

How to see the forest for the trees

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Abstract

One of the major starting points of discrete optimization is the theorem of Nash-Williams and Tutte on the existence of k disjoint spanning trees of a graph along with its counterpart on the existence of k forests covering all edges of the graph. These elegant results triggered a comprehensive research that gave rise to far-reaching generalizations and found applications at seemingly far-fetched areas. There are well over a thousand papers in the literature, including quite a few brand-new ones.

Our first goal is to enlighten some aspects and links of these developments with the hope that the melody finds its way to non-experts. But we hope that experts will also find some novelties in our orchestration.

1 Introduction

Graphs (directed, undirected, mixed, hyper-) are fundamental mathematical tools for modelling problems of real-life networks. For example, a GPS seeks to find the fastest/shortest path from one location of a map to another.

For managing practical problems, the very first step is to capture (define) formally a certain intuitive feature of the network. For example, the feeling that a graph $G = (V, E)$ with node-set V and edge-set E ‘consists of one piece/component’ or that G ‘is connected’ can be formally defined by requiring that V cannot be cut (partitioned) into two non-empty parts in such a way that no edge connects the two parts. Another possibility is to require that there is a path connecting any two nodes. These two formal definitions of connectedness are easily seen to be equivalent.

A **tree** $G = (V, T)$ is a critically (or minimally) connected graph in the sense that deleting any of its edges destroys connectivity. This is equivalent to requiring that G is connected and includes no circuit. Yet another equivalent definition is that G is connected and every non-empty subset $X \subseteq V$ of nodes is sparse in the sense that X induces at most $|X| - 1$ edges. The connectivity of a graph G is equivalent to requiring that G includes a spanning tree. A **forest** is a graph whose components are trees. An equivalent definition is that the graph includes no circuit, that is, the graph is sparse.

Node, edge, degree, path, circuit, loop, cut, tree, forest, arborescence, branching, (connected) component: these are some of the most important basic concepts of graph theory. An easy property is that an undirected graph $G = (V, E)$ with an n -element node-set V and edge-set E admits an st -path (a path connecting two specified nodes $s, t \in V$) if and only if

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there is no empty cut separating s and t , that is, the number $d_G(X)$ of edges between X and $V - X$ is positive whenever $s \in X \subseteq V - t$. There are simple efficient algorithms for checking local connectivity (between nodes s and t) that either find an $s\bar{t}$ -set X with $d_G(X) = 0$ or find an st -path. G is called (globally) connected if there is a path between any two nodes $s, t \in V$.

A basic graph optimization problem is finding a spanning tree of G which is cheapest with respect to a cost-function on E . A simple greedy algorithm, due to Kruskal, does this job. By Dijkstra's classic algorithm, the problem of finding algorithmically a cheapest st -path is also efficiently manageable, at least for non-negative cost-functions. However, the sensitivity of this class of problems is nicely reflected by the fact that the cheapest (or shortest) path problem is NP-hard for general cost-functions. On the positive side, when negative components are allowed and only circuits with negative total costs are forbidden, then there is a polynomial algorithm, which relies, however, on deeper ideas.

The need for capturing intuitively more 'massive/strong/solid' connectivity concepts appeared as early as 1927, when Menger characterized those graphs admitting k openly disjoint st -paths. Here it is straightforward to figure out a natural necessary condition: no $k - 1$ nodes (distinct from s and t) can block all st -paths. The deep part of Menger's theorem is the proof of the sufficiency of this condition.

Naturally, there are several other variants for extending connectivity concepts. For example, a variation of Menger's theorem is about the existence of k edge-disjoint st -paths, which is often referred to as the (undirected) edge-Menger theorem. These results gave rise to a huge discipline: network flows (in directed graphs), in particular, the max-flow min-cut theorem. It should be emphasized that in this area it is rather typical that figuring out a necessary condition for the existence of the investigated property is easy, and the essence is finding a (possibly algorithmic) proof of sufficiency (see, for example, Hall's theorem on the existence of a perfect matching in a bipartite graph, or Dilworth's theorem on the partitionability of a poset into k chains).

Global connectivity has also been extended to higher connectivities. A graph G is **k -edge-connected** if $d_G(X) \geq k$ whenever $\emptyset \subset X \subset V$, which is equivalent, by the edge-Menger theorem, to requiring that there are k edge-disjoint st -paths for every pair $\{s, t\}$ of nodes. G is **k -node-connected** if it has at least $k + 1$ nodes and G remains connected after leaving out any subset of less than k nodes.

On this way of possible extensions of connectivity concepts, a natural question arises: what would be a right generalization of the above mentioned trivial statement that a graph is connected if and only if it has a spanning tree?

A graph is called **k -tree-connected** if it contains k edge-disjoint spanning trees. Following the melody of the edge-Menger theorem, one may surmise at first sight that a graph is k -edge-connected if and only if it is k -tree-connected. However, this is plain wrong already for $k = 2$, as is shown by a triangle. This tiny example indicates that one must also require that the graph should have at least $k \cdot (|V| - 1)$ edges. However, even this strengthened necessary condition is not sufficient, and the question arises: what is a necessary and sufficient condition for a graph to be k -tree-connected?

For Menger (also for Hall or for Dilworth), it was easy to formulate a necessary condition, for the problem of finding k disjoint spanning trees, however, just figuring out a promising conjecture is already a great challenge. A bit surprisingly, the answer was provided by two authors, W.T. Tutte and C.St.J.A Nash-Williams, independently of each other. Their papers appeared in the same issue (number 36) of the Journal of London Mathematical Society, in 1961. The paper of Tutte was submitted on April 29, 1960, and the paper of Nash-Williams was submitted on December 4, 1960.

Tutte writes on page 223 of his paper:

”Note. -Dr. C. St. J. A. Nash-Williams has sent me another proof of Theorem I (obtained quite independently), which does not use the concept of a multiple graph.”

Nash-Williams writes at the beginning of his paper:

”Tutte [3] proved a theorem equivalent to Theorem 1 of the present paper. A few months later, the present author (unaware of Tutte’s work) obtained a different proof, which is the object of this paper to present.”

By using notation fitting to the present overview, Theorem I of Tutte and Theorem 1 of Nash-Williams are as follows.

THEOREM 1.1 (Tutte [49], Theorem I) *An undirected graph $G = (V, E)$ includes k edge-disjoint spanning trees if and only if*

$$|F| \geq k \cdot (q(G, F) - 1), \quad (1.1)$$

holds for every subset $F \subseteq E$ of edges, where $q(G, F)$ denotes the number of connected components of the subgraph of G induced by $E - F$.

We hasten to emphasize that Tutte proved this result actually in a significantly more general form (Theorem II), see below Theorem 2.1.

THEOREM 1.2 (Nash-Williams [42], Theorem 1) *A graph G has k edge-disjoint spanning trees if and only if*

$$e_G(\mathcal{P}) \geq k \cdot (|\mathcal{P}| - 1) \quad (1.2)$$

for every partition $\mathcal{P}$ of $V(G)$, where $e_G(\mathcal{P})$ denotes the number of edges connecting distinct members of $\mathcal{P}$.

We assume throughout that the graphs are always loopless. It is also assumed that a partition consists of non-empty sets and has at least two members. In what follows, we shall refer to the edges connecting distinct parts of a partition $\mathcal{P}$ as **cross-edges** of $\mathcal{P}$.

In an equivalent formulation, the condition in the theorem is that the number of cross-edges of every q -partite ($q = 2, 3, \dots, n$) partition of $V(G)$ is at least $k \cdot (q - 1)$. Such graphs are sometimes called **k -partition-connected**.

It is seen immediately, without referring to the theorems themselves, that the conditions in the two theorems are equivalent. Namely, the Tutte condition may be viewed as a redundant form of the Nash-Williams condition. In fact, even the latter one is redundant a bit since it suffices to require (1.2) only for partitions where each part induces a $(k + 1)$ -edge-connected subgraph.

The form of Theorem 1.2 immediately implies that a $2k$ -edge-connected graph $G = (V, E)$ is k -tree-connected and actually it remains so after deleting an arbitrary subset K of at most k edges. Indeed, for a partition $\{V_1, \dots, V_q\}$ of V , the number of cross-edges in $G' := (V, E - K)$ is $\sum_i d_{G'}(V_i)/2 \geq \sum_i d_G(V_i)/2 - k = q(2k)/2 - k = k \cdot (q - 1)$, and Theorem 1.2 can be applied to G' in place of G .

The theorem of Tutte and Nash-Williams can be formulated in a concise way, as follows.

THEOREM 1.3 (Tree-packing theorem of Tutte and Nash-Williams) *A graph G is k -tree-connected if and only if G is k -partition-connected.*

For later generalization, it is useful to introduce a natural extension of k -partition-connectivity [18]. Beside the non-negative integer k , let l be a non-negative integer. A graph $G = (V, E)$ is called (k, l) -**partition-connected** if

$$e_G(\mathcal{P}) \geq k \cdot (|\mathcal{P}| - 1) + l \quad (1.3)$$

holds for any partition $\mathcal{P}$ of V . Obviously, $(k, 0)$ -partition-connectivity is the same as k -partition-connectivity, $(0, l)$ -partition-connectivity is equivalent to l -edge-connectivity, and (k, k) -partition-connectivity is $2k$ -edge-connectivity, or more generally, for $k \leq l$, (k, l) -partition-connectivity is equivalent to $k + l$ -edge-connectivity. Theorem 1.3 immediately shows for a given non-negative integer l that a graph G remains k -tree-connected after leaving out any subset of l edges if and only if G is (k, l) -partition-connected.

This observation on $2k$ -edge-connected graphs has far-reaching applications, for example, it served as a starting point for the design of min cut algorithms [28] and remained the basis for many others. Karger [28] observed that given a maximum set of edge-disjoint spanning trees, a minimum cut intersects at least one of them in at most 2 edges, thus to find a minimum cut it is enough to find the minimum of those cuts containing at most 2 edges from one of the given trees.

A natural counterpart of the disjoint-tree theorem investigates the problem of decomposing the edge-set of a graph $G = (V, E)$ into k forests, which is equivalent to cover E by k spanning trees when G is connected. This was solved by Nash-Williams in a (one-page long) paper that also appeared in the Journal of London Math. Society.

THEOREM 1.4 (Tree-covering theorem of Nash-Williams [43]) *The edge-set of a connected graph $G = (V, E)$ can be covered by k spanning trees (or equivalently, E can be partitioned into k forests) if and only if G is k -forest-sparse (or just k -sparse) in the sense that*

$$i_G(X) \leq k \cdot (|X| - 1) \quad (1.4)$$

holds for every non-empty subset X of V , where $i_G(X)$ denotes the number of edges induced by X .

Proof outline The necessity is straightforward, while sufficiency follows once we observe that G may be assumed to be saturated in the sense that adding any new edge destroys k -sparsity. For such a graph the supermodularity of i_G implies that $|E| = k \cdot (|V| - 1)$ in which case k -sparsity is just equivalent to k -partition-connectivity, and then the tree-packing theorem implies the tree-covering theorem. ■

For integers $0 \leq l < k$, we call a graph $G = (V, E)$ (k, l) -**forest-sparse** if the graph obtained from G by adding any set of l new edges is decomposable into k forests. By Theorem 1.4 of Nash-Williams, this is equivalent to requiring that

$$i_G(X) \leq k \cdot (|X| - 1) - l \quad (1.5)$$

holds for every subset X of V with $|X| \geq 2$. In the literature (e.g. [33]) a graph is called (k, l) -sparse if $i_G(X) \leq k \cdot |X| - l$ holds for every nonempty subset $X \subseteq V$ (where $0 \leq l \leq 2k - 1$). Clearly, a graph is (k, l) -forest-sparse if and only if it is $(k, k + l)$ -sparse.

It should be noted that there is yet another closely related paper that appeared in the Journal of London Math. Society, actually earlier than the above-mentioned works of Nash-Williams and Tutte. A. Horn proved the following.

THEOREM 1.5 (Horn [25]) *A (finite) set S of vectors can be partitioned into k linearly independent subsets if and only if*

$$k \cdot r(X) \geq |X| \tag{1.6}$$

holds for every subset X of S , where $r(X)$ denotes the rank of X .

A bit surprisingly, the same theorem was published in the same journal by R. Rado in 1962 [44], without referring to Horn's paper. (For a profound historic overview, see section 42.6f of Schrijver's book. [46]).

Actually, it suffices to require (1.6) only for closed and non-separable subsets X . It is also known for a graph $G = (V, E)$ that there is a matrix M_G whose columns correspond to the edges of G in such a way that a subset $X \subseteq E$ is a forest if and only if the set of columns of M_G corresponding to X is linearly independent. These two observations show that Horn's theorem implies Theorem 1.4.

These starting results of Nash-Williams and Tutte on trees and forests triggered an immense advancement in various related areas: matroid optimization, directed counterparts, hypergraph extensions. A rich overview of the area can be found, for example, in the books of Frank [15] and Schrijver [46], and there are thousands of papers referring to the above-mentioned results of Nash-Williams and Tutte. This interest has not quieted down in recent times either. Google Scholar, for example, found more than 300 papers of the past 15 years that cited the article [49] of Tutte. Obviously, the present summary cannot be appropriate for providing a general overview, and our goal is only to pick up and outline some of the most exciting related results.

1.1 Some graph problems triggered by the tree-packing and tree-covering theorems

We close this introductory session by a list of naturally emerging questions related to the tree-packing and the tree-covering theorems, which will be answered in the remaining parts. Some notions used here are also defined later.

Problem 1.1 How can we find (algorithmically) k disjoint spanning trees, when they exist? How can we find a deficient partition (that is, one violating (1.2)) when no k disjoint spanning trees exist? In the latter case, what is the minimum number (total cost) of new edges whose addition makes the graph k -tree-connected? •

Problem 1.2 Given k edge-disjoint forests $F_1, \dots, F_k$ of a graph $G = (V, E)$, when is it possible to extend these forests (A) to k edge-disjoint spanning trees, (B) to k forests covering all edges? More generally, for given edge-sets $E_1, \dots, E_k$ with $F_i \subseteq E_i$, one may expect that the extension of F_i be a subset of E_i ($i = 1, \dots, k$). •

Problem 1.3 Given non-negative lower bounds $f_1, \dots, f_k$ and upper bounds $g_1, \dots, g_k$, when do there exist k disjoint forests $F_1, \dots, F_k$ in G for which $f_i \leq |F_i| \leq g_i$ for $i = 1, \dots, k$? •

Problem 1.4 What is the maximum cardinality K of the union of k forests of G ? Clearly, $K = |E|$ precisely when G can be decomposed into k forests, and $K = k \cdot (|V| - 1)$ when G includes k spanning trees. In an equivalent formulation, what is the minimum number of new edges which are parallel to existing ones, whose addition to G results in a k -tree-connected graph? •

Problem 1.5 Given a non-negative weight-function on E , what is the maximum total weight of the union of k forests? Given a non-negative cost-function on E , what is the minimum total cost of the union of k disjoint spanning trees? •

Problem 1.6 Given k cost-functions $c_1, \dots, c_k$ on E , find k disjoint spanning trees $F_1, \dots, F_k$ for which the total cost $\tilde{c}_1(F_1) + \tilde{c}_2(F_2) + \dots + \tilde{c}_k(F_k)$ is minimum, where $\tilde{c}_i(F_i) := \sum [c_i(e) : e \in F_i]$. •

Problem 1.7 What are the counterparts of the theorems of Nash-Williams and Tutte concerning hypergraphs? •

Problem 1.8 What are the counterparts of the theorems of Nash-Williams and Tutte concerning directed and mixed graphs? •

Problem 1.9 (A directed counterpart of the second part of Problem 1.5) For a given cost-function, how can we compute a cheapest rooted k -arc-connected subgraph of a digraph? •

Problem 1.10 (A directed counterpart of Problem 1.6) For given cost-function $c_1, \dots, c_k$, how can we compute k arc-disjoint spanning arborescences $F_1, \dots, F_k$ with given root of a digraph for which $\tilde{c}_1(F_1) + \dots + \tilde{c}_k(F_k)$ is minimum? •

2 Matroids in the background

As already indicated, Tutte actually proved the tree-packing theorem in a more general form. He called a set of graphs $G_1 = (V_1, E, \varphi_1), \dots, G_k = (V_k, E, \varphi_k)$ a **multiple graph** where $\varphi_i(e) = \{u, v\}$ denotes the end-nodes of $e \in E$ in V_i .

THEOREM 2.1 (Tutte [49], Theorem II.) *Suppose that the graphs $G_1 = (V_1, E, \varphi_1), \dots, G_k = (V_k, E, \varphi_k)$ form a multiple graph. There exist k disjoint edge-sets $F_1 \subseteq E, \dots, F_k \subseteq E$ for which F_i is a spanning tree of G_i ($i = 1, \dots, k$) if and only if*

$$|F| \geq \sum_{i=1}^k (q(G_i, F) - 1) \quad (2.1)$$

holds for every subset $F \subseteq E$ of edges, where $q(G_i, F)$ denotes the number of (connected) components of the subgraph $(V_i, E - F, \varphi_i)$ of G_i induced by $E - F$.

This theorem rather easily implies, for example, a solution to Part (A) of Problem 1.2. A bit surprisingly, despite the numerous works in the literature citing the paper [49] of Tutte, only very few of them mention Theorem 2.1. A reason of this may be that the notion of Tutte's multiple graph is not particularly intuitive.

As it is so typical in mathematics, a phenomenon is more naturally understandable in an appropriately generalized framework. In the present case, this wonder framework is matroids. Already the paper [8] of Edmonds contains the following extension of Theorem 1.4 of Nash-Williams.

THEOREM 2.2 (Edmonds [8]) *The ground-set of a matroid $M = (S, r)$ with rank function r can be partitioned into k independent sets if and only if $k \cdot r(X) \geq |X|$ holds for every subset $X \subseteq S$.*

As noted after Theorem 1.5 of Horn, it suffices to require the inequality only for closed and non-separable sets X . When M is a graphic matroid, we are back at Nash-Williams' theorem, and when M is a matroid representable over a field, we are back at Theorem 1.5 of Horn. This theorem immediately provides a characterization of graphs whose edge-set can be partitioned into k forests, each consisting of at most l edges.

However, a straight extension of Tutte's Theorem II to a matroidal framework was described only in the fundamental paper [14] of Edmonds and Fulkerson, who proved the following result (formulated here in our terms).

THEOREM 2.3 (Edmonds and Fulkerson [14], Theorem 2c) *Let $M_1, \dots, M_k$ be matroids on a common ground-set S . There is a sub-partition $\{B_1, \dots, B_k\}$ of S for which B_i is a basis of M_i ($i = 1, \dots, k$) if and only if*

$$|X| \geq \sum_{i=1}^k t_i(X),$$

holds for every subset $X \subseteq S$, where t_i denotes the co-rank function of M_i defined by

$$t_i(X) := \min\{|X \cap B| : B \text{ a basis of } M_i\}.$$

After this theorem, Edmonds and Fulkerson explicitly mention that in the special case when each M_i is a graphic matroid, Theorem 2.3 is nothing but Theorem II of Tutte (= Theorem 2.1).

Edmonds and Fulkerson also proved a significant extension of Theorem 2.2, which result may be viewed as the covering counterpart of Theorem 2.3.

THEOREM 2.4 (Edmonds and Fulkerson [14], Theorem 1c) *Let $M_1, \dots, M_k$ be matroids on a common ground-set S . There is a partition $\{I_1, \dots, I_k\}$ of S for which I_i is an independent set of M_i ($i = 1, \dots, k$) if and only if*

$$|X| \leq \sum_{i=1}^k r_i(X),$$

holds for every subset $X \subseteq S$, where r_i denotes the rank function of M_i .

In the special case, when the matroids are graphic, this theorem implies not only Theorem 1.4 but its extension as well when, for given upper bounds $g_1, \dots, g_k$, we want to find a decomposition of E into forests $F_1, \dots, F_k$ for which $|F_i| \leq g_i$ ($i = 1, \dots, k$).

It should be emphasized that there are constructive proofs of the two theorems of Edmonds and Fulkerson which give rise to polynomial algorithms to compute the k independent sets or the k bases in question.

These general matroidal results of Edmonds and Fulkerson are extremely helpful in properly understanding the background of the tree-covering and tree-packing theorems but the genuine recognition behind the real power of the matroidal view is the following theorem of Edmonds from 1968.

THEOREM 2.5 (Matroid-sum theorem of Edmonds [10], Theorems 1 and 2) *Let $M_1 = (S, r_1), \dots, M_k = (S, r_k)$ be k matroids on a common ground-set S , and let $\mathcal{F}_i$ denote the set of independent sets of M_i . Let*

$$\mathcal{F}_\Sigma := \{F_1 \cup \dots \cup F_k : F_i \in \mathcal{F}_i\}. \quad (2.2)$$

Then $\mathcal{F}_\Sigma$ forms the set of independent sets of a matroid (called the sum or union of matroids $M_1, \dots, M_k$). The rank function r_Σ of M_Σ is as follows.

$$r_\Sigma(Z) = \min\{|Z - X| + \sum_{i=1}^k r_i(X) : X \subseteq Z\}. \quad (2.3)$$

Finally, it is useful to recall that Kruskal's algorithm has also been extended to matroids to compute a maximum weight (or minimum cost) basis of a matroid, see the paper [11] of Edmonds.

In this light, we can safely say that the theorems of Nash-Williams and Tutte became the root and a major driving force of the huge area of matroid optimization.

At this point, it is useful to remark, that a fundamental feature of matroids is that the rank-function of a matroid is a submodular function (that is, $r(X) + r(Y) \geq r(X \cap Y) + r(X \cup Y)$ holds whenever $X, Y \subseteq S$). We note that submodular functions (coming from graphs) were already used by Tutte and Nash-Williams in their proofs.

2.1 Some consequences on graphs

The matroidal background outlined above gave rise to solutions to the first six problems listed in Section 1.1. Here we mention explicitly only one concrete extension of Theorem 1.3 that can be derived from the matroid results.

THEOREM 2.6 *For integers $k \geq 1$ and $h \geq 0$, a graph $G = (V, E)$ can be made k -tree-connected by adding at most h new edges if and only if*

$$e_G(\mathcal{P}) \geq k \cdot (|\mathcal{P}| - 1) - h \quad (2.4)$$

holds for every partition $\mathcal{P}$ of V .

By relying on the matroid-sum theorem of Edmonds, one can observe that, for any spanning tree (V, F) of the complete graph on V , the h new edges in the theorem whose addition to G makes the graph k -tree-connected can be chosen to be parallel to some edges of F . In particular, the new edges may be chosen in such a way that they form a star (or that they are parallel to some edges of a spanning tree of G).

For a partition $\mathcal{P}$, the value $(k(|\mathcal{P}| - 1) - e_G(\mathcal{P}))^+$ is called the k -**deficit** of $\mathcal{P}$, where $x^+ := \max\{0, x\}$. We call the largest k -deficit over all partitions of V the k -**partition-deficiency** of G and denote it by $\Pi_k(G)$.

Let us consider a partition $\mathcal{P}$ with maximum k -deficit. It can be shown that each node-set $V_i \in \mathcal{P}$ induces a k -tree-connected graph, and the contraction of sets V_i gives a k -sparse graph. Jackson and Jordán [26] showed that there is a unique partition $\mathcal{P}^*$ among them with $|\mathcal{P}^*|$ minimum (which they called the brick-partition of G). These observations give an alternative proof of the following theorem.

THEOREM 2.7 *The minimum number of new edges whose addition to a graph $G = (V, E)$ results in a k -tree-connected graph is $\Pi_k(G)$. The maximum cardinality of the union of spanning trees of a connected graph G is $k(n - 1) - \Pi_k(G)$.*

To demonstrate how actively investigated this area is even nowadays, we cite a recent theorem of Akrami, Raj and Végh [1] stating that if the ground-set S of a matroid M can be partitioned into k bases, then, for any subset Z of S , there is a partition $\{B_1, \dots, B_k\}$ of S into bases which is Z -equitable in the sense that $\lfloor \frac{|Z|}{k} \rfloor \leq |B_i \cap Z| \leq \lceil \frac{|Z|}{k} \rceil$ for $i = 1, \dots, k$. For graphic matroids, this means that if a graph is decomposable into k spanning trees, then there is an equitable decomposition for any subset of edges.

3 Hypergraphs and directed graphs

After exploring and understanding the matroidal background of the tree-packing and tree-covering theorems, the question naturally emerges: how can the connectivity concepts and

theorems concerning undirected graphs be extended to hypergraphs, to directed graphs and to hypergraphs (= directed hypergraphs.)

3.1 Hypergraphs

Let $H = (V, \mathcal{E})$ be a hypergraph with node-set V where $\mathcal{E}$ denotes the set of hyperedges. We assume throughout that each hyperedge has at least two elements. By **trimming** a hyperedge X to a graph-edge, we mean the operation of deleting $|X| - 2$ elements of X , or in other words, replacing X with a graph-edge connecting two elements of X .

We say that a hypergraph H is a **hyperforest** if its hyperedges can be trimmed to a forest. Lovász [37] proved that H is a hyperforest if and only if the union of any $j > 0$ hyperedges has at least $j + 1$ elements. A **spanning hypertree** is a hyperforest having exactly $|V| - 1$ hyperedges.

A natural extension of the notion of graph connectivity is as follows. H is **connected** if $d_H(X) \geq 1$ for every non-empty proper subset X of V , where $d_H(X)$ is the number of hyperedges intersecting both X and $V - X$. H is **partition-connected** if, for every partition $\mathcal{P}$ of V , the number of hyperedges intersecting at least two members of $\mathcal{P}$ is at least $|\mathcal{P}| - 1$. While these two connectivity concepts are equivalent for graphs, a hypergraph on a 3-element node-set V with a single hyperedge V is connected but not partition-connected.

THEOREM 3.1 (Lorea [35]) *The hyperforests of a hypergraph $H = (V, \mathcal{E})$ form the independent sets of a matroid on the ground-set $\mathcal{E}$ of hyperedges.*

A matroid arising in this way is called the **hypergraphic** matroid of H .

THEOREM 3.2 (Whiteley [51]) *The rank of the hypergraphic matroid of a hypergraph $H = (V, \mathcal{E})$ is equal to*

$$\min\{|V| - |\mathcal{P}| + e_H(\mathcal{P}) : \mathcal{P} \text{ a partition of } V\}$$

where $e_H(\mathcal{P})$ denotes the number of hyperedges intersecting at least two members of the partition.

This implies that a hypergraph $H = (V, \mathcal{E})$ is partition-connected if and only if the rank of its hypergraphic matroid is $|V| - 1$. The following theorem may be considered as a straight generalization of the tree-packing and tree-covering theorems of Tutte and Nash-Williams (Theorems 1.3 and 1.4) to hypergraphs.

THEOREM 3.3 (Frank, Király, Kriesell [18]) *Let $H = (V, \mathcal{E})$ be a hypergraph.*

- (A) *$\mathcal{E}$ can be decomposed into k spanning partition-connected subhypergraphs (or, equivalently, H includes k disjoint spanning hypertrees) if and only if H is k -partition-connected.*
- (B) *$\mathcal{E}$ can be decomposed into k hyperforests if and only if we have*

$$i_{\mathcal{E}}(X) \leq k \cdot (|X| - 1)$$

for every non-empty subset X of V , where $i_{\mathcal{E}}(X)$ denotes the number of hyperedges included in X .

This theorem provides a solution to Problem 1.7. As for algorithmic aspects, see the paper [3] of Baïou and Barahona. Note that the problem of decomposing a hypergraph into k connected hypergraphs is **NP**-hard already for $k = 2$ [18].

3.2 Directed and mixed graphs

A directed tree is an **arborescence** or r_0 -**arborescence** if it has a node r_0 of in-degree 0 (called the root-node) and the in-degree of every other node is one. The union of node-disjoint arborescences is a **branching**. A digraph $D = (V, A)$ is called **root-connected** (from a root-node $r_0 \in V$) if the in-degree $\varrho_D(X)$ of every non-empty set $X \subseteq V - r_0$ is positive. This is equivalent to requiring that D has a spanning r_0 -arborescence, or that every node of D is reachable from r_0 along a directed path. D is **rooted k -arc-connected** if $\varrho_D(X) \geq k$ for every non-empty set $X \subseteq V - r_0$. This is equivalent (by Menger) to requiring that there are k arc-disjoint r_0v -dipaths for each $v \in V - r_0$.

The following fundamental theorem of Edmonds may be viewed as the directed counterpart of the tree-packing theorem of Tutte and Nash-Williams.

THEOREM 3.4 (Arborescence-packing theorem of Edmonds [12]) (Weak form) *A digraph $D = (V, A)$ with a root-node r_0 contains k arc-disjoint spanning r_0 -arborescences if and only if D is rooted k -arc-connected. (Strong form) Let $(V, A_1), \dots, (V, A_k)$ be k arc-disjoint r_0 -arborescences of D , and let $A_0 := A - (A_1 \cup \dots \cup A_k)$. They can be extended to k arc-disjoint spanning arborescences if and only if $\varrho_{A_0}(X)$ plus the number of arborescences A_i entering X is at least k for every non-empty subset $X \subseteq V - r_0$.*

Lovász [38] found a stunningly short and simple proof of this theorem, but a major difference between the directed and the undirected cases is that, unlike the tree-packing theorem of Tutte and Nash-Williams, no matroidal result is known that implies Edmonds' theorem.

We show how Edmonds' theorem helps solve Problem 1.9. Let c be a non-negative cost-function on the arc-set of an r_0 -rooted k -arc-connected digraph $D = (V, A)$ with no arc entering r_0 . Let $G = (V, E)$ denote the undirected graph obtained from D by de-orienting each arc. Let M_1 be the matroid on A in which a set of arcs is a basis of the corresponding set of edges in G is the union of k disjoint trees (that is, M_1 may be viewed as k times the graphic matroid of G). Let M_2 be a partition matroid on A in which a subset of arcs is a basis if the in-degree of every non-root node is k . By relying on Theorem 3.4 of Edmonds, it is not difficult to see for a subset $J \subseteq A$ of arcs that the subgraph (V, J) of D is a minimally (with respect to arc deletion) rooted k -arc-connected digraph if and only if it is a common basis of M_1 and M_2 . Therefore an algorithm for computing a minimum cost common basis of two matroids can be applied [13].

An essential difference between the directed and the undirected cases is that, while there is an efficient algorithm (via matroids) for solving Problem 1.6, its directed counterpart in Problem 1.10 is NP-hard already for $k = 2$.

As far as the directed counterpart of Nash-Williams tree-covering theorem is concerned, there are two results.

THEOREM 3.5 (Vidyasankar [50]) *Let r_0 be a root-node of a digraph $D = (V, A)$ such that no arc enters r_0 . The arc-set of D can be covered by k spanning r_0 -arborescences if and only if (a) $\varrho_D(v) \leq k$ for every $v \in V - r_0$ and (b) $k - \varrho_D(X) \leq \sum [k - \varrho_D(v) : v \in \Gamma^-(X)]$ holds for every non-empty set $X \subseteq V - r_0$ where $\Gamma^-(X)$ denotes the set of heads of the arcs entering X .*

THEOREM 3.6 (Frank [17]) *The arc-set of a digraph $D = (V, A)$ can be covered by k branchings if and only if (a) the in-degree of each node is at most k and (b) $i_D(X) \leq k \cdot (|X| - 1)$ holds for every set $\emptyset \subset X \subseteq V$, where $i_D(X)$ denotes the number of arcs induced by X .*

It is possible to merge the tree-packing and the arborescence-packing theorems. A **mixed graph** $M = (V, A + E)$ is a graph with possible directed and undirected edges. A mixed r_0 -arborescence is a mixed tree in which it is possible to orient the undirected edges to get an arborescence [17].

THEOREM 3.7 ([17]) *In a mixed graph $M = (V, A + E)$ with root-node r_0 , there are k edge-disjoint spanning mixed r_0 -arborescences if and only if $e_G(\mathcal{P}) \geq \sum [k - \varrho_A(V_i) : i = 1, \dots, q]$ holds for every partition $\mathcal{P} = \{V_0, V_1, \dots, V_q\}$ of V , where $r_0 \in V_0$ and $e_G(\mathcal{P})$ denotes the number of edges from E connecting distinct parts of $\mathcal{P}$.*

Note that these results provide an answer to Problem 1.8.

3.3 Dypergraphs

Analogously to digraphs, one may consider directed hypergraphs or dypergraphs. A **dyper-edge** (directed hyperedge) (Z, z) is a pair of a set Z with $|Z| \geq 2$ and an element z of Z , where z is called the **head** of Z while the remaining elements of Z are its **tails**. A dyper-edge (Z, z) is said to enter a set $X \subset V$ if $z \in X$ and $Z - X$ is non-empty. A dypergraph $D = (V, \mathcal{A})$ consists of a node-set V and a family $\mathcal{A}$ of dyperedges. The in-degree $\varrho_D(X)$ of a subset $X \subseteq V$ is the number of dyperedges entering X .

Given a root-node r_0 , we say that a dypergraph is **out-rooted k -arc-connected** if the in-degree of every non-empty set $X \subseteq V - r_0$ is at least k . When $k = 1$, D is out-rooted arc-connected. The dypergraphic extension of Edmonds' arborescence-packing theorem is as follows.

THEOREM 3.8 ([18]) *Suppose that every dyperedge of a dypergraph $D = (V, \mathcal{A})$ has at least two elements. Let $r_0 \in V$ be a given root-node. Then D can be decomposed into k out-rooted arc-connected dypergraphs if and only if D is out-rooted k -arc-connected.*

4 Orienting graphs and hypergraphs

4.1 Graphs

Orienting an undirected edge $e = uv$ means that we replace e with one of the two directed edges (= arcs) uv and vu . Orienting a hyperedges Z means that we replace Z by a dyperedge (Z, z) for some $z \in Z$.

Graph (hypergraph) orientation problems serve as a bridge between directed graphs and undirected graphs (hypergraphs). For example, the weak form of Nash-Williams' orientation theorem [41] states that a graph has a k -arc-connected orientation if and only if G is $2k$ -edge-connected. A general graph orientation result [16] implies the following.

THEOREM 4.1 *A graph G has a rooted k -arc-connected orientation if and only if G is k -partition-connected.*

On one-hand, if G is k -partition-connected, then, by Theorem 1.3, there are k disjoint spanning trees, and we can orient them separately to obtain k disjoint spanning arborescences. On the other hand, this orientation theorem and the disjoint arborescence theorem of Edmonds immediately implies Theorem 1.3.

For non-negative integers k and l , a digraph or a dypergraph D on node-set V with a specified root-node $r_0 \in V$ is **rooted (k, l) -arc-connected** if $\varrho_D(X) \geq k$ and $\varrho_D(V - X) \geq l$ whenever $\emptyset \subset X \subseteq V - r_0$.

THEOREM 4.2 ([16]) *For non-negative integers k and l , an undirected graph G with a root-node r_0 has a rooted (k, l) -arc-connected orientation if and only if G is (k, l) -partition-connected.*

Incidentally, we note that the concept of k -edge-connectivity is simpler than k -tree-connectivity. However, if we want a concise certificate for a graph to be k -edge-connected, then the simplest one (so far) is a set of k edge-disjoint r_0v paths for every node $v \in V - r_0$. But there is a more concise certificate implied by Theorem 4.1.

THEOREM 4.3 *A graph $G = (V, E)$ is k -edge-connected if and only if, for an arbitrarily specified node $r_0 \in V$, the digraph $D = (V, A)$ arising from G by replacing each edge by two oppositely directed parallel edges contains k disjoint spanning arborescences of root r_0 .*

4.2 Hypegraphs

As for orientation problems of hypergraphs are concerned, one has the following.

THEOREM 4.4 ([18]) (A) *A hypergraph $H = (V, \mathcal{E})$ has a k -arc-connected orientation if and only if $e_H(\mathcal{P}) \geq k \cdot |\mathcal{P}|$ holds for every partition $\mathcal{P}$ of V , where $e_H(\mathcal{P})$ denotes the number of hyperedges intersecting at least two members of $\mathcal{P}$. (B) H has an out-rooted k -arc-connected orientation (with respect to a specified root-node) if and only if H is k -partition-connected.*

A trivial observation is that a digraph is rooted (k, l) -arc-connected if and only if the digraph obtained by reversing all arcs is (l, k) -arc-connected. Because of the asymmetric role of tails and heads in a dypergraph, this observation does not extend to dypergraph. For example a dypergraph on node-set $V := \{r_0, s, t\}$ having the single hyperedge V has a $(0, 1)$ -partition-connected orientation but it has no $(1, 0)$ -partition-connected orientation.

In a paper of Guo et al. [24] a hypergraph H is called **h -weakly-partition-connected** if $\sum [d_{\mathcal{P}}(Z) - 1 : Z \in \mathcal{E}] \geq h \cdot (|\mathcal{P}| - 1)$ for every partition $\mathcal{P}$ of V , where $d_{\mathcal{P}}(Z)$ denotes the number of members of $\mathcal{P}$ intersecting Z . By relying on this concept, they formulated a general conjecture concerning Reed-Solomon codes. The question whether there is a good characterization of this property is answered in a more general form by the following result.

THEOREM 4.5 ([19]) *Let $H = (V, \mathcal{E})$ be a hypergraph with a root-node r_0 , and let k and l be non-negative integers.*

(A) *In case $l \leq k$, H has a (k, l) -arc-connected orientation if and only if*

$$e_H(\mathcal{P}) \geq k \cdot (|\mathcal{P}| - 1) + l \quad (4.1)$$

holds for every partition $\mathcal{P}$ of V . In the special cases when $l = k$ or $l = 0$, we are back at Parts (A) and (B), respectively, of Theorem 4.4.

(B) *In case $l > k$, H has a (k, l) -arc-connected orientation if and only if (4.1) holds and*

$$\sum_{Z \in \mathcal{E}} [d_{\mathcal{P}}(Z) - 1] \geq l \cdot (|\mathcal{P}| - 1) + k \quad (4.2)$$

holds for every partition $\mathcal{P}$ of V , where $d_{\mathcal{P}}(Z)$ denotes the number of members of $\mathcal{P}$ intersecting Z . In the special case when $k = 0$, we obtain that H has an orientation in which the out-degree of every non-empty subset of $V - r_0$ is at least l if and only if H is l -weakly-partition-connected.

We remark that the special case $k = 0$ of Part (B) (for characterizing l -weakly-partition-connected hypergraphs) was applied in a recent paper of Alrabiah et al. [2] in the theory of Reed-Solomon codes.

The proof of Theorem 4.5 gives rise to a polynomial algorithm that finds either a (k, l) -arc-connected orientation or a partition $\mathcal{P}$ violating the conditions.

5 Constructive characterizations

By a constructive characterization of a graph with a certain property, we mean a simple construction to build up the graph with the given property step by step. For example, a graph is connected if and only if it can be built up from a node by adding consecutively new edges for which at least one end-node is an already existing nodes. Constructive characterizations may be useful for deriving theorems concerning the graph property in question.

THEOREM 5.1 (Lovász [36]) *An undirected graph is $2k$ -edge-connected if and only if it can be built up from an initial node by consecutively applying the following two operations.*

- (A) *Add a new edge (possibly a loop) connecting two existing nodes.*
- (B) *Pinch together k existing edges with a new node.*

Note that this constructive characterization immediately implies a theorem of Nash-Williams stating that a $2k$ -edge-connected graph has a k -arc-connected orientation.

THEOREM 5.2 (Mader [39]) *A digraph is k -arc-connected if and only if it can be built up from an initial node by consecutively applying the following two operations.*

- (A) *Add a new arc (possibly a loop) connecting two existing nodes.*
- (B) *Pinch together k existing arcs with a new node.*

This result rather easily implies the following.

THEOREM 5.3 *A digraph $D = (V, A)$ with a root-node r_0 is rooted k -arc-connected if and only if it can be built up from r_0 by consecutively applying the following two operations.*

- (A) *Add a new arc (possibly a loop) connecting two existing nodes.*
- (B) *Add a new node z along with k arcs entering z for which their tails are existing nodes.*
- (C) *Pinch together j ($0 < j \leq k$) existing arcs with a new node z and add $k - j$ new arcs entering z whose tails are existing nodes.*

By combining Theorems 5.3 and 4.1, one obtains the following.

THEOREM 5.4 *An undirected graph G is k -partition-connected if and only if it can be built up from a node by consecutively applying the following operations.*

- (A) *Add a new edge (possibly a loop) connecting two existing nodes.*
- (B) *Add a new node z along with k edges connecting z with existing nodes.*
- (C) *Pinch together j existing edges ($0 < j \leq k$) with a new node z and add $k - j$ (possibly parallel) new edges connecting z with existing nodes.*

This result rather easily implies the tree-packing theorem of Tutte and Nash-Williams (Theorem 1.3). A significant extension of Theorem 5.3 is as follows.

THEOREM 5.5 (Kovács and Végő [30]) *Let $0 \leq l \leq k - 1$ be integers. A digraph D is (k, l) -arc-connected with respect to a root-node r_0 if and only if D can be built up from r_0 by consecutively applying the following operations.*

- (A) *Add a new arc connecting two existing nodes.*
- (B) *Pinch together j existing arcs ($l \leq j \leq k - 1$) with a new node z and add $k - j$ (possibly parallel) new arcs entering z for which their tails are existing nodes.*

By combining Theorems 4.2 and 5.5, one gets the following.

THEOREM 5.6 *Let $0 \leq l < k$ be integers. An undirected graph G is (k, l) -partition-connected if and only if it can be built up from a node by consecutively applying the following operations.*

- (A) *Add a new edge (possibly a loop) connecting two existing nodes.*
- (B) *Pinch together j existing edges ($l \leq j \leq k$) with a new node z and add $k - j$ (possibly parallel) new edges connecting z with existing nodes.*

The proof of this theorem is deep and difficult, however, a significantly simpler proof is available for its special case $l = 1$ in the paper of Frank and Szegő [20]. This special case has an interesting application in rigidity theory, see Section 6.

6 Tree-connectivity in rigidity theory

Tree-connectivity finds applications in seemingly distant areas such as rigidity theory. Rigidity examines the stability of structures modeled as graphs embedded in the Euclidean space with straight-line edges. The primary goal is to determine whether these structures (called frameworks) can be deformed without changing the edge lengths. If such a deformation does not exist, the framework is called rigid. In a bar-and-joint framework, nodes of a graph (the joints) are embedded in a d -dimensional space, where the edges (the bars) have fixed lengths. Intuitively, such a framework is rigid if a continuous deformation of the framework preserves pairwise distances of the nodes. An embedding of a graph is generic if all coordinates of the node positions are algebraically independent over the rational numbers. Rigidity is a generic property [51], and in some cases, generic rigid graphs can be characterized by sparsity. Here we call a graph $G = (V, E)$ (k, l) -forest-tight if it is (k, l) -forest-sparse and $|E| = k \cdot (|V| - 1) - l$. Laman [32] proved that $(2, 1)$ -forest-tight graphs are the generic minimally rigid graphs for bar-and-joint frameworks in the plane, and a graph is rigid in the plane if and only if it contains a $(2, 1)$ -forest-tight subgraph.

A body-and-bar framework is similar to a bar-and-joint framework, except the nodes of a graph represent full-dimensional bodies that are connected by fixed-length bars such that the attachment points of the bars on the bodies are pairwise distinct. Tay showed [47] that a multi-graph (a graph with possibly parallel edges but no loops) can be realized by a rigid body-bar framework in dimension d if and only if it contains $d(d+1)/2$ edge-disjoint spanning trees. Informally, a framework is globally rigid if for every other embedding with the same bar lengths also the distances of every pair of nodes are the same. A graph G is highly k -tree-connected if $e_G(\mathcal{P}) \geq k \cdot (|\mathcal{P}| - 1) + 1$. There is also a strong relation between generic global rigidity of body-bar frameworks and highly tree-connected graphs. A framework is called redundantly rigid if it is rigid and remains rigid after the removal of any edge. For a multi-graph H , a so-called body-bar graph G_H can be associated with H . (Its definition can be found for example in [6].)

Connelly et al. [6] showed that the following are equivalent: (a) G_H is generically globally rigid in $\mathbf{R}^d$, (b) G_H is generically redundantly rigid in $\mathbf{R}^d$, (c) H is highly $\binom{d+1}{2}$ -tree-connected. Their proof relies on the constructive characterization of highly k -tree-connected graphs, that is, $(k, 1)$ -partition-connected graphs (see Theorem 5.6 for the special case $l = 1$). We note that generic globally rigid body-hinge frameworks can also be characterized by highly tree-connected graphs [27].

We saw above that connectivity and constructive characterizations are closely related, and the latter turned out to be a useful tool in rigidity. Interestingly, this connection is bidirectional: recent results in rigidity were the key to prove some long-standing conjectures

about connectivity. Kriesell conjectured that a graph with high node-connectivity can always be partitioned into a spanning tree and a k -connected subgraph [40]. His conjecture has recently been proven by Garamvölgyi et al. [23]. They actually prove a stronger statement about subgraphs with higher dimensional rigidity.

THEOREM 6.1 (Garamvölgyi, Jordán, Király, Villányi [23]) *Every $(t \cdot 10d(d+1))$ -connected graph contains t edge-disjoint d -rigid (and hence d -connected) spanning subgraphs.*

They use a probabilistic approach in their proof. We note that with this tool they also answer an old conjecture of Thomassen [48] about highly connected orientations.

THEOREM 6.2 (Garamvölgyi, Jordán, Király, Villányi [23]) *Every $(320 \cdot k^2)$ -connected graph has a k -connected orientation.*

For a more detailed description of these recent results see [7].

7 Switching games

Shannon (see [22]) introduced a two-person game on connected undirected graphs with two specified nodes s and t . The players, Short and Cut, alternately tag an untagged edge, as long as there are untagged edges. Short wins if she tagged all edges of an st -path, and Cut wins if she tagged all edges of a cut separating s and t .

A special case of the switching game is the Bridge-It game of Gale played on a square grid [21]. A matroidal extension is due to Lehman [34] while Edmonds [9] described a general framework to explore the link between Lehman's results and the theorem 1.3 of Nash-Williams and Tutte.

For technical simplicity, here we discuss a version of Shannon's switching game, where no specified nodes s and t are designated, Short wins if she tagged all the edges of a spanning tree of G and Cut wins if she tagged all the edges of a cut of G . Clearly, at most one of the two players can win, and an immediate observation shows that one of them will definitely win.

Actually, there are two variants of the game, depending on who the first player is. Since these two variants are close to each other, we assume that Cut tags first.

THEOREM 7.1 *Suppose that $G = (V, E)$ is a connected graph with $|V| \geq 3$.*

(A) *There is a winning strategy for Short if and only if G is 2-tree-connected.*

(B) *There is a winning strategy for Cut if and only if V has a deficient partition $\mathcal{P} = \{V_1, \dots, V_q\}$ for $k = 2$, that is, one for which $e_G(\mathcal{P}) < 2q - 2$.*

Proof. (outline) By Theorem 1.3, G is 2-tree-connected if and only if there is no deficient partition. Therefore, exactly one of the two configurations can occur.

If F_1 and F_2 are two disjoint spanning trees, and Cut tags an edge e , say, from F_1 , then Short should tag an edge $f \in F_2$ that connects the two components of $F_1 - e$. (If e is not in $F_1 \cup F_2$, then any edge f will do). Now the graph G' arising from $G - e$ by contracting f is also 2-tree-connected, and by using iteratively this approach, Short can win.

If there is a deficient partition $\mathcal{P}$, a winning strategy for Cut is that she always tags an untagged cross-edge of $\mathcal{P}$. After her first move, there are at most $2q - 4$ untagged cross-edges. During the entire game Short can tag at most half of these edges, but every spanning tree contains at least $q - 1$ cross-edges, therefore Short will loose, and Cut wins. ■

It is not difficult to formulate an analogous theorem for the case when Short tags first. Roughly, in this case, there is a winning strategy for Cut if and only if there is a partition

with 2-deficit at least 2 (which is equivalent to requiring that every two spanning trees have at least two edges in common), and there is a winning strategy for Short if and only if there are two spanning trees with at most one edge in common (which is equivalent to requiring that the 2-deficiency of G is at most 1).

For the original Shannon switching game (for cutting or connecting s and t) when Cut is again the first player, one has the following.

THEOREM 7.2 (A) *There is a winning strategy for Short if and only if there exists a subset U of nodes containing s and t for which $G|U$ is 2-tree-connected, where $G|U$ denotes the restriction of G to U .*

(B) *There is a winning strategy for Cut if and only if V has a partition $\mathcal{P} = \{V_1, \dots, V_q\}$ separating s and t such that the graph arising from G by contracting each V_i to a single node can be decomposed into two forests F_1 and F_2 where s and t belong to distinct components of F_1 .*

This theorem is a consequence of Edmonds' [9] general results concerning games on matroids. From an algorithmic aspect, it is useful mentioning that if the unique maximum 2-deficit partition $\mathcal{P}^*$ ensured by Jackson and Jordán [26] (that is the brick partition of G) happens to separate s and t , then $\mathcal{P}^*$ will meet the requirements in Part (B).

Theorem 7.2 gives a characterization of the winning cases of player Cut when she starts the game. However, the result itself does not provide an algorithm for deciding whether property (A) or (B) holds. It can be decided by finding a pair of maximally disjoint spanning trees [4]. The approach of Bruno and Weinberg [5] uses the structural result of Kishi and Kajitani [29] on maximally disjoint trees, which is motivated by an application in electrical networks, where the above distances of the spanning trees correspond to certain degrees of freedom of a given electric network [31, 45].

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