GROTHENDIECK TOPOLOGIES WITH LOGARITHMIC MODIFICATIONS

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Abstract. Many concepts in logarithmic geometry are invariant under log blowups. To formalize this invariance, we introduce the m-open, m-étale, m-smooth, m-fppf, and m-fpqc topologies for fs log schemes. These refine the standard topologies from scheme theory by treating log modifications as covers. In constructing them, we identify and correct errors in the definitions of log modifications and the full log étale topology. Our m-topologies are variants of those introduced by Nizioł and Park; specifically, the m-étale topology is a subtopology of Kato's full log étale topology, characterized by a stronger lifting property than for log étale maps. This strengthening ensures the functoriality of the corresponding small site. We also characterize the sheaves for all these sites and connect the m-open site to Kato's valuative space.

Contents

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1. Introduction

1.1. **Logarithmic modifications.** Logarithmic modifications are fundamental to logarithmic geometry, with log Gromov-Witten theory, log Chow theory, log Hilbert schemes and log Hochschild (co)homology all being invariant under log modifications (see, e.g., [1, 2, 3, 4, 6, 8, 9, 13, 18, 22, 23, 24, 25, 34]).

For some special cases, many authors define logarithmic modifications as proper, birational log étale morphisms (see, e.g., [1, Section 4], [4, Definition 2.6.1], [13, Section 1.3]). This definition, however, is too narrow for the general case, as it fails to include all log blow-ups, which are not necessarily birational.

As far as we are aware, the first and most general definition is [15, Definition 3.14][1], which goes as follows:

> A morphism $f\colon X \to Y$ of locally Noetherian, fine and saturated log schemes is called a *logarithmic modification* if for any point $y \in Y$, there exists an étale neighbourhood V of y and a log blow-up $Z \to X \times_Y V$ such that the composition $Z \to X \times_Y V \to V$ is also a log blow-up.

However, Example 3.6 due to *Chikara Nakayama* raises a crucial issue: Such a morphism is not always log étale or even log flat. Inspired by the work of Doosung Park [33], we therefore propose the following definition, which generalizes all other definitions that we are aware of:

Definition 1.1. A morphism $f\colon X \to Y$ of fs log schemes is called a *logarithmic modification* if f is a universally surjective log étale monomorphism.

It turns out that any such morphism is automatically proper (see Proposition 2.6). In particular, this is equivalent to the notion of *dividing cover* in [7, 32, 33]. In particular, log modifications are stable under base change and composition. Furthermore, toric modifications and log blow-ups are all log modifications (see [33, Example 2.7] and [31, Theorem III.2.6.3(3)]).

Throughout this paper, we call a morphism of fs log schemes universally surjective if all its base changes in the category of integral log schemes are surjective. This is strictly stronger than the usual universal surjectivity in the category of fs log schemes (see Subsection 1.4 for more details).

1.2. **M-type morphisms.** Since many log moduli spaces satisfy descent along log modifications as well as the classical topologies of scheme theory (see e.g. [18, 19, 22, 23, 25, 26]), we would like to combine both of these. To formalize this, we will introduce what we call *m-type* morphisms. First, we have the *m-open* morphisms:

Definition 1.2. A morphism $X \to Y$ of fs log schemes is called *m-open* if one can cover X by opens $U \subset X$ so that $U \hookrightarrow X \to Y$ is a log étale monomorphism.

M-open morphisms are thus a combination of strict open immersions and log modifications (see Remark 2.5 for details). Moreover, it turns out that m-open morphisms between fs log schemes with Zariski log structure can be described by an explicit chart criterion (see Proposition 2.1). This motivates our definition of *m-étale*, *m-smooth*, and *m-flat* morphisms:

[1] Note that only the first version of [15] contains that definition.

Definition 1.3. A morphism $f : X \to Y$ of fs log schemes is called *m-étale* (resp. *m-smooth*, *m-flat*) if for any $x \in X$ and strict étale neighborhood $V \to Y$ of $f(x)$ admitting a chart $Q \to \Gamma(V, \mathcal{M}_Y)$, there exists a strict étale neighborhood $U \to X$ of x such that $U \to X \to Y$ factors over a morphism $U \to V$ and admits a morphism of charts

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "\theta"'] & \Gamma(V, \mathcal{M}_Y) \arrow[d] \\ P \arrow[r] & \Gamma(U, \mathcal{M}_X) \end{tikzcd}$$

such that:

(1) The morphism θ induces an isomorphism $Q^{\mathrm{gp}} \cong P^{\mathrm{gp}}$.
(2) The induced morphism $U \to V \times_{\mathbb{A}_Q} \mathbb{A}_P$ is strict étale (resp. smooth, flat).

Similarly, a morphism is m-étale (resp. m-smooth, m-flat) if and only if it admits a strict étale local factorization as a strict étale (resp. smooth, flat) morphism and a log blow-up (see Proposition 2.8 for a precise statement).

Rather surprisingly, m-étale and m-smooth morphisms can also be characterized by an infinitesimal lifting criterion.

Definition 1.4. Let $f : X \to Y$ be a morphism of fs log schemes. We say f is *formally m-étale* (resp. m-smooth) if for any given outer commutative square

(1.1)
$$\begin{tikzcd} T' \arrow[r] \arrow[d, hook, "i"] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

so that

(1) the underlying morphism of schemes $\underline{i}\colon \underline{T'} \hookrightarrow \underline{T}$ is a first order thickening over Y;
(2) the morphism $i\colon T' \hookrightarrow T$ is exact, i.e. the diagram

(1.2)
$$\begin{tikzcd} i^*\mathcal{M}_T \arrow[r] \arrow[d] & \mathcal{M}'_T \arrow[d] \\ i^*\mathcal{M}_T^{gp} \arrow[r] & \mathcal{M}_{T'}^{gp} \end{tikzcd}$$

is cartesian,

there is strict étale locally in T *exactly one* (resp. *at least one*) dotted arrow making the diagram commute.

Note that this is a strengthening of the classical definition of log étale and log smooth (see [17, Section 3.2 and 3.3]) - the difference being that the thickenings $i\colon T' \hookrightarrow T$ are not assumed to be strict. In analogy with (log) étale morphisms, we have:

Theorem A. A morphism of fs log schemes is m-étale (resp. m-smooth) if and only if it is formally m-étale (resp. m-smooth) and locally of finite presentation.

1.3. **M-type topologies.** M-type morphisms naturally yield corresponding Grothendieck topologies:

Definition 1.5. For each $\tau \in \{\text{m-open}, \text{m-étale}, \text{m-smooth}, \text{m-fppf}, \text{m-fpqc}\}$, a family of morphisms of fs log schemes $\{U_i \to U\}_{i\in I}$ is called a *τ-covering* if and only if $f\colon \coprod_{i\in I} U_i \to U$ is universally surjective and

1. for $\tau \in \{\text{m-open}, \text{m-étale}, \text{m-smooth}\}$, we require that f is of type τ.
2. for $\tau = \text{m-fppf}$, we require that f is m-flat and locally of finite presentation
3. for $\tau = \text{m-fpqc}$, we require that f is m-flat and any quasi-compact open of U is the image of a quasi-compact open along f.

These topologies also come with associated sites:

Definition 1.6. Let S be an fs log scheme. For each $\tau \in \{\text{m-open}, \text{m-étale}, \text{m-smooth}, \text{m-fppf}, \text{m-fpqc}\}$ and each $\tau' \in \{\text{m-open}, \text{m-étale}\}$ we define

1. the *big τ-site* $(\mathsf{fsLogSch}/S)_\tau$ of S as the category $\mathsf{fsLogSch}/S$ equipped with the τ-topology.
2. the *small τ'-site* $S_{\tau'}$ of S as the full subsite of $(\mathsf{fsLogSch}/S)_\tau$ spanned by those objects that are τ'-type over S.

In analogy to classical scheme theory, we will not consider small sites for the other topologies since they are not functorial, see Remark 4.2.

The following theorem characterizes sheaves on our sites:

Theorem B. Let S be an fs log scheme, for any $\tau \in \{\text{m-open}, \text{m-étale}, \text{m-smooth}, \text{m-fppf}, \text{m-fpqc}\}$ (resp. $\tau' \in \{\text{m-open}, \text{m-étale}\}$), let $\mathcal{F}$ be a presheaf on $(\mathsf{fsLogSch}/S)_\tau$ (resp. $S_{\tau'}$) valued in a category $\mathcal{C}$ with all limits. Then $\mathcal{F}$ is a sheaf if and only if

1. for any strict τ-cover (τ'-cover) $\{U_i \to U\}_{i\in I}$ in $\mathsf{fsLogSch}/S$ (resp. $S_{\tau'}$), we have
$$\mathcal{F}(U) = \mathrm{Eq}\left(\prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I}\mathcal{F}(U_i\times_U U_j)\right);$$
2. restriction along a log blow-up $U \to V$ in $\mathsf{fsLogSch}/S$ (resp. $S_{\tau'}$) induces $\mathcal{F}(V) = \mathcal{F}(U)$.

1.4. **Subtleties on universal surjectivity.** Our usage of “universal surjectivity” above deviates from the standard definition. In the literature, a morphism $f\colon X \to Y$ of fs log schemes is universally surjective if all of its pullbacks $f_T\colon X\times_Y T \to T$ along all morphisms of fs fs log schemes $T \to Y$ are surjective.

This property is well-known from the full log étale topology, which takes as its covers precisely those families of log étale morphisms that are jointly universally surjective in this conventional sense (cf. [11, 14, 28]). However, in Section 5.1, we show the following:

Theorem C. Let S be an fs log scheme and X an object in the small (or big) log étale site of S. Then X is a quasi-compact object in this site if and only if

1. its underlying scheme is quasi-compact and
2. for all $x \in X$, the group $\overline{\mathcal{M}}^{gp}_{X,\overline{x}}$ is of rank ≤ 1.

This contradicts [28, Lemma 3.14], where condition (2) was not mentioned. As a result, the log étale site has relatively few quasi-compact objects, which seems to break most of [28]. In particular, the log étale topology will very rarely be generated by Kummer étale covers and log blowups (see Corollary 5.3). This further contradicts [28, Proposition 3.9] and, as far as we are aware, impacts virtually every paper that uses the log étale topology.

In Section 5.1, we show that these issues go away if one instead uses our definition of universal surjectivity, which requires that every base change along any *integral* log scheme is surjective.

1.5. **Sheaves on valuative log spaces.** Recall from [16] that one can associate to any fs log scheme X a certain log locally ringed space X^{val} called its *valuative log space*. It turns out that the valuative log space is closely related to our m-open site:

Theorem D. There is a natural equivalence of topoi

$$\mathrm{Shv}(X^{\mathrm{val}}) = \mathrm{Shv}\left(X_{\mathrm{mop}}\right).$$

We will use Theorem D to deduce a vanishing result for the sheaf cohomology on X_{mop}. To this end, we introduce the following definition:

Definition 1.7. For any fs log scheme X we define its *logarithmic dimension* as the Krull dimension of its valuative log space

$$\text{log-dim}(X) \coloneqq \dim(X^{\mathrm{val}}).$$

A different notion of log dimension is presented in [20, Definition 2.3], defined for closed subschemes of log smooth fs log schemes over $\mathbb{C}$. However, this definition differs from ours; for example, the standard log point $X = \mathrm{Spec}(\mathbb{N} \to \mathbb{C})$ has log dimension 1 in their sense, but dimension 0 in ours.

Corollary 1.8. For any quasi-compact and quasi-separated fs log scheme X and sheaf $\mathcal{F}$ of abelian groups on X_{mop}, we have $H^p(X, \mathcal{F}) = 0$ if $p > \text{log-dim}(X)$.

In order to make Corollary 1.8 usable in practice, for a locally noetherian fs log scheme, we give the following more explicit description of the logarithmic dimension.

Theorem E. If X is a locally noetherian fs log scheme, then we have

$$\text{log-dim}(X) = \sup_{x \in X}\{\dim \overline{\{x\}} + \max\left(\mathrm{rank}(\overline{\mathcal{M}}^{gp}_{X,\bar{x}}) - 1, 0\right)\}. \tag{1.3}$$

1.6. **Conventions.** Throughout this paper, we ignore all the set-theoretical issues associated with large sites. We use fs to denote fine and saturated and log schemes will be endowed with étale logarithmic structures. We denote the category of such logarithmic schemes by $\mathsf{fsLogSch}$ and the category of schemes by Sch. All charts in this paper are fine and saturated unless stated otherwise. Moreover, we will say that the log structure of a fs log scheme is *Zariski* if it can be covered by open subsets that admit global charts. Because of [31, Proposition III.1.4.1] we may (and will) identify these with log structures defined on the Zariski topology.

We largely follow the notation established in the standard reference [31]. For a logarithmic scheme X, we denote by $\underline{X}$ its underlying scheme, by $\mathcal{M}_X$ its logarithmic structure, and by $\overline{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{O}_X^{\times}$ its characteristic sheaf. We also use $\mathrm{Spec}(P \to A)$

to denote the scheme $\mathrm{Spec}(A)$ equipped with the logarithmic structure induced by a morphism of monoids $P \to A$. In particular, we write $\mathbb{A}_P = \mathrm{Spec}(P \to \mathbb{Z}[P])$ for short.

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2. Logarithmic modifications and m-type morphisms

The main goal of this section is to prove basic properties of m-type morphisms and log modifications. In particular, we show that log modifications are proper and m-type morphisms are locally compositions of log blow-ups and strict morphisms of the same type. Furthermore, exact m-type morphisms are strict.

2.1. **Logarithmic modification and m-open morphisms.** It turns out that the m-open morphisms admit a chart criterion if the target has Zariski log structure.

Proposition 2.1. Let $f\colon X \to Y$ be a morphism of fs log schemes where the log structure of Y is Zariski. Then f is m-open if and only if it satisfies the following chart criterion:
For any $x \in X$ and open neighborhood $f(x) \in V \subset Y$ of $f(x)$ admitting a chart $Q \to \Gamma(V, \mathcal{M}_Y)$ there exists an open neighborhood $x \in U \subset X$ so that $f(U) \subset V$ there is a morphism of charts

$$\begin{tikzcd} Q \arrow[r] \arrow[d, "\theta"'] & \Gamma(V, \mathcal{M}_Y) \arrow[d] \\ P \arrow[r] & \Gamma(U, \mathcal{M}_X) \end{tikzcd}$$

such that:

(1) the morphism θ induces an isomorphism $Q^{\mathrm{gp}} \cong P^{\mathrm{gp}}$;
(2) the induced morphism $U \to V \times_{\mathbb{A}_Q} \mathbb{A}_P$ is a strict open immersion.

Before we give the proof, we first establish the following technical result:

Proposition 2.2. Let $f\colon X \to Y$ be a monomorphism between fs log schemes. If the log structure on Y is Zariski, then the log structure of X is also Zariski.

Proof. Let $\underline{X}^{\text{ét}}$ and $\underline{X}^{\mathrm{Zar}}$ be the small étale and Zariski sites of the underlying scheme of X and $\epsilon_*\colon \mathrm{LogStr}(\underline{X}^{\text{ét}}) \to \mathrm{LogStr}(\underline{X}^{\mathrm{Zar}})$ and $\epsilon^*\colon \mathrm{LogStr}(\underline{X}^{\mathrm{Zar}}) \to \mathrm{LogStr}(\underline{X}^{\text{ét}})$ be the associated pushforwards and pullbacks of log structures. Denoting by X^{Zar} the scheme $\underline{X}$ equipped with the log structure $\mathcal{M}_X^{\mathrm{Zar}} := \epsilon^*\epsilon_*\mathcal{M}_X$, there is a natural morphism $\phi\colon X \to X^{\mathrm{Zar}}$ induced by the counit $\mathcal{M}_X^{\mathrm{Zar}} \to \mathcal{M}_X$. Since ϵ^* is fully faithful, one can see that $\mathcal{M}_X^{\mathrm{Zar}}$ is terminal among Zariski log structures mapping to $\mathcal{M}_X$. As a result, X^{Zar} is initial among Zariski log schemes under X. Since Y is Zariski it follows that f factors over ϕ and hence ϕ is also a monomorphism[2].

In order to show that ϕ is an isomorphism, it suffices by [29, Proposition 2.8] to show that for any $x \in X$, there is no non-trivial automorphism of $\overline{\mathcal{M}}_{X,\overline{x}}$ that fixes $\overline{\mathcal{M}^{\mathrm{Zar}}}_{X,\overline{x}} \to \overline{\mathcal{M}}_{X,\overline{x}}$. Since ϕ is a monomorphism, the base change $\overline{x}^{\text{ét}} \to \overline{x}^{\mathrm{Zar}}$ of ϕ along the geometric point $\overline{x}^{\mathrm{Zar}} = \mathrm{Spec}(k) \to X^{\mathrm{Zar}}$ equipped with the induced log structure is also a monomorphism. By [31, Remark III.1.5.3], the log structure on $\overline{x}^{\text{ét}}$ splits into $\mathcal{M}_{\overline{x}^{\text{ét}}} = k^\times \oplus \overline{\mathcal{M}}_{X,\overline{x}}$. Hence any automorphism of $\overline{\mathcal{M}}_{X,\overline{x}}$ that fixes $\overline{\mathcal{M}^{\mathrm{Zar}}}_{X,\overline{x}}$ can be lifted to an automorphism of $\mathcal{M}_{\overline{x}^{\text{ét}}}$ that fixes $\mathcal{M}_{\overline{x}^{\mathrm{Zar}}}$. But since $\overline{x}^{\text{ét}} \to \overline{x}^{\mathrm{Zar}}$ is a monomorphism, this automorphism must have been the identity, which concludes the proof. □

Remark 2.3. In particular, this shows that if $f\colon X \to Y$ is m-open and Y has a Zariski log structure, then X has a Zariski log structure as well.

Proof of Proposition 2.1. If f is m-open, it is is enough to check the chart criterion on an open cover of X, hence we may assume that f is a log étale monomorphism. By Proposition 2.2, X will then also have a Zariski log structure and [7, Lemma A.11.3] gives us the desired charts. Conversely, the chart criterion implies that f admits Zariski-locally a factorization of the shape $X \hookrightarrow Y \times_{\mathbb{A}_Q} \mathbb{A}_P \to Y$ with $P^{gp} = Q^{gp}$, where the first map is a strict open immersion. Such a composite is log étale and to show that it is a monomorphism, one can use the construction of the fiber product in the category of fs log schemes or alternatively Lemma 2.4. □

Lemma 2.4. Let $Q \to R$ be a pre-log ring and $Q \to P$ a morphism of fine, saturated monoids which induces an isomorphism $Q^{gp} \cong P^{gp}$. The resulting morphism $f\colon \mathrm{Spec}(P \to R \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) \to \mathrm{Spec}(Q \to R)$ can be factored into the composition of an open immersion and a log blow-up.

Proof. Since $Q \to P$ induces an isomorphism $Q^{gp} \cong P^{gp}$ and Q is integral, we know that $Q \subset P$. Since P is finitely generated and $P \subset Q^{gp}$, we may choose generators $\frac{f_1}{s}, ..., \frac{f_n}{s}$ of P, where $f_i, s \in Q$. Let I be the ideal generated by $f_1, f_2, ..., f_n, s$. As a result, we get $P = \bigcup_{n\geq 0} \frac{I^n}{s^n}$ and by the construction of the log blow-up (see [28, Section 2.1] for details), the morphism f factors as desired:

$$\mathrm{Spec}(P \to R \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) = D_+(s) \hookrightarrow \mathrm{Bl}_I(\mathrm{Spec}(Q \to R)) \to \mathrm{Spec}(Q \to R).$$

□

Remark 2.5. Using Lemma 2.4, one can rephrase Proposition 2.1 in the following way: If Y is Zariski, then f is m-open if and only if Zariski locally in X and Y, f can be written as the composition of a strict open immersion followed by a log blowup.

[2]Note that $\mathcal{M}_X^{\mathrm{Zar}}$ may not be quasi-coherent, so ϕ only exists in a much bigger category of log schemes. By ϕ being a monomorphism we merely mean that for any fs log scheme T and morphisms $T \underset{g}{\overset{f}{\rightrightarrows}} X$ that equalize ϕ, we must have $f = g$.

Definitions of log modifications usually require properness as an extra assumption. We show that this is already automatic from our definition.

Proposition 2.6. Any log modification is proper.

Proof. Since both notions are strict étale local in the target (for properness, see [36, p. 02L1]), it suffices to consider the case where the target Y of the log modification $f\colon X \to Y$ is quasi-compact and admits a global chart $P \to \Gamma(Y, \mathcal{M}_Y)$. By Proposition 2.1, we can find a Zariski cover $\{U_i \to X\}_{i \in I}$ so that each $U_i \to Y$ factors by Lemma 2.4 as a composition $U_i \to \widetilde{Y}_i \to Y$ of an open immersion and the log blow-up at an ideal $I_i \subset P$. Since f is universally surjective, the $U_i \to Y$ form an m-open cover of Y. By Proposition 4.3 and quasi-compactness of Y, there is a finite $J \subset I$ so that $\{U_i \to Y\}_{i \in J}$ is again a cover.

Now let $\widetilde{Y} \to Y$ be the log blow-up at $I = \prod_{j \in J} I_j$ so that $\widetilde{Y} \to Y$ factors over $\widetilde{Y}_i \to Y$ for all $i \in J$. Since log blow-ups are monomorphisms, it follows that each base change $\widetilde{U}_i := U_i \times_Y \widetilde{Y} \to \widetilde{Y}$ is an open immersion. Hence, $\widetilde{U} := \bigcup_{j \in J} \widetilde{U}_j \subset \widetilde{X} := X \times_Y \widetilde{Y}$ is an open subset such that $\widetilde{U} \to \widetilde{X} \to \widetilde{Y}$ is a surjective local isomorphism that is also a monomorphism and hence an isomorphism. As a result, $\widetilde{X} \to \widetilde{Y}$ is a monomorphism with a section, which forces it to be an isomorphism as well. Thus, the composite $\widetilde{X} \to X \to Y$ is a log blow-up, which by [36, 09MQ (4)] means that $X \to Y$ is separated and by [36, 03GN] universally closed. Since $X \to Y$ is log étale, it is locally of finite type. Lastly, X is also quasi-compact since it is dominated by $\widetilde{X}$, which is quasi-compact as it is a log blow-up of Y and Y was quasi-compact. Therefore, $X \to Y$ is proper as desired. □

Remark 2.7.
(1) Since we appealed to Proposition 4.3, this proof does not work unless one uses our notion of universal surjectivity. Indeed, one can use the covering in Proposition 5.1 to construct examples of (weakly) universally surjective log étale monomorphisms that are not proper.
(2) The above proof shows that any log modification in our sense is also a log modification in Kato's sense [15].

2.2. **Basic properties.** Here we collect a few of the basic properties of m-type morphisms.

Proposition 2.8.
(1) Every m-open morphism is m-étale.
(2) Any m-étale (resp. m-smooth, m-flat) morphism is also log étale (resp. log smooth, log flat).
(3) Local isomorphisms (resp. strict étale, smooth, flat morphisms) are m-open (resp. m-étale, m-smooth, m-flat).
(4) The conditions of being m-open, m-étale, m-smooth and m-flat are stable under composition and base change.
(5) A morphism $f\colon X \to Y$ is m-étale (resp. m-smooth, m-flat) if and only if it can strict étale locally in X and Y be written as a composition of a strict étale (resp. strict smooth, strict flat) morphism followed by a log blow-up. In particular, being m-étale (resp. m-smooth, m-flat) is strict étale local in source and target.

Proof. Properties (2) and (3) and stability under composition in (4) follow directly from the definitions. Stability under base change is immediate for m-open morphisms and follows from (5) for m-étale, m-smooth and m-flat morphisms. It furthermore follows from (5) that any m-open morphism is m-étale since one can replace the target by an étale cover with Zariski log structure and then apply Remark 2.5. Hence, it only remains to show (5).

For that, we first note that any m-étale (resp. m-smooth, m-flat) morphism is of the described shape since one can cover Y by charts $U \to Y$ for which we can find a compatible chart $V \to X$ so that $U \to V$ factors by Lemma 2.4 in the desirable way. For the converse, first note that log blow-ups are m-étale as can be seen from their explicit construction. Together with (3), this shows that any composition as in (5) is m-étale (resp. m-smooth, m-flat). It therefore suffices to show that being m-étale (resp. m-smooth, m-flat) are étale local properties.

Indeed, étale locality in the source follows directly from the definition. For étale locality in the target we let $Y' \twoheadrightarrow Y$ be an étale cover so that $X' = X \times_Y Y' \to Y'$ is m-étale (resp. m-smooth, m-flat). Now let $x \in X$ and $V \to Y$ an étale neighborhood of $f(x)$ with a chart $Q \to \Gamma(V, \mathcal{M}_Y)$ and let $x' \in X'$ be some preimage of x. By assumption, there is an étale neighborhood $U \to X'$ of x' with a chart $P \to \Gamma(U, \mathcal{M}_{X'})$ so that $U \to X' \to Y'$ factors over $V' = V \times_Y Y' \to Y'$. We further get that there is a morphism of charts $Q \to P$ so that $Q^{gp} = P^{gp}$ and the resulting $U \to V'' := V' \times_{\mathbb{A}_Q} \mathbb{A}_P$ is strict étale (resp. smooth, flat). However, $V'' \to V''' := V \times_{\mathbb{A}_Q} \mathbb{A}_P$ is strict étale as it is a base change of $Y' \to Y$ and so the resulting composite $U \to V'' \to V'''$ is again strict étale (resp. smooth, flat) and hence the chart $U \to X' \to X$ is the desired one. $\square$

Example 2.9. The following example illustrates why m-open morphisms are different than the other m-type morphisms. Let $X = \operatorname{Spec}(\mathbb{N}^2 \xrightarrow{0} \mathbb{C})$ and $Y = \operatorname{Spec}(\mathbb{R})$ equipped with the log structure descended from X by imposing that complex conjugation swaps the two factors of $\mathbb{N}^2$. Hence all morphisms in the cartesian square

$$\begin{tikzcd} X \sqcup X = X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

are strict étale covers and the upper horizontal morphism is a local isomorphism and thus m-open. However, $X \to Y$ is not a monomorphism and thus not m-open.

Hence being m-open is not étale local in the target (unlike the other m-type morphisms). Moreover, this demonstrates why we did not define m-open morphisms via a chart criterion as the natural analog of Definition 1.3 also holds for $X \to Y$. Hence this does not capture the notion of being a mix between log modifications and open immersions.

We now show that m-type morphisms occupy the opposite extreme from Kummer type morphisms.

Proposition 2.10. Let $f\colon X \to Y$ be a m-open (resp. m-étale, m-smooth, m-flat) morphism. If f is also exact, then f is a strict local isomorphism (resp. strict étale, strict smooth, strict flat). In particular, a m-open (resp. m-étale, m-smooth, m-flat) morphism of Kummer type is a strict local isomorphism (resp. strict étale, strict smooth, strict flat).

Proof. For m-open morphisms, this follows from the fact that an exact log étale monomorphism has to be strict open (see [33, Proposition 2.8]).

For the other cases, we recall from Proposition 2.8(2) that a strict m-étale (resp. m-smooth, m-flat) morphism also a strict log étale (resp. log smooth, log flat) and by [31, Proposition IV.3.1.6] and [31, Proposition IV.4.1.2(2)], this forces it to be strict étale (resp. smooth, flat).

Now assume that f is exact. By the previous discussion, it suffices to show that f is strict i.e. for any $x \in X$, we have $\overline{\mathcal{M}}_{X,\overline{x}} = \overline{\mathcal{M}}_{Y,\overline{f(x)}}$. By Proposition 2.8(5), we have the following diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ V \arrow[r] & Y. \end{tikzcd} \tag{2.1}$$

Here, U and V are étale neighborhoods of x and $f(x)$ and the horizontal morphisms are strict étale. Moreover, $U \to V$ factors as $U \xrightarrow{g} \widetilde{V} \xrightarrow{h} V$, where g is a strict étale (resp. smooth, flat) morphism and h is a log blow-up. Since $U \to V$ is exact at x and g is strict, it follows that h is exact at $g(x)$. By [31, Proposition I.4.2.1(5)], it follows that $\overline{\mathcal{M}}_{V,\overline{g(x)}} \to \overline{\mathcal{M}}_{\widetilde{V},\overline{f(x)}}$ is injective. Since h is a log blow-up, we moreover have that $\overline{\mathcal{M}}^{gp}_{V,\overline{g(x)}} \to \overline{\mathcal{M}}^{gp}_{\widetilde{V},\overline{f(x)}}$ is surjective, which by exactness also implies that $\overline{\mathcal{M}}_{V,\overline{g(x)}} \to \overline{\mathcal{M}}_{\widetilde{V},\overline{f(x)}}$ is surjective. This implies the desired strictness.

A morphism of Kummer type is also exact (see [31, Chapter I, Corollary 4.3.10]), hence, an m-type morphism of Kummer type is a strict morphism of the same type. □

3. The lifting criterion for m-étale and m-smooth morphisms

The main goal of this section is to prove Theorem A. We start this section with some preliminary results on formally m-étale (m-smooth) morphisms.

3.1. **Strict étale morphisms and log blow-ups are formally m-étale.**

As for the classical notions of formal étaleness and formal smoothness, we have:

Lemma 3.1. (1) Being formally m-étale or m-smooth is stable under base change and composition.
(2) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in of fs log schemes. If the composition $g \circ f\colon X \to Z$ is formally m-étale (resp. m-smooth) and g is formally m-étale, then f is formally m-étale (resp. m-smooth).

Proof. The proofs of both statements are identical to the standard argument for formally étale and smooth in Sch. □

Lemma 3.2. A strict morphism is formally m-étale (resp. m-smooth) if and only if the underlying morphism of schemes is formally étale (resp. m-smooth).

Proof. For any strict morphism $f\colon X \to Y$ of log schemes, the following diagram

$$\begin{tikzcd} X \arrow[r, "p_X"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\mathrm{triv}} \arrow[d, "f^{\mathrm{triv}}"] \\ Y \arrow[r, "p_Y"] & Y^{\mathrm{triv}} \end{tikzcd} \tag{3.1}$$

is cartesian by [31, III, Proposition 1.2.5], where X^{triv} and Y^{triv} are $\underline{X}$ and $\underline{Y}$ equipped with trivial log structures and p_X and p_Y are the canonical morphisms. Then a lifting of a morphism $T \to Y$ to X is the same as a lifting of $T^{\mathrm{triv}} \to Y^{\mathrm{triv}}$ to X^{triv}. □

Proposition 3.3. Let $f\colon X \to Y$ be a log blow-up. For any commutative diagram of the following form

$$\begin{tikzcd} T' \arrow[r, "l"] \arrow[d, "i"] & X \arrow[d, "f"] \\ T \arrow[r, "g"] \arrow[ur, dashed, "\exists!\, h"] & Y \end{tikzcd}$$

where i is exact and surjective, there is a unique lift h as indicated in the above diagram. In particular, f is formally m-étale.

Proof. Let $\mathcal{I} \subset \mathcal{M}_Y$ be the coherent ideal sheaf such that $X = \mathrm{Bl}_I(Y)$. By the universal property of the log blow-up, the lift h exists if and only if $g^*\mathcal{I}$ is étale locally principal and h is then also unique. By [31, III, Proposition 2.6.1] this must show that $g^*\mathcal{I}_{\bar{t}}$ is generated by one element for every $t \in T$. As i is surjective, there is a $t' \in T'$ with $i(t') = t$. Since $i^*g^*\mathcal{I}_{T',\bar{t}'}$ is principal and generated by $i_{log}(g^*\mathcal{I}_{\bar{t}})$, we can find a generator of the shape $i_{log}(a)$ for $a \in \mathcal{M}_{T,\bar{t}}$. We now claim that a also generates $g^*\mathcal{I}_{\bar{t}}$ itself. Indeed, for any $b \in g^*\mathcal{I}_{\bar{t}}$ we have $i_{log}(b) \in i_{log}(a) \cdot \mathcal{M}_{T',\bar{t}'}$ or equivalently $i_{log}(b)/i_{log}(a) \in \mathcal{M}_{T',\bar{t}'}$. By exactness of i this implies that $b/a \in \mathcal{M}_{T,t}$ as desired. □

3.2. **Proof of Theorem A and consequences.** In order to prove Theorem A, we introduce the following technical lemma:

Lemma 3.4. Let $f\colon X \to Y$ be a morphism in $\mathsf{fsLogSch}$ and $\beta\colon Q \to \Gamma(Y, \mathcal{M}_Y)$ is a chart. For any point $x \in X$, there is an étale neighborhood U of x and a morphism of charts

$$\begin{tikzcd} Q \arrow[r, "\beta"] \arrow[d, "\theta"] & \Gamma(Y, \mathcal{M}_Y) \arrow[d, "f^{log}"] \\ P \arrow[r, "\alpha"] & \Gamma(U, \mathcal{M}_X) \end{tikzcd} \tag{3.2}$$

with the following properties:

1. α is exact at x or equivalently, $\overline{P} \xrightarrow{\overline{\alpha}_{\overline{x}}} \overline{\mathcal{M}}_{X,\overline{x}}$ is an isomorphism.
2. θ is injective.

Proof. Let $U \to X$ be any étale neighborhood of x together with a chart for f given by $\alpha'\colon P' \to \Gamma(U, \mathcal{M}_X)$ and $Q \xrightarrow{\theta'} P'$. Note that $P' \oplus Q^{gp} \xrightarrow{\pi_{P'}} P' \xrightarrow{\alpha'} \Gamma(U, \mathcal{M}_X)$ is again a chart at x and $Q \xrightarrow{(\theta', \mathrm{incl})} P' \oplus Q^{gp}$ is injective, hence we may assume that θ' is already injective.

It remains to ensure that the chart is exact at x. For that we note that $S :=(\alpha'_{\overline{x}})^{-1}\mathcal{O}^{\times}_{X,\overline{x}} \subset P$ is a finitely generated submonoid as it is generated by a subset of the generators of P. By shrinking U we may therefore assume that $S = (\alpha')^{-1}\mathcal{O}_X^\times(U)$ and hence we get an induced morphism $\alpha\colon S^{-1}P \to \Gamma(U, \mathcal{M}_X)$ that is again a chart so that $S^{-1}P \to \mathcal{M}_{X,\overline{x}}$ is exact. Thus, $P = S^{-1}P'$ and $\theta : Y \overset{\theta'}{\hookrightarrow} P' \hookrightarrow P$ are as desired. $\square$

Proof of Theorem A. By Proposition 2.8(5), m-étale (resp. m-smooth) are formally m-étale (resp. m-smooth) as strict étale (resp. smooth) morphisms and log blow-ups are formally m-étale (resp. m-smooth).

For the converse, let $f\colon X \to Y$ be locally of finite presentation and formally m-étale (resp. m-smooth) and let Y be a global chart $Q \to \Gamma(Y, \mathcal{M}_Y)$. For any $x \in X$ there is by Lemma 3.4 an étale neighborhood $U \to X$ of x and a chart of f given by $Q \hookrightarrow P'$ and $P' \to \Gamma(U, \mathcal{M}_X)$ that is exact at x.

Let $P := P' \cap Q^{gp}$, which by [31, Theorem I.2.1.17(2)] is fine and saturated. Since P' is an exact chart at $\overline{x}$ and $P \hookrightarrow P'$ is an exact morphism of monoids, it follows from [31, Proposition III.2.2.2(1)] that the induced morphism of log structures $P^a \to P'^a = \mathcal{M}_X|_U$ is exact at x and P is an exact chart of P^a at x. In particular, we have the following diagram

$$\begin{tikzcd} \overline{P} \arrow[r, hook, "\phi"] \arrow[d, "\overline{\alpha}_{\overline{x}}"', "\cong"] & \overline{P'} \arrow[d, "\overline{\alpha'}_{\overline{x}}"', "\cong"] \\ \overline{P^a}_{\overline{x}} \arrow[r, hook, "\phi_{\overline{x}}"] & \overline{P'^a}_{\overline{x}}. \end{tikzcd} \tag{3.3}$$

The morphisms $\overline{\alpha}_{\overline{x}}$ and $\overline{\alpha'}_{\overline{x}}$ are isomorphisms by the assumption of exactness. Since both morphisms ϕ and $\phi_{\overline{x}}$ are exact and both monoids $\overline{P}$ and $\overline{P'}$ are sharp, it follows from [27, Proposition I.2.1.16(2)] that $\phi_{\overline{x}}$ and $\phi_{\overline{x}}$ are injective.

By pulling back those log structures to $\mathrm{Spec}(k(x))$, we get the following diagram

$$\begin{tikzcd} \mathrm{Spec}(P' \to k(x)) \arrow[r, hook] \arrow[d, "i"] & X \arrow[d, "f"] \\ \mathrm{Spec}(P \to k(x)) \arrow[r, "\tilde{f}"] \arrow[ur, dashed, "g"] & Y \end{tikzcd} \tag{3.4}$$

where $\tilde{f}(x) = f(x)$. As i is an exact morphism (as it is a pullback of $(U, P'^a) \to (U, P^a)$, which is exact at x) and also a (trivial) nilpotent thickening, we get a lift g as shown in the diagram. By commutativity of the upper triangle, this gives a section for the injection $\overline{P^a}_{\overline{x}} \hookrightarrow \overline{P'^a}_{\overline{x}}$ and hence $\overline{P^a} \cong \overline{P'^a}$. By (3.3), we get $\overline{P} \cong \overline{P'}$, which implies that P is also a chart of $\mathcal{M}_X$ in some possibly smaller étale neighborhood of x, which we might as well assume to be U. Hence, we found a chart

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & \Gamma(Y, \mathcal{M}_Y) \arrow[d] \\ P \arrow[r] & \Gamma(U, \mathcal{M}_X) \end{tikzcd} \tag{3.5}$$

for f which satisfies the desired condition $Q \hookrightarrow P$ and $Q^{gp} = P^{gp}$.

Finally, we consider the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "f_1"] \arrow[rr, bend left, "f_2"] \arrow[dr, "f_4"'] & Y \times_{\mathbb{A}_Q} \mathbb{A}_P \arrow[r, "f_3"] \arrow[d, "f_5"] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "f_7"] \\ & \mathbb{A}_P \arrow[r, "f_6"] & \mathbb{A}_Q. \end{tikzcd} \tag{3.6}$$

By Lemma 2.4, f_6 is m-étale, hence, f_3 is also m-étale. As given f_2 is m-étale (resp. m-smooth), by Lemma 3.1(2), f_1 is formally m-étale (resp. m-smooth). However, f_1 is strict as f_4 is strict and f_5, being the base change of f_7, is also strict. By Lemma 3.2, this implies that f_1 is formally étale (resp. smooth) and since f and f_3 are locally of finite presentation, it follows from [36, 02FV] that f_1 is locally of finite presentation. This concludes the proof. □

Corollary 3.5. Given morphisms of fs log schemes $f\colon X \to Y$ and $g\colon Y \to Z$ such that g and $g \circ f$ are m-étale (resp. m-open), then f must be m-étale (resp. m-open).

Proof. For m-étale morphisms, this follows directly from Theorem A, Lemma 3.1(2) and [36, 02FV]. Now we will deal with the case of m-open morphisms. Given a point $x \in X$, let $x \in U \subset X$ be an open neighborhood so that $g \circ f|_U$ is a log étale monomorphism. Since g is log étale, it follows that $f|_U$ is also a log étale monomorphism. This shows that f is also m-open. □

Moreover, for étale, smooth, and flat morphisms in the category of schemes, the following property holds: For any sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

if $g \circ f$ is étale (resp. smooth), f is smooth and surjective, and g is locally of finite presentation, then g is necessarily étale (resp. smooth). It is natural to ask whether this useful property also holds for m-type morphisms. However, this is not true by the following example due to Chikara Nakayama:

Example 3.6. [31, Example IV.4.3.4] Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"] \arrow[dr, phantom, "\lrcorner"] & Y := \operatorname{Spec}(\mathbb{N}^2 \to \mathbb{C}[x,y]/(x^2,y^2,xy)) \arrow[d, hook, "g"] \\ \widetilde{Z} \arrow[r, "\pi'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z := \operatorname{Spec}(\mathbb{N}^2 \to \mathbb{C}[x,y]/(x^2,y^2)) \arrow[d, hook, "i"] \\ \mathrm{Bl}_0(\mathbb{A}^2) \arrow[r, "\pi"] & \mathbb{A}^2. \end{tikzcd}$$

The spaces $\mathbb{A}^2$ and $\mathrm{Bl}_0(\mathbb{A}^2)$ carry standard toric logarithmic structures, and π is a toric blow-up (hence a log blow-up). Since log blow-ups are stable under base change, it follows that f and π' are also log blow-ups and thus universally surjective by (see Proposition 5.7(3)). Either by considering the universal property of the log blow-up or through a direct computation, one can verify that h is an isomorphism and hence $g \circ f$ is a log blow-up. However, g is not even log flat since it is strict and its underlying morphism of schemes is not flat.

4. M-type topologies

We start by showing that the m-type sites are functorial. For this, we need the following fact:

Proposition 4.1. All big and small m-type sites over S in Definition 1.6 have all finite limits and they agree with the limits in $\mathsf{fsLogSch}/S$.

Proof. Since all big and small sites have a terminal object, it is enough to show the existence of fiber products. For the big sites, the claim is immediate. To see that this also works for the small m-open (resp. m-étale) site, one uses Corollary 3.5, which implies that the morphisms in that site are also m-open (resp. m-étale). Hence, the usual fiber products are indeed objects in this site. □

Remark 4.2. Let $\tau \in \{\text{m-open}, \text{m-étale}, \text{m-smooth}, \text{m-fppf}, \text{m-fpqc}\}$ and $\tau' \in \{\text{m-open}, \text{m-étale}\}$. For any morphism $f\colon X \to Y$ of fs log schemes, the pullback functor $f^*\colon \mathsf{LogSch}/Y \to \mathsf{LogSch}/X$ preserves finite limits. By Proposition 4.1, the sites $(\mathsf{LogSch}/Y)_\tau$ and $(\mathsf{LogSch}/X)_\tau$ (resp. $Y_{\tau'}$ and $X_{\tau'}$) admit all finite limits and the resulting pullback functors $f^*\colon (\mathsf{LogSch}/Y)_\tau \to (\mathsf{LogSch}/X)_\tau$ (resp. $Y_{\tau'} \to X_{\tau'}$) preserve them. Consequently, f^* induces a morphism of sites between all our big and small sites and hence of their topoi. (see [21, Theorem VII.10.2]).

The following proposition will be important for the proof of Theorem B.

Proposition 4.3. An object in any of the m-type sites is quasi-compact (c.f. [36, 090H]) if and only if its underlying topological space is quasi-compact.

Proof. Since all m-type topologies contain strict open covers, an fs log scheme is quasi-compact if it is so in one of the m-type sites. For the converse, let $\{U_i \to U\}_{i\in I}$ be a cover in any of our topologies where the underlying scheme $\underline{U}$ is quasi-compact. In the m-fpqc case, it follows from the definition that we can find a finite subcover. Otherwise, the cover is m-flat and locally finitely presented, which by Proposition 2.8(5), Lemma 6.5(2) and the fact that fppf morphisms are universally open implies that all morphisms $U_i^{\mathrm{val}} \to U^{\mathrm{val}}$ have open image. Theorem 6.3(1) now implies that U^{val} must be quasi-compact and hence there is a finite $I' \subset I$ so that $\coprod_{i'\in I'} U_{i'}^{\mathrm{val}} \to U^{\mathrm{val}}$ is still universally surjective. By Lemma 6.5(1), the family $\{U_{i'} \to U\}_{i'\in I'}$ is our desired subcover. □

From this, we immediately get:

Corollary 4.4. Every m-type cover is also an m-fpqc cover.

We are now ready to prove Theorem B

Proof of Theorem B. It is enough to show that any m-type covering in any of the big and small m-sites admits a refinement that is a covering in the Grothendieck topology generated by strict m-type covers and log blow-ups (see [37, Section 2.3.5]).

Indeed, let S be a fs log scheme and $\{U_i \to S\}_{i\in I}$ an m-type covering. By [29, Theorem 5.4], we may replace S with a log blow-up of S that has a Zariski log structure. By taking an open cover, we may further assume that S is quasi-compact and admits a global chart. Proposition 4.3 now implies that there is a finite subcovering $\{U_{i'} \to S\}_{i'\in I'}$, where we may assume that the $U_{i'}$ are quasi-compact. Furthermore, according to [28, Lemma 3.10], there is a log blow-up $S' \to S$ such that $\coprod_{i' \in I'} U_{i'} \times_S S' \to S'$ is exact and by Proposition 2.10 therefore also strict. Hence, $\{U_{i'} \times_S S' \to U_{i'} \to S\}_{i' \in I'}$ is the desired refinement, which concludes the proof. $\square$

Remark 4.5. On the category $\mathsf{fsLogSch}/S$, the m-open (resp. m-étale, m-smooth, m-fppf) topology is equivalent to (i.e. has the same covering sieves as) the *valuative Zariski (resp. étale, smooth, flat) topology* defined in [30, Definition 2.23] and the *dividing Zariski (resp. étale) topology* of the same type defined in [7, Definition 3.1.5].

5. Subtleties regarding universal surjectivity

In this section, we prove the Theorem C and show how to fix the problem that this theorem raises.

5.1. **Proof of Theorem C and consequences.** We will first recall some terminology:

A monoid P is *valuative* if it is integral and for any $x \in P^{gp}$, we have $x \in P$ or $x^{-1} \in P$. An fs log scheme X is called *valuative* if all stalks of its log structure $\mathcal{M}_X$ are valuative monoids. By [31, Proposition I.2.4.2], this is equivalent to $\overline{\mathcal{M}}^{gp}_{X,\overline{x}}$ being of rank ≤ 1 for all $x \in X$.

The following proposition will be essential for the proof of Theorem C.

Proposition 5.1. Let X be an fs log scheme with a global chart $P \to \Gamma(X, \mathcal{M}_X)$ and let $\mathcal{V}$ be the set of all finitely generated valuative submonoids $P \subset V \subset P^{gp}$. Then the morphism

$$f_{\mathcal{V}} \colon \coprod_{V \in \mathcal{V}} X \times_{\mathbb{A}_P} \mathbb{A}_V \to X$$

is universally surjective in fs log schemes.

Proof. We need to show that any base change of $f_{\mathcal{V}}$ along a morphism $g \colon T \to X$ of fs log schemes is surjective. Since the question is local in T, we may assume that T is quasi-compact. By [28, Lemma 3.10], there is a non-empty ideal $I \subset P$ such that the base change of g along $\mathrm{Bl}_I X \to X$ is exact. Moreover, by [27, (2.2.2)], base change under exact morphisms preserves surjectivity. Therefore, we may assume that g is a log blow-up. By locality, it is furthermore enough to consider g of the shape $g \colon X \times_{\mathbb{A}_P} \mathbb{A}_Q \to X$ for some fine, saturated submonoid $P \subset Q \subset P^{gp}$. It now follows from Lemma 5.2 that any point of $X \times_{\mathbb{A}_P} \mathbb{A}_Q$ is in the image $X \times_{\mathbb{A}_P} \mathbb{A}_V \to X \times_{\mathbb{A}_P} \mathbb{A}_Q$ for some finitely generated valuative monoid $Q \subset V \subset P^{gp}$. $\square$

Lemma 5.2. For any pre-log ring $P \xrightarrow{\alpha} k$ with P an integral (not necessarily finitely generated) monoid and k a field, there is some inclusion of pre-log rings

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, hook] & k \arrow[d, hook] \\ V \arrow[r, "\beta"] & l \end{tikzcd}$$

so that $P \subset V \subset P^{gp}$ is valuative and l is a field. Moreover, if P is finitely generated, then we can choose V to be finitely generated as well.

Proof. We may assume that k is algebraically closed. Let $P' = \alpha^{-1}(k^\times)^{-1}P$ be the localization and $\widetilde{\alpha} \colon P' \to k$ the resulting morphism so that $\widetilde{\alpha}^{-1}(k^\times) = P'^\times$. By [31, Proposition I.2.4.1], there is a valuative submonoid $P' \subset V \subset P^{gp}$ so that $V^\times \cap P' = P'^\times$. Moreover, if P is finitely generated, then P' is finitely generated and V can be chosen to be finitely generated. Since k is algebraically closed, it follows that $k^\times$ is divisible and hence injective. This allows us to extend $\widetilde{\alpha}\colon P'^\times \to k^\times$ to $\beta\colon V^\times \to k^\times$ and further to all of V by declaring $\beta(v) = 0$ if $v \in V$ is not a unit. Since $V^\times \cap P' = P'^\times = \widetilde{\alpha}^{-1}(k^\times)$, we get that $\beta|_{P'} = \widetilde{\alpha}$ as desired. □

We are now ready to prove Theorem C:

Proof of Theorem C. We can cover any fs log scheme X by affine étale charts $U_i \to X$. By Proposition 5.1, we can cover U_i by affine valuative fs log schemes. If this composite cover has a finite subcover, then, by base change, each U_i must also have a quasi-compact log étale cover $V_i \to U_i$ where V_i is valuative. By [28, Lemma 3.11], we can factor this into a Kummer surjection (actually strict in our case) $V_i \to \widetilde{U}_i$ followed by a log blow-up $\widetilde{U}_i \to U_i$. By the definition of Kummer morphisms and [31, Proposition III.2.6.6], this however implies that all U_i are valuative and hence X is quasi-compact and valuative.

Conversely, if X is quasi-compact and valuative, then the small log étale of X agrees with the small Kummer log étale site of X (see [28, Proposition 3.7]) and then the claim follows from the fact that Kummer morphisms are open [14, Corollary 3.7]. □

The following corollary contradicts one of the fundamental results regarding the full log étale topology (see [14, Section 9.1]).

Corollary 5.3. For any fs log scheme X, the following are equivalent:

(1) The topology of the small log étale site $X_{\text{lét}}$ is generated by log blowups and Kummer étale covers.
(2) X is valuative.

Proof. If X is valuative, then any log étale morphism $U \to X$ is automatically Kummer, hence (2) implies (1). For the converse, it suffices by Theorem C to show that every quasi-compact open $U \subset X$ is a quasi-compact object of $X_{\text{lét}}$. Since Kummer morphisms are open by [14, Corollary 3.7] and log blow-ups are quasi-compact, it follows by an inductive argument that every cover of U that is a repeated composite of Kummer covers and log blow-ups admits a finite refinement. This concludes the proof. □

5.2. **The correct definition of universal surjectivity.** In order to fix the problem mentioned in the last section, we will enlarge the category of fs log schemes with which we are testing universal surjectivity. As a result, we will adopt a slightly different terminology from the rest of this paper.

By a *log scheme* we now mean a scheme equipped with an étale log structure that locally admits charts. We, however, do not assume morphisms of log schemes to be covered by compatible charts. An integral (resp. saturated, valuative, fs) log scheme is a log scheme that is covered by charts of integral (resp. saturated, valuative, fs) monoids.

Lemma 5.4. Let $f\colon X \to Y$ be a morphism of fs log schemes. Then the following are equivalent:

(1) For any integral log scheme T and any morphism $T \to Y$, the pullback $f_T \colon X \times_Y T \to T$ in the category of integral fs log schemes is surjective.
(2) For any saturated log scheme T and any morphism $T \to Y$, the pullback $f_T \colon X \times_Y T \to T$ in the category of saturated log schemes is surjective.
(3) For any valuative log scheme T with underlying scheme $\underline{T} = \operatorname{Spec}(k)$ for k an algebraically closed field, the base change $X \times_Y T$ in the category of integral (or saturated) log schemes is non-empty.

Remark 5.5. By [5, Lemma 3.1.9] all of the above fiber products actually exist. This is because Y is an fs log scheme and thus any morphism from an integral fs log scheme to Y admits charts, which one can use to construct the fiber product.

Proof. For any integral log scheme X, the canonical morphism $X^{\mathrm{sat}} \to X$ from its saturation is surjective (see e.g. [5, Proposition 3.1.10]). Hence, (1) and (2) are equivalent as well as the integral and saturated versions of (3). It thus remains to show that (3) implies (1). For this, let T be an integral log scheme and $T \to Y$ a morphism of log schemes. Since being surjective is strict étale local in the target, we may assume that the log structure of T is Zariski. By Lemma 5.2, every point of T is now the image of some valuative log point, which reduces to (3). $\square$

Thus we define:

Definition 5.6. A morphism $f \colon X \to Y$ of fs log schemes is *universally surjective* if it satisfies the equivalent conditions of Lemma 5.4.

We will now close this section by showing that our above definition still captures the expected examples. Together with Lemma 6.5(1), this will fix the proof of [28, Lemma 3.14], which is very similar to the proof of Proposition 4.3, and thus the rest of that paper.

Proposition 5.7. If $f \colon X \to Y$ is a morphism of fs log schemes and one of the following:

(1) exact and surjective,
(2) quasi-compact and universally surjective in fs log schemes,
(3) a log blow-up,

then it is also universally surjective. Moreover, universal surjections are stable under composition and base change in fs log schemes.

Proof. The stability under composition and base change follow directly from the definition. By Lemma 5.4(3) we can check the universal surjectivity of log blow-ups on valuative log points. Indeed, it follows from [5, Lemma 2.5.5] that any morphism from a valuative log point lifts along a log blow-up. This gives (3). For (2), we use that universal surjectivity is strict étale local in Y, which allows us to assume that Y and thus X is quasi-compact and Y admits a global chart. By [28, Lemma 3.10], there is hence a log blow-up $\widetilde{Y} \to Y$ so that $X \times_Y \widetilde{Y} \to \widetilde{Y}$ is exact and surjective. Since log blow-ups are universally surjective, this reduces (2) to (1). For (1) we note that any log point T in Lemma 5.4(3) has a canonical chart given by the global sections $Q = \Gamma(T, \mathcal{M}_T)$, which is exact at the unique point of T. Moreover, for any morphism $g \colon T \to Y$ any chart P of Y canonically fits into a chart $P \to Q$ for g. With these preparations in place, the proof of [31, Proposition III.2.2.3] still works in our more general setting. □

6. Sheaves on the valuative log spaces

In this section, we prove the results claimed in Section 1.5.

6.1. **Valuative log spaces.** We first need to define the so-called *valuative log space* associated to an fs log scheme. This space turns out to be neither a scheme, nor is its log structure quasi-coherent. Hence we need to start with some general discussion:

Definition 6.1. A *log locally ringed space* X is a locally ringed space $\underline{X}$ equipped with a log structure $\mathcal{M}_X \to \mathcal{O}_X$ where $\mathcal{M}_X$ is a sheaf of integral monoids on $\underline{X}$. Morphisms of log locally ringed spaces are defined accordingly and we denote the resulting category by LogLRS. We call X *valuative* if all stalks of $\mathcal{M}_X$ are valuative monoids.

Remark 6.2. (1) Any integral Zariski log scheme can be naturally regarded as a log locally ringed space.

(2) The category LogLRS is all cofiltered limits. Indeed, if $\{(X_i, \mathcal{O}_{X_i}, \mathcal{M}_{X_i})\}_{i\in I}$ in LogLRS is a cofiltered system, then the limit is given by

$$\lim_{i\in I}(X_i, \mathcal{O}_{X_i}, \mathcal{M}_{X_i}) = (\lim_{i\in I} X_i, \operatorname*{colim}_{i\in I} \pi_i^{-1}\mathcal{O}_{X_i}, \operatorname*{colim}_{i\in I} \pi_i^{-1}\mathcal{M}_{X_i}),$$

where $\pi_i \colon \lim_{j\in I} X_j \to X_i$ is the i-th projection. Using [12], one can show that LogLRS actually has arbitrary limits, but we will not need this fact.

The following theorem is stated in [16], where its proof is omitted. We provide a proof here for the reader's convenience.

Theorem 6.3. [16, Theorem 1.3.1] For any integral Zariski log scheme X, there is a log locally ringed space $\pi\colon X^{\mathrm{val}} \to X$ over X with valuative log structure that is terminal among all valuative log locally ringed spaces over X. Moreover, we have:

(1) The morphism $\pi\colon X^{\mathrm{val}} \to X$ is quasi-compact, quasi-separated, closed and surjective. Moreover, if X is quasi-compact and quasi-separated, then X^{val} is spectral[3].

(2) If X has a global chart $P \to \Gamma(X, \mathcal{M}_X)$, then we have

$$X^{\mathrm{val}} = \lim_{I\subset P} \mathrm{Bl}_I X.$$

The limit goes over the cofiltered poset of non-empty, finitely generated ideals $I \subset P$, where we let $I \leq J$ if there is a finitely generated ideal $K \subset P$ so that $I = J \cdot K$.

(3) For any strict morphism of integral Zariski log schemes $f\colon X \to Y$, the induced morphism $f^{\mathrm{val}}\colon X^{\mathrm{val}} \to Y^{\mathrm{val}}$ of valuative spaces is a pullback in LogLRS

$$\begin{tikzcd} X^{\mathrm{val}} \arrow[r, "f^{\mathrm{val}}"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y^{\mathrm{val}} \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{6.1}$$

[3]Recall that a topological space T is *spectral* if it is sober, quasi-compact, quasi-separated and its quasi-compact opens form a base of its topology.

Proof. For a strict morphism of integral log schemes $f\colon X \to Y$ so that Y^{val} exists, it follows that the pullback $X \times_Y Y^{\mathrm{val}} \to Y^{\mathrm{val}}$ in LRS equipped with the log structure pulled back from Y^{val} is valuative and fulfills the universal property of the valuative log space of X. To conclude the proof of (3), it suffices to construct X^{val} for any integral Zariski log scheme X. Indeed, it suffices to construct it for all members of an open covering of X since the intersections will glue by uniqueness and (3). As a result, we may assume that X has a global chart $P \to \Gamma(X, \mathcal{M}_X)$, and it suffices to show (2).

We now prove (2). First, one can show that the universal property of the log blow-up also holds in LogLRS, hence any valuative log locally ringed space mapping to X admits a unique lifting to $X' = \lim_{I\subset P} \mathrm{Bl}_I X$. It therefore suffices to show that X' is valuative. For that, let $x \in X'$ and $m \in \mathcal{M}^{gp}_{X',x}$. By Remark 6.2 we have $\mathcal{M}_{X',x} = \mathrm{colim}_{I\subset P} \mathcal{M}_{\mathrm{Bl}_I X, \pi_I(x)}$. Since group completion is a left adjoint, we also get $\mathcal{M}^{gp}_{X',x} = \mathrm{colim}_{I\subset P} \mathcal{M}^{gp}_{\mathrm{Bl}_I X, \pi_I(x)}$, which is a filtered colimit and hence agrees with the colimit taken in the category of sets. Now let $I \subset P$ be a finitely generated ideal so that $m \in \mathcal{M}^{gp}_{\mathrm{Bl}_I X, \pi_I(x)}$. By the construction of the log blow-up, there is a morphism $P^{gp} \to \mathcal{M}^{gp}_{\mathrm{Bl}_I X, \pi_I(x)}$ coming from a local chart whose image along with $\mathcal{O}^{\times}_{\mathrm{Bl}_I X, \pi_I(x)}$ generates $\mathcal{M}^{gp}_{\mathrm{Bl}_I X, \pi_I(x)}$. As a result, there is an element $\frac{a}{b} \in P^{gp}$ that maps to m modulo $\mathcal{O}^{\times}_{\mathrm{Bl}_I X, \pi_I(x)}$, which implies that $m \in \mathcal{M}_{\mathrm{Bl}_J X, \pi_J(x)}$ or $m^{-1} \in \mathcal{M}_{\mathrm{Bl}_J X, \pi_J(x)}$ if $J = (a,b)\cdot I$ with $(a,b) \subset P$ the ideal generated by a and b. This finally shows (2) and the existence of X^{val}.

To show the claims in (1), we may assume that $X = \mathrm{Spec}(P \to A)$, in which case the claim follows from Lemma 6.8 and the fact that log blow-ups are proper and surjective. $\square$

Remark 6.4. If X is an fs étale log scheme, then by [29, Theorem 5.4] there is a log blow-up $\widetilde{X} \to X$ so that the log structure of $\widetilde{X}$ is Zariski. Thus, we may define $X^{\mathrm{val}} := \widetilde{X}^{\mathrm{val}}$, which admits a projection $\pi\colon X^{\mathrm{val}} \to \widetilde{X} \to X$ to X that only exists as a morphism of locally ringed spaces. It is easy to verify that this is independent of the choice of $\widetilde{X}$ and the conclusions of Theorem 6.3(1) also hold for π. Using this construction, any morphism $f\colon X \to Y$ of fs étale log schemes also induces a morphism $f^{\mathrm{val}}\colon X^{\mathrm{val}} \to Y^{\mathrm{val}}$ in a functorial way.

Lemma 6.5. Let $f\colon X \to Y$ be a morphism of fs log schemes. Then:

(1) f is universally surjective if and only if f^{val} is surjective.
(2) If f is m-strict in the sense that strict étale locally in X and Y we can factor $f\colon X \xrightarrow{g} \widetilde{Y} \xrightarrow{h} Y$ with g strict and universally open and h a log blowup, then $f^{\mathrm{val}}\colon X^{\mathrm{val}} \to Y^{\mathrm{val}}$ is open.
(3) If f is m-open, then f^{val} is a local isomorphism.

Proof. By Proposition 5.7(3), log blow-ups are universally surjective. This and [29, Theorem 5.4] reduces (1) to the case of X and Y having Zariski log structures. Note that any point $y \in Y^{\mathrm{val}}$ is valuative when equipped with the induced log structure. Hence, f^{val} is surjective if and only if for every such log point $T \hookrightarrow Y^{\mathrm{val}}$, there is a commutative square

$$\begin{tikzcd} \exists\, T' \arrow[r, dashed] \arrow[d, dashed] & T \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd} \tag{6.2}$$

where T' is another valuative log point. Indeed, this occurs if and only if $X \times_Y T \neq \emptyset$, since, by Lemma 5.2, any log point can be dominated by a valuative log point. It follows that f^{val} is surjective if f is universally surjective.

For the converse, let $T_0 \to Y$ be a morphism from a valuative log point. By the above discussion and the universal property of the valuative space, we can factor $T_0 \to T \to Y$ so that $T \to Y$ fits into a square as in (6.2). We thus get a morphism $T' \times_T T_0 \to X \times_Y T_0$, where we take fiber products in the category of integral log schemes. Indeed, note that $T' \times_T T_0$ exists as valuative log points admit canonical charts coming from their global sections, which enables one to use the standard construction of the fiber product. However, since T is valuative, it follows from [31, Proposition I.4.6.3(5)] that $T' \to T$ is integral and hence $\underline{T' \times_T T_0} = \underline{T'} \times_{\underline{T}} \underline{T_0}$ is non-empty, which implies that $X \times_Y T_0$ is also non-empty.

For both (2) and (3) we may base change by a log blow-up of Y that has Zariski log structure, which reduces to the case that Y is Zariski.

For (2), we will first assume that f is strict. We may furthermore require that Y has a global chart. It now follows from Theorem 6.3 that $Y^{\mathrm{val}} = \lim_{\widetilde{Y} \to Y} \widetilde{Y}$ is the limit of all log blow-ups and $X^{\mathrm{val}} = X \times_Y Y^{\mathrm{val}} = \lim_{\widetilde{Y} \to Y} X \times_Y \widetilde{Y}$. Thus, any point $x \in X^{\mathrm{val}}$ admits a base of neighborhoods that are pulled back from some open in $X \times_Y \widetilde{Y}$. By universal openness of f, each $X \times_Y \widetilde{Y} \to \widetilde{Y}$ is open and thus the image of such neighborhoods in Y^{val} is indeed open. It thus follows that f^{val} being open is étale local in X and Y and so we may assume that $f = h \circ g$ as above, in which case openness of $f^{\mathrm{val}} = g^{\mathrm{val}}$ follows from the strict case.

Claim (2) follows from Remark 2.5 and the fact that f^{val} is an open immersion if f is an open immersion. □

For any fs log scheme X, we obtain a morphism of sites

$$\Phi \colon X_{\mathrm{mop}} \longrightarrow X^{\mathrm{val}}_{\mathrm{Zar}}$$
$$(T \to X) \mapsto \left(T^{\mathrm{val}} \to X^{\mathrm{val}}\right),$$

where for any locally ringed space Y we let $Y_{\mathrm{Zar}} \subset \mathsf{LRS}/Y$ be the site of locally ringed spaces with local isomorphisms $T \to Y$ and coverings $\{T_i \to T\}_{i \in I}$ consisting of jointly surjective morphisms over Y. Note that there is a canonical equivalence $\mathrm{Shv}(Y_{\mathrm{Zar}}) = \mathrm{Shv}(Y)$. We are now ready to give a more precise statement and proof of Theorem D.

Theorem D. The induced pushforward

$$\Phi_* \colon \mathrm{Shv}(X^{\mathrm{val}}) \xrightarrow{\cong} \mathrm{Shv}(X_{\mathrm{mop}})$$

is an equivalence.

Proof of Theorem D. Again, by [29, Theorem 5.4], we may assume that X has a Zariski log structure. Moreover, both categories of sheaves glue along any open cover of X

in a way that is compatible with Φ_*. Hence, we may assume that X admits a global chart $P \to \Gamma(X, \mathcal{M}_X)$.

By Theorem 6.3, the opens $U \subset X^{\mathrm{val}}$ that are pulled back from some blow-up form a topology base and are in the essential image of Φ. This shows that Φ_* is faithful. It thus suffices to show that the unit $\mathrm{Id} \to \Phi_*\Phi^*$ is an isomorphism. For this, it is enough to check that for any sheaf $\mathcal{F}$ on X_{mop} and m-open $U \to X$ with U quasi-compact and quasi-separated with a global chart, we have $(\Phi^*\mathcal{F})(U^{\mathrm{val}}) = \mathcal{F}(U)$. Indeed, we will check that the following map is an isomorphism

$$\mathcal{F}(U) \to \operatorname*{colim}_{U^{\mathrm{val}} \to V^{\mathrm{val}}} \mathcal{F}(V),$$

where the colimit runs over m-open $V \to X$ with morphisms $U^{\mathrm{val}} \to V^{\mathrm{val}}$ over X^{val}. By [16, Proposition 1.4.2], any such morphism comes from a morphism of the shape $\widetilde{U} \to V$, where $\widetilde{U} \to U$ is a log blow-up. Thus, such $\widetilde{U}$ are cofinal in the above colimit, which implies the claim since $\mathcal{F}(\widetilde{U}) = \mathcal{F}(U)$ by the sheaf property. □

Proof of Corollary 1.8. By Theorem 6.3(1), the underlying topological space of X^{val} is spectral. Hence, the vanishing follows from Theorem D and [35, Corollary 4.6]. □

Remark 6.6. A similar cohomology vanishing also occurs for the Kummer log étale and full log étale topologies, see [28, Theorem 7.2]. More generally, we expect that the ∞-topos $\mathrm{Shv}(X_{\mathrm{mop}}; \mathcal{S})$ of sheaves valued in spaces has homotopy dimension $\leq$ log-dim(X). One piece of evidence for this is [10, Theorem 2.4.15], which is a very analogous statement for the cdh site of any quasi-compact, quasi-separated scheme.

