# Classifying strict discrete opfibrations with lax morphisms

*In honour of Robert Paré on the occasion of his 80th birthday.*


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**Abstract**

We study how discrete opfibration classifiers in a(n enhanced) 2-category can be endowed with the structure of a $\boldsymbol{T}$-algebra and thereby lift to the enhanced 2-category of 2-algebras and lax morphisms. To support this study, we give a definition of discrete opfibration classifier in the enhanced setting in which tight (e.g. strict) discrete opfibrations are classified by loose (e.g. lax) maps.
We then single out conditions on the 2-monad $\boldsymbol{T}$ and the classifier that make this possible, and observe these hold in a wide range of examples: double categories (recovering the results of Parè and Lambert), (symmetric) monoidal categories, and all structures encoded by familial 2-monads. We also prove the properties needed on such 2-monads are stable under replacement by pseudo-algebra coclassifiers (when sufficient exactness conditions hold), allowing us to replace a pseudo-algebra structure on the classifier by a strict one.
To get to our main theorem, we introduce the concepts of *cartesian maps* and *cartesian objects* of a 2-algebra, which generalize various other notions in category theory such as cartesian monoidal categories, extensive categories, categories with descent, and more. As a corollary, we characterize when representable copresheaves are pseudo rather than lax in terms of the cartesianity at their representing object.



**Keywords:** 2-topos, 2-algebras, discrete opfibrations, cartesianity

**MSC Classification:** 18N15 , 18D70

## 1 Introduction

To define Bob Parè's work on double category theory *influential* would be a huge understatement: together with Grandis, he did nothing short of kick-starting the entire field, moving it from Ehresmann's curiosity to a powerful and central theory. In *Yoneda theory for double categories* Paré (2011), argues that $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ is the correct codomain for double presheaves as the natural recipient of the representables. However, these representable double functors are *lax*.

Lambert would later bring further evidence of the adequacy of such definitions by showing that presheaves over $\mathbb{A}$ are in fact in correspondence with 'double discrete opfibrations' over $\mathbb{A}$: this is (the dual of) Theorem 2.27 in Lambert (2021), which, formally, exhibits an equivalence of categories

$$\mathcal{dOpfib}(\mathbb{A}) \simeq \mathcal{Dbl}^{\mathrm{lx}}(\mathbb{A}, \mathbb{S}\mathrm{pan}(\mathrm{Set})) \tag{1.1}$$

where on the left we have the category of discrete double opfibrations in the sense of Lambert (Definition 2.12 in Lambert (2021)) and on the right the hom-category of lax functors and tight natural transformations of (weak) double categories. We note that Lambert's double discrete opfibrations are *strict* double functors; they are, in fact, *strict* discrete opfibrations in the 2-category of double categories, lax functors, and tight transformations.

In fact, both Parè and Lambert go on showing theorems that are both stronger and more significant to double category theory than the one above, in that they promote the equivalence of 1-categories to an equivalence of virtual double categories. But here, we focus on the 2-categorical aspect of this result, following Parè ((Paré 2011, 437)) in noticing that $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ is (almost) a *discrete opfibration classifier* in the sense of Weber (2007b) in the 2-category of double categories and lax functors.

A discrete opfibration classifier is a higher-dimensional version of a subobject classifier; where a subobject classifier is a subobject (whose fibers have category level $-1$) which is universal in the sense that every other subobject appears as pullback of it, a discrete opfibration classifier is a discrete opfibration (whose fibers have category level 0, i.e. are discrete) which is universal in the same sense. It follows from Lambert's correspondence that $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ is a *strict* discrete opfibration classifier, only 2-classifiers in the 2-category of double categories and lax functors whose underlying double functor is strict.

In this paper, we will show that the case of double categories is far from unique. In a wide variety of 2-algebraic theories, spans or something like them will classify *strict* discrete fibrations via lax morphisms.

How does the double category of $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ enter the story? As Parè mentions (Paré 2011, 437), double categories (of a particular unbiased flavor) are pseudoalgebras for the free-category monad $T$ on the 2-category Graph of graphs of categories. This monad is, famously, *cartesian* in the sense of Leinster (Definition 4.1.1 of Leinster (2004)) and moreover preserves arrow objects; it therefore preserves discrete fibrations. In particular, if $u : \Omega_\bullet \to \Omega$ is the 2-classifier of Graph, then $Tu : T\Omega_\bullet \to \Omega$ is also a discrete opfibration and is therefore classified by a map $\omega : T\Omega \to \Omega$. We will show that under these or weaker conditions, $\omega$ gives $\Omega$ the structure of a pseudo-$T$-algebra which classifies *strict* discrete opfibrations in the 2-category of pseudo-$T$-algebras and *lax* morphisms. In this case, of course, this means double categories.

But what is the discrete opfibration classifier of Graph? In Mesiti (2024), the author gives a recipe for constructing the discrete opfibration classifier of a 2-category of $[C, \mathcal{C}at]$ which neatly generalizes the construction of the subobject classifier of a 2-category of Set-valued (co)presheaves. Specifically,

$$\Omega(c) := [c/C, \mathrm{Set}] \tag{1.2}$$

is the discrete opfibration classifier of $[C, \mathcal{C}at]$, and the 2-classifier is the evident projection from $\Omega_\bullet(c) := [c/C, \mathrm{Set}_\bullet]$. Following Mesiti's recipe for $\mathrm{Graph}(\mathcal{C}at) = [\{e \rightrightarrows v\}, \mathcal{C}at]$, we find that:

$$\Omega(e) = \left[\left\{ \begin{array}{ccc} & \mathrm{id}_e & \\ \swarrow & & \searrow \\ s & & t \end{array} \right\}, \mathrm{Set}\right] = \mathrm{Set}^{\ulcorner} \qquad \Omega(v) = [\{\mathrm{id}_v\}, \mathrm{Set}] \cong \mathrm{Set} \tag{1.3}$$

with $\Omega(s)$ and $\Omega(t)$ given by the evident restrictions. That is, $\Omega \in \mathrm{Graph}$ is the (large) graph of spans of sets, and $\Omega_\bullet$ consists of spans of pