AN EXTENSION OF VIENNOT'S SHADOW TO ROOK PLACEMENTS VIA ORBIT HARMONICS

JASPER M. LIU AND HAI ZHU

Abstract. For fixed positive integers n, m, r, let $\mathrm{Mat}_{n\times m}(\mathbb{C})$ be the affine space of $n \times m$ complex matrices with coordinate ring $\mathbb{C}[\mathbf{x}_{n\times m}]$. We define a homogeneous ideal $I_{n,m,r}$, where the graded quotient $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$ is obtained from the orbit harmonics deformation of the matrix loci corresponding to all rook placements of size at least r. By extending rook placements to elements in $\mathfrak{S}_{n+m-r}$ and applying Viennot's shadow line avatar of the Schensted correspondence, we compute the standard monomial basis of the quotient $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$ with respect to diagonal monomial orders. We also determine the graded $\mathfrak{S}_n \times \mathfrak{S}_m$-module structure of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$.

1. Introduction

Let $\mathbf{x}_N = (x_1, x_2, \ldots, x_N)$ be a collection of N variables and let $\mathbb{C}[\mathbf{x}_N] := \mathbb{C}[x_1, x_2, \ldots, x_N]$ be the polynomial ring over these variables. For a finite locus of points $\mathcal{Z} \subseteq \mathbb{C}^N$, its vanishing ideal is

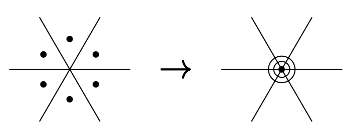
$$\mathbf{I}(\mathcal{Z}) := \{f \in \mathbb{C}[\mathbf{x}_N] : f(z) = 0 \text{ for all } z \in \mathcal{Z}\}. \tag{1.1}$$

The coordinate ring $\mathbb{C}[\mathcal{Z}]$ can then be identified with $\mathbb{C}[\mathbf{x}_N]/\mathbf{I}(\mathcal{Z})$. The method of *orbit harmonics* replaces the ideal $\mathbf{I}(\mathcal{Z})$ with its associated graded ideal $\mathrm{gr}\,\mathbf{I}(\mathcal{Z})$, extending this identification of (ungraded) vector spaces to

$$\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[\mathbf{x}_N]/\mathbf{I}(\mathcal{Z}) \cong \mathbb{C}[\mathbf{x}_N]/\mathrm{gr}\,\mathbf{I}(\mathcal{Z}) =: R(\mathcal{Z}). \tag{1.2}$$

If $\mathcal{Z}$ is invariant under the action of a finite matrix group $G \subseteq \mathrm{GL}_N(\mathbb{C})$, this isomorphism can be treated as an isomorphism of G-modules, and $R(\mathcal{Z})$ has the additional structure as a graded G-module.

Geometrically, the method of orbit harmonics linearly deforms the locus $\mathcal{Z}$ to a scheme of multiplicity $|\mathcal{Z}|$ supported at the origin. In the picture below, this deformation is drawn schematically for $\mathcal{Z}$, where $\mathcal{Z}$ is a locus of six points stable under the action of $\mathfrak{S}_3$ generated by reflections across the three lines.



The method of orbit harmonics dates back to Kostant [6]. It has seen its application in numerous settings, including cohomology theory [3], Macdonald theory [4, 5], cyclic sieving [10], Donaldson–Thomas invariants [11], and Ehrhart theory [12]. In cases where the locus $\mathcal{Z}$ has nice combinatorial structures or symmetries, one can usually expect that the algebraic properties of $R(\mathcal{Z})$ are governed by the combinatorial properties of $\mathcal{Z}$.

Let $\mathbf{x}_{n\times m} = (x_{i,j})_{1\leq i\leq n, 1\leq j\leq m}$ be an $n \times m$ matrix of variables. Let $\mathbb{C}[\mathbf{x}_{n\times m}]$ be the polynomial ring over these variables and $\mathrm{Mat}_{n\times m}(\mathbb{C})$ be the affine space of $n \times m$ complex matrices. The application of orbit harmonics to finite matrix loci $\mathcal{Z} \subseteq \mathrm{Mat}_{n\times m}(\mathbb{C})$ was initiated by Rhoades [13]. He considered the case $n = m$ with the point locus $\mathcal{Z} = \mathfrak{S}_n \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$ of $n \times n$ permutation matrices. This permutation matrix locus carries an action of $\mathfrak{S}_n \times \mathfrak{S}_n$ by left and right multiplication. Algebraic properties of $R(\mathfrak{S}_n)$ are governed by the longest increasing

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subsequences in $\mathfrak{S}_n$ and Viennot's shadow line construction. Liu [7] extended this work to the locus $\mathcal{Z} = \mathbb{Z}_r \wr \mathfrak{S}_n$ of r-colored permutation matrices. Liu, Ma, Rhoades, Zhu [8] further studied the matrix loci $\mathcal{Z} = \{w \in \mathfrak{S}_n : w^2 = 1\} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C})$ of involutions and the matrix loci $\mathcal{Z} = \mathcal{PM}_n = \{w \in \mathcal{M}_n : w(i) \neq i \text{ for all } i\}$ of fixed-point-free involutions.

In this paper, we consider a matrix locus formed by rook placements. For any positive integer n, let $[n]$ be the set of integers $\{1, 2, \ldots, n\}$. For positive integers n and m, a *rook placement* $\mathcal{R}$ on the $n \times m$ board is a subset of $[n] \times [m]$ such that $(i, j) \in \mathcal{R}$ and $(i', j') \in \mathcal{R}$ implies $i \neq i'$ and $j \neq j'$. We call each element $(i, j) \in \mathcal{R}$ a *rook* of $\mathcal{R}$. Define the *size* $|\mathcal{R}|$ of $\mathcal{R}$ by

$$|\mathcal{R}| \coloneqq \text{the number of rooks of } \mathcal{R}.$$

Identify each rook placement $\mathcal{R}$ on the $n \times m$ board with the $n \times m$ matrix $(a_{i,j})_{1\leq i\leq n,\, 1\leq j\leq m}$ given by

$$a_{i,j} = \begin{cases} 1, & \text{if } (i,j) \in \mathcal{R} \\ 0, & \text{otherwise.} \end{cases}$$

Write $\mathcal{Z}_{n,m,r} \subseteq \mathrm{Mat}_{n\times m}(\mathbb{C})$ for the set of all rook placements on the $n \times m$ board of size r, where $r \leq \min\{m, n\}$. We are interested in the finite locus of all rook placements with at least r rooks:

$$\mathcal{UZ}_{n,m,r} \coloneqq \bigsqcup_{r'=r}^{\min\{m,n\}} \mathcal{Z}_{n,m,r'} \subseteq \mathrm{Mat}_{n\times m}(\mathbb{C}).$$

The locus $\mathcal{UZ}_{n,m,r}$ is closed under the action of $\mathfrak{S}_n \times \mathfrak{S}_m$ by row and column permutation, and the quotient $R(\mathcal{UZ}_{n,m,r})$ is a graded $\mathfrak{S}_n \times \mathfrak{S}_m$-module. Our results on this quotient include:

- We give an explicit generating set of $\mathrm{gr}\,\mathbf{I}(\mathcal{UZ}_{n,m,r})$. See Definition 3.1 and Theorem 4.12.
- We show (Theorem 4.12) that $R(\mathcal{UZ}_{n,m,r})$ admits a monomial basis $\mathfrak{cs}(\mathcal{R})$ indexed by rook placements in $\mathcal{UZ}_{n,m,r}$ (see Definition 4.8), which is defined by extending each rook placement to a permutation in $\mathfrak{S}_{n+m-r}$ and applying Viennot's shadow construction. We further show that this is the standard monomial basis with respect to diagonal monomial orders (Definition 4.7).
- We describe the graded $\mathfrak{S}_n \times \mathfrak{S}_m$-module structure of $R(\mathcal{UZ}_{n,m,r})$.

The rest of the paper is structured as follows. In Section 2 we give background material on Gröbner theory, orbit harmonics, and representations of $\mathfrak{S}_n$. In Section 3 we define the ideal $I_{n,m,r}$, and study its algebraic structure. In Section 4 we define Viennot's shadow line construction on $\mathfrak{S}_n$ and our extension of this construction to $\mathcal{UZ}_{n,m,r}$. We show that this construction gives the standard monomial basis of $R(\mathcal{UZ}_{n,m,r})$ and prove that the ideal we define in Section 3 is exactly $\mathrm{gr}\,\mathbf{I}(\mathcal{UZ}_{n,m,r})$. In Section 5 we discuss the structure of the graded $\mathfrak{S}_n \times \mathfrak{S}_m$-module $R(\mathcal{UZ}_{n,m,r})$. We close in Section 6 with conjectures and possible directions of future research.

2. Background

2.1. **Gröbner Theory.** Let $\mathbf{x}_N = (x_1, x_2, \ldots x_N)$ be a sequence of N variables, and let $\mathbb{C}[\mathbf{x}_N]$ be the polynomial ring over these variables. A total order $<$ on the set of monomials in $\mathbb{C}[\mathbf{x}_N]$ is called a *monomial order* if

- $1 \leq m$ for every monomial $m \in \mathbb{C}[\mathbf{x}_N]$
- for monomials m, m_1, m_2, we have that $m_1 < m_2$ implies $mm_1 < mm_2$.

Let $<$ be a monomial order. For any nonzero polynomial $f \in \mathbb{C}[\mathbf{x}_N]$, the *initial monomial* $\mathrm{in}_< f$ of f is the largest monomial in f with respect to $<$. Given an ideal $I \subseteq \mathbb{C}[\mathbf{x}_N]$, its *initial ideal* is defined to be

$$\mathrm{in}_< I = \langle \mathrm{in}_< f : f \in I, f \neq 0 \rangle. \tag{2.1}$$

A monomial $m \in \mathbb{C}[\mathbf{x}_N]$ is called a *standard monomial* of I if it is not an element of $\mathrm{in}_< I$. It is well known that

$$\{m + I : m \text{ is a standard monomial of } I\} \tag{2.2}$$

forms a basis for the vector space $\mathbb{C}[\mathbf{x}_N]/I$. This is called the *standard monomial basis*. See [2] for further details on Gröbner theory.

2.2. **Orbit Harmonics.** Let $\mathcal{Z} \subset \mathbb{C}^N$ be a finite locus of points in affine N-space. The *vanishing ideal* $\mathbf{I}(\mathcal{Z}) \subseteq \mathbb{C}[\mathbf{x}_N]$ is the ideal

$$\mathbf{I}(\mathcal{Z}) = \{f \in \mathbb{C}[\mathbf{x}_N] : f(z) = 0 \text{ for all } z \in \mathcal{Z}\}. \tag{2.3}$$

Identifying $\mathbb{C}[\mathcal{Z}]$ with the vector space of all functions from $\mathcal{Z}$ to $\mathbb{C}$, we obtain

$$\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[\mathbf{x}_N]/\mathbf{I}(\mathcal{Z}). \tag{2.4}$$

Given a nonzero polynomial $f \in \mathbb{C}[\mathbf{x}_N]$, write $f = f_0 + f_1 + \cdots + f_d$, where f_i is homogeneous of degree i and $f_d \neq 0$. Denote by $\tau(f)$ the top degree homogeneous part of f, that is, $\tau(f) = f_d$. For any ideal $I \subseteq \mathbb{C}[\mathbf{x}_N]$, its *associated graded ideal* is

$$\mathrm{gr}\, I = \langle \tau(f) : f \in I \rangle. \tag{2.5}$$

Remark 2.1. *Given a set of generators $I = \langle f_1, f_2, \ldots, f_r \rangle$, we have the containment of ideals $\langle \tau(f_1), \tau(f_2), \ldots, \tau(f_r) \rangle \subseteq \mathrm{gr}\, I$. This containment is strict in general. On the other hand, if $<$ is a monomial order such that $f \leq g$ if and only if $\deg(f) \leq deg(g)$ and $\{f_1, f_2, \ldots, f_r\}$ forms a Gröbner basis of I with respect to $<$, we have the equality $\langle \tau(f_1), \tau(f_2), \ldots, \tau(f_r) \rangle = \mathrm{gr}\, I$.*

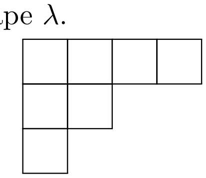
The isomorphism of ungraded vector spaces in Equation (2.4) can be extended to

$$\mathbb{C}[\mathcal{Z}] \cong \mathbb{C}[\mathbf{x}_N]/\mathbf{I}(\mathcal{Z}) \cong \mathbb{C}[\mathbf{x}_N]/\mathrm{gr}\,\mathbf{I}(\mathcal{Z}) =: R(\mathcal{Z}). \tag{2.6}$$

Since the generators of $\mathrm{gr}\,\mathbf{I}(\mathcal{Z})$ are all homogeneous, $R(\mathcal{Z})$ has the additional structure of a graded vector space. Furthermore, if $\mathcal{Z}$ is stable under the action of a matrix group $G \subseteq GL_N(\mathbb{C})$, Equation (2.6) is also an isomorphism of ungraded G-modules, and $R(\mathcal{Z})$ has the additional structure of a graded G-module.

2.3. **Schensted Correspondence.** Let n be any positive integer. A *partition* λ of n is a sequence of non-increasing positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum_{i=1}^k \lambda_i = n$. We write $\lambda \vdash n$ to indicate that λ is a partition of n.

For a partition $\lambda \vdash n$, its *Young diagram* is a diagram with n left-justified boxes, with λ_i boxes on the i-th row. A *standard Young tableau* of shape λ is a filling of $[n]$ into the boxes in the Young diagram of λ, so that the numbers increase across each row and down each column. Below on the left is the Young diagram of the partition $\lambda = (4, 2, 1) \vdash 7$ and on the right is an example of a standard Young tableau of shape λ.



$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Denote by $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of shape λ. The *Schensted correspondence* [15] is a bijection

$$\mathfrak{S}_n \xrightarrow{\sim} \bigsqcup_{\lambda \vdash n} \{(P, Q) : P, Q \in \mathrm{SYT}(\lambda)\} \tag{2.7}$$

that takes a permutation in $\mathfrak{S}_n$ to a pair of standard Young tableaux of the same shape. This bijection is usually achieved through an insertion algorithm, details of which can be found in [14]. In Section 4.1, we will introduce an alternate description of the Schensted correspondence discovered by Viennot [16].

2.4. **Representation Theory.** Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be the graded algebra of symmetric functions in an infinite set of variables $\mathbf{x} = (x_1, x_2, \dots)$ over the ground field $\mathbb{C}(q)$. For example, the *complete homogeneous symmetric function* of degree n is $h_n := \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \dots x_{i_n} \in \Lambda$ and the *elementary symmetric function* of degree n is $e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n}$. Both $\{h_1, h_2, \dots\}$ and $\{e_1, e_2, \dots\}$ are algebraically independent generating sets of Λ.

The degree-n component Λ_n of Λ has bases indexed by partitions $\lambda \vdash n$. For example, the *complete homogeneous basis* of Λ_n is $\{h_\lambda : \lambda \vdash n\}$ where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \dots$ for $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$. We will also need the *Schur basis* $\{s_\lambda : \lambda \vdash n\}$ (see [9] for its definition). If $\lambda = (n)$ one has $s_{(n)} = h_n$.

The algebra of *doubly symmetric functions* is the tensor product $\Lambda \otimes_{\mathbb{C}(q)} \Lambda$, which is naturally doubly graded. The Schur basis of the degree (n, m) piece is $\{s_\lambda \otimes s_\mu : \lambda \vdash n\,,\ \mu \vdash m\}$.

A symmetric function $F \in \Lambda$ is *Schur-positive* if it can be written as $F = \sum_\lambda c_\lambda(q) \cdot s_\lambda$ with coefficients $c_\lambda(q) \in \mathbb{R}_{\geq 0}[q]$. Similarly, a doubly symmetric function $F' \in \Lambda \otimes_{\mathbb{C}(q)} \Lambda$ is *Schur-positive* if it can be written as $F' = \sum_{\lambda,\mu} c_{\lambda,\mu}(q) \cdot s_\lambda \otimes s_\mu$ with coefficients $c_{\lambda,\mu} \in \mathbb{R}_{\geq 0}[q]$. For $F, G \in \Lambda$ (resp. $F, G \in \Lambda \otimes_{\mathbb{C}(q)} \Lambda$), we write $F \leq G$ if and only if $G - F$ is Schur-positive.

We introduce the *truncation operator* $\{-\}_P$ where P is a condition that partitions may or may not satisfy. Given $F = \sum_\lambda c_\lambda s_\lambda \in \Lambda$ with $c_\lambda \in \mathbb{C}(q)$, we define $\{F\}_P$ by

$$\{F\}_P \coloneqq \sum_{\lambda \text{ satisfies } P} c_\lambda s_\lambda.$$

Given $G = \sum_{\lambda,\mu} c_{\lambda,\mu} s_\lambda \otimes s_\mu \in \Lambda \otimes_{\mathbb{C}(q)} \Lambda$ with $c_{\lambda,\mu} \in \mathbb{C}(q)$ for all partitions λ, μ, we similarly define $\{G\}_P$ by

$$\{G\}_P \coloneqq \sum_{\lambda, \mu \text{ both satisfy } P} c_{\lambda,\mu} s_\lambda \otimes s_\mu.$$

For instance, we have

$$\{G\}_{\lambda_1 \leq L} = \sum_{\substack{\lambda_1 \leq L \\ \mu_1 \leq L}} c_{\lambda,\mu} s_\lambda \otimes s_\mu.$$

Irreducible representations of $\mathfrak{S}_n$ are indexed by partitions $\lambda \vdash n$. We write V^λ for the irreducible $\mathfrak{S}_n$-module indexed by λ. Given an $\mathfrak{S}_n$-module V, there exists a unique decomposition $V \cong \bigoplus_{\lambda \vdash n} c_\lambda V^\lambda$ with $c_\lambda \geq 0$. The *Frobenius image* of V is defined to be the symmetric function

$$\operatorname{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda, \tag{2.8}$$

obtained by replacing irreducible representation V^λ with the Schur polynomial s_λ. If $V = \bigoplus_{d \geq 0} V_d$ is a graded $\mathfrak{S}_n$-module, the *graded Frobenius image* of V is

$$\operatorname{grFrob}(V, q) := \sum_{d \geq 0} \operatorname{Frob}(V_d) \cdot q^d. \tag{2.9}$$

For integers $n, m \geq 0$, irreducible representations of $\mathfrak{S}_n \times \mathfrak{S}_m$ are of the form $V^\lambda \otimes V^\mu$, where $\lambda \vdash n$ and $\mu \vdash m$. Each $\mathfrak{S}_n \times \mathfrak{S}_m$-module V has a unique decomposition $V \cong \bigoplus c_{\lambda,\mu} V^\lambda \otimes V^\mu$ with $c_{\lambda,\mu} \geq 0$, and we extend the definition of Frobenius images to $\mathfrak{S}_n \times \mathfrak{S}_m$-modules by letting

$$\operatorname{Frob}(V) := \sum_{\substack{\lambda \vdash n \\ \mu \vdash m}} c_{\lambda,\mu} s_\lambda \otimes s_\mu, \tag{2.10}$$

and define the graded Frobenius images for graded $\mathfrak{S}_n \times \mathfrak{S}_m$-modules analogously.

We need the following result about symmetrizers acting on irreducible representations of $\mathfrak{S}_n$. Let $j < n$ be a positive integer and embed $\mathfrak{S}_j \subseteq \mathfrak{S}_n$ by letting it act on the first j letters. This induces an embedding of group algebras $\mathbb{C}[\mathfrak{S}_j] \subseteq \mathbb{C}[\mathfrak{S}_n]$. Let $\eta_j \in \mathbb{C}[\mathfrak{S}_j] \subseteq \mathbb{C}[\mathfrak{S}_n]$ be the element that symmetrizes over $\mathfrak{S}_j$, i.e.

$$\eta_j = \sum_{w \in \mathfrak{S}_j} w. \tag{2.11}$$

If V is an $\mathfrak{S}_n$-module, η_j acts as an operator on V, and we have the following lemma which characterizes when η_j annihilates irreducible $\mathfrak{S}_n$-modules.

Lemma 2.2. *For $j \leq n$ and $\lambda \vdash n$, we have that $\eta_j \cdot V^\lambda \neq 0$ if and only if $\lambda_1 \geq j$.*

Proof. For any $\mathfrak{S}_n$-module V^λ, we have

$$\eta_j \cdot V^\lambda = (V^\lambda)^{\mathfrak{S}_j} = \{v \in V^\lambda \,:\, w \cdot v = v \text{ for all } w \in \mathfrak{S}_j\}. \tag{2.12}$$

Thus, $\eta_j \cdot V^\lambda \neq 0$ if and only if $(\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j} V^\lambda)^{\mathfrak{S}_j} \neq 0$. This is true if and only if the trivial representation $V^{(j)}$ appears with positive multiplicity in the decomposition of $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j} V^\lambda$. The branching rule of $\mathfrak{S}_n$-modules (see [9, 14]) states that

$$\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} V^\lambda = \bigoplus_{\substack{\mu \vdash n-1, \\ \mu_i \leq \lambda_i \text{ for all } i}} V^\mu, \tag{2.13}$$

iterating which implies that $V^{(j)}$ has positive multiplicity in $\operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_j} V^\lambda$ if and only if $\lambda_1 \geq j$, completing the proof. $\square$

3. The ideal $I_{n,m,r}$ and its structure

Throughout the remainder of the paper, let n, m be positive integers and r a nonnegative integer with $r \leq \min\{n, m\}$. As in the introduction, $\mathcal{Z}_{n,m,r}$ is the finite locus of rook placements of size r on the $n \times m$ board. We can view $\mathcal{Z}_{n,m,r} \subseteq \operatorname{Mat}_{n \times m}(\mathbb{C})$ as the locus of $n \times m$ complex matrices defined by the following conditions:

- every entry in the matrix is 0 or 1,
- there is at most one nonzero entry in each row or column,
- there are r ones in total.

The locus $\mathcal{Z}_{n,m,r}$ itself is hard to analyze using orbit harmonics. However, it turns out that the locus of *upper rook placements*, defined as

$$\mathcal{UZ}_{n,m,r} := \bigsqcup_{r'=r}^{\min\{m,n\}} \mathcal{Z}_{n,m,r'} \subseteq \operatorname{Mat}_{n \times m}(\mathbb{C}) \tag{3.1}$$

the union of all rook placements of size at least r, behaves nicely with the method of orbit harmonics. To study the orbit harmonics quotient $R(\mathcal{Z}_{n,m,r})$, we define the following ideal, which we later prove to be $\operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r})$.

Definition 3.1. *Let $\mathbf{x}_{n \times m}$ be an $n \times m$ matrix of variables $(x_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ and consider the polynomial ring $\mathbb{C}[\mathbf{x}_{n \times m}]$ over these variables. Let $I_{n,m,r} \subseteq \mathbb{C}[\mathbf{x}_{n \times m}]$ be the ideal generated by*

- *any product $x_{i,j} \cdot x_{i,j'}$ $(1 \leq i \leq n;\ 1 \leq j, j' \leq m)$ of variables in the same row,*
- *any product $x_{i,j} \cdot x_{i',j}$ $(1 \leq i, i' \leq n;\ 1 \leq j \leq m)$ of variables in the same column,*
- *any product $\prod_{k=1}^{n-r+1} \left(\sum_{j=1}^m x_{i_k,j} \right)$ of $n-r+1$ distinct row sums for $1 \leq i_1 < \cdots < i_{n-r+1} \leq n$, and*
- *any product $\prod_{k=1}^{m-r+1} \left(\sum_{i=1}^n x_{i,j_k} \right)$ of $m-r+1$ distinct column sums for $1 \leq j_1 < \cdots < j_{m-r+1} \leq m$.*

To study this ideal, we will also need the following definition, which associates a monomial with any rook placement:

Definition 3.2. *For any rook placement $\mathcal{R}$ on the $n \times m$ board, the monomial $\mathfrak{m}(\mathcal{R}) \in \mathbb{C}[\mathbf{x}_{n\times m}]$ is defined to be*

$$\mathfrak{m}(\mathcal{R}) := \prod_{(i,j)\in\mathcal{R}} x_{i,j}.$$

In the following subsection, we first handle the special case $r = 0$, which prepares the reader for the more technical Section 3.2.

3.1. $r = 0$ **case.** When $r = 0$, write $\mathcal{Z}_{n,m} := \mathcal{U}\mathcal{Z}_{n,m,0}$ for the locus of all rook placements on the $n \times m$ board. The locus $\mathcal{Z}_{n,m}$ is a finite *shifted locus*, which was defined and studied by Reiner and Rhoades [12, Sec. 3.4]. Lemma 3.3 and Proposition 3.4 below can be derived directly from the results by Reiner and Rhoades, but we still include their proofs as they will promote understanding of later sections.

Write $I_{n,m} := I_{n,m,0}$ for the ideal generated by

- any product $x_{i,j} \cdot x_{i,j'}$ for $1 \le i \le n$ and $1 \le j, j' \le m$ of variables in the same row,
- any product $x_{i,j} \cdot x_{i',j}$ for $1 \le i, i' \le n$ and $1 \le j \le m$ of variables in the same column.

Note that the products of row sums and column sums are no longer present as it's impossible to have $n+1$ distinct rows or $m+1$ distinct columns. We start by giving a spanning set of the vector space $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$:

Lemma 3.3. *The family of monomials $\{\mathfrak{m}(\mathcal{R}) : \mathcal{R} \in \mathcal{Z}_{n,m}\}$ descends to a spanning set of the vector space $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$.*

Proof. Note that the generators $x_{i,j} \cdot x_{i,j'}$ and $x_{i,j} \cdot x_{i',j}$ of $I_{n,m}$ above imply that all the monomials containing variables from the same column or row of $\mathbf{x}_{n\times m}$ vanish in the quotient ring $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$. Consequently, the monomials with no two variables from the same row or column span $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$. □

We next strengthen Lemma 3.3 to yield a basis of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$.

Proposition 3.4. *The family of monomials $\{\mathfrak{m}(\mathcal{R}) : \mathcal{R} \in \mathcal{Z}_{n,m}\}$ descends to a $\mathbb{C}$-linear basis of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$. Furthermore, we have $\operatorname{gr}\mathbf{I}(\mathcal{Z}_{n,m}) = I_{n,m}$.*

Proof. Since a rook placement contains no rooks in the same row or column, it is clear that $x_{i,j} \cdot x_{i,j'} \in \mathbf{I}(\mathcal{Z}_{n,m})$ and $x_{i,j} \cdot x_{i',j} \in \mathbf{I}(\mathcal{Z}_{n,m})$. We thus have $x_{i,j} \cdot x_{i,j'} \in \operatorname{gr}\mathbf{I}(\mathcal{Z}_{n,m})$ and $x_{i,j} \cdot x_{i',j} \in \operatorname{gr}\mathbf{I}(\mathcal{Z}_{n,m})$. This implies that $I_{n,m} \subseteq \operatorname{gr}\mathbf{I}(\mathcal{Z}_{n,m})$, which naturally induces a linear surjection

$$\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m} \twoheadrightarrow R(\mathcal{Z}_{n,m}).$$

We now compare dimensions. Lemma 3.3 implies that

$$\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}) \le |\mathcal{Z}_{n,m}| = \dim_{\mathbb{C}}(R(\mathcal{Z}_{n,m})),$$

while the surjection above implies $\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}) \ge \dim_{\mathbb{C}}(R(\mathcal{Z}_{n,m}))$. Hence, all inequalities are equalities, and the surjection above is a linear isomorphism. This forces $I_{n,m} = \operatorname{gr}\mathbf{I}(\mathcal{Z}_{n,m})$ and ensures that the spanning set $\{\mathfrak{m}(\mathcal{R}) : \mathcal{R} \in \mathcal{Z}_{n,m}\}$ in Lemma 3.3 is a basis. □

The monomial basis from Proposition 3.4 inherits the $\mathfrak{S}_n \times \mathfrak{S}_m$-action on $\mathcal{Z}_{n,m}$, allowing us to compute the graded module structure of $R(\mathcal{Z}_{n,m}) = \mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$.

Proposition 3.5. *The graded Frobenius image of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$ is given by*

$$\operatorname{grFrob}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}; q) = \sum_{d=0}^{\min\{m,n\}} q^d \cdot \Bigg(\sum_{\mu \vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}) \Bigg).$$

Proof. Note that the monomial basis in Proposition 3.4 is compatible with the $\mathfrak{S}_n \times \mathfrak{S}_m$-action on $\mathcal{Z}_{n,m}$. Therefore, we have an isomorphism of $\mathfrak{S}_n \times \mathfrak{S}_m$-modules:

$$(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m})_d \cong \mathbb{C}[\mathcal{Z}_{n,m,d}]$$

for $0 \leq d \leq \min\{m,n\}$, and

$$(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m})_d = \{0\}$$

for $d > \min\{m,n\}$.

It suffices to study the module structure of $\mathbb{C}[\mathcal{Z}_{n,m,d}]$ for $0 \leq d \leq \min\{m,n\}$. Consider $\mathfrak{S}_d \times \mathfrak{S}_{n-d}$ (resp. $\mathfrak{S}_d \times \mathfrak{S}_{m-d}$) as a subgroup of $\mathfrak{S}_n$ (resp. $\mathfrak{S}_m$) by letting $\mathfrak{S}_d$ and $\mathfrak{S}_{n-d}$ (resp. $\mathfrak{S}_{m-d}$) separately act on $\{1,\ldots,d\}$ and $\{d+1,\ldots,n\}$ (resp. $\{d+1,\ldots,m\}$). Thus, we embed $(\mathfrak{S}_d \times \mathfrak{S}_{n-d}) \times (\mathfrak{S}_d \times \mathfrak{S}_{m-d})$ into $\mathfrak{S}_n \times \mathfrak{S}_m$ as a subgroup. Note that $(\mathfrak{S}_d \times \mathfrak{S}_{n-d}) \times (\mathfrak{S}_d \times \mathfrak{S}_{m-d})$ acts on $\mathbb{C}[\mathfrak{S}_d]$ by

$$((g_1,h_1),(g_2,h_2)) \cdot g = g_1 g g_2^{-1}$$

which restricts to the $\mathfrak{S}_d \times \mathfrak{S}_d$-action. Artin-Wedderburn Theorem gives this $\mathfrak{S}_d \times \mathfrak{S}_d$-module structure by

$$\mathbb{C}[\mathfrak{S}_d] \cong \bigoplus_{\mu \vdash d} V^\mu \otimes V^\mu \tag{3.2}$$

where $\mathfrak{S}_d \times \mathfrak{S}_d$ acts on each $V^\mu \otimes V^\mu$ by $(g_1,g_2)\cdot(v_1 \otimes v_2) = (g_1 \cdot v_1) \otimes (g_2 \cdot v_2)$. Then we have that, as $\mathfrak{S}_n \times \mathfrak{S}_m$-modules,

$$\begin{aligned}
\mathbb{C}[\mathcal{Z}_{n,m,d}] &\cong \mathrm{Ind}_{(\mathfrak{S}_d\times\mathfrak{S}_{n-d})\times(\mathfrak{S}_d\times\mathfrak{S}_{m-d})}^{\mathfrak{S}_n\times\mathfrak{S}_m} \mathbb{C}[\mathfrak{S}_d] \\
&\cong \mathrm{Ind}_{(\mathfrak{S}_d\times\mathfrak{S}_{n-d})\times(\mathfrak{S}_d\times\mathfrak{S}_{m-d})}^{\mathfrak{S}_n\times\mathfrak{S}_m} \left(\bigoplus_{\mu\vdash d} (V^\mu \otimes V^{(n-d)}) \otimes (V^\mu \otimes V^{(m-d)}) \right) \\
&\cong \bigoplus_{\mu\vdash d} \mathrm{Ind}_{(\mathfrak{S}_d\times\mathfrak{S}_{n-d})\times(\mathfrak{S}_d\times\mathfrak{S}_{m-d})}^{\mathfrak{S}_n\times\mathfrak{S}_m} \big((V^\mu \otimes V^{(n-d)}) \otimes (V^\mu \otimes V^{(m-d)}) \big) \\
&\cong \bigoplus_{\mu\vdash d} \left(\mathrm{Ind}_{\mathfrak{S}_d\times\mathfrak{S}_{n-d}}^{\mathfrak{S}_n} (V^\mu \otimes V^{(n-d)}) \right) \otimes \left(\mathrm{Ind}_{\mathfrak{S}_d\times\mathfrak{S}_{m-d}}^{\mathfrak{S}_m} (V^\mu \otimes V^{(m-d)}) \right)
\end{aligned}$$

where the second isomorphism arises from Equation (3.2). Therefore, the Frobenius image of $\mathbb{C}[\mathcal{Z}_{n,m,d}]$ as an $\mathfrak{S}_n \times \mathfrak{S}_m$-module is given by

$$\mathrm{Frob}(\mathbb{C}[\mathcal{Z}_{n,m,d}]) = \sum_{\mu\vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}) \tag{3.3}$$

and hence

$$\mathrm{Frob}\big((\mathbb{C}[\mathbf{x}_{n\times m}])_d\big) = \sum_{\mu\vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}),$$

completing our proof. □

3.2. **Structure of $I_{n,m,r}$.** In order to analyze the standard monomial basis of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$ in Section 4 and the graded module structure of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$ in Section 5, we start by studying strategic elements of the ideal $I_{n,m,r}$. On a first reading, it may be better to skip all the technical proofs in this section and accept Corollary 3.9 on faith.

For induction convenience, we introduce a new ideal $J_{n,m,r} \subseteq I_{n,m,r}$ by removing some generators of $I_{n,m,r}$ in Definition 3.1:

Definition 3.6. *Let $J_{n,m,r} \subseteq \mathbb{C}[\mathbf{x}_{n\times m}]$ be the ideal generated by*

- *all products $x_{i,j} \cdot x_{i,j'}$ for $1 \leq i \leq n$ and $1 \leq j,j' \leq m$ of variables in the same row, and*
- *all products $\prod_{k=1}^{m-r+1} \left(\sum_{i=1}^n x_{i,j_k} \right)$ of $m-r+1$ distinct column sums for $1 \leq j_1 < \cdots < j_{m-r+1} \leq m$.*

We start by modifying the products of column sums in Definition 3.6 so they are easier to use:

Lemma 3.7. *If $m-r<t\leq m$ and $b_1,\dots,b_t\in[m]$ are distinct integers, then*

$$\sum_{\substack{i_1,\dots,i_t\in[n]\\ \text{distinct}}}\prod_{j=1}^{t}x_{i_j,b_j}\in J_{n,m,r}$$

summing over all ordered sequences of distinct integers in $[n]$ of length t.

Proof. Since any product of the form $x_{i,j}\cdot x_{i,j'}$ belongs to $J_{n,m,r}$, we have that

$$\sum_{\substack{i_1,\dots,i_t\in[n]\\ \text{distinct}}}\prod_{j=1}^{t}x_{i_j,b_j}\equiv\sum_{i_1,\dots,i_t\in[n]}\prod_{j=1}^{t}x_{i_j,b_j}=\prod_{j=1}^{t}\sum_{i=1}^{n}x_{i,b_j}\mod J_{n,m,r}.$$

Note that $t>m-r$. Then the generators of $J_{n,m,r}$ of the form $\prod_{k=1}^{m-r+1}\left(\sum_{i=1}^{n}x_{i,j_k}\right)$ imply that $\prod_{j=1}^{t}\sum_{i=1}^{n}x_{i,b_j}\in J_{n,m,r}$, completing the proof. □

We will then prove the most technical result in this section. As we have the $\mathbb{C}$-algebra identification $\mathbb{C}[\mathfrak{S}_n\times\mathfrak{S}_m]\cong\mathbb{C}[\mathfrak{S}_n]\otimes_{\mathbb{C}}\mathbb{C}[\mathfrak{S}_m]$, we write $\eta_p\otimes 1$ for $\sum_{w\in\mathfrak{S}_p}(w,1)$.

Lemma 3.8. *Let p and d be two integers such that $0\leq d\leq\min\{m,n\}$ and $n+m-d-r<p\leq n$. Then, for any rook placement $\mathcal{R}\in\mathcal{Z}_{n,m,d}$, we have*

$$(\eta_p\otimes 1)\cdot\mathfrak{m}(\mathcal{R})\in J_{n,m,r}.$$

Proof. We prove the lemma by inducting on n. The base case $n=1$ is trivial. Assume that Lemma 3.8 holds for any positive integer $n'<n$, we prove it for n.

Write $\mathfrak{m}(\mathcal{R})=\prod_{k=1}^{d}x_{a_k,b_k}$ where $b_1,\dots,b_d\in[m]$ are distinct and $1\leq a_1<\cdots<a_d\leq n$. Since $p+d>n+m-r\geq n$, it follows that $[p]\cap\{a_1,\dots,a_d\}\neq\varnothing$. Therefore, we can choose the largest $t\in[d]$ such that $a_t\in[p]$.

Claim: $t>m-r$.

In fact, the maximality of t implies that $[p]\cap\{a_{t+1},\dots,a_d\}=\varnothing$, so we have that $[p]\sqcup\{a_{t+1},\dots,a_d\}\subseteq[n]$ and hence $p+d-t\leq n$. Consequently, $t\geq p+d-n>m-r$, which completes the proof of the claim.

Back to the proof of Lemma 3.8. We have that

$$(\eta_p\otimes 1)\cdot\mathfrak{m}(\mathcal{R})=(\eta_p\otimes 1)\cdot\prod_{k=1}^{d}x_{a_k,b_k}=(p-t)!\cdot\left(\sum_{\substack{i_1,\dots,i_t\in[p]\\ \text{distinct}}}x_{i_1,b_1}\dots x_{i_t,b_t}\right)\cdot\prod_{k=t+1}^{d}x_{a_k,b_k}$$

$$\equiv(p-t)!\cdot\left(-\sum_{\substack{\text{distinct } i_1,\dots,i_t\in[n],\\ \{i_1,\dots,i_t\}\not\subseteq[p]}}x_{i_1,b_1}\dots x_{i_t,b_t}\right)\cdot\prod_{k=t+1}^{d}x_{a_k,b_k}$$

$$\equiv(p-t)!\cdot\left(-\sum_{\substack{\text{distinct } i_1,\dots,i_t\in[n]\setminus\{a_{t+1},\dots,a_d\},\\ \{i_1,\dots,i_t\}\not\subseteq[p]}}x_{i_1,b_1}\dots x_{i_t,b_t}\right)\cdot\prod_{k=t+1}^{d}x_{a_k,b_k}\mod J_{n,m,r}$$

where the first congruence follows from the claim above, and the second from the relation $x_{i,j}\cdot x_{i,j'}\in J_{n,m,r}$. Now it remains to show that

$$\left(\sum_{\substack{\text{distinct } i_1,\dots,i_t\in[n]\setminus\{a_{t+1},\dots,a_d\},\\ \{i_1,\dots,i_t\}\not\subseteq[p]}}x_{i_1,b_1}\dots x_{i_t,b_t}\right)\cdot\prod_{k=t+1}^{d}x_{a_k,b_k}\in J_{n,m,r}. \tag{3.4}$$

If the summation in (3.4) is empty, there is nothing to prove. Otherwise, we decompose the summation into sub-summations with smaller index sets as follows. Note that the summation in (3.4) requires that $\{i_1,\ldots,i_t\} \not\subseteq [p]$, so we assume, without loss of generality, that $\{i_1,\ldots,i_t\} \setminus [p] = \{i_{t-s+1},\ldots,i_t\}$ where $0 < s < t$. (Here, we have $s < t$ because $\{i_1,\ldots,i_t\} \cap [p] \neq \varnothing$. Otherwise, $\{i_1,\ldots,i_t\} \sqcup [p] \sqcup \{a_{t+1},\ldots,a_d\} \subseteq [n]$ implies that $t+p+d-t \leq n$, i.e. $p \leq n-d$. However, $p > n+m-d-r \geq n-d$, a contradiction.) Summing over such $i_1,\ldots,i_t$ for fixed $i_{t-s+1} = c_{t-s+1},\ldots,i_t = c_t$, we obtain a sub-summation of the summation in (3.4). It suffices to show (3.4) after replacing its summation with these sub-summations, since adding them up yields (3.4) itself. Consequently, it remains to show that

$$\left(\sum_{\substack{i_1,\ldots,i_{t-s}\in[p]\setminus\{a_{t+1},\ldots,a_d\}\\ \text{distinct}}} x_{i_1,b_1}\cdots x_{i_{t-s},b_{t-s}}\right)\cdot \prod_{k=t-s+1}^{t} x_{c_k,b_k}\cdot \prod_{k=t+1}^{d} x_{a_k,b_k} \in J_{n,m,r}.$$

Recall that the choice of t implies that $[p]\cap\{a_{t+1},\ldots,a_d\} = \varnothing$, so it remains to show that

$$\left(\sum_{\substack{i_1,\ldots,i_{t-s}\in[p]\\ \text{distinct}}} x_{i_1,b_1}\cdots x_{i_{t-s},b_{t-s}}\right)\cdot \prod_{k=t-s+1}^{t} x_{c_k,b_k}\cdot \prod_{k=t+1}^{d} x_{a_k,b_k} \in J_{n,m,r}. \tag{3.5}$$

Note that this expression possesses the following form:

$$c\cdot((\eta_p\otimes 1)\cdot \mathfrak{m}(\mathcal{R}'))\cdot \prod_{k=t-s+1}^{t} x_{c_k,b_k} \tag{3.6}$$

where $c\in\mathbb{C}$, and $\mathcal{R}'$ is a rook placement of size $d' = d-s$ on the sub-board $([n]\setminus\{c_{t-s+1},\ldots,c_t\})\times([m]\setminus\{b_{t-s+1},\ldots,b_t\})$ of the original board $[n]\times[m]$. For the sake of induction, consider restricting Definition 3.6 to a smaller variable set:

- $n' = n-s$, $m' = m-s$, and $r' = r-s$;
- the restricted variable matrix given by

$$\mathbf{x}_{n',m'} := (x_{i,j})_{i\in[n]\setminus\{c_{t-s+1},\ldots,c_t\},\, j\in[m]\setminus\{b_{t-s+1},\ldots,b_t\}};$$

- the restricted counterpart $J'_{n',m',r'} \subseteq \mathbb{C}[\mathbf{x}]$ of $J_{n,m,r}$ generated by
  - any product $x_{i,j}\cdot x_{i,j'}$ for $i\in[n]\setminus\{c_{t-s+1},\ldots,c_t\}$ and $j,j'\in[m]\setminus\{b_{t-s+1},\ldots,x_t\}$, and
  - any product $\prod_{k=1}^{m'-r'+1}(\sum_{i\in[n]\setminus\{c_{t-s+1},\ldots,c_t\}} x_{i,j_k})$ for distinct $j_1,\ldots,j_{m'-r'+1}\in[m]\setminus\{b_{t-s+1},\ldots,b_t\}$.

Note that $p > n+m-d-r = (n-s)+(m-s)-(d-s)-(r-s) = n'+m'-d'-r'$, and hence the induction assumption reveals that $(\eta_p\otimes 1)\cdot\mathfrak{m}(\mathcal{R}') \in J'_{n',m',r'}$. Therefore, in order to show (3.5) using Expression (3.6), it suffices to show that

$$J'_{n',m',r'}\cdot \prod_{k=t-s+1}^{t} x_{c_k,b_k} \subseteq J_{n,m,r}.$$

We verify this containment using the generators of $J'_{n',m',r'}$ above. For any product $x_{i,j}\cdot x_{i,j'}$ for $i\in[n]\setminus\{c_{t-s+1},\ldots,c_t\}$ and $j,j'\in[m]\setminus\{b_{t-s+1},\ldots,x_t\}$, it is clear that $x_{i,j}\cdot x_{i,j'}\cdot\prod_{k=t-s+1}^{t} x_{c_k,b_k}\in J_{n,m,r}$. For any product $\prod_{k=1}^{m'-r'+1}(\sum_{i\in[n]\setminus\{c_{t-s+1},\ldots,c_t\}} x_{i,j_k})$ for distinct $j_1,\ldots,j_{m'-r'+1}\in[m]\setminus$

$\{b_{t-s+1}, \dots, b_t\}$, $m' - r' + 1 = (m-s) - (r-s) + 1 = m - r + 1$ implies that

$$\begin{aligned}
&\prod_{k=1}^{m'-r'+1} \left(\sum_{i \in [n] \setminus \{c_{t-s+1}, \dots, c_t\}} x_{i,j_k} \right) \cdot \prod_{k=t-s+1}^{t} x_{c_k, b_k} \\
&= \prod_{k=1}^{m-r+1} \left(\sum_{i \in [n] \setminus \{c_{t-s+1}, \dots, c_t\}} x_{i,j_k} \right) \cdot \prod_{k=t-s+1}^{t} x_{c_k, b_k} \\
&\equiv \prod_{k=1}^{m-r+1} \left(\sum_{i \in [n]} x_{i,j_k} \right) \cdot \prod_{k=t-s+1}^{t} x_{c_k, b_k} \equiv 0 \mod J_{n,m,r}
\end{aligned}$$

which completes the proof. □

Let $0 \le d \le \min\{m, n\}$ be an integer. For any set $S \subseteq [n]$ and $T \subseteq [m]$ such that $|S|, |T| > n + m - d - r$, let $\mathfrak{S}_S \subseteq \mathfrak{S}_n$ (resp. $\mathfrak{S}_T \subseteq \mathfrak{S}_m$) be the symmetric group of bijections $S \to S$ (resp. $T \to T$). We have that the symmetrizers of $\mathfrak{S}_S$ and $\mathfrak{S}_T$ annihilate $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$.

Corollary 3.9. *For any rook placement $\mathcal{R} \in \mathcal{Z}_{n,m,d}$, we have*

$$\sum_{w \in \mathfrak{S}_S} (w, 1) \cdot \mathfrak{m}(\mathcal{R}) \in I_{n,m,r}$$

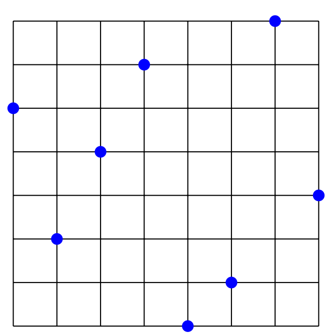
and

$$\sum_{w \in \mathfrak{S}_T} (1, w) \cdot \mathfrak{m}(\mathcal{R}) \in I_{n,m,r}.$$

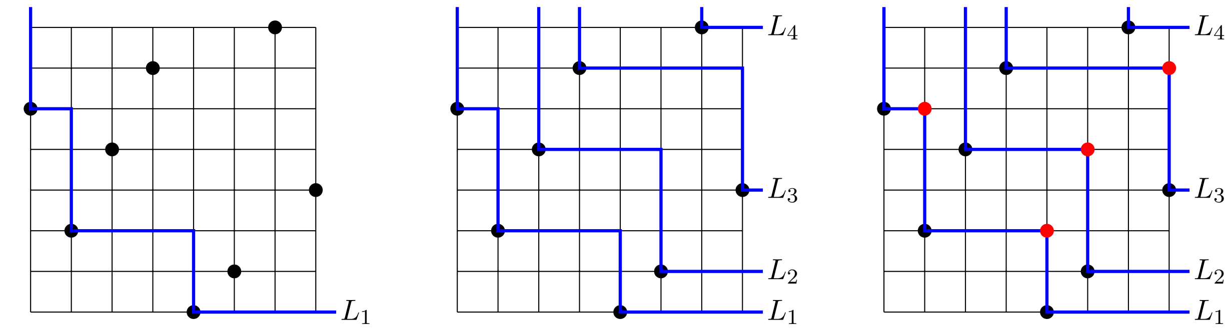
Proof. write $p = |S|$ and $q = |t|$. Since $J_{n,m,r} \subseteq i_{n,m,r}$, Lemma 3.8 implies that $(\eta_p \otimes 1) \cdot \mathfrak{m}(\mathcal{R}) \in I_{n,m,r}$. Analogously, we have $(1 \otimes \eta_q) \cdot \mathfrak{m}(\mathcal{R}) \in I_{n,m,r}$. The proof is thus complete as $I_{n,m,r}$ is closed under row and column permutations. □

4. Hilbert series and standard monomial basis

4.1. **Viennot Shadow.** We begin this section by describing Viennot's shadow line construction [16]. Let $w = [w(1), w(2), \dots, w(n)] \in \mathfrak{S}_n$ be a permutation. We draw its diagram on an $n \times n$ grid by putting a point at $(i, w(i))$ for $i \in [n]$. For example, the permutation $w = [6, 3, 5, 7, 1, 2, 8, 4] \in \mathfrak{S}_8$ has the digram shown below.



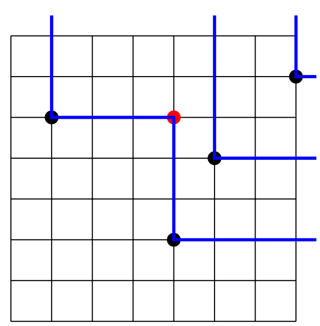
Place an imaginary light source at the bottom left corner of the diagram, shining northeast. Each point $(i, w(i))$ in the diagram of w blocks the regions to its north and to its east. We take the boundary of the shaded region and call it *the first shadow line* L_1 (see the left diagram below). Removing all points on the first shadow line and iterating this process yields the *second shadow line* L_2, *third shadow line* L_3, and so on (see the center diagram below).



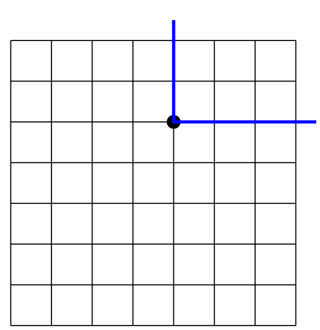
Suppose that $w \to (P(w), Q(w))$ under the Schensted correspondence, and that the shadow lines of w are $L_1, L_2, \ldots, L_k$. Viennot proved [16] that the entries in the first row of $P(w)$ are the coordinates of the infinite horizontal rays of the shadow lines, and the entries in the first row of $Q(w)$ are the coordinates of the infinite vertical rays. In the example above, we have that the first row of $P(w)$ is $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 8 \\ \hline\end{array}$ and the first row of $Q(w)$ is $\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline\end{array}$.

Definition 4.1. *Given a permutation $w \in \mathfrak{S}_n$, the* shadow set $\mathcal{S}(w)$ *of w is the collection of points that lie on the northeast corners of the shadow lines.*

In the above example, $\mathcal{S}(w) = \{(2,6), (5,3), (6,5), (8,7)\}$, as depicted in red in the right diagram above. We can iterate the shadow line construction on $\mathcal{S}(w)$, as depicted in the diagram below. Viennot proved [16] that the coordinates of the infinite horizontal rays of the new shadow lines correspond to the second row of $P(w)$, and the x coordinates of the infinite vertical rays correspond to the second row of $Q(w)$. In our example, the second row of $P(w)$ is $\begin{array}{|c|c|c|}\hline 3 & 5 & 7 \\ \hline\end{array}$ and the second row of $Q(w)$ is $\begin{array}{|c|c|c|}\hline 2 & 6 & 8 \\ \hline\end{array}$.



This iterated process produces the iterated shadow set $\mathcal{S}(\mathcal{S}(w))$, which consists of a single point as drawn in red above. Repeating this process, we obtain a single shadow line and an empty shadow set,



from which we conclude that $P(w)$ and $Q(w)$ are respectively given by

1	2	4	8
3	5	7	
6			

and

1	3	4	7
2	6	8	
5			

From the construction, it is immediate that the shadow set $\mathcal{S}(w)$ of any permutation w is a rook placement. However, not all rook placements are shadow sets of permutations. In [13], Rhoades described the following algorithm to determine whether a rook placement is the shadow set of some permutation. In the next section, we present a generalized version of this algorithm.

Lemma 4.2. *Let $\mathcal{R}$ be a rook placement on the $n \times n$ board and apply the Viennot shadow construction to $\mathcal{R}$, yielding shadow lines $L_1, \ldots, L_p$. Define two sequences $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^m$ over the alphabet $\{-1, 0, 1\}$ by*

$$x_i = \begin{cases} 1, & \text{if one of the shadow lines } L_1, \ldots, L_p \text{ has a vertical ray at } x = i, \\ -1, & \text{if the vertical line } x = i \text{ does not meet } \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases}$$

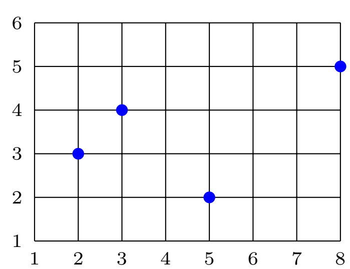
and

$$y_j = \begin{cases} 1, & \text{if one of the shadow lines } L_1, \ldots, L_p \text{ has a horizontal ray at } y = j, \\ -1, & \text{if the horizontal line } y = j \text{ does not meet } \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases}$$

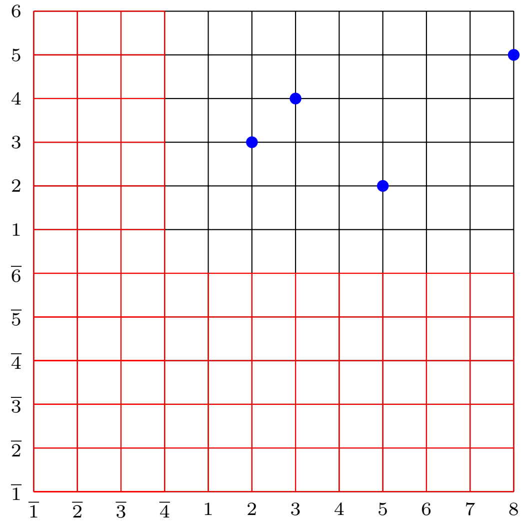
Then $\mathcal{R} = \mathcal{S}(w)$ is the extended shadow set of some permutation $w \in \mathfrak{S}_n$ if and only if for all $1 \leq i \leq n$ $x_1 + x_2 + \cdots + x_i \leq 0$ and $y_1 + y_2 + \cdots + y_j \leq 0$.

4.2. **Extended shadow sets.** Recall that $\mathcal{UZ}_{n,m,r}$ is the set of rook placements with at least r rooks. Given a rook placement $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$, we define an algorithm that extends $\mathcal{R}$ to a permutation $\mathcal{EX}(\mathcal{R}) \in \mathfrak{S}_{n+m-r}$. The shadow set of this permutation, denoted by $\mathcal{ES}(R)$, plays a key role in analyzing the orbit harmonics quotient $R(\mathcal{UZ}_{n,m,r})$.

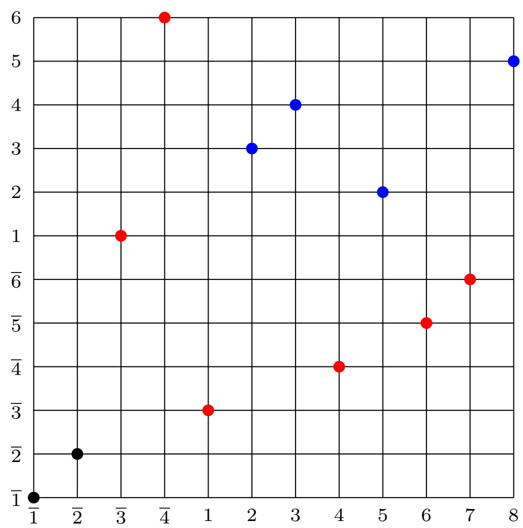
To start, let $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$ with $r' \geq r$ rooks. Draw the diagram of the rook placement on the $[n] \times [m]$ grid, with a point at (i, j) if $(i, j) \in R$. Below is the diagram of $\mathcal{R} = \{(2, 3), (3, 4), (5, 2), (8, 5)\} \in \mathcal{UZ}_{8,6,2}$.



We extend the diagram to an $(n+m-r) \times (n+m-r)$ grid by adding $(m-r)$ columns on the left and $(n-r)$ rows on the bottom. The newly added rows and columns are drawn in red in the diagram below. Label the added rows as $\{\overline{1}, \overline{2}, \ldots, \overline{n-r}\}$ from bottom to top and label the added columns as $\{\overline{1}, \overline{2}, \ldots, \overline{m-r}\}$ from left to right, as shown in the diagram.



For $i \leq r' - r$, we put a point at $(\bar{i}, \bar{i})$, as drawn in black in the diagram below. Finally, let $i_1 < i_2 < \cdots < i_{n-r'}$ be the indices of columns that do not contain a rook, and let $j_1 < j_2 < \cdots < j_{m-r'}$ be the indices of rows that do not contain a rook. For $1 \leq l \leq m - r'$, add a point at $(\overline{r' - r + l}, j_l)$; for $1 \leq k \leq n - r'$, add a point at $(i_k, \overline{r' - r + k})$. These added points are drawn in red in the diagram below.



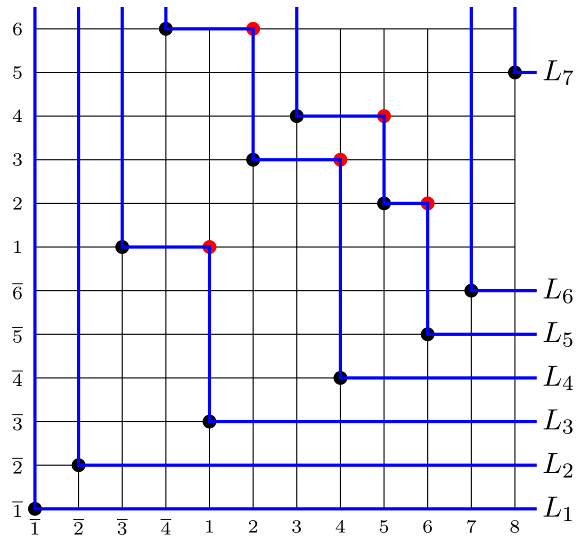
It can be easily checked that the diagram obtained this way has exactly one point in each row and each column, hence forming a permutation in $\mathfrak{S}_{n+m-r}$. We write $\mathcal{EX}(\mathcal{R})$ for this *extended permutation* arising from $\mathcal{R}$, identified with the diagram on the extended board.

Remark 4.3. *There is another way to define the extended permutation $\mathcal{EX}(\mathcal{R})$. Define an ordering of row and column indices by*

- *if $i < j$ then $\bar{i} < \bar{j}$*
- *$\bar{i} < j$ for all i, j.*

$\mathcal{EX}(\mathcal{R})$ is the unique extension of $\mathcal{R}$ to a permutation in the $(n+m-r) \times (n+m-r)$ board such that the vertical coordinates of the points in the first $m - r$ columns form an increasing pattern and the horizontal coordinates of the points in the first $n - r$ rows form an increasing pattern.

Apply Viennot's shadow line construction to $\mathcal{EX}(R)$. From the construction (and by Remark 4.3), we have that the shadow lines $L_1, L_2, \ldots, L_{n-r}$ will have infinite horizontal coordinates $\overline{1}, \overline{2}, \ldots, \overline{n-r}$, and the shadow lines $L_1, L_2, \ldots, L_{m-r}$ will have infinite vertical coordinates $\overline{1}, \overline{2}, \ldots, \overline{m-r}$. Let $\mathcal{ES}(\mathcal{R})$ be the shadow set of $\mathcal{EX}(\mathcal{R})$, we call it the *extended shadow set* of $\mathcal{R}$. See the diagram for an example.



The following lemma is an important fact about $\mathcal{ES}(\mathcal{R})$.

Lemma 4.4. *Let $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$, we have that $\mathcal{ES}(\mathcal{R})$ is also a rook placement in $[n] \times [m]$. That is, the shadow set of $\mathcal{EX}(\mathcal{R})$ is contained in the $n \times m$ box in the upper right corner of the diagram, with column indices in $[n]$ and row indices in $[m]$.*

Proof. Let (P, Q) be the pair of tableaux corresponding to $\mathcal{EX}(\mathcal{R})$ in the Schensted correspondence. We adopt the notion in Remark 4.3 so the content of P is $\{\overline{1}, \overline{2}, \ldots, \overline{n-r}, 1, 2, \ldots m\}$ and the content of Q is $\{\overline{1}, \overline{2}, \ldots, \overline{m-1}, 1, 2, \ldots n\}$. According to Viennot's construction in [16], we have that $\overline{1}, \overline{2}, \ldots, \overline{n-r}$ are in the first row of P as they are the coordinates of the infinite horizontal rays of shadow lines $L_1, L_2, \ldots, L_{n-r}$. Similarly, $\overline{1}, \overline{2}, \ldots, \overline{m-r}$ are in the first row of Q. As the horizontal coordinates of points in the shadow set are entries in P beyond the first row, and the vertical coordinates of points in the shadow set are entries in Q beyond the shadow set, we conclude that $\mathcal{ES}(\mathcal{R}) = \mathcal{S}(\mathcal{EX}(\mathcal{R})) \subseteq [n] \times [m]$. $\square$

Given a rook placement $\mathcal{R} \subseteq [n] \times [m]$, the following proposition generalizes 4.2. It characterizes whether $\mathcal{R}$ is the extended shadow set of some other rook placement $\tilde{\mathcal{R}}$ with size at least r.

Proposition 4.5. *Let $\mathcal{R}$ be a rook placement on the $n \times m$ board and apply the Viennot shadow construction to $\mathcal{R}$, yielding shadow lines $L_1, \ldots, L_p$. Define two sequences $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^m$ over the alphabet $\{-1, 0, 1\}$ by*

$$x_i = \begin{cases} 1, & \text{if one of the shadow lines } L_1, \ldots, L_p \text{ has a vertical ray at } x = i, \\ -1, & \text{if the vertical line } x = i \text{ does not meet } \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$y_j = \begin{cases} 1, & \text{if one of the shadow lines } L_1, \ldots, L_p \text{ has a horizontal ray at } y = j, \\ -1, & \text{if the horizontal line } y = j \text{ does not meet } \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{R} = \mathcal{ES}(\tilde{\mathcal{R}})$ is the extended shadow set of some rook placement $\tilde{\mathcal{R}} \in \mathcal{UZ}_{n,m,r}$ if and only if for all $1 \le i \le n$ and all $1 \le j \le m$ we have $x_1 + x_2 + \cdots + x_i \le m - r$ and $y_1 + y_2 + \cdots + y_j \le n - r$.

Proof. Suppose $\mathcal{R} = \mathcal{ES}(\tilde{\mathcal{R}})$ for some $\tilde{\mathcal{R}} \in \mathcal{UZ}_{n,m,r}$. Let $\mathcal{EX}(\tilde{\mathcal{R}})$ be the extended permutation of $\tilde{\mathcal{R}}$. Using the same notion as in the proof of Lemma 4.4, if $\mathcal{EX}(\tilde{\mathcal{R}}) \mapsto (P, Q)$ under the Schensted correspondence, the horizontal rays of $L_1, L_2, \ldots, L_p$ give the second row of P and the vertical rays of $L_1, L_2, \ldots, L_p$ give the second row of Q. The y-coordinates that do not appear in $\mathcal{R}$ together with $\overline{1}, \overline{2}, \ldots, \overline{n-r}$ form the first row of P, where the first $n-r$ entries are $\overline{1}, \overline{2}, \ldots, \overline{n-r}$. Similarly, the x-coordinates that do not appear in $\mathcal{R}$ together with $\overline{1}, \overline{2}, \ldots, \overline{m-r}$ form the first row of Q, and the first $m-r$ entries are $\overline{1}, \overline{2}, \ldots, \overline{m-r}$. Since P and Q are standard, we must have all prefix sums of the sequence $x_1 x_2 \ldots x_n$ are at most $m-r$ and all prefix sums of the sequence $y_1 y_2 \ldots y_m$ are at most $n-r$.

Now assume that all prefix sums of $x_1 x_2 \ldots x_n$ are at most $m-r$ and all prefix sums of $y_1 y_2 \ldots y_m$ are at most $n-r$. Embed $\mathcal{R}$ in top right corner of the $(n+m-r) \times (n+m-r)$ board, and apply Viennot's shadow construction to the set R to get a pair (P', Q') of partial standard Young tableaux where entries of P' are the x coordinates of points in $\mathcal{R}$ and entries in Q' are the y coordinates of points in $\mathcal{R}$. Add a first row consisting of entries $\overline{1}, \overline{2}, \ldots, \overline{n-r}$ together with y coordinates not appearing in $\mathcal{R}$ to P', and add a first row consisting of entries $\overline{1}, \overline{2}, \ldots, \overline{n-r}$ together with x coordinates not appearing in $\mathcal{R}$ to Q'. The pair of tableaux (P, Q) obtained this way is standard by the conditions on prefix sums. Consider the diagram of the permutation $w \in \mathfrak{S}_{n+m-r}$ on the $(n+m-r) \times (n+m-r)$ board. From our construction of the first row of P and Q, it must be the case that the vertical coordinates of the points in the first $m-r$ columns form an increasing pattern and the horizontal coordinates of the points in the first $n-r$ rows form an increasing pattern. Hence, let $\tilde{\mathcal{R}}$ be the rook placement in the upper-right $n \times m$ board in the diagram of w, it must be the case that $w = \mathcal{EX}(\tilde{\mathcal{R}})$ and that $\mathcal{R} = \mathcal{ES}(\tilde{\mathcal{R}})$ by Remark 4.3. $\square$

4.3. **Extended shadow monomials and spanning.** Our next goal is to obtain a basis for $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$ using the combinatorics of the previous section.

Lemma 4.6. *The family of monomials $\{\mathfrak{m}(\mathcal{R}) : \mathcal{R} \in \bigsqcup_{d=0}^{\min\{m,n\}} \mathcal{Z}_{n,m,d}\}$ descends to a spanning set of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$.*

Proof. This follows immediately since $I_{n,m,r}$ contains all the products $x_{i,j} \cdot x_{i,j'}$ and $x_{i,j} \cdot x_{i',j}$ of variables in the same row or column. $\square$

However, the spanning set given by Lemma 4.6 is far from a basis. To extract a basis from this spanning set, we introduce a class of special monomial orders, namely the *diagonal monomial orders*. Recall that the *lexicographical order* in an ordered set of variables $y_1 > y_2 > \cdots > y_N$ is given by $y_1^{a_1} \ldots y_N^{a_N} < y_1^{b_1} \ldots y_N^{b_N}$ if and only if there exists an integer $1 \le M \le N$ such that $a_i = b_i$ for $1 \le i < M$ and $a_M < b_M$.

Definition 4.7. *Observe that any total order $<$ on the variable set $\mathbf{x}_{n\times m}$ induces a lexicographical order on the monomials in $\mathbb{C}[\mathbf{x}_{n\times m}]$. Such a monomial order is called* diagonal *if the total order $<$ satisfies $x_{i,j} < x_{i',j'}$ whenever $i + j < i' + j'$.*

It is easy to construct a diagonal monomial order. For instance, the variable order

$$x_{1,1} > x_{2,1} > x_{1,2} > x_{3,1} > x_{2,2} > x_{1,3} > \cdots > x_{n,m-1} > x_{n-1,m} > x_{n,m}$$

induces a diagonal monomial order.

From now on, we fix a diagonal monomial order $<_{\text{Diag}}$ on the set of monomials in $\mathbb{C}[\mathbf{x}_{n\times m}]$.

Definition 4.8. *Let $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$. The* extended shadow monomial $\mathfrak{es}(\mathcal{R}) \in \mathbb{C}[\mathbf{x}_{n\times m}]$ *is defined as $\mathfrak{es}(\mathcal{R}) \coloneqq \mathfrak{m}(\mathcal{ES}(\mathcal{R}))$.*

We now prove the last technical lemma to obtain a smaller spanning set.

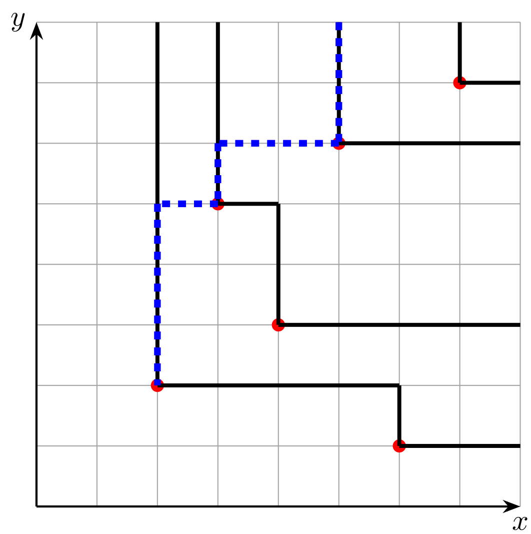
Lemma 4.9. *For $0 \le d \le \min\{m, n\}$ and a rook placement $\mathcal{R} \in \mathcal{Z}_{n,m,d}$ that is not the extended shadow set of any rook placement in $\mathcal{UZ}_{n,m,r}$, we have*

$$\mathfrak{m}(\mathcal{R}) \in \operatorname{span}\{\mathfrak{m}(\mathcal{R}') : \mathcal{R}' \in \mathcal{Z}_{n,m,d},\ \mathfrak{m}(\mathcal{R}') <_{Diag} \mathfrak{m}(\mathcal{R})\} + I_{n,m,r}.$$

Proof. Applying Viennot's shadow construction to $\mathcal{R}$, we obtain shadow lines $L_1, \dots, L_p$, together with sequences $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^m$ as in Proposition 4.5. Because $\mathcal{R}$ is not the extended shadow set of any rook placements in $\mathcal{UZ}_{n,m,r}$, Proposition 4.5 implies that either $\sum_{i=1}^k x_i > m - r$ for some $1 \le k \le n$ or $\sum_{j=1}^l y_j > n - r$ for some $1 \le l \le m$.

Without loss of generality, we assume that $\sum_{i=1}^k x_i > m-r$ for some $1 \le k \le n$ and further require k to be the smallest integer with this property. Then $x_k > 0$ and thus $x_k = 1$, implying that some shadow line L_t has a vertical ray at $x = k$. We define a size t subset $\mathcal{R}_1 = \{(i_1, j_1), \dots, (i_t, j_t)\} \subseteq \mathcal{R}$ as follows. Start at the vertical ray of L_t and go south. Let (i_t, j_t) be the first rook of $\mathcal{R}$ encountered (in particular, we have $i_t = k$). Then go west from (i_t, j_t) until we encounter a vertical segment of the shadow line L_{t-1} and turn left. Go south until we encounter a rook $(i_{t-1}, j_{t-1}) \in \mathcal{R}$. Repeating this process, we obtain a sequence of rooks $(i_t, j_t), (i_{t-1}, j_{t-1}), \dots, (i_1, j_1)$ sequentially, such that (i_u, j_u) is on the shadow line L_u $(u = 1, \dots, t)$, with $i_1 < i_2 < \cdots < i_t$, and $j_1 < j_2 < \cdots < j_t$. Let $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1$.

For example, let $n = m = 8$ and $r = 7$. Let $\mathcal{R} = \{(2, 2), (3, 5), (4, 3), (5, 6), (6, 1), (7, 7)\} \in \mathcal{Z}_{8,8,6}$ be a rook placement on the 8×8 board. We mark all rooks of $\mathcal{R}$ using red points and draw black shadow lines $L_1, \dots, L_4$. The sequence $(x_i)_{i=1}^8$ is $-1, 1, 1, 0, 1, 0, 1, -1$ where the smallest k such that $\sum_{i=1}^k x_i > m-r$ is $k = 5$. Since $x = 5$ corresponds to the vertical ray of the shadow line L_3, we start from this vertical ray and go south, then west, and so on, yielding $\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$. Our track is shown in blue as follows.



Consequently we obtain $(i_1, j_1) = (2, 2)$, $(i_2, j_2) = (3, 5)$, and $(i_3, j_3) = (5, 6)$, so

$$\mathcal{R}_1 = \{(2, 2), (3, 5), (5, 6)\}$$

and

$$\mathcal{R}_2 = \mathcal{R} \setminus \{(2, 2), (3, 5), (5, 6)\} = \{(4, 3), (6, 1), (7, 7)\}$$

in the example above.

Back to our proof. Define $S \subseteq [n]$ by

$$S = \{i_1, i_2, \dots, i_t\} \sqcup \{i \in \{i_t + 1, \dots, n\} : \text{the vertical line } x = i \text{ does not meet } \mathcal{R}\}.$$

Claim: $|S| > n + m - d - r$.

In fact, $\sum_{i=1}^{k} x_i > m - r$ implies that

$$t > |\{i \in [i_t] \,:\, \text{the vertical line } x = i \text{ does not meet } \mathcal{R}\}| + m - r$$

and hence

$$\begin{aligned}|S| =& t + |\{i \in \{i_t+1, \dots, n\} \,:\, \text{the vertical line } x = i \text{ does not meet } \mathcal{R}\}| \\ >& |\{i \in [i_t] \,:\, \text{the vertical line } x = i \text{ does not meet } \mathcal{R}\}| + m - r + \\ & |\{i \in \{i_t+1, \dots, n\} \,:\, \text{the vertical line } x = i \text{ does not meet } \mathcal{R}\}| \\ =& |\{i \in [n] \,:\, \text{the vertical line } x = i \text{ does not meet } \mathcal{R}\}| + m - r \\ =& n - d + m - r = n + m - d - r\end{aligned}$$

which completes the proof of the claim above.

Then Corollary 3.9 reveals that

$$\sum_{w \in \mathfrak{S}_S} (w, 1) \cdot \mathfrak{m}(\mathcal{R}) \in I_{n,m,r}. \tag{4.1}$$

Note that $\mathfrak{m}(\mathcal{R}) = \mathfrak{m}(\mathcal{R}_1) \cdot \mathfrak{m}(R_2)$ where for all $w \in \mathfrak{S}_S$ the action of $(w, 1)$ does not change $\mathfrak{m}(\mathcal{R}_2)$. Therefore, we have that

$$\sum_{w \in \mathfrak{S}_S} (w, 1) \cdot \mathfrak{m}(\mathcal{R}) = \left(\sum_{w \in \mathfrak{S}_S} (w, 1) \cdot \mathfrak{m}(\mathcal{R}_1) \right) \cdot \mathfrak{m}(\mathcal{R}_2)$$

where $\mathfrak{m}(\mathcal{R}_1)$ is the initial monomial of $\sum_{w \in \mathfrak{S}_S} (w, 1) \cdot \mathfrak{m}(\mathcal{R}_1)$ under the monomial order $<_{\text{Diag}}$, indicating that $\mathfrak{m}(\mathcal{R}_1) \cdot \mathfrak{m}(\mathcal{R}_2) = \mathfrak{m}(\mathcal{R})$ is the initial monomial of $\sum_{w \in \mathfrak{S}_S} (w, 1) \cdot \mathfrak{m}(\mathcal{R})$. As a result, the initial monomial of the expression on the left-hand side of Equation (4.1) is $\mathfrak{m}(\mathcal{R})$, finishing the proof. ☐

Now we are ready to provide a smaller spanning set than Lemma 4.6. In the next section, we will see that this spanning set is actually the standard monomial basis of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$ with respect to $<_{\text{Diag}}$

Lemma 4.10. *The family of monomials $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ descends to a spanning set of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$.*

Proof. By Lemma 4.6, it suffices to show that for all the rook placements $\mathcal{R}$ on the $n \times m$ board we have that

$$\mathfrak{m}(\mathcal{R}) \in \operatorname{span}\{\mathfrak{es}(\mathcal{R}') \,:\, \mathcal{R}' \in \mathcal{UZ}_{n,m,r}\} + I_{n,m,r}.$$

This is immediately from Lemma 4.9 which enables us to do induction with respect to the monomial order $<_{\text{Diag}}$ on $\{\mathfrak{m}(\mathcal{R}) \,:\, \mathcal{R} \text{ is a rook placement on the } n \times m \text{ board}\}$. ☐

Remark 4.11. *In fact, the proof of Lemma 4.10 gives a slightly stronger result than Lemma 4.10 itself: For each rook placement $\mathcal{R}$ on the $n \times m$ board, we have that*

$$\mathfrak{m}(\mathcal{R}) \in \operatorname{span}\{\mathfrak{es}(\mathcal{R}') \,:\, \mathcal{R}' \in \mathcal{UZ}_{n,m,r},\ \mathfrak{es}(\mathcal{R}') \leq_{Diag} \mathfrak{m}(\mathcal{R})\} + I_{n,m,r}.$$

4.4. **Standard monomial basis and Hilbert series.** As promised, we strengthen Lemma 4.10 to obtain the standard monomial basis of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$.

Theorem 4.12. *The family of monomials $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ descends to the standard monomial basis of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$. Furthermore, we have $\operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r}) = I_{n,m,r}$.*

Remark 4.13. *Consider the case $r = 0$. Recall that Proposition 3.4 provides a basis $\{\mathfrak{m}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{Z}_{n,m}\}$ which contains the standard monomial basis $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,0} = \mathcal{Z}_{n,m}\}$ given by Theorem 4.12. Therefore, we have*

$$\{\mathfrak{m}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{Z}_{n,m}\} = \{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,0} = \mathcal{Z}_{n,m}\},$$

indicating that

$$\mathcal{Z}_{n,m} = \{\mathcal{ES}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{Z}_{n,m}\}.$$

In fact, this fact is also immediate from Proposition 4.5.

In order to prove Theorem 4.12, we need the following lemma on ideal containment.

Lemma 4.14. $I_{n,m,r} \subseteq \operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r})$.

Proof. For any rook placement $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$, each column (resp. row) of $[n] \times [m]$ has at most one rook. Consequently, for all $1 \leq i \leq n$ and $1 \leq j, j' \leq m$ we have $x_{i,j} \cdot x_{i,j'} \in \mathbf{I}(\mathcal{UZ}_{n,m,r})$ and thus $x_{i,j} \cdot x_{i,j'} \in \operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r})$ (resp. for all $1 \leq i, i' \leq n$ and $1 \leq j \leq m$ we have $x_{i,j} \cdot x_{i',j} \in \operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r})$). Additionally, since $\mathcal{R}$ has at least r rooks, any union of $n - r + 1$ distinct columns of $[n] \times [m]$ must meet $\mathcal{R}$. Thus, for all $1 \leq i_1 < \cdots < i_{n-r+1} \leq n$ we have

$$\prod_{k=1}^{n-r+1} \left(\sum_{j=1}^{m} x_{i_k,j} - 1 \right) \in \mathbf{I}(\mathcal{UZ}_{n,m,r}),$$

which implies that

$$\prod_{k=1}^{n-r+1} \left(\sum_{j=1}^{m} x_{i_k,j} \right) \in \operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r}).$$

Similarly, for all $1 \leq j_1 < \cdots < j_{m-r+1} \leq m$ we have

$$\prod_{k=1}^{m-r+1} \left(\sum_{i=1}^{n} x_{i,j_k} \right) \in \operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r}).$$

The proof is thus complete as all generators of $I_{n,m,r}$ are in $\operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r})$. $\square$

With the lemma, we now give the proof of Theorem 4.12

Proof of Theorem 4.12. Lemma 4.14yields a surjective $\mathbb{C}$-linear map

$$\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r} \twoheadrightarrow R(\mathcal{UZ}_{n,m,r}),$$

implying that $\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}) \geq \dim_{\mathbb{C}}(R(\mathcal{UZ}_{n,m,r}))$. However, Lemma 4.10 implies that $\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}) \leq |\mathcal{UZ}_{n,m,r}| = \dim_{\mathbb{C}}(R(\mathcal{UZ}_{n,m,r}))$. Therefore, we conclude that

$$\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}) = \dim_{\mathbb{C}}(R(\mathcal{UZ}_{n,m,r})),$$

so the spanning set $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ in Lemma 4.10 is indeed a basis of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$. Furthermore, the surjection $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r} \twoheadrightarrow R(\mathcal{UZ}_{n,m,r})$ above is actually a linear isomorphism, and hence

$$I_{n,m,r} = \operatorname{gr} \mathbf{I}(\mathcal{UZ}_{n,m,r}).$$

As we have shown that $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ descends to a basis of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$, it remains to show that $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ descends to the standard monomial basis of $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$. Assume, for the sake of contradiction, that there exists some $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$ such that $\mathfrak{es}(\mathcal{R})$ is the initial monomial of some $f \in I_{n,m,r}$. Then we have that

$$\mathfrak{es}(\mathcal{R}) \equiv \sum_{\substack{\mathcal{R}' \in \mathcal{Z}_{n,m,|\mathcal{R}|} \\ \mathfrak{m}(\mathcal{R}') < \mathfrak{es}(\mathcal{R})}} c_{\mathcal{R}'} \cdot \mathfrak{m}(\mathcal{R}') \mod I_{n,m,r} \tag{4.2}$$

where $c_{\mathcal{R}'} \in \mathbb{C}$. However, Remark 4.11 implies that the right-hand side of Equation 4.2 can be replaced with a linear combination of some extended shadow monomials

$$\{\mathfrak{es}(\mathcal{R}') \,:\, \mathcal{R}' \in \mathcal{UZ}_{n,m,r},\ \mathfrak{es}(\mathcal{R}') <_{\mathrm{Diag}} \mathfrak{es}(\mathcal{R})\},$$

which means that extended shadow monomials $\{\mathfrak{es}(\mathcal{R}) \,:\, \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ are linearly dependent in $\mathbb{C}[\mathbf{x}_{n \times m}]/I_{n,m,r}$. This yields a contradiction because we have shown the family of monomials

$\{\mathfrak{es}(\mathcal{R}) : \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ descends to a basis of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$. Therefore, the family of monomials $\{\mathfrak{es}(\mathcal{R}) : \mathcal{R} \in \mathcal{UZ}_{n,m,r}\}$ descends to the standard monomial basis of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$. $\square$

Theorem 4.12 immediately gives the Hilbert series of $R(\mathcal{UZ}_{n,m,r}) = \mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$. Given a permutation $w \in \mathfrak{S}_{n+m-r}$, a *increasing subsequence of length* k is a sequence $1 \leq i_1 < \cdots < i_k \leq n+m-r$ of integers such that $w(i_1) < \cdots < w(i_k)$. We write

$$\mathrm{lis}(w) \coloneqq \text{length of the longest increasing subsequence of } w.$$

Corollary 4.15. *The Hilbert series of $R(\mathcal{UZ}_{n,m,r})$ is*

$$\mathrm{Hilb}(R(\mathcal{UZ}_{n,m,r}); q) = \sum_{\mathcal{R}\in\mathcal{UZ}_{n,m,r}} q^{n+m-r-\mathrm{lis}(\mathcal{EX}(\mathcal{R}))}. \tag{4.3}$$

Proof. By Theorem 4.12, it suffices to show that $\deg(\mathfrak{es}(\mathcal{R})) = n+m-r-\mathrm{lis}(\mathcal{EX}(\mathcal{R}))$ for $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$, which is equivalent to

$$|\mathcal{ES}(w)| = n+m-r-\mathrm{lis}(\mathcal{EX}(\mathcal{R})).$$

Recall that $\mathcal{ES}(\mathcal{R}) = \mathcal{S}(\mathcal{EX}(\mathcal{R}))$ is the shadow set of $\mathcal{EX}(\mathcal{R}) \in \mathfrak{S}_{n+m-r}$. Consequently, it remains to show that for all $w \in \mathfrak{S}_{n+m-r}$ we have

$$|\mathcal{S}(w)| = n+m-r-\mathrm{lis}(w). \tag{4.4}$$

Let k be the number of shadow lines of w that we use to construct $\mathcal{S}(w)$. Write $w \mapsto (P,Q)$ according to the Schensted correspondence, where P,Q have the same shape λ. The shadow line construction in Subsection 4.1 implies that $k = \lambda_1$. In addition, Schensted correspondence possesses the standard property [15, Thm. 1] $\lambda_1 = \mathrm{lis}(w)$, so we have $k = \mathrm{lis}(w)$ and thus Equation (4.4) holds. $\square$

Surprisingly, although $\mathcal{UZ}_{n,m,r}$ looks larger than $\mathfrak{S}_n$, our extension operation $\mathcal{EX}$ embeds $\mathcal{UZ}_{n,m,r}$ into $\mathfrak{S}_{n+m-r}$, so that we can substitute $n = N-a, m = N-b, r = N-a-b$ and hence embed $\mathcal{UZ}_{n,m,r}$ into $\mathfrak{S}_N$. Therefore, while [13, Corollary 3.13] states that

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_{N\times N}]/I_{N,N,N}; q) = \sum_{w\in\mathfrak{S}_N} q^{N-\mathrm{lis}(w)}, \tag{4.5}$$

the following modified version of Corollary 4.15 refines the index set of Equation (4.5).

Corollary 4.16. *For $N \in \mathbb{Z}_{>0}$ and $0 \leq a, b < N$, we have*

$$\mathrm{Hilb}(\mathbb{C}[\mathbf{x}_{(N-a)\times(N-b)}]/I_{N-a,N-b,N-a-b}; q) = \sum_{w} q^{N-\mathrm{lis}(w)}$$

where w ranges over $\{w \in \mathfrak{S}_N : w(1) < w(2) < \cdots < w(a)$ and $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(b)\}$.

Proof. Write $\mathrm{Inc} \coloneqq \{w \in \mathfrak{S}_N : w(1) < w(2) < \cdots < w(a) \text{ and } w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(b)\}$ and let $n = N-a, m = N-b, r = N-a-b$, then we have

$$\mathrm{Inc} = \{w \in \mathfrak{S}_{n+m-r} : w(1) < w(2) < \cdots < w(m-r) \text{ and } w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n-r)\}.$$

According to Corollary 4.15, it suffices to show that

$$\begin{aligned} \mathcal{EX} : \mathcal{UZ}_{n,m,r} &\longrightarrow \mathrm{Inc} \\ \mathcal{R} &\longmapsto \mathcal{EX}(\mathcal{R}) \end{aligned}$$

is a bijection.

Since the diagram of $\mathcal{R}$ is contained in the diagram of $\mathcal{EX}(\mathcal{R})$, it immediately follows that $\mathcal{EX}$ is injective. It remains to show that $\mathcal{EX}$ is surjective. For any $w \in \mathrm{Inc}$, choose maximal indices $1 \leq i_0 \leq m-r, 1 \leq j_0 \leq n-r$ such that $w(i_0) \leq n-r$ and $w^{-1}(j_0) \leq m-r$. If such i_0 (resp. j_0) does not exist, let $i_0 = 0$ (resp. $j_0 = 0$). Then, we have $[i_0] \cap w^{-1}(\{j_0+1, \ldots, n-r\}) = \varnothing$ and hence $w(i_0) \leq j_0$. Similarly, we have $[j_0] \cap w(\{i_0+1,\dots,m-r\}) = \varnothing$ and hence $j_0 \leq w(i_0)$. These two inequalities force $w(i_0) = j_0$. Therefore, $w(1) < w(2) < \cdots < w(i_0)$ indicates that $i_0 \leq j_0$, while $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(j_0)$ indicates that $j_0 \leq i_0$. As a result, we have $i_0 = j_0$ and $w(i) = i$ for all $i \in [i_0]$, so $w = \mathcal{EX}(\mathcal{R})$ where $\mathcal{R} \in \mathcal{UZ}_{n,m,r}$ arises from the removal of the first $m-r$ columns and the first $n-r$ rows from the diagram of w. $\square$

5. Module structure

We first provide some combinatorial results about symmetric functions. For convenience, we write $\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F$ for the coefficient of $s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}}$ in the Schur expansion of any $F \in \Lambda \otimes_{\mathbb{C}(q)} \Lambda$. Similarly, we write $\langle q^i \rangle f$ for the coefficient of q^i in any generating function $f \in \mathbb{C}[[q]]$.

Lemma 5.1. *For $d, a, b, p, q \in \mathbb{Z}_{\geq 0}$, consider*

$$F := \sum_{\mu \vdash d} \{s_\mu \cdot h_a\}_{\lambda_1 = p} \otimes \{s_\mu \cdot h_b\}_{\lambda_1 = q} \in \Lambda \otimes_{\mathbb{C}(q)} \Lambda$$

and let $\lambda^{(1)} \vdash a+d$ and $\lambda^{(2)} \vdash b+d$ be partitions.
If all the following conditions hold:

- $\lambda^{(1)}_1 = p$,
- $\lambda^{(2)}_1 = q$, *and*
- *for all $i \geq 1$ we have* $\min\{\lambda^{(1)}_i, \lambda^{(2)}_i\} \geq \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}$.

Then we have

$$\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F = \left\langle q^{d - \sum_{i=1}^{\infty} \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}} \right\rangle \left(\prod_{i=1}^{\infty} \left(\sum_{j=0}^{\min\{\lambda^{(1)}_i, \lambda^{(2)}_i\} - \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}} q^j \right) \right).$$

Otherwise, we have

$$\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F = 0.$$

Proof. If $s_{\lambda^{(1)}_1} \otimes s_{s_{\lambda^{(2)}_1}}$ appears in the Schur expansion of F, we must have $\lambda^{(1)}_1 = p$ and $\lambda^{(2)}_1 = q$ by the definition of F. Additionally, we must have $\min\{\lambda^{(1)}_i, \lambda^{(2)}_i\} \geq \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}$ for all $i \geq 1$, since both $\lambda^{(1)}$ and $\lambda^{(2)}$ arise from attaching a horizontal strip to the same partition μ. In order to calculate $\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F$, we only need to count all the partitions $\mu \vdash d$ such that: $\max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\} \leq \mu_i \leq \min\{\lambda^{(1)}_i, \lambda^{(2)}_i\}$ for all $i \geq 1$. Therefore, we have that

$$\begin{aligned}
\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F &= \langle q^d \rangle \left(\prod_{i=1}^{\infty} \left(\sum_{j=\max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}}^{\min\{\lambda^{(1)}_i, \lambda^{(2)}_i\}} q^j \right) \right) \\
&= \langle q^{d - \sum_{i=1}^{\infty} \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}} \rangle \left(\prod_{i=1}^{\infty} \left(\sum_{j=\max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}}^{\min\{\lambda^{(1)}_i, \lambda^{(2)}_i\}} q^{j - \sum_{i=1}^{\infty} \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}} \right) \right) \\
&= \left\langle q^{d - \sum_{i=1}^{\infty} \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}} \right\rangle \left(\prod_{i=1}^{\infty} \left(\sum_{j=0}^{\min\{\lambda^{(1)}_i, \lambda^{(2)}_i\} - \max\{\lambda^{(1)}_{i+1}, \lambda^{(2)}_{i+1}\}} q^j \right) \right).
\end{aligned}$$

$\square$

Lemma 5.2. *Let $d, a, b, p, q \in \mathbb{Z}_{\geq 0}$ be nonnegative integers. We have*

$$\sum_{\mu \vdash d} \{s_\mu \cdot h_a\}_{\lambda_1 = p} \otimes \{s_\mu \cdot h_b\}_{\lambda_1 = q} = \sum_{\mu \vdash \left(d+a+b-\max\{p,q\}\right)} \{s_\mu \cdot h_{\max\{p,q\}-b}\}_{\lambda_1=p} \otimes \{s_\mu \cdot h_{\max\{p,q\}-a}\}_{\lambda_1=q}.$$

Proof. We denote the left-hand side (resp. right-hand side) by F (resp. G). It suffices to show that

$$\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F = \langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle G$$

for any pair of partitions $\lambda^{(1)} \vdash a+d$ and $\lambda^{(2)} \vdash b+d$.

Consider three conditions:

- $\lambda_1^{(1)} = p$,
- $\lambda_1^{(2)} = q$, and
- for all $i \geq 1$ we have $\min\{\lambda_i^{(1)}, \lambda_i^{(2)}\} \geq \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}$.

If one of them is not satisfied, Lemma 5.1 implies that $\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F = 0 = \langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle G$. We henceforth suppose that all these three conditions hold. Then Lemma 5.1 implies that

$$\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle F = \left\langle q^{d - \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}} \right\rangle f$$

and

$$\langle s_{\lambda^{(1)}} \otimes s_{\lambda^{(2)}} \rangle G = \left\langle q^{d+a+b-\max\{p,q\} - \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}} \right\rangle f$$

where $f = \prod_{i=1}^{\infty} \left(\sum_{j=0}^{\min\{\lambda_i^{(1)}, \lambda_i^{(2)}\} - \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}} q^j \right)$. Therefore, it remains to show that

$$\left\langle q^{d - \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}} \right\rangle f = \left\langle q^{d+a+b-\max\{p,q\} - \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}} \right\rangle f.$$

Note that f is actually a product of finitely many palindromic polynomials in q, f itself is also a palindromic polynomial in q. Thus, it suffices to show that

$$\left(d - \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\} \right) + \left(d + a + b - \max\{p,q\} - \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\} \right) = \deg(f),$$

which is equivalent to

$$2d + a + b = \max\{p,q\} + \sum_{i=1}^{\infty} \min\{\lambda_i^{(1)}, \lambda_i^{(2)}\} + \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\}. \tag{5.1}$$

Since the right-hand side of Equation (5.1) equals

$$\begin{aligned}
&\max\{\lambda_1^{(1)}, \lambda_1^{(2)}\} + \sum_{i=1}^{\infty} \min\{\lambda_i^{(1)}, \lambda_i^{(2)}\} + \sum_{i=1}^{\infty} \max\{\lambda_{i+1}^{(1)}, \lambda_{i+1}^{(2)}\} \\
=& \sum_{i=1}^{\infty} \min\{\lambda_i^{(1)}, \lambda_i^{(2)}\} + \sum_{i=1}^{\infty} \max\{\lambda_i^{(1)}, \lambda_i^{(2)}\} \\
=& \sum_{i=1}^{\infty} \left(\min\{\lambda_i^{(1)}, \lambda_i^{(2)}\} + \max\{\lambda_i^{(1)}, \lambda_i^{(2)}\} \right) \\
=& \sum_{i=1}^{\infty} \left(\lambda_i^{(1)} + \lambda_i^{(2)} \right) = |\lambda^{(1)}| + |\lambda^{(2)}| = (a+d) + (b+d) = 2d + a + b,
\end{aligned}$$

which equals the left-hand side of Equation (5.1). Therefore, Equation (5.1) holds, completing our proof. ∎

Corollary 5.3.

(5.2)
$$\sum_{d=0}^{\min\{m,n\}} \left\{ \sum_{\mu\vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}) \right\}_{\lambda_1 \leq n+m-d-r} = \sum_{r'=r}^{\min\{m,n\}} \sum_{\mu\vdash r'} (s_\mu \cdot h_{n-r'}) \otimes (s_\mu \otimes h_{m-r'})$$

Proof. Pieri's rule and Lemma 5.2 imply that the left-hand side equals

$$\sum_{d=0}^{\min\{m,n\}} \sum_{\mu\vdash d} \{s_\mu \cdot h_{n-d}\}_{n-d\leq\lambda_1\leq n+m-d-r} \otimes \{s_\mu \cdot h_{m-d}\}_{m-d\leq\lambda_1\leq n+m-d-r}$$

$$= \sum_{\substack{(d,p,q)\in\mathbb{Z}^3 \\ 0\leq d\leq\min\{m,n\} \\ n-d\leq p\leq n+m-d-r \\ m-d\leq q\leq n+m-d-r}} \sum_{\mu\vdash d} \{s_\mu \cdot h_{n-d}\}_{\lambda_1=p} \otimes \{s_\mu \cdot h_{m-d}\}_{\lambda_1=q}$$

$$= \sum_{\substack{(d,p,q)\in\mathbb{Z}^3 \\ 0\leq d\leq\min\{m,n\} \\ n-d\leq p\leq n+m-d-r \\ m-d\leq q\leq n+m-d-r}} \sum_{\mu\vdash(m+n-d-\max\{p,q\})} \{s_\mu \cdot h_{\max\{p,q\}-m+d}\}_{\lambda_1=p} \otimes \{s_\mu \cdot h_{\max\{p,q\}-n+d}\}_{\lambda_1=q}$$

(5.3)
$$= \sum_{r'=r}^{\min\{m,n\}} \sum_{\substack{(d,p,q)\in\mathbb{Z}^3 \\ 0\leq d\leq\min\{m,n\} \\ p\geq n-d \\ q\geq m-d \\ n+m-d-\max\{p,q\}=r'}} \sum_{\mu\vdash(m+n-d-\max\{p,q\})} \{s_\mu \cdot h_{\max\{p,q\}-m+d}\}_{\lambda_1=p} \otimes \{s_\mu \cdot h_{\max\{p,q\}-n+d}\}_{\lambda_1=q}$$

where the second equal sign arises from Lemma 5.2, and the last equal sign arises from the following two facts:

- $p, q \leq n+m-d-r$ if and only if $n+m-d-\max\{p,q\} \geq r$, and
- $p \geq n-d$ and $q \geq m-d$ together imply that $n+m-d-\max\{p,q\} \leq \min\{m,n\}$.