6.2. **Proof of Theorem E.** We begin with several lemmas that will be used in the proof of Theorem E.

Lemma 6.7. Let $f\colon X \to Y$ be a continuous map between sober topological spaces.

(1) If f has discrete fibers, then $\dim(X) \leq \dim(Y)$.
(2) If f is closed and surjective, then $\dim(X) \geq \dim(Y)$.
(3) If X and Y are spectral and f is quasi-compact, open and surjective, then $\dim(X) \geq \dim(Y)$.

Proof. If f has discrete fibers, then for any specialization $x_1 \rightsquigarrow x_2$ in X with $f(x_1) = f(x_2)$ we must already have $x_1 = x_2$. Hence, any chain $x_n \rightsquigarrow x_{n-1} \rightsquigarrow \ldots \rightsquigarrow x_0$ of proper specializations in X maps to a chain of proper specializations in Y. This shows $\dim(X) \leq \dim(Y)$ in case (1).

For (2), note that for any such closed f and $x \in X$ we have $f(\overline{\{x\}}) = \overline{\{f(x)\}}$. This means that f satisfies going up i.e. for any specialization $y_1 \rightsquigarrow y_2$ in Y and $x_1 \in f^{-1}\{y_1\}$, there is a $x_2 \in f^{-1}\{y_2\}$ with $x_1 \rightsquigarrow x_2$. This and surjectivity allow us to lift any chain of proper specializations in Y to one in X, which shows $\dim(X) \geq \dim(Y)$.

Similarly, in case (3), it suffices to show that f satisfies going down i.e. for any specialization $y_1 \rightsquigarrow y_2$ in Y and $x_2 \in f^{-1}\{y_2\}$, there is a $x_1 \in f^{-1}\{y_1\}$ with $x_1 \rightsquigarrow x_2$. This is equivalent to showing that the intersection $\bigcap_{x_2 \in U} U \cap f^{-1}\{y_1\}$ is non-empty. Indeed, since f is open, each individual term in the intersection is non-empty. Moreover, it suffices to only consider quasi-compact U in which case $U \hookrightarrow X \xrightarrow{f} Y$ is a quasi-compact morphism between spectral spaces. By [36, 0A2S, 0G1J], this implies that $U \cap f^{-1}\{y_1\}$ is spectral and the inclusion $U \cap f^{-1}\{y_1\} \hookrightarrow U$ is quasi-compact. This implies that the quasi-compact opens in $U \cap f^{-1}\{y_1\}$ are precisely the pre-images of the quasi-compact opens in U. Hence, for any inclusion $U \subset V$ of quasi-compact opens, the resulting inclusion $U \cap f^{-1}\{y_1\} \hookrightarrow V \cap f^{-1}\{y_1\}$ is a quasi-compact morphism. By [36, 0A2W], this implies that $\bigcap_{x_2 \in U} U \cap f^{-1}\{y_1\} \neq \emptyset$ as desired. ☐

Lemma 6.8. Let $\{X_i\}_{i \in I}$ be a cofiltered system of spectral spaces with quasi-compact transition maps $\pi_{i,j} \colon X_i \to X_j$. Let furthermore $X = \lim_{i \in I} X_i$ be its limit and $\pi_i \colon X \to X_i$ the projections.

(1) X is also spectral and the π_i are quasi-compact.
(2) If all the $\pi_{i,j}$ are closed, then the π_i are closed.
(3) If all the $\pi_{i,j}$ are surjective, then the π_i are surjective.

Proof. If the X_i are spectral and the $\pi_{i,j}$ are quasi-compact, then X is also spectral by [36, 0A2Z] and the projections $\pi_i \colon X \to X_i$ are quasi-compact, which concludes the proof of (1). If all transition maps are surjective, then by [36, 0A2W(2), 0902, 0A2S(1)] the π_i are also surjective, which finishes the proof of (3).

It remains to show (2). Let $C \subset X$ be a closed subset and $x \in X_i$ in the complement of $\pi_i(C)$. Then we have $\pi_i^{-1}\{x\} \cap C = \emptyset$ and hence we can cover $\pi_i^{-1}\{x\}$ by open neighborhoods pulled back from some X_j that are disjoint with C. By [36, 0A2S(2)], $\pi_i^{-1}\{x\}$ is quasi-compact, hence we may find a $j \leq i$ and $U \subset X_j$ disjoint from $\pi_j(C)$ so that $\pi_i^{-1}\{x\} \subset \pi_j^{-1}(U)$. Now letting $C' = X_j \setminus U \subset X_j$, we get that $\pi_i(C) \subset \pi_{j,i}(C') \not\ni x$. Since $\pi_{j,i}$ is closed, this implies that $x \notin \overline{\pi_i(C)}$. As a result, $\pi_i(C)$ must have been closed and we are done. ☐

Lemma 6.9. Let $\{X_i\}_{i \in I}$ be a cofiltered system of topological spaces and $X = \lim_{i \in I} X_i$ its limit. Then we have

$$\dim(X) \leq \sup_{i \in I} \dim(X_i)$$

and equality if the X_i are spectral and the transition maps $\pi_{i,j} \colon X_i \to X_j$ are surjective, quasi-compact and closed.

Proof. Let $Z_1 \subsetneq Z_2 \subset X$ be a proper inclusion of irreducible closed subsets. Since opens pulled back from one of the projections $\pi_i \colon X \to X_i$ form a topology base of X, we can choose an $x \in Z_2 \setminus Z_1$ and a neighborhood $\pi_i(x) \in U \subset X_i$ for some $i \in I$ so that $Z_1 \cap \pi_i^{-1}(U) = \emptyset$. This shows that $\overline{\pi_j(Z_1)} \subsetneq \overline{\pi_j(Z_2)}$ for all $j \leq i$ and hence any finite chain of irreducible closed subsets on X descends to one of the same length on some X_i. This proves the desired inequality. Combining Lemma 6.8 and Lemma 6.7(2), we get the desired equality of dimensions. ☐

Now, we are ready to present the proof of Theorem E.

Proof of Theorem E. Throughout this proof we will write $r_x = \max\left(\operatorname{rank}(\overline{\mathcal{M}}^{gp}_{X,\overline{x}}) - 1, 0\right)$ for any $x \in X$. First, we want to show that

$$\text{log-dim}(X) = \sup_{U \to X} \dim(U) \tag{6.3}$$

where the $U \to X$ are log blow-ups of open subsets of X. Indeed, for any such morphism, Theorem 6.3 implies that $U^{\mathrm{val}} \to U$ is closed and surjective and $U^{\mathrm{val}} \to X^{\mathrm{val}}$ is an open immersion. This gives $\dim(U) \leq \dim(U^{\mathrm{val}}) \leq \dim(X^{\mathrm{val}})$, where we use

Lemma 6.7(2) for the first inequality. This shows "$\geq$" in (6.3). The other inequality follows from the locality of Krull dimension, Theorem 6.3(2) and Lemma 6.9.

Having established (6.3), we first try to bound the right hand side of (6.3) by the right hand side of (1.3). For this, let $p\colon U \to X$ be as above with image $U' \subset X$. For any $x \in U'$ we can choose a neat chart $P \to \Gamma(V, \mathcal{M}_X)$ at x for some étale neighborhood $V \to U'$. By the construction of the log blow-up, there is a cartesian square

$$\begin{tikzcd} U \times_{U'} V \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & W \arrow[d, "\pi'"] \\ V \arrow[r] & \mathbb{A}_P, \end{tikzcd}$$

where π' is a toric blow-up. Any fiber of π' is also a fiber of some base change $\pi'_k\colon W_k \to \mathrm{Spec}(k[P])$ for some prime field k. But since π'_k is birational, it follows that said fiber must have dimension at most $\max(\dim k[P] - 1, 0) = r_x$, where we used [31, Proposition I.3.4.1]. By [36, 02FY], this means that the fibers of $U \to U'$ over the image of V have at most this dimension. By [36, 02FX, 0BAG] it therefore follows that $\dim(p^{-1}\overline{\{x\}}) \leq \dim \overline{\{x\}} + r_x$, which proves the desired first inequality.

To prove the converse inequality, we again choose a point $x \in X$ and a neat chart $P \to \Gamma(V, \mathcal{M}_X)$ at x on an affine étale neighborhood $p\colon V \to X$. Note that the preimage $p^{-1}\overline{\{x\}} \subset V$ maps into the irrelevant set $Z := V(P \setminus \{1\}) \subset \mathbb{A}_P$ and hence, the log blow-up $\pi\colon \widetilde{V} \to V$ resulting from

$$\begin{tikzcd} \widetilde{V} \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Bl}_Z\mathbb{A}_P \arrow[d] \\ V \arrow[r] & \mathbb{A}_P \end{tikzcd}$$

restricts to a flat morphism of relative pure dimension $\max(\mathrm{rank}(P^{gp}) - 1, 0) = r_x$ over $p^{-1}\overline{\{x\}}$. Hence, by [36, 0AFE], we have

$$\dim(\widetilde{V}) \geq \dim \pi^{-1}p^{-1}\overline{\{x\}} = \dim p^{-1}\overline{\{x\}} + r_x \geq \dim \overline{\{x\}} + r_x,$$

where the last inequality used Lemma 6.7(3). Moreover, by (6.3) applied to $\widetilde{V}$, we have $\dim(V^{\mathrm{val}}) \geq \dim \overline{\{x\}} + r_x$ and since $V^{\mathrm{val}} \to X^{\mathrm{val}}$ has discrete fibers by Theorem 6.3(3), we get $\dim(X^{\mathrm{val}}) \geq \dim \overline{\{x\}} + r_x$ by Lemma 6.7(1). This concludes the proof. ☐

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