deformation rigidity of projective manifolds with semiample canonical line bundles. In particular, also by E. Viehweg–K. Zuo's result on the minimal number of singular fibers in a family and the first author–X. Liu's isotriviality result, if a projective family over $\mathbb{P}^1$ or an elliptic curve has one fiber with the big and nef (or more generally semiample) canonical line bundle, then all fibers are isomorphic to this fiber.

Next, much inspired by M. Andreatta–T. Peternell's deformation theoretical approach, we prove that, in a smooth Kähler family of threefolds, if the canonical line bundle of one fiber is not nef, then none of its small deformations admits a nef canonical line bundle either. This partially confirms a problem posed by F. Campana–T. Peternell and the global stability of semiampleness of canonical line bundles of threefolds under a Kähler smooth deformation.

1. Introduction: main results and corollaries

Let $\pi : \mathcal{X} \to B$ be a smooth proper morphism between complex manifolds with fiber $X_t := \pi^{-1}(t)$ for any $t \in B$. Denote by $K_{\mathcal{X}}$ the canonical line bundle on $\mathcal{X}$ (and similarly for the fibers and other complex manifolds). Then $K_{X_t} = K_{\mathcal{X}}|_{X_t}$ by adjunction formula. Recall that additionally if $\pi : \mathcal{X} \to B$ is a projective morphism, then π is called a *smooth projective family.*

For a smooth projective family over a projective manifold, J. A. Wiśniewski [Wi91b, Wi09] proves that if the canonical line bundle of one fiber is not nef, then the ones of all other fibers are not nef, either. More complex analytically, for a smooth projective family over a complex manifold, M. Andreatta–T. Peternell [AP97] prove that if the canonical line bundle of the central fiber is not nef, then none of its small deformations admits a nef canonical line bundle under the restriction on extremal contractions over the central fiber.

Similar to [AP97], our work is carried out in the complex analytic setting, focusing on the deformation behavior of nef adjoint canonical line bundles. For convenience, we use additive notation to describe the tensor operation on line bundles, and always denote by $\pi : \mathcal{X} \to \Delta$ a smooth family over the unit disk Δ in $\mathbb{C}$.

1.1. Deformation of nefness for projective families.

One main theorem of this paper is:

Theorem 1.1 (=Theorem 3.9). *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and L a π-semiample line bundle on $\mathcal{X}$. If $K_{X_0} + L_0$ is nef, then $K_{X_t} + L_t$ is nef for any $t \in \Delta$. Here and henceforth, denote by L_t the restriction $L|_{X_t}$.*

In fact, much inspired by the nef-value function method of Wiśniewski [Wi91b, Wi09], we use the minimal model program of projective morphism between complex analytic spaces by N.

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Nakayama [Na87], O. Fujino [Fu22] and O. Das–C. Hacon–M. Păun [DHP24] to obtain that for a relatively semiample line bundle L on $\mathcal{X}$, the *nef locus* of $K_{\mathcal{X}} + L$

$$\mathcal{N}(K_{\mathcal{X}} + L) := \{t \in \Delta \mid K_{X_t} + L_t \text{ is nef}\}$$

of adjoint canonical line bundle in a smooth projective family over the unit disk is empty or the whole unit disk.

Combining Theorem 1.1 with Siu's remarkable deformation invariance of semipositively twisted plurigenera [Si02, Corollary 0.2] and Kawamata's characterization of semiampleness of adjoint canonical line bundles [Ka85, Theorem 6.1], we get the global deformation stability of semiampleness of adjoint canonical line bundles.

Corollary 1.2 (= Corollary 4.2)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and L a π-semiample line bundle on $\mathcal{X}$. If $K_{X_0} + L_0$ is semiample, then for any $t \in \Delta$, $K_{X_t} + L_t$ is semiample and $K_{\mathcal{X}} + L$ is thus π-semiample near X_t.*

As a direct application of Corollary 1.2, one obtains the deformation invariance of generalized plurigenera:

Theorem 1.3 (=Theorem 4.8)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family. If K_{X_0} is semiample, then for any $i \geq 0$ and $m \geq 1$, the generalized m-genus $P^i_m(X_t) := \dim_{\mathbb{C}} H^i(X_t, K_{X_t}^{\otimes m})$ is independent of $t \in \Delta$.*

Applying Kawamata–Viehweg vanishing theorem and Grauert's upper semi-continuity theorem for higher cohomology, we have a slight generalization of [Ca91, Proposition 3.16]:

Corollary 1.4 (=Corollary 4.14)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family. If the central fiber X_0 is a minimal manifold of general type, then all fibers are minimal manifolds of general type.*

Recall that if there is a smooth Kähler form ω on $\mathcal{X}$, i.e., ω is a smooth d-closed positive $(1, 1)$-form, then the smooth family $\pi : \mathcal{X} \to \Delta$ is called a *smooth Kähler family* here.

We also improve the first author–X. Liu's recent deformation rigidity [LL24, Theorem 1.2] of projective manifolds with semiample canonical line bundles.

Set the *S-locus* of the family $\pi : \mathcal{X} \to B$ over the base B as

$$\mathcal{S} := \{t \in B : X_t \cong S\}.$$

Theorem 1.5 (=Theorem 4.18)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family, and S a projective manifold with the semiample canonical line bundle. Then $\mathcal{S}$ is either at most a discrete subset of Δ or the whole Δ.*

[LRW25, Example 1.2] presents a smooth family of genus $g \geq 2$ curves which contains countably (but not finitely) many isomorphic fibers. So there indeed exists a smooth projective family with discretely (but not finitely) many isomorphic fibers.

As a direct application of Theorem 1.5, one has:

Corollary 1.6. *Let $\pi : \mathcal{X} \to Y$ be a smooth Kähler family over a smooth algebraic curve Y, and S a projective manifold with the semiample canonical line bundle. Then $\mathcal{S}$ is either at most finite or the whole Y.*

In particular, using Corollary 1.4, E. Viehweg–K. Zuo's minimal number [VZ01, Theorem 0.1] of singular fibers in a family and the first author–X. Liu's isotriviality [LL24, Theorem 7.1], one obtains:

Corollary 1.7 (=Corollary 4.21)**.** *Let $\pi : \mathcal{X} \to Y$ be a smooth Kähler family over Y, where Y is isomorphic to $\mathbb{P}^1$ or an elliptic curve. Let S be a projective manifold with the big and nef canonical line bundle. Then $\mathcal{S}$ is either empty or the whole Y.*

Upon closer examination, we are able to establish a more general rigidity theorem.

Theorem 1.8 (=Theorem 4.22)**.** *Let $\pi : \mathcal{X} \to Y$ be a smooth projective family over Y, where Y is isomorphic to $\mathbb{P}^1$ or an elliptic curve. Let S be a projective manifold with the semiample canonical line bundle. Then $\mathcal{S}$ is either empty or the whole Y.*

This can also be deduced from Corollary 1.6 and [Dn22, Theorem A] or [DLSZ24, Theorem B].

1.2. **Deformation of nefness for Kähler families.** H.-Y. Lin [Ln24, Theorem 1.1] proves that every 3-dimensional compact Kähler manifold X has an algebraic approximation, i.e., there is a smooth family $\mathcal{X} \to \Delta$ of threefolds with central fiber $X_0 = X$ and there is a sequence of points $\{t_\mu\} \to 0$ as $\mu \to \infty$ such that X_{t_μ} are projective for all large μ. And [Pe98, Theorem 4.1] by Peternell implies that if K_{X_0} is not nef, then $K_{X_{t_\mu}}$ is not nef for μ large enough. This motivates the following question.

Question 1.9. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family of n-folds. If K_{X_0} is not nef, is it true that K_{X_t} is not nef for any sufficiently small t?*

See also [CP99, Problem 3.14], where π is assumed to be a smooth family of compact Kähler manifolds. As the other main theorem of this paper, much inspired by Andreatta–Peternell's deformation theoretical approach [AP97], we use the classification of divisorial contractions of 3-dimensional compact Kähler manifolds [Pe98, Main Theorem] and the characterization of the nefness of the canonical line bundle to answer Question 1.9 in the case of $n \leq 3$.

Theorem 1.10 (= Theorem 3.24)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family of n-folds with $n \leq 3$. If K_{X_0} is not nef, then no K_{X_t} is nef for any sufficiently small t.*

Recall the famous abundance theorem of Kähler threefolds proved by F. Campana–A. Höring–Peternell [CHP16, CHP23] (and also O. Das–W. Ou [DO23, DO24]):

Theorem 1.11. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is nef. Then K_X is semiample.*

As a direct corollary of Theorems 1.10, 1.11 and M. Levine's deformation invariance of plurigenera [Lv83, Corollary 1.10], one obtains the global stability of semiampleness of canonical line bundles and deformation invariance of generalized plurigenera by Nakayama's argument [Na87].

Corollary 1.12 (=Corollary 4.10)**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family of threefolds. If K_{X_0} is nef (or equivalently semiample), then for any $t \in \Delta$, K_{X_t} is semiample and $K_{\mathcal{X}}$ is π-semiample over $\pi^{-1}(U_t)$, where U_t is a Zariski neighborhood of t. Furthermore, for any $i \geq 0$ and $m \geq 1$, the generalized m-genus $P^i_m(X_t)$ is independent of $t \in \Delta$.*

Based on these, it is reasonable to propose:

Conjecture 1.13. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family of compact Kähler manifolds or even compact complex manifolds in the Fujiki class $\mathcal{C}$ (i.e., bimeromorphic to compact Kähler manifolds). If K_{X_0} is nef, then for any $t \in \Delta$, K_{X_t} is semiample and $K_{\mathcal{X}}$ is π-semiample over $\pi^{-1}(U_t)$, where U_t is a Zariski neighborhood of t. Furthermore, for any $i \geq 0$ and $m \geq 1$, the generalized m-genus $P^i_m(X_t)$ is independent of $t \in \Delta$.*

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2. Preliminaries: positivities, moduli and minimal model program

In this section, we introduce the preliminaries on the positivities of line bundles, moduli space of polarized manifolds and minimal model program of projective morphism between complex analytic spaces, to be used in this paper.

2.1. **Positivities of line bundles.** Let us review various positivities of holomorphic line bundles in the absolute and relative cases.

2.1.1. *Absolute case.* Let X be a connected compact complex manifold of dimension n and L a holomorphic line bundle on X. Recall that the *Kodaira dimension* of L is defined to be

$$\kappa(L) := \limsup_{m\to\infty} \frac{\log h^0(X, L^{\otimes m})}{\log m}.$$

Then $\kappa(L) \in \{-\infty, 0, \cdots, n\}$. Denote by K_X the canonical line bundle of X and define $\kappa(X)$ as $\kappa(K_X)$. If $\kappa(L) = n$, then L is called *big.* It is well known that L is big if and only if the first Chern class $c_1(L)$ of L contains a Kähler current representative. If X admits a big line bundle, then X is a *Moishezon* manifold. In particular, if K_X is big, then X is said to be of *general type.*

If $c_1(L)$ contains a closed positive (1,1)-current, then L is called *pseudo-effective.* Let ω_X be a smooth Hermitian metric on X. For any $\epsilon > 0$, if there is a smooth representative $\alpha_\epsilon \in c_1(L)$ such that $\alpha_\epsilon \geq -\epsilon\omega_X$, then L is called *nef.* If L is nef, the *numerical dimension* of L is defined as

$$\nu(L) := \max\{k \in \mathbb{N} \mid c_1(L)^k \neq 0\}.$$

Moreover, if $\nu(L) = \kappa(L)$, then L is called *good.* In particular, if K_X is nef, we define $\nu(X)$ by $\nu(K_X)$. If L is big and nef, then L is good. And Kawamata [Ka85, Theorem 1.1] proves that K_X is semiample if and only if K_X is nef and good. Let $\{s_1, \cdots, s_m\}$ be a basis of $H^0(X, L)$. Then the *Kodaira map* associated to L is defined to be:

$$\phi_L : X \dashrightarrow \mathbb{P}(H^0(X, L)^*), x \mapsto [s_1(x) : \cdots : s_m(x)].$$

If $H^0(X, L^{\otimes m}) = 0$ for any integer $m > 0$, then $\kappa(L) = -\infty$. Otherwise, it is well known that

$$\kappa(L) = \max_m\{\dim_{\mathbb{C}} \phi_{L^{\otimes m}}(X)\}.$$

Then the *base locus* $\mathrm{Bs}(L)$ of L is defined to be

$$\mathrm{Bs}(L) = \{x \in X \mid s(x) = 0 \text{ for any } s \in H^0(X, L)\}.$$

If $\mathrm{Bs}(L) = \emptyset$, then L is called *globally generated.* If $L^{\otimes m}$ is globally generated for some $m > 0$, then L is called *semiample.* If L is globally generated and ϕ_L is an embedding, then L is called *very ample.* If $L^{\otimes m}$ is very ample for some $m > 0$, then L is called *ample.* So if X admits an ample line bundle, then X is a *projective* manifold. By Kodaira's embedding theorem, L is ample if and only if $c_1(L)$ is a Kähler class.

When X is Moishezon or in particular projective, L is nef if and only if $L \cdot C \geq 0$ for any curve $C \subset X$ by [Pa98, Corollary 1]. For Kähler manifolds, this is not right in general but for the canonical line bundle.

2.1.2. *Relative case.* Let $\pi : X \to S$ be a proper holomorphic map between complex manifolds with fiber $X_s := \pi^{-1}(s)$ for $s \in S$ and L a holomorphic line bundle on X.

If the canonical morphism

$$\phi : \pi^*\pi_*L \to L$$

is surjective, then L is called *π-globally generated.* If $L^{\otimes m}$ is π-globally generated for some $m > 0$, then L is called *π-semiample.* If L is π-globally generated and induces the embedding

$$X \to \mathbb{P}(\pi_*L)$$

over S, then L is called *π-very ample.* If $L^{\otimes m}$ is π-very ample for some $m > 0$, then L is called *π-ample.* It is well known that L is π-ample if and only if $L|_{X_s}$ is ample for any $s \in S$. If X admits a π-ample line bundle, then π is called *projective.*

If $L|_{X_s}$ is big (resp. nef) for any $s \in S$, then L is called *π-big* (resp. *π-nef*). If X admits a π-big line bundle, then π is called Moishezon. Obviously, π is Moishezon if and only if π is bimeromorphic to a projective morphism. By definition, π is called *Moishezon* if π is bimeromorphic to a projective morphism over the same base S. Furthermore, if π is smooth over Δ, this definition amounts to the existence of a π-big line bundle by the bimeromorphic embedding [RT21, Theorem 1.4], [RT22, Theorem 1.8].

If π is a smooth projective family over the unit disk in $\mathbb{C}$, Siu [Si98, Si02] proves that for any positive integer m, the m-genus of the fibers X_s (i.e., the dimension $h^0(X_s, K_{X_s}^{\otimes m})$ of the global section of m-canonical line bundle over the fibers X_s) is locally constant and thus $\kappa(X_s)$ is locally constant. The fiberwise Moishezon case is obtained in [RT22, Theorem 1.2.(i)].

2.2. **Moduli of polarized manifolds.** For a polynomial $h \in \mathbb{Q}[T]$, let $\mathcal{P}_h$ be the fibered category over the category of schemes, such that for a scheme B, the groupoid $\mathcal{P}_h(B)$ is as follows

$$\begin{aligned}\mathcal{P}_h(B) = \{(f : X \to B) \mid\ & f \text{ is a smooth morphism}; \mathcal{H} \text{ is a line bundle on } X;\\ & (X_b, \mathcal{H}_b) \text{ is a polarized manifold, } K_{X_b} \text{ is semiample, for any } b \in B;\\ & h(m) = \chi(\mathcal{H}_b^{\otimes m}),\ m \in \mathbb{Z}\}.\end{aligned}$$

The arrow of $\mathcal{P}_h(B)$ from $(f : X \to B, \mathcal{H})$ to $(f' : X' \to B, \mathcal{H}')$ is an isomorphism $\tau : X \to X'$ such that $\mathcal{H}_b$ and $\tau^*\mathcal{H}'|_{X_b}$ are numerically equivalent for all $b \in B$. By [Vi91, Theorem 4.6], there exists a coarse separated moduli algebraic space P_h for $\mathcal{P}_h$. Note that P_h is the quotient of the moduli space of polarized manifolds by compact equivalence relations by [Vi91].

If $(\phi : X \to B, \mathcal{H})$ is a smooth family of polarized manifolds with semiample canonical line bundles and Hilbert polynomial $h(m) = \chi(\mathcal{H}_b^{\otimes m})$ for all $m \in \mathbb{Z}$ and $b \in B$, then we have an induced morphism

$$\mu_\phi : B \to P_h$$

satisfying that $\mu_\phi(b) = (X_b, \mathcal{H}_b) \in P_h$ for all $b \in B$.

2.3. **Minimal model program of projective morphisms between complex analytic spaces.** We refer to [Fu22, Na87, Na04] for the minimal model program of projective morphisms between complex analytic spaces.

Let X be a normal complex analytic variety. The *canonical sheaf* ω_X is the unique reflexive sheaf whose restriction to X_{sm} is isomorphic to the sheaf $\Omega^{\dim_{\mathbb{C}} X}_{X_{sm}}$, where X_{sm} is the smooth locus of X. Note that unlike the case of algebraic varieties, ω_X here does not necessarily correspond to a Weil divisor K_X such that $\omega_X \cong \mathcal{O}_X(K_X)$. However, by abuse of notation we will say that K_X is a canonical divisor when we actually mean the canonical sheaf ω_X. This doesn't create any problem in general as running the minimal model program involves intersecting subvarieties with ω_X.

Let $\pi : X \to Y$ be a proper surjective morphism between complex analytic spaces and W a Stein compact subset of Y. Recall that a compact subset on a complex analytic space is said to be *Stein compact* if it admits a fundamental system of Stein open neighborhoods.

And $\pi : X \to Y$ and W satisfy the *conditions* (P) if

- (P1) X is a normal complex variety;
- (P2) Y is a Stein space;
- (P3) W is a Stein compact subset of Y;
- (P4) $W \cap Z$ has only finitely many connected components for any analytic subset Z which is defined over an open neighborhood of W.

Remark 2.1. Assume that Y is Stein. And we take a π-ample line bundle $\mathcal{A}$ on X. Since Cartan's theorem A implies that there exists a sufficiently large positive integer m such that

$$H^0(X, \omega_X \otimes \mathcal{A}^{\otimes m}) \cong H^0(Y, \pi_*(\omega_X \otimes \mathcal{A}^{\otimes m})) \neq 0$$

and

$$H^0(X, \mathcal{A}^{\otimes m}) \cong H^0(Y, \pi_*(\mathcal{A}^{\otimes m})) \neq 0,$$

we can always take a Weil divisor K_X on X satisfying $\omega_X \cong \mathcal{O}_X(K_X)$. As usual, we call it the *canonical divisor* of X. More generally, let $\mathcal{L}$ be a line bundle (resp. reflexive sheaf of rank one) on X. By the same argument as above, we can take a Cartier (resp. Weil) divisor D on X such that $\mathcal{L} \cong \mathcal{O}_X(D)$.

Theorem 2.2 (Cartan's theorem A)**.** *Let Y be a Stein space and $\mathcal{F}$ a coherent analytic sheaf over Y. Then $\mathcal{F}$ is globally generated by its sections, i.e., for every point $y \in Y$, the stalk $\mathcal{F}_y$ is generated by the germs at z of global sections of $\mathcal{F}$ over Y.*

Let

$$D := \Sigma a_i D_i$$

be an effective $\mathbb{Q}$-divisor on the normal complex analytic space X, where D_i are prime divisors, such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\mu : \widetilde{X} \to X$ be a proper bimeromorphic morphism between normal complex varieties, and

$$K_{\widetilde{X}} + \mu_*^{-1} D \sim_{\mathbb{Q}} \mu^*(K_X + D) + \sum_i b_i E_i,$$

where E_i are prime exceptional divisors and $\mu_*^{-1} D$ is the strict transform of D. The pair (X, D) is called *pure log terminal* (*plt*) if for any $\mu : \widetilde{X} \to X$ and every μ-exceptional divisor E_i, the *log discrepancies* $a(E_i, X, D) := b_i > -1$. Furthermore, if the round-down $\lfloor D \rfloor$ of D is 0, then the pair (X, D) is called *Kawamata log terminal* (*klt*). Moreover, if there exists *log resolution* of the pair (X, D) (i.e., a proper bimeromorphic morphism $\mu : \widetilde{X} \to X$ from a non-singular complex variety $\widetilde{X}$ such that the exceptional locus $\mathrm{Exc}(\mu)$ and $\mu_*^{-1}(D) \cup \mathrm{Exc}(\mu)$ are simple normal crossing divisors on $\widetilde{X}$), and the *log discrepancies* $a(E_i, X, D) := b_i > -1$ for every μ-exceptional divisor E_i, then (X, D) is called *divisorial log terminal* (*dlt*).

Let V be an $\mathbb{R}$-vector space. A subset $N \subset V$ is called a *cone* if $0 \in N$ and N is closed under multiplication by positive scalars. A subcone $M \subset N$ is called *extremal* if $u, v \in N, u + v \in M$ implies that $u, v \in M$. A one-dimensional extremal subcone is called an *extremal ray.*

The free abelian group $Z_1(X/Y; W)$ is generated by the projective integral curves C on X such that $\pi(C)$ is a point of W. Take C_1, $C_2 \in Z_1(X/Y; W) \otimes_{\mathbb{Z}} \mathbb{R}$. If $C_1 \cdot L = C_2 \cdot L$ holds for every $L \in \mathrm{Pic}(\pi^{-1}(U))$ and every open neighborhood U of W, then we write $C_1 \equiv_W C_2$. Set

$$N_1(X/Y; W) := Z_1(X/Y; W) \otimes_{\mathbb{Z}} \mathbb{R} / \equiv_W .$$

Then $(P4)$ implies that $N_1(X/Y; W)$ is a finite $\mathbb{R}$-vector space. The *Kleiman–Mori cone*

$$\overline{NE}(X/Y; W)$$

is defined to be the closure of the convex cone in $N_1(X/Y; W)$ generated by projective integral curves C on X such that $\pi(C)$ is a point in W.

Nakayama [Na87], Fujino [Fu22] and Das–Hacon–Păun [DHP24] develop the theory of projective morphisms between complex analytic spaces. We just list a few of them to be used later, although they hold more generally for boundary $\mathbb{R}$-divisors and mostly dlt pairs.

Let $\pi : X \to Y$ be a projective surjective morphism between complex analytic spaces, W a Stein compact subset of Y and (X, Γ) a klt pair.

Theorem 2.3 (Rationality theorem, [Na87, Theorem 4.11], [Fu22, Theorem 7.1]). *Assume that $\pi : X \to Y$ and W satisfy conditions (P). Let H be a π-ample Cartier divisor on X. Assume that $K_X + \Gamma$ is not π-nef over W. Then*

$$r := \max\{t \in \mathbb{R} \mid H + t(K_X + \Gamma) \textit{ is } \pi\textit{-nef over } W\}$$

is a positive rational number.

Theorem 2.4 (Basepoint-free theorem, [Na87, Theorem 4.10], [Fu22, Theorem 6.2]). *Let D be a Cartier divisor on X and nef over W. Assume that $aD - (K_X + \Gamma)$ is π-ample for some positive real number a. Then there exists an open neighborhood U of W and a positive integer m_0 such that*

$$\pi^* \pi_* \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)$$

is surjective over U for any $m \geq m_0$.

Theorem 2.5 (Cone and contraction theorem, [Na87, Theorem 4.12], [Fu22, Theorem 7.2]). *Assume that $\pi : X \to Y$ and W satisfy conditions (P). Then we have the decomposition of Kleiman–Mori cone*

$$\overline{NE}(X/Y; W) = \overline{NE}(X/Y; W)_{K_X + \Gamma \geq 0} + \Sigma_j R_j$$

with the following properties. Let R be a $(K_X+\Gamma)$-negative extremal ray. Then, after one shrinks the base Y around W suitably, there exists a contraction morphism $\phi_R : X \to Z$ over Y such that

1. *Let C be an integral projective curve on X such that $\pi(C)$ is a point in W. Then $\phi_R(C)$ is a point if and only if $[C] \in R$;*
2. *$\phi_{R_*}(\mathcal{O}_X) \cong \mathcal{O}_Z$;*
3. *Let L be a line bundle on X such that $L \cdot C = 0$ for every curve C with $[C] \in R$. Then there is a line bundle M on Z such that $\phi_R^* M \cong L$.*

3. Global stability of nefness: Proofs of Main Theorems 1.1 and 1.10

Much inspired by the nef-value function method of Wiśniewski [Wi91b, Wi09], we use the contraction theorem of projective morphism between analytic spaces to obtain the invariance of the nef-value function $\eta(t)$ and thus the density of the nef locus of the adjoint canonical line bundle, to conclude the proof of Theorem 1.1. And much inspired by Andreatta–Peternell's deformation theoretical approach [AP97], we use the contraction theorem in Kähler threefolds and deform the K_X-negative rational curve to the nearby fibers to prove Theorem 1.10. We will divide this section into two subsections according to these two cases.

3.1. **Stability of nefness for smooth projective families.** Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective morphism with fiber $X_t := \pi^{-1}(t)$ for $t \in \Delta$ and D a $\mathbb{Q}$-Cartier divisor which contains no fibers of π on $\mathcal{X}$ such that $(\mathcal{X}, D)$ is a klt pair. And let A be a smooth π-very ample divisor on $\mathcal{X}$. Denote by $K_{X_t} + D_t$ the restriction $(K_{\mathcal{X}} + D)|_{X_t}$ and similarly for others. The *nef-value function* of $K_{\mathcal{X}} + D$ on Δ respect to A is defined by

$$\eta(t) = \inf\{s \in \mathbb{Q}_{\geq 0} \mid K_{X_t} + D_t + sA_t \text{ is ample}\}.$$

Then the rationality Theorem 2.3 tells us that $\eta(t)$ is a rational number. And it is easy to check:

Lemma 3.1. *For any $t \in \Delta$, $K_{X_t} + D_t$ is not nef if and only if $\eta(t) > 0$.*

Lemma 3.2. *Let D be π-nef over a compact subset W of Δ and R a $(K_{\mathcal{X}}+D)$-negative extremal ray of Kleiman–Mori cone $\overline{NE}(\mathcal{X}/\Delta; W)$. Let F be an irreducible component of a non-trivial fiber of the contraction of R. Then*

$$\dim F + \dim(\mathcal{L}(R)) \geq \dim \mathcal{X} = \dim X_0 + 1,$$

where $\mathcal{L}(R)$ is the locus of curves from R.

Proof. The proof is almost the same as those of [In, (0.4) Theorem] and [Wi91a, (1.1) Theorem] (based on Mori's seminal work [Mo79]), but for reader's convenience, we include a proof here.

Let C be a rational curve in $\mathcal{X}$. Then $\pi|_C : C \to \Delta$ is a constant holomorphic map by maximal principle, i.e., any rational curve lies in one fiber of π. Let $x \in F$ be a general point of F and $C_0 \subset F$ a rational curve through x such that the intersection number $-(K_{\mathcal{X}}+D)\cdot C_0$ is minimal among all rational curves contained in F and containing x. Denote by T the irreducible variety parameterizing deformations of the curve C_0. Set

$$V = \{(x,t) \in \mathcal{X} \times T : x \text{ lies on the rational curve parameterized by } t\}$$

and then the diagram

$$\begin{tikzcd} V \arrow[r, "q"] \arrow[d, "p"'] & T \\ \mathcal{X} & \end{tikzcd}$$

follows. Since D is π-nef, $D \cdot C_0 \geq 0$. So [Ko96, 1.14 Theorem, 1.17 Remark of Chapter II] give

$$\dim T \geq \dim \mathcal{X} - K_{\mathcal{X}} \cdot C_0 - 3 = \dim \mathcal{X} - (K_{\mathcal{X}} + D) \cdot C_0 + D \cdot C_0 - 3$$

which means

$$\dim T \geq \dim \mathcal{X} - 2.$$

And as the fibers of q are of dimension 1, we get

$$\dim V \geq \dim \mathcal{X} - 1.$$

Let E be an irreducible component of the fiber $p^{-1}(x)$. Then the curves parameterized by the set $q(E)$ pass through x and are contained in F. If among curves parameterized by $q(E)$ we find a continuous family of curves passing through a point x' different from x, then all these rational curves and x' are in the smooth projective fiber $\pi^{-1}(x)$. So the famous bend and break theorem by Mori implies that the curve C_0 could be broken into a sum of curves lying in F and $-(K_{\mathcal{X}} + D) \cdot C_0$ would not be minimal. Therefore, the map p restricted to $q^{-1}(q(E))$ is generically finite into F and

$$\dim E = \dim q^{-1}(q(E)) - 1 \leq \dim F - 1.$$

Finally, since the map p from the irreducible variety V has a fiber of dimension $\leq \dim F - 1$ it follows that

$$\dim \mathcal{L}(R) \geq \dim V - \dim F + 1 \geq \dim \mathcal{X} - \dim F.$$

□

Remark 3.3. With the same setting as Lemma 3.2, it is easy to see the inequality from the above proof

$$\dim F + \dim(\mathcal{L}(R)) \geq \dim \mathcal{X} + \ulcorner l(R) \urcorner - 1,$$

where $\ulcorner l(R) \urcorner$ is the round up of the length

$$l(R) := \min\{-(K_{\mathcal{X}} + D) \cdot C : C \text{ rational curve and it numerical class } [C] \in R\}$$

of the extremal ray R.

Lemma 3.4. *Let D_0 be nef on X_0 and R a $(K_{\mathcal{X}} + D)$-negative extremal ray of Kleiman–Mori cone $\overline{NE}(\mathcal{X}/\Delta; W)$ with $W := \{t \in \mathbb{C} \mid |t| \leq 1/2\}$. Assumes that the locus of curves from R has non-empty intersection with the fiber X_0. Then after one shrinks Δ around W, the locus of curves from R dominates some small open neighborhood of the origin* 0.

Proof. Obviously,

$$\{t \in \mathbb{C} \mid |t| < s\}_{s>1/2}$$

form a fundamental system of Stein open neighborhoods of W. And W also satisfies the condition $(P4)$. Hence, $\pi : \mathcal{X} \to \Delta$ and W satisfy the conditions (P). Let $\phi_R : \mathcal{X} \to \mathcal{Y}$ be the contraction of the ray R by the cone and contraction Theorem 2.5 and H a good supporting divisor of R, i.e., the pull-back of a relative ample divisor of $\mathcal{Y}$.

Assume that the locus $\mathcal{L}(R)$ of curves from R doesn't dominate any small open neighborhood of the origin 0. Set $\mathring{W} := \{t \in \mathbb{C} \mid |t| < 1/2\}$ as the interior of W. As $\mathcal{L}(R) \cap \pi^{-1}(\mathring{W})$ is an analytic subset, $\pi(\mathcal{L}(R)) \cap \mathring{W}$ is a proper analytic subset of $\mathring{W}$, by Remmert's proper mapping theorem. So for any small $t \neq 0$ the restriction ϕ_R to the fibers X_t is finite to one and then H_t is ample on X_t. Then for an irreducible component $Z_0 \subset X_0$ of the exceptional set of ϕ_R, the nefness of D_0 and the proof of Lemma 3.2 give rise to

$$2 \dim_{\mathbb{C}} Z_0 - \dim_{\mathbb{C}} \phi_R(Z_0) \geq \dim_{\mathbb{C}} \mathcal{X} = \dim_{\mathbb{C}} X_0 + 1,$$

while [Wi91b, Lemma 1.1] gives

$$2 \dim_{\mathbb{C}} Z_0 - \dim_{\mathbb{C}} \phi_R(Z_0) \leq \dim_{\mathbb{C}} X_0.$$

This is obviously a contradiction. □

Lemma 3.5. *If $\eta(0) > 0$ and D_0 is nef on X_0, then $\eta(t)$ is constant around some open neighborhood of* 0.

Proof. Let A be a smooth π-very ample divisor on $\mathcal{X}$. Then for any $r > 0$,

$$\{t \in \Delta \mid \eta(t) < r\} = \bigcup_{\mathbb{Q}_{\geq 0} \ni s < r} \{t \in \Delta \mid K_{X_t} + D_t + sA_t \text{ is ample}\}$$

is a Zariski open subset of Δ and thus, $\eta(t) \leq \eta(0)$ for any t near 0. Assume that for any $0 \neq |t| \leq 1/2$, $\eta(t) < \eta(0)$. Then the base-point free Theorem 2.4 and the rationality Theorem 2.3 imply that for some positive integer m,

$$L := m(K_{\mathcal{X}} + D + \eta(0)A)$$

is a π-globally generated divisor over $W := \{t \in \mathbb{C} \mid |t| \leq 1/2\}$. Since the unit disk is Stein and W is a Stein compact subset of Δ, the cone and contraction Theorem 2.5 tells us that after one shrinks the base Δ around W, there is a contraction morphism $\phi_R : \mathcal{X} \to \mathcal{Y}$ over Δ associated to L. Since L_t is very ample for any small $t \neq 0$, the exceptional locus of the contraction morphism ϕ_R is contained in the central fiber X_0, which contradicts Lemma 3.4. □

Remark 3.6. In [FH11, Remark 4.5 and Example 5.3], de Fernex–Hacon present two smooth projective families with non-constant nef-value functions of the adjoint canonical line bundles, whose boundary divisors admit no sufficient positivities on X_0. So the nefness assumption of D_0 in Lemma 3.5 is essential.

Proposition 3.7. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and L a π-semiample line bundle on $\mathcal{X}$. If the adjoint canonical line bundles $K_{X_t} + L_t$ are nef for all $t \neq 0$, then $K_{X_0} + L_0$ is still nef.*

Proof. Since L is a π-semiample line bundle, $L^{\otimes m}$ is π-globally generated for some integer $m \geq 1$. And since Δ is a Stein manifold, Cartan's theorem A (Theorem 2.2) implies that $L^{\otimes m}$ is globally generated and L is thus semi-positive. Moreover, Bertini's theorem implies that there is a general smooth section $D \in H^0(\mathcal{X}, mL)$ such that any fiber X_t is not contained in Supp D. Hence, the pair $(\mathcal{X}, \frac{1}{m}D)$ is klt.

If $K_{X_0} + L_0$ is not nef, i.e., $K_{X_0} + \frac{1}{m}D_0$ is not nef, then the nef value $\eta(0) > 0$ by Lemma 3.1. After one shrinks the base Δ around 0, the nef-value function $\eta(t)$ is constant by Lemma 3.5. Hence, $\eta(t) > 0$ for t near 0 which means that $K_{X_t} + \frac{1}{m}D_t$ is not nef by Lemma 3.1 again. This is a contradiction. □

Remark 3.8. With the setting of Proposition 3.7, it suffices to assume that there is a sequence of points $\{t_i\}_{i\in\mathbb{N}}$ converging to 0 such that $K_{X_{t_i}} + L_{t_i}$ are nef for all i, to conclude that $K_{X_0} + L_0$ is nef. Hence, the *nef locus*

$$\mathcal{N}(K_{\mathcal{X}} + L) := \{t \in \Delta \mid K_{X_t} + L_t \text{ is nef}\}$$

is closed.

Theorem 3.9. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and L a π-semiample line bundle on $\mathcal{X}$. If $K_{X_0} + L_0$ is nef, then $K_{X_t} + L_t$ is nef for any $t \in \Delta$.*

Proof. As in Proposition 3.7, there is a general smooth section $D \in H^0(\mathcal{X}, mL)$ such that any fiber X_t is not contained in Supp D. Hence, the pair $(\mathcal{X}, \frac{1}{m}D)$ is klt.

The set $\{t \in \Delta \mid \eta(t) < 1/k\}$ is a dense Zariski open subset of Δ for any positive integer k. Baire's theorem implies that the nef locus

$$\bigcap_{k\geq 1} \{t \in \Delta \mid \eta(t) < 1/k\}$$

of adjoint canonical line bundle $K_{\mathcal{X}} + \frac{1}{m}D$ is a dense subset of Δ. By Remark 3.8, the nef locus of the adjoint canonical line bundle $K_{\mathcal{X}} + \frac{1}{m}D$ is closed and so it is the whole unit disk, i.e., $K_{X_t} + L_t$ is nef for any $t \in \Delta$. □

Remark 3.10. Theorem 3.9 can also be proved by the nefness openness of the adjoint canonical line bundle with respect to the countable Zariski topology on the base Δ of the deformation (via the relative Barlet cycle space theory) and the nefness closedness of the adjoint canonical line bundle in Remark 3.8.

Remark 3.11. One obtains a divisor analogue of Theorem 3.9 intrinsically: Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and D a π-nef $\mathbb{Q}$-Cartier divisor containing no fibers of π on $\mathcal{X}$ such that $(\mathcal{X}, D)$ is a klt pair. If $K_{X_0} + D_0$ is nef, then $K_{X_t} + D_t$ is nef for any $t \in \Delta$. This should also hold for a dlt pair.

Remark 3.12 (cf. [Le14, Theorem 1.2])**.** There exists a projective surjective morphism of algebraic varieties $\pi : X \to Y$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor D on X such that $\{y \in Y \mid D|_{X_y} \text{ is nef}\}$ is not Zariski open.

Corollary 3.13. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family. If the canonical line bundle K_{X_0} of the central fiber is nef, then $K_{\mathcal{X}/\Delta}$ is nef.*

Proof. Let $L = \mathcal{O}_{\mathcal{X}}$. Then K_{X_t} is nef for any $t \in \Delta$ by the proof of Theorem 3.9. And then [Pa17, Corollary 1.3] implies that the relative canonical line bundle $K_{\mathcal{X}/\Delta}$ is nef. □

Example 3.14. Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and L a π-semiample line bundle on $\mathcal{X}$. If some fiber of π is a compact hyperbolic complex manifold, then $K_{X_t} + L_t$ is nef for any $t \in \Delta$ and $K_{\mathcal{X}/\Delta}$ is nef. Recall a complex manifold X is called *(Brody) hyperbolic* if any holomorphic map $\mathbb{C} \to X$ is constant. Mori's breakthrough [Mo79] shows that the hyperbolicity of the projective X implies the nefness of its canonical line bundle, due to the absence of rational curves.

Remark 3.15. Corollary 3.13 implies that the nef locus of the canonical line bundle in a smooth projective family over the unit disk is empty or the whole disk.

In general, the limit H_0 of the ample lines (H_t) for $t \neq 0$ fails to be ample and is even not nef. But we can say something about the smooth member in $|mH_0|$ as follows which removes the restriction on the dimension in [AP97].

Corollary 3.16. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and $\mathcal{H}$ a line bundle on $\mathcal{X}$ such that $H_t := \mathcal{H}|_{X_t}$ is ample for $t \neq 0$ but H_0 is not nef. For $m \gg 0$, let $s \in H^0(\mathcal{X}, m\mathcal{H})$ such that $(s_t = 0) \subset X_t$ is smooth for $t \neq 0$, where $s_t := s|_{X_t}$. Then $(s_0 = 0) \subset X_0$ is singular.*

Proof. It follows from Remark 3.15 and the proof of [AP97, second Corollary]. □

Remark 3.17. By the upper semicontinuity of the dimension $h^0(X_t, H_t)$ of sections, the line bundle H_0 is still big in Corollary 3.16. And if $D_0 \in H^0(X_0, mH_0)$ is a smooth section, then D_0 can not occur in a family (D_t), where $D_t \in H^0(X_t, mH_t)$.

3.2. **Stability of nefness for Kähler family of threefolds.** Let X be a 3-dimensional compact Kähler manifold. If K_X is not pseudo-effective, M. Brunella [Br06] proves that X is uniruled and there is a rational curve C such that $K_X \cdot C < 0$. As K_X is pseudo-effective, Höring–Peternell [HP16] prove that if K_X is not nef, then there is a rational curve C such that $K_X \cdot C < 0$. See [HP15, HP16, HP24] by Höring–Peternell for more details about 3-dimensional terminal Kähler minimal model program. And Das–Hacon and J. I. Yáñez [DH25, DH24, DHY25] develop the minimal model program of klt and generalized klt Kähler 3-folds.

Let X be a normal $\mathbb{Q}$-factorial compact Kähler space with terminal singularities, and let $\mathbb{R}^+[\Gamma_i]$ be a K_X-negative extremal ray in the generalized Mori cone $\overline{NA}(X)$. A *contraction of the extremal ray* $\mathbb{R}^+[\Gamma_i]$ is a morphism $\phi : X \to Y$ onto a normal compact Kähler space such that $-K_X$ is ϕ-ample and a curve $C \subset X$ is contracted if and only if $[C] \in \mathbb{R}^+[\Gamma_i]$. And if $\dim_{\mathbb{C}} Y < \dim_{\mathbb{C}} X$, then ϕ is called *fiber type*. If $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X$ and ϕ contracts divisors, then it is called a *divisorial contraction*. If $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X$ and ϕ contracts no divisors, then it is called a *small contraction*.

Höring–Peternell [HP15] proves the contraction theorem of Kähler manifolds of dimension 3 when K_X is not pseudo-effective. Then Höring–Peternell [HP16] proves the contraction theorem of Kähler manifolds of dimension 3 when K_X is pseudo-effective but not nef, in the two cases of contracting divisors or curves to points. And Höring–Peternell [HP24] improves the contraction theorem of Kähler manifolds of dimension 3 when K_X is pseudo-effective but not nef in the case of contracting divisors to curves.

Theorem 3.18 ([HP15], [HP16] and [HP24])**.** *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with terminal singularities. And let $\mathbb{R}^+[\Gamma_i]$ be a K_X-negative extremal ray in $\overline{NA}(X)$. Then the contraction of $\mathbb{R}^+[\Gamma_i]$ exists in the Kähler category.*

Lemma 3.19. *Let X be a smooth compact Kähler 3-fold. Then K_X is not nef if and only if there is a rational curve C such that $K_X \cdot C < 0$.*

Proof. If K_X is not pseudo-effective, then there is a K_X-negative rational curve by [Br06]. If K_X is pseudo-effective but not nef, then there is a K_X-negative rational curve by [HP16, Corollary 4.2]. $\square$

Remark 3.20. In the case of dimension of X larger than 3, W. Ou [Ou25, Theorem 1.1] proves that a compact Kähler manifold X is uniruled if and only if its canonical line bundle K_X is not pseudo-effective (see also J. Cao–Păun [CP25]). So if K_X is not pseudo-effective, then there is a K_X-negative rational curve. And by also [CH20, Theorem 1.3] by Cao–Höring, if K_X is pseudo-effective but not nef, then there is a K_X-negative rational curve. Above all, if K_X is not nef, then there exists a K_X-negative rational curve.

Remark 3.21. The pseudo-effectiveness of the canonical line bundle is stable under smooth Kähler deformation by Ou [Ou25, Theorem 1.1], A. Fujiki [Fu81, Proposition 2.3] and M. Levine [Le81].

Lemma 3.22. *Let $f : X \to Y$ be the divisorial contraction from a three-dimensional compact Kähler manifold X. Then $R^i f_* \mathcal{O}_X(E) = 0$ for $i > 0$, where E is the f-exceptional divisor.*

Proof. We just replace by the analytic relative Kawamata–Viehweg vanishing theorem [DH25, Theorem 2.21] and analytic Grauert–Riemenschneider's vanishing theorem [Ta85, Theorem I] in the proof of [AP97, Proposition on p. 2]. $\square$

Remark 3.23. Mori [Mo82] classifies the smooth projective threefolds by extremal contractions and obtains that the smooth projective threefolds has no small extremal contractions. F. Campana and Höring–Peternell [HP15, HP16, HP24, CHP16, CHP23] and Das–Ou [DO23, DO24] prove that every smooth Kähler 3-folds admits a Mori fiber space or a good minimal model. So based on these and Peternell [Pe98], one classifies the smooth non-algebraic Kähler threefolds by extremal contractions in Kähler category and concludes that every smooth non-algebraic Kähler threefold only has divisorial extremal contractions or fiber type contractions.

Theorem 3.24. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family of n-folds with $n \leq 3$. If K_{X_0} is not nef, then no K_{X_t} is nef for any sufficiently small t.*

Proof. If $n = 1$, then all fibers are smooth projective curves. If K_{X_0} is not nef, then $\int_{X_t} c_1(X_t) = \int_{X_0} c_1(K_{X_0}) < 0$ and so K_{X_t} is not nef.

If $n = 2$, then all fibers are smooth compact Kähler surfaces. If K_{X_0} is not nef, then there is a K_{X_0}-negative rational curve C_0 which is a (-1)-curve or K_{X_0} is not pseudo-effective by [BPV84, Corollary 2.4]. On the one hand, the (-1)-curve C_0 can be deformed to the nearby fibers X_t by [Wi78, Proposition 3.2]. Therefore, K_{X_t} is not nef for small t. On the other hand, if K_{X_0} is not pseudo-effective, the proof is the same as the first case of $n = 3$ below.

If $n = 3$, by Theorem 3.18, there is an extremal contraction $\phi : X_0 \to Y_0$. Since X_0 is a smooth Kähler threefold, ϕ is not small by Remark 3.23. Firstly, we assume that $\dim_{\mathbb{C}} Y_0 < \dim_{\mathbb{C}} X_0$. Then X_0 is uniruled and so is X_t by Fujiki [Fu81, Proposition 2.3] and Levine [Le81]. And [Br06] implies that K_{X_t} is not pseudo-effective and thus not nef. Secondly, we assume that ϕ is a divisorial contraction and E is the ϕ-exceptional divisor which is the union of some K_{X_0}-negative curves. Then [Mo82, Theorem 3.3] and [Pe98, Main Theorem] implies that E with its normal bundle $N_{E|X_0}$ is one of the following

$$(\mathbb{P}^2, \mathcal{O}(-1)), (\mathbb{P}^2, \mathcal{O}(-2)), (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-1,-1)), (Q_0, \mathcal{O}(-1)),$$

where Q_0 is the quadric cone. In every case, the normal bundle $N_{E|X_0}$ is anti-ample. And

$$N_{E|\mathcal{X}} = N_{E|X_0} \oplus \mathcal{O}$$

and therefore E can deformed to every X_t, so that K_{X_t} is not nef by Lemma 3.19. $\square$

Remark 3.25. We can still follow the arguments of [AP97] by using Lemma 3.22 of Kähler version and Remark 3.23 to prove Theorem 3.24.

4. Applications: global stability of semiampleness and deformation rigidity

In this section, we obtain several applications of the main theorems: the global stability of semiampleness of the (adjoint) canonical line bundles, the stability of minimal manifolds of general type and deformation rigidity of projective manifolds with semiample canonical line bundles under smooth Kähler deformation.

4.1. Global stability of semiampleness of (adjoint) canonical line bundles.

We start with:

Lemma 4.1. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth family and fix some fiber X_t. If $(\mathcal{X}, X_t + \Sigma_i a_i D_i)$ is a log smooth pair where $0 < a_i < 1$ and $X_t + \Sigma_i D_i$ is a simple normal crossing divisor, then (X_t, D_t) is a klt pair where $D_t := (\Sigma_i a_i D_i)|_{X_t}$.*

Proof. Since $(\mathcal{X}, X_t + \Sigma_i a_i D_i)$ is a log smooth pair, [KM98, Corollary 2.31] implies that

$$\operatorname{disrep}(\mathcal{X}, X_t + \Sigma_i a_i D_i) \geq \min_i \{-a_i\} > -1.$$

Then $(\mathcal{X}, X_t + \Sigma_i a_i D_i)$ is a plt pair and hence (X_t, D_t) is a klt pair according to the adjunction formula [KM98, Theorem 5.50 (1)]. □

Corollary 4.2. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family and L a π-semiample line bundle on $\mathcal{X}$. If $K_{X_0} + L_0$ is semiample, then for any $t \in \Delta$, $K_{X_t} + L_t$ is semiample and $K_{\mathcal{X}} + L$ is thus π-semiample over $\pi^{-1}(U_t)$, where U_t is a Zariski neighborhood of t in Δ.*

Proof. Fix a fiber X_t for any $t \in \Delta$. Since L is a π-semiample line bundle, $L^{\otimes m}$ is π-globally generated for some positive integer $m \geq 1$. And since Δ is a Stein manifold, Cartan's theorem A (Theorem 2.2) implies that $L^{\otimes m}$ is globally generated. So L is semi-positive and there is a general smooth section

$$D \in H^0(\mathcal{X}, L^{\otimes m})$$

such that X_t is not contained in Supp D and $X_t + D$ is a simple normal crossing divisor. So $(\mathcal{X}, X_t + \frac{1}{m}D)$ is a log smooth pair and Lemma 4.1 implies that $(X_t, \frac{1}{m}D_t)$ is klt.

The semiampleness of $K_{X_0} + L_0$ implies that

$$\kappa(K_{X_0} + L_0) = \nu(K_{X_0} + L_0)$$

and $K_{X_0} + L_0$ is nef, which in turn implies that $K_{X_t} + L_t$ is nef for any $t \in \Delta$ by Theorem 3.9. For any positive integer m, Siu's invariance of semipositively twisted plurigenera [Si02, Corollary 0.2] implies that the dimension $h^0(X_t, mK_{X_t} + mL_t)$ of global sections of $mK_{X_t} + mL_t$ is independent of $t \in \Delta$. Thus, for any $t \in \Delta$, the Kodaira–Iitaka dimensions satisfy

$$\kappa(K_{X_t} + L_t) = \kappa(K_{X_0} + L_0)$$

and the numerical dimensions also satisfy

$$\nu(K_{X_t} + L_t) = \nu(K_{X_0} + L_0)$$

since π is a smooth projective family and $K_{X_t} + L_t$ is nef by [Na04, § 2.a of Chapter V]. So

$$\kappa(K_{X_t} + L_t) = \nu(K_{X_t} + L_t)$$

for $t \in \Delta$. Since $K_{X_t} + L_t$ is nef and good, [Ka85, Theorem 6.1] implies that $K_{X_t} + L_t$ is semiample for any $t \in \Delta$.

Hence, $K_{\mathcal{X}} + L$ is π-semiample near X_t since $K_{X_t} + L_t$ is semiample and $h^0(X_t, mK_{X_t} + mL_t)$ is independent of $t \in \Delta$ by Siu's invariance of semipositively twisted plurigenera [Si02, Corollary 0.2]. Take $0 \in \Delta$ as an example. In fact, by the semiampleness of $K_{X_0} + L_0$, there exists some positive integer m such that $m(K_{X_0} + L_0)$ is generated by global sections $H^0(X_0, m(K_{X_0} + L_0))$, i.e., the elements of $H^0(X_0, m(K_{X_0} + L_0))$ generate the stalks $\mathcal{O}_{X_0}(m(K_{X_0} + L_0))_x$, which are

$$(\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + L))|_{X_0})_x = (\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + L)))_x / \hat{\mathfrak{m}}_0 (\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + L)))_x$$

for all $x \in X_0$. Here $\hat{\mathfrak{m}}_0$ is the ideal sheaf of $\mathcal{O}_{\mathcal{X}}$ generated by the inverse image of the maximal ideal $\mathfrak{m}_0$ of $\mathcal{O}_{\Delta,0}$ under the canonical map $\pi^*\mathfrak{m}_0 \to \mathcal{O}_{\mathcal{X}}$. Then Siu's invariance of semipositively twisted plurigenera [Si02, Corollary 0.2] implies that $h^0(X_t, mK_{X_t} + mL_t)$ is independent of $t \in \Delta$. So Grauert's base change theorem and the adjunction formula give rise to

$$(\pi_*\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}}+L)))_0/\mathfrak{m}_0(\pi_*\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}}+L)))_0 = H^0(X_0, \mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}}+L))|_{X_0}) = H^0(X_0, m(K_{X_0}+L_0)).$$

Moreover, Nakayama lemma implies that the natural morphism of sheaves

$$\rho : \pi^*\pi_*\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + L)) \to \mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + L))$$

is surjective in the points of X_0. Hence, ρ will be surjective in a Zariski neighbourhood $\pi^{-1}(U_0)$ of X_0 for some Zariski neighborhood $U_0 \subset \Delta$ of 0. This argument should be traced back to [Gr61, THÉORÈME 2.1]. □

Remark 4.3. Let $L = \mathcal{O}_{\mathcal{X}}$, and then one still sees that the semiampleness of the canonical line bundle is (globally) stable under a smooth projective deformation.

Remark 4.4. An alternate proof of Corollary 4.2 can proceed via Siu's invariance of semipositively twisted plurigenera [Si02, Corollary 0.2] and Kawamata's relative basepoint-free theorem (cf. [KMM87, Theorem 6-1-11], [Na87, Theorem 5.8] or [Fu11, Theorem 1.1]). A much related argument can be found in the proof of Corollary 4.10 and more details are left to interested readers. Recall Kawamata's relative basepoint-free theorem: Let $\pi : X \to S$ be a proper morphism from a normal complex variety onto a normal variety and (X, Γ) a klt pair. Assume the following conditions:

(1) H is a π-nef $\mathbb{Q}$-Cartier divisor on X;
(2) $H - (K_X + \Gamma)$ is π-nef and π-*abundant* (i.e.,
$$\kappa(X_\eta, (H - (K_X + \Gamma))|_{X_\eta}) = \nu(X_\eta, (H - (K_X + \Gamma))|_{X_\eta})$$
for any general point $\eta \in S$;
(3) for any general point $\eta \in S$, $\kappa(X_\eta, (aH - (K_X + \Gamma))|_{X_\eta}) \geq 0$ and
$$\nu(X_\eta, (aH - (K_X + \Gamma))|_{X_\eta}) = \nu(X_\eta, (H - (K_X + \Gamma))|_{X_\eta})$$
for some $1 < a \in \mathbb{Q}$;
(4) every component C of any special fiber X_0 is compact complex variety in the *Fujiki class* $\mathcal{C}$ (i.e., bimeromorphic to a compact Kähler manifold) and $H|_C, (H - (K_X + \Gamma))|_C$ are *quasi-nef* (i.e., its pullback is nef under the bimeromorphic morphism just mentioned)

Then there exists positive integers p and m_0 such that

$$\pi^*\pi_*\mathcal{O}_X(mpH) \to \mathcal{O}_X(mpH)$$

is surjective near X_0 for all $m \geq m_0$.

Remark 4.5. Fujino [Fu22, Theorem 23.2] proves the following abundance theorem by using the local finite generation of log canonical rings, to reduce the abundance conjecture for projective morphisms of complex analytic spaces to the original abundance conjecture for projective varieties. Assume that *abundance conjecture* for a projective klt pair (X, D) holds in dimension n, i.e., if $K_X + \Delta$ is nef, then $K_X + D$ is semiample. Let $\pi : X \to Y$ be a projective surjective morphism of normal complex varieties with $\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} Y = n$, and (X, D) a klt pair. Assume that $K_X + D$ is π-nef. Let W be a Stein compact subset of Y such that $\Gamma(W, \mathcal{O}_Y)$ is noetherian. Then $K_X + D$ is π-semiample over some open neighborhood of W. By Fujino's proof, it seems that the relative semiampleness part of Corollary 4.2 can be refined to state that $K_{\mathcal{X}} + L$ is π-semiample over some open neighborhood of W as above.

Two more important propositions will be used in the proof of deformation invariance Theorem 4.8 of generalized plurigenera.

Proposition 4.6 ([Na87, Corollary 3.15])**.** *Let X be a normal complex variety with only klt singularities and $\pi : X \to \Delta$ a proper surjective morphism. Assume that every irreducible component of X_0 is a variety in the Fujiki class $\mathcal{C}$ and that K_X is π-semiample. Then $R^i\pi_*\mathcal{O}_X(K_X^{\otimes m})$ is locally free at 0 for any $i \geq 0$ and $m \geq 1$.*

Proposition 4.7 ([Ha77, Theorem 12.11 (Cohomology and Base Change)]). *Let $f : X \to Y$ be a projective morphism of noetherian schemes, and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Let y be a point of Y. Then:*

(a) If the natural map

$$\varphi^i(y) : R^i f_*(\mathcal{F}) \otimes k(y) \longrightarrow H^i(X_y, \mathcal{F}(y)) \tag{4.1}$$

is surjective, then it is an isomorphism, and the same is true for any y' in a suitable neighborhood of y.

(b) Assume that $\varphi^i(y)$ is surjective. Then the following two conditions are equivalent:
- *(i) $\varphi^{i-1}(y)$ is also surjective;*
- *(ii) $R^i f_*(\mathcal{F})$ is locally free in a neighborhood of y.*

Proposition 4.7 also holds in the complex analytic setting. Recall that $\mathcal{F}$ is *cohomologically flat in dimension q* if $\varphi^i(y)$ in (4.1) are bijective for any $y \in Y$ and $i = q, q-1$, or if $R^q f_*(\mathcal{F})$ is locally free and $\varphi^q(y)$ in (4.1) are bijective for any $y \in Y$. This notion plays a core role in Grauert's direct image theory and when Y is reduced, it equivalent to the local constancy of the function $y \mapsto \dim_{\mathbb{C}} H^q(X_y, \mathcal{F}(y))$.

Theorem 4.8. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth projective family of n-folds. If K_{X_0} is semiample, then for any $i \geq 0$ and $m \geq 1$, the generalized m-genus $P^i_m(X_t) := \dim_{\mathbb{C}} H^i(X_t, K^{\otimes m}_{X_t})$ is independent of $t \in \Delta$.*

Proof. By Corollary 4.2 and Proposition 4.6, $R^i\pi_*\mathcal{O}_{\mathcal{X}}(K^{\otimes m}_{\mathcal{X}})$ is locally free at any $t \in \Delta$ for any $i \geq 0$ and $m \geq 1$. Run Proposition 4.7.(b) first with $i = n+1$, to obtain that

$$\varphi^n(t) : R^n\pi_*\mathcal{O}_{\mathcal{X}}(K^{\otimes m}_{\mathcal{X}}) \otimes \mathbb{C}(t) \longrightarrow H^n(X_t, K^{\otimes m}_{X_t})$$

is surjective and thus an isomorphism by Proposition 4.7.(a), and then with $i = n$, to obtain that

$$\varphi^{n-1}(t) : R^{n-1}\pi_*\mathcal{O}_{\mathcal{X}}(K^{\otimes m}_{\mathcal{X}}) \otimes \mathbb{C}(t) \longrightarrow H^{n-1}(X_t, K^{\otimes m}_{X_t})$$

is an isomorphism. By induction, for any $i \geq 0$,

$$\varphi^i(t) : R^i\pi_*\mathcal{O}_{\mathcal{X}}(K^{\otimes m}_{\mathcal{X}}) \otimes \mathbb{C}(t) \longrightarrow H^i(X_t, K^{\otimes m}_{X_t})$$

is an isomorphism. Hence, the local freeness of $R^i\pi_*\mathcal{O}_{\mathcal{X}}(K^{\otimes m}_{\mathcal{X}})$ gives the deformation invariance of $P^i_m(X_t)$. □

Remark 4.9. In [HL06], N. Hao–L. Li show that the higher cohomology of the pluricanonical bundle is not deformation invariant by computing the dimensions of the first and second cohomology groups of all the pluricanonical bundles for Hirzebruch surfaces, and the dimensions of the first and second cohomology groups of the second pluricanonical bundles for a blow-up of a projective plane along finite many distinct points. Notice that the anti-canonical line bundle of the Hirzebruch surface $\mathbb{F}_m := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$ (with $m \geq 0$) is big for any $m \geq 0$, nef if and only if $m \leq 2$, ample if and only if $m < 2$.

Similarly, one obtains the global stability of semiampleness of canonical line bundles and deformation invariance of generalized plurigenera of threefolds under a Kähler smooth deformation.

Corollary 4.10. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family of threefolds. If K_{X_0} is nef (or equivalently semiample), then for any $t \in \Delta$, K_{X_t} is semiample and $K_{\mathcal{X}}$ is π-semiample over $\pi^{-1}(U_t)$, where U_t is a Zariski neighborhood of t. Furthermore, for any $i \geq 0$ and $m \geq 1$, the generalized m-genus $P^i_m(X_t)$ is independent of $t \in \Delta$.*

Proof. Although the semiampleness part is analogous to that of Corollary 4.2, we still include a proof here since there indeed exist several slight differences between them.

By the proof of [Lv83, Corollary 1.10], the canonical line bundle of each generic fiber in π is semiample. In fact, by the assumption of the semiampleness of K_{X_0}, there exists some positive integer m such that $K_{X_0}^{\otimes m}$ is generated by global sections $H^0(X_0, K_{X_0}^{\otimes m})$, i.e., the elements of $H^0(X_0, K_{X_0}^{\otimes m})$ generate the stalks

$$(K_{X_0}^{\otimes m})_x = (K_{\mathcal{X}}^{\otimes m}|_{X_0})_x = (K_{\mathcal{X}}^{\otimes m}/\hat{\mathfrak{m}}_0 K_{\mathcal{X}}^{\otimes m})_x = K_{\mathcal{X},x}^{\otimes m}/\hat{\mathfrak{m}}_0 K_{\mathcal{X},x}^{\otimes m}$$

for all $x \in X_0$. Then by Bertini's theorem and [Lv83, Theorem 1.9], $P_m(X_t)$ is independent of $t \in U$, where U is a Zariski neighbourhood of 0. So Grauert's base change theorem and the adjunction formula give rise to

$$(\pi_* K_{\mathcal{X}}^{\otimes m})_0/\mathfrak{m}_0(\pi_* K_{\mathcal{X}}^{\otimes m})_0 = H^0(X_0, K_{\mathcal{X}}^{\otimes m}|_{X_0}) = H^0(X_0, K_{X_0}^{\otimes m}).$$

Moreover, Nakayama lemma implies that the natural morphism of sheaves

$$\pi^*\pi_* K_{\mathcal{X}}^{\otimes m} \to K_{\mathcal{X}}^{\otimes m}$$

is surjective in the points of X_0. Hence, it will be surjective in a Zariski neighbourhood $\pi^{-1}(U')$ of X_0 in $\pi^{-1}(U)$ for some dense Zariski open subset $U' \subset U$. This in turn implies that $K_{X_t}^{\otimes m}$ is generated by global sections $H^0(X_t, K_{X_t}^{\otimes m})$ for any $t \in U'$.

Therefore, the semiampleness of the corollary follows from the closedness Theorem 3.24 of nefness and the abundance Theorem 1.11 of Kähler threefolds, while the deformation invariance of any generalized plurigenera follows from the same argument as in Theorem 4.8. □

Remark 4.11. On the deformation invariance of plurigenera, compared with [Lv83, Corollary 1.10] and [Na87, Corollary 6.4], Corollary 4.10 assumes only one fiber to admit the semiample canonical line bundle in the smooth Kähler family of threefolds.

4.2. **Stability of minimal manifolds of general type.** We need a relative version of the projectivity criterion of smooth Moishezon and Kähler manifolds.

Lemma 4.12. *Let $\pi : X \to Y$ be a smooth Kähler and Moishezon morphism over a Stein manifold Y and fix a Stein compact subset $W \subset Y$ which satisfies the condition (P). Then there is an open neighborhood U of W such that $\pi|_{\pi^{-1}(U)}$ is projective.*

Proof. The proof is the same as [CH24, Theorem 1.1] by B. Claudon–Höring. Just replace the MMP-steps by the minimal model program of projective morphism over a Stein analytic space [Fu22, Theorem 1.7]. □

And a special case of [RT21, Theorem 1.4], [RT22, Theorem 1.8] is needed here.

Proposition 4.13 (bimeromorphic embedding)**.** *For a smooth Kähler family $\pi : \mathcal{X} \to \Delta$, if there exists a (global) line bundle L over $\mathcal{X}$ such that the restrictions of L to uncountably many fibers are big or equivalently there are uncountably many Moishezon fibers in π, then for some $N \in \mathbb{N}$, there exists a bimeromorphic map (over Δ)*

$$\Phi : \mathcal{X} \dashrightarrow \mathcal{Y}$$

to a subvariety $\mathcal{Y}$ of $\mathbb{P}^N \times \Delta$ and thus π is a Moishezon family.

Corollary 4.14. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family. If the central fiber X_0 is a minimal manifold of general type, then all fibers are minimal manifolds of general type.*

Proof. As X_0 is of general type and Kähler, X_0 is projective by Moishezon's theorem [Mo66]. Since K_{X_0} is big and nef, $H^i(X_0, K_{X_0}^{\otimes m}) = 0$ for any positive integers $m \geq 2$ and $i \geq 1$ by Kawamata–Viehweg vanishing theorem. Grauert's upper semi-continuity theorem implies that there is a dense Zariski open subset $U_m \ni 0$ such that $H^i(X_t, K_{X_t}^{\otimes m}) = 0$ for any $t \in U_m$, $m \geq 2$ and $i \geq 1$. And by Riemann–Roch theorem,

$$\dim_{\mathbb{C}} H^0(X_0, K_{X_0}^{\otimes m}) = \dim_{\mathbb{C}} H^0(X_t, K_{X_t}^{\otimes m})$$

for any $t \in U_m$ and $m \geq 2$. So

$$\kappa(X_t) = \kappa(X_0) = \dim_{\mathbb{C}}(X_0) = \dim_{\mathbb{C}}(X_t)$$

for any $t \in \bigcap_{m>1} U_m$. Then one obtains uncountably many fibers X_t of general type. So the canonical line bundle K_{X_t} is big for any $t \in \Delta$ according to the bigness extension [RT21, Corollary 4.3]. Hence, π is a Moishezon family by the bimeromorphic embedding Proposition 4.13.

Choose the Stein compact subset

$$W_r := \{t \in \mathbb{C} \mid |t| \leq r\},$$

where $0 < r < 1$. Then there is an open neighborhood

$$U_{r,\epsilon_r} := \{t \in \mathbb{C} \mid |t| < r + \epsilon_r\}$$

of W_r for small ϵ_r such that $\pi|_{\pi^{-1}(U_{r,\epsilon_r})}$ is projective by Lemma 4.12. Since X_0 is minimal, K_{X_t} are nef for all $t \in U_{r,\epsilon_r}$ by the proof of Corollary 3.13. And taking the limit $r \to 1$, we conclude that all fibers are minimal manifolds of general type. □

Remark 4.15. Corollary 4.14 can be regarded as a slight generalization of [Ca91, Proposition 3.16]. Moreover, J. Kollár [Ko21, Theorem 1] proves that in a flat proper family, if the central fiber is projective of general type with canonical singularity, then its nearby fibers are projective and of general type. Combining this with [RT21, Corollary 4.3] also gives a proof of the first part of Corollary 4.14.

4.3. **Rigidity of projective manifolds with semiample canonical line bundles.** We improve the recent deformation rigidity result by the first author–Liu [LL24] of projective manifolds with semiample canonical line bundles.

We need Lemma 4.16 to construct a global line bundle on $\mathcal{X}$ which restricts on the special fibers to be ample.

Lemma 4.16. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family, and S a projective manifold. Assume that there is a subset $E \subseteq \Delta$ with accumulation points in Δ such that $X_t \cong S$ for any $t \in E$. Then there exists a global line bundle H on $\mathcal{X}$ such that $H|_{X_t}$ is ample for any $t \in E$.*

Proof. Compare the proofs of [RT21, Lemma 3.6] and [RT22, Lemma 4.25]. For reader's convenience, we sketch a proof here.

By the exponential sequence and Ehresmann's theorem, one has the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) & \longrightarrow & H^2(\mathcal{X}, \mathbb{Z}) & \longrightarrow & H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow \cong & & \downarrow & \\ \cdots \longrightarrow & H^1(X_t, \mathcal{O}^*_{X_t}) & \longrightarrow & H^2(X_t, \mathbb{Z}) & \longrightarrow & H^2(X_t, \mathcal{O}_{X_t}) & \longrightarrow \cdots \end{array} \tag{4.2}$$

for any $t \in \Delta$. The Leray spectral sequence to π for the sheaves $\mathcal{O}_{\mathcal{X}}$ gives rise to

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong H^0(\Delta, R^2\pi_*\mathcal{O}_{\mathcal{X}})$$

since Δ is a holomorphic domain. Since π is a smooth Kähler morphism, any pure-type Hodge number of X_t is constant by Hodge decomposition and the upper semi-continuity of Hodge number. So the map

$$R^2\pi_*\mathcal{O}_{\mathcal{X}}(t) := (R^2\pi_*\mathcal{O}_{\mathcal{X}})_t \otimes \mathbb{C}(t) \to H^2(X_t, \mathcal{O}_{X_t}) \tag{4.3}$$

is bijective for each $t \in \Delta$ by Grauert's base change theorem, and $R^2\pi_*\mathcal{O}_{\mathcal{X}}$ is locally free over Δ by Grauert's continuity theorem.

Take any point t of E. And run the commutative Diagram 4.2 and use the base change (4.3), the local freeness of $R^2\pi_*\mathcal{O}_{\mathcal{X}}$ to obtain a global line bundle over the total space $\mathcal{X}$ such that its restriction to X_t for any $t \in E$ is ample, since $E \subseteq \Delta$ is a subset with accumulation points in Δ such that X_t is isomorphic to a fixed projective manifold for any $t \in E$. □

Remark 4.17. In general, we can not obtain a relative ample line bundle on total space of a smooth fiberwise projective family. For example, Hopf surface is an elliptic fibration, which is a smooth fiberwise projective family but not projective, over $\mathbb{P}^1$. Indeed, if it was a projective morphism, then Hopf surface would be projective.

Now, let us prove an equivalent form of Theorem 1.5:

Theorem 4.18. *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family, and S a projective manifold with the semiample canonical line bundle. Assume that there is a subset $E \subseteq \Delta$ with accumulation points in Δ such that $X_t \cong S$ for any $t \in E$. Then all fibers $X_t \cong S$ for $t \in \Delta$.*

Proof. By Lemma 4.16, one obtains a global line bundle H over the total space $\mathcal{X}$ such that $H|_{X_t}$ for any $t \in E$ is ample. By Grothendieck's Zariski openness of the ampleness of the line bundle [Gr61], there is a dense Zariski open subset $U \subset \Delta$ such that H_{X_t} is ample for any $t \in U$ and $E \subset U$. On this, we can also argue as [RT22, Lemma 4.25, Remark 4.26]. So π is a Moishezon family by the bimeromorphic embedding Proposition 4.13. Just as the proof of Corollary 4.14, we can choose the Stein compact subset

$$W_r := \{t \in \mathbb{C} \mid |t| \leq r\}$$

with $0 < r < 1$, and there is an open neighborhood

$$U_{r,\epsilon_r} := \{t \in \mathbb{C} \mid |t| < r + \epsilon_r\}$$

of W_r for small ϵ_r such that $\pi|_{\pi^{-1}(U_{r,\epsilon_r})}$ is projective by Lemma 4.12. Then, take r large enough such that U_{r,ϵ_r} contains a accumulation point of E in Δ.

Since K_S is semiample, K_{X_t} is semiample for any $t \in U_{r,\epsilon_r}$ by Remark 4.3. So the family

$$(\pi_{U_{r,\epsilon_r}} : \pi^{-1}(U_{r,\epsilon_r}) \to U_{r,\epsilon_r}, H_{U_{r,\epsilon_r}})$$

induces a morphism

$$\mu := \mu_{\pi_{U_{r,\epsilon_r}}} : U_{r,\epsilon_r} \to P_h$$

to the coarse moduli space P_h of polarized manifolds, where $h(m) := \chi(H_t^{\otimes m})$ for any $m \in \mathbb{Z}$ and $t \in U_{r,\epsilon_r}$. For any $t_1, t_2 \in E$, H_{t_1} and H_{t_2} are numerically equivalent, so $\mu(t_1) = \mu(t_2) = o$, where o is a point in P_h.

Since the coarse moduli space P_h is separated and μ is holomorphic, $\mu^{-1}(o)$ is a closed analytic subset of U_{r,ϵ_r} which contains a subset of U_{r,ϵ_r} with accumulation points in U_{r,ϵ_r}. Thus,

$$\mu^{-1}(o) = U_{r,\epsilon_r},$$

which means that the morphism μ is constant. Therefore, $X_t \cong S$ for all $t \in U_{r,\epsilon_r}$. Hence, take $r \to 1$ for U_{r,ϵ_r} and apply the same moduli argument above to conclude the proof, i.e., $X_t \cong S$ for all $t \in \Delta$. □

As a direct result of Theorem 4.18, we have:

Corollary 4.19 ([LL24, Theorem 1.2])**.** *Let $\pi : \mathcal{X} \to \Delta$ be a smooth Kähler family, and S a projective manifold with the semiample canonical line bundle. If all the fibers X_t with $t \neq 0$ are biholomorphic to S, then the central fiber X_0 is biholomorphic to S.*

Proof. We sketch a new proof by combining those of [LL24, Theorem 1.2] and Theorem 4.18. By a Hodge theoretic argument in [LL24, Theorem 1.2], one obtains a global line bundle $\mathcal{A}$ over $\mathcal{X}$ such that $c_1(\mathcal{A}|_{X_0})$ is an ample class on X_0 and thus the restriction $\mathcal{A}|_{X_0}$ is ample. So the Zariski openness of the ampleness of the line bundle reduces our proof to the claim: if all the fibers X_t with $t \neq 0$ in a smooth projective family over a (small) disk are biholomorphic to a projective manifold S with the semiample canonical line bundle, then X_0 is biholomorphic to S, which directly follows from the global stability of semiampleness of the canonical line bundles in Remark 4.3 and the moduli argument in the proof of Theorem 4.18. □

Finally, let us come to the proof of Theorem 1.8. We need E. Viehweg–K. Zuo's result [VZ01, Theorem 0.1] on the minimal number of singular fibers in a family of varieties:

Theorem 4.20 ([VZ01, Theorem 0.1])**.** *Let Y be a non-singular curve and X a projective manifold. Let $f : X \to Y$ be a surjective morphism with connected general fiber F. Fix a reduced divisor D on Y containing the discriminant divisor of f, and set*

$$f_0 = f|_{X_0} : X_0 \longrightarrow Y_0,$$

where $Y_0 = Y \setminus D$ and $X_0 = f^{-1}(Y_0)$. Then f_0 is smooth. Assume that f is not birationally isotrivial, and that one of the following conditions holds:

a) $\kappa(F) = \dim(F)$.
b) *F has a minimal model F' with $K_{F'}$ semiample.*

Then f has at least

i) *three singular fibers if $Y = \mathbb{P}^1$.*
ii) *one singular fiber if Y is an elliptic curve.*

Recall that $f : X \to Y$ is called birationally isotrivial*, if $X \times_Y$ Spec $\overline{\mathbb{C}(Y)}$ is birational to $F \times$ Spec $\overline{\mathbb{C}(Y)}$.*

We first prove an equivalent form of Corollary 1.7:

Corollary 4.21. *Let $\pi : \mathcal{X} \to Y$ be a smooth Kähler family over Y, where Y is isomorphic to $\mathbb{P}^1$ or an elliptic curve. Let S be a projective manifold with the big and nef canonical line bundle. If there exists one fiber isomorphic to S, then $\mathcal{S} := \{t \in Y : X_t \cong S\}$ is the whole Y.*

Proof. By Corollary 4.14, all fibers of π are minimal and of general type if so is one fiber of π. Then Claudon–Höring's criterion [CH24, 1.1 Theorem] of projective morphisms shows that π is projective. So the total space $\mathcal{X}$ of the family π is projective. Theorem 4.20 implies that the smooth family π is birationally isotrivial. Hence, [LL24, Theorem 7.1] tells us that π is actually isotrivial since all fibers of π are good minimal. This concludes the proof. □

Upon closer examination, we are able to establish a more general rigidity theorem.

Theorem 4.22. *Let $\pi : \mathcal{X} \to Y$ be a smooth projective family over Y, where Y is isomorphic to $\mathbb{P}^1$ or an elliptic curve. Let S be a projective manifold with the semiample canonical line bundle. Then $\mathcal{S}$ is either empty or the whole Y.*

Proof. As mentioned after Theorem 1.8, this theorem can be deduced from Corollary 1.6 and [Dn22, Theorem A] or [DLSZ24, Theorem B]. Recall that [Dn22, Theorem A] proves the Brody hyperbolicity of coarse moduli spaces for polarized manifolds with semiample canonical sheaves. More precisely, consider the moduli functor $\mathcal{P}_h$ of polarized manifolds with semiample canonical sheaves introduced by Viehweg [Vi91, § 7.6] with the Hilbert polynomial h associated to the polarization $\mathcal{H}$, and then for some quasi-projective manifold V, there exists a smooth family

$$(f : U \to V, \mathcal{H}) \in \mathcal{P}_h(V)$$

for which the induced moduli map $V \to P_h$ to the quasi-projective coarse moduli scheme for $\mathcal{P}_h$ is quasi-finite over its image. Then the base space V is Brody hyperbolic. However, neither of the two bases of the family in Theorem 4.22 is Brody hyperbolic.

Actually, we can also prove Theorem 4.22 as follows. Notice that in the proof of Corollary 4.21, the bigness of K_S is mainly used to guarantee the projectivity of π and Remark 4.3 gives the (global) stability of semiampleness, which is either one condition of Viehweg–Zuo's birational isotriviality Theorem 4.20 of a smooth projective family. Hence, the first author–Liu's isotriviality theorem [LL24, Theorem 7.1] concludes the proof. □

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