Note that we can eliminate d in the expression (5.3) using the condition $n+m-d-\max\{p,q\} = r'$ under the summation, i.e., we can substitute $d = n+m-\max\{p,q\}-r'$ in the expression (5.3). Therefore, the left-hand side of Equation (5.2) equals

$$\sum_{r'=r}^{\min\{m,n\}} \sum_{\substack{(p,q)\in\mathbb{Z}^2 \\ 0\leq n+m-\max\{p,q\}-r'\leq\min\{m,n\} \\ p\geq\max\{p,q\}+r'-m \\ q\geq\max\{p,q\}+r'-n}} \sum_{\mu\vdash r'} \{s_\mu \cdot h_{n-r'}\}_{\lambda_1=p} \otimes \{s_\mu \cdot h_{m-r'}\}_{\lambda_1=q}. \tag{5.4}$$

Claim: All three inequalities under the second summation in (5.4) can be discarded.

In fact, for any non-empty term $\{s_\mu \cdot h_{n-r'}\}_{\lambda_1=p} \otimes \{s_\mu \cdot h_{m-r'}\}_{\lambda_1=q}$ in (5.4), Pieri's rule yields that

$$\max\{n-r', \mu_1\} \leq p \leq \mu_1 + n - r' \ \ (\leq n) \tag{5.5}$$

and

$$\max\{m-r', \mu_1\} \leq q \leq \mu_1 + m - r' \ \ (\leq m). \tag{5.6}$$

Then

$$\max\{m,n\} - r' \leq \max\{p,q\} \leq \max\{m,n\}. \tag{5.7}$$

Therefore, $0 \le \min\{m,n\} - r' = n+m-\max\{m,n\} - r' \le n+m-\max\{p,q\} - r' \le n+m-(\max\{m,n\}-r') - r' = \min\{m,n\}$, so we have $0 \le n+m-\max\{p,q\}-r' \le \min\{m,n\}$ and hence the first equality under the summation in (5.4) can be discard. Now we suppose that $p \le q$ without loss of generality. Then (5.6) implies that $\max\{p,q\}+r'-m = q+r'-m \le \mu_1$, and (5.5) implies that $p \ge \mu_1$. Consequently, we have $p \ge \max\{p,q\}+r'-m$, so we can discard the second inequality under the summation of (5.4). Additionally, $\max\{p,q\}+r'-n = q+r'-n \le q$ because $r' \le \min\{m,n\}$. Thus, we can discard the last inequality under the summation of (5.4). Therefore, the claim above holds.

By this claim, the expression (5.4) equals the right-hand side of Equation (5.2), finishing the proof. $\square$

Recall that we have studied the module structure of $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m}$ in Section 3.1 and note that $I_{n,m} \subseteq I_{n,m,r}$. This observation reveals an $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant surjection $\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m} \twoheadrightarrow \mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$. Thanks to this surjection, we finally obtain the graded $\mathfrak{S}_n \times \mathfrak{S}_n$-module structure of $R(\mathcal{UZ}_{n,m,r}) = \mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$:

Theorem 5.4.

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r};q) = \sum_{d=0}^{\min\{m,n\}} q^d \cdot \left\{ \sum_{\mu \vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}) \right\}_{\lambda_1 \le n+m-d-r}$$

Proof. The containment $I_{n,m} \subseteq I_{n,m,r}$ implies an $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant surjection

$$\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m} \twoheadrightarrow \mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r},$$

which, together with Proposition 3.5, means that

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r};q) \le \mathrm{grFrob}(\mathbb{C}[\mathbf{x}_{n\times m}])/I_{n,m};q) = \sum_{d=0}^{\min\{m,n\}} q^d \cdot \sum_{\mu \vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}).$$

Then Lemma 2.2 and Corollary 3.9 implies that

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r};q) \le \sum_{d=0}^{\min\{m,n\}} q^d \cdot \left\{ \sum_{\mu \vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}) \right\}_{\lambda_1 \le n+m-d-r}. \tag{5.8}$$

However, Inequality (5.8) implies that

$$\begin{aligned} \mathrm{Frob}(\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}) &\le \sum_{d=0}^{\min\{m,n\}} \left\{ \sum_{\mu \vdash d} (s_\mu \cdot h_{n-d}) \otimes (s_\mu \cdot h_{m-d}) \right\}_{\lambda_1 \le n+m-d-r} \\ &= \sum_{r'=r}^{\min\{m,n\}} \sum_{\mu \vdash r'} (s_\mu \cdot h_{n-r'}) \otimes (s_\mu \otimes h_{m-r'}) \\ &= \mathrm{Frob}\left(\bigoplus_{r'=r}^{\min\{m,n\}} \mathbb{C}[\mathcal{Z}_{n,m,r'}] \right) = \mathrm{Frob}(\mathbb{C}[\mathcal{UZ}_{n,m,r}]) \end{aligned} \tag{5.9}$$

where the first equal sign derives from Corollary 5.3, and the second equal sign arises from Equation (3.3). Now Theorem 4.12 tells us $R(\mathcal{UZ}_{n,m,r}) = \mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r}$, so orbit harmonics yields an $\mathfrak{S}_n \times \mathfrak{S}_m$-module isomorphism

$$\mathbb{C}[\mathbf{x}_{n\times m}]/I_{n,m,r} \cong \mathbb{C}[\mathcal{UZ}_{n,m,r}],$$

forcing all the inequalities in (5.9) to be equalities, and hence forcing Inequality (5.8) to be the equality that we desire. $\square$

6. Conclusion

We provide an open problem regarding log-concavity. Recall that a sequence of positive real numbers $(a_1, a_2, \ldots, a_N)$ is *log-concave* if for all $1 < i < N$ we have $a_i^2 \geq a_{i-1} \cdot a_{i+1}$. Chen conjectured [1] that the sequence $(a_{n,1}, a_{n,2}, \ldots, a_{n,n})$ is log-concave where $a_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ with longest subsequence of length k. Using the terminology of orbit harmonics, Chen's conjecture means that the Hilbert series of the orbit harmonics ring $R(\mathfrak{S}_n)$ of the permutation matrix locus $\mathfrak{S}_n$ is log-concave.

We can generalize the definition of log-concavity by incorporating group action. Let G be a group and $(V_1, V_2, \ldots, V_N)$ is a sequence of G-modules. We say that $(V_1, V_2, \ldots, V_N)$ is *G-equivariant log-concave* if for all $1 < i < N$ we have a G-module surjection $V_i \otimes V_i \twoheadrightarrow V_{i-1} \otimes V_{i+1}$ where the G-action is diagonal, i.e., $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$. A graded G-module $V = \bigoplus_{d=0}^{N} V_d$ is *G-equivariant log-concave* if the sequence $(V_1, V_2, \ldots, V_N)$ is G-equivariant log-concave. Rhoades conjectured [13] that $R(\mathfrak{S}_n)$ is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant log-concave, which generalizes Chen's conjecture.

Our locus $\mathcal{UZ}_{n,m,r}$ coincides with the permutation matrix locus $\mathfrak{S}_n$ visited by Rhoades if $n = m = r$. Therefore, we raise the following conjecture, which further generalizes Rhoades's conjecture.

Conjecture 6.1. *The graded $\mathfrak{S}_n \times \mathfrak{S}_m$-module $R(\mathcal{UZ}_{n,m,r})$ is $\mathfrak{S}_n \times \mathfrak{S}_m$-log-concave.*

Conjecture 6.1 has been checked by coding for $n \leq 8, m \leq 10$. The more flexibility provided by Conjecture 6.1 than Rhoades's conjecture, such as three parameters n, m, r rather than n itself, may permit more potential induction strategies and make the proof of log-concavity easier.

Another interesting future direction is to study the orbit harmonics ring $R(\mathcal{Z}_{n,m,r})$. Since $\mathcal{Z}_{n,m,r} \subseteq \mathcal{UZ}_{n,m,r}$, we need to add some elements to the generating set in Definition 3.1 to obtain the defining ideal $\operatorname{gr} \mathbf{I}(\mathcal{Z}_{n,m,r})$, such as $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j}$ and $\mathfrak{m}(\mathcal{R})$ for $\mathcal{R} \in \mathcal{UZ}_{n,m,r+1}$. Some relevant results are presented in Zhu [17] focusing on the graded module structure.

Finally, we may also consider extending the results to colored rook placement loci. Let N, k be positive integers, the *k-colored permutation group* $\mathfrak{S}_{N,k}$ is the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_N$ which can be embedded into $\mathrm{Mat}_{N\times N}(\mathbb{C})$ by

$$\mathfrak{S}_{N,k} = \left\{ X \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \; : \; \begin{array}{c} X \text{ has exactly one nonzero entry in each row and column,} \\ \text{and nonzero entries of } X \text{ are } k\text{-th roots-of-unity} \end{array} \right\}.$$

Define the *k-colored rook placement loci* $\mathcal{Z}^{(k)}_{n,m,r}$ by

$$\mathcal{Z}^{(k)}_{n,m,r} := \left\{ X \in \mathrm{Mat}_{n\times m}(\mathbb{C}) \; : \; \begin{array}{c} X \text{ has at most one nonzero entry in each row and column,} \\ \text{nonzero entries of } X \text{ are } k\text{-th roots-of-unity,} \\ \text{and } X \text{ possesses exactly } r \text{ nonzero entries} \end{array} \right\}$$

and let

$$\mathcal{UZ}^{(k)}_{n,m,r} := \bigsqcup_{r'=r}^{\min\{m,n\}} \mathcal{Z}^{(k)}_{n,m,r'}.$$

Problem 6.2. *Find the graded $\mathfrak{S}_{n,k} \times \mathfrak{S}_{m,k}$-module structures of $R(\mathcal{Z}^{(k)}_{n,m,r})$ and $R(\mathcal{UZ}^{(k)}_{n,m,r})$.*

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References

[1] William Y. C. Chen. *Log-concavity and q-Log-convexity Conjectures on the Longest Increasing Subsequences of Permutations*. 2008. arXiv: `0806.3392 [math.CO]`. URL: `https://arxiv.org/abs/0806.3392`.

[2] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, 3/e (Undergraduate Texts in Mathematics).* Berlin, Heidelberg: Springer-Verlag, 2007. ISBN: 0387356509.

[3] A.M Garsia and C Procesi. "On certain graded Sn-modules and the q-Kostka polynomials". In: *Advances in Mathematics* 94.1 (1992), pp. 82–138. ISSN: 0001-8708. DOI: `https://doi.org/10.1016/0001-8708(92)90034-I`. URL: `https://www.sciencedirect.com/science/article/pii/000187089290034I`.

[4] Sean T. Griffin. *Ordered set partitions, Garsia-Procesi modules, and rank varieties*. 2021. arXiv: `2004.00788 [math.CO]`. URL: `https://arxiv.org/abs/2004.00788`.

[5] James Haglund, Brendon Rhoades, and Mark Shimozono. "Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture". In: *Advances in Mathematics* 329 (2018), pp. 851–915. ISSN: 0001-8708. DOI: `https://doi.org/10.1016/j.aim.2018.01.028`. URL: `https://www.sciencedirect.com/science/article/pii/S0001870816314591`.

[6] Bertram Kostant. "Lie Group Representations on Polynomial Rings". In: *Bulletin of the American Mathematical Society* 69 (1963), pp. 518–526. URL: `https://api.semanticscholar.org/CorpusID:7228322`.

[7] Jasper Liu. "Viennot shadows and graded module structure in colored permutation groups". In: *Combinatorial Theory* 5 (July 2025). DOI: `10.5070/C65265409`.

[8] Jasper Liu et al. "Involution matrix loci and orbit harmonics". In: *Mathematische Zeitschrift* 310 (Apr. 2025). DOI: `10.1007/s00209-025-03736-4`.

[9] I.G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford classic texts in the physical sciences. Clarendon Press, 1998. ISBN: 9780198504504. URL: `https://books.google.com/books?id=srv9OXiUbZoC`.

[10] Jaeseong Oh and Brendon Rhoades. "Cyclic sieving and orbit harmonics". In: *Mathematische Zeitschrift* 300 (Jan. 2022), pp. 639–660. DOI: `10.1007/s00209-021-02800-z`.

[11] Markus Reineke, Brendon Rhoades, and Vasu Tewari. "Zonotopal Algebras, Orbit Harmonics, and Donaldson–Thomas Invariants of Symmetric Quivers". In: *International Mathematics Research Notices* 2023.23 (Mar. 2023), pp. 20169–20210. ISSN: 1073-7928. DOI: `10.1093/imrn/rnad033`. eprint: `https://academic.oup.com/imrn/article-pdf/2023/23/20169/54019853/rnad033.pdf`. URL: `https://doi.org/10.1093/imrn/rnad033`.

[12] Victor Reiner and Brendon Rhoades. *Harmonics and graded Ehrhart theory*. 2024. arXiv: `2407.06511 [math.CO]`. URL: `https://arxiv.org/abs/2407.06511`.

[13] Brendon Rhoades. "Increasing subsequences, matrix loci and Viennot shadows". In: *Forum of Mathematics, Sigma*. Vol. 12. Cambridge University Press. 2024, e97.

[14] Bruce E. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN: 9781475768046. URL: `https://books.google.com/books?id=Y6vTBwAAQBAJ`.

[15] C. Schensted. "Longest Increasing and Decreasing Subsequences". In: *Canadian Journal of Mathematics* 13 (1961), 179–191. DOI: `10.4153/CJM-1961-015-3`.

[16] G. Viennot. "Une forme geometrique de la correspondance de Robinson-Schensted". In: *Combinatoire et Représentation du Groupe Symétrique*. Ed. by Dominique Foata. Berlin, Heidelberg: Springer Berlin Heidelberg, 1977, pp. 29–58. ISBN: 978-3-540-37385-8.

[17] Hai Zhu. *Rook placements and orbit harmonics*. 2025. arXiv: `2510.25106 [math.CO]`. URL: `https://arxiv.org/abs/2510.25106`.

Department of Mathematics, UC San Diego, La Jolla, CA, 92093, USA
Email address: `(mol008, haz138)@ucsd.edu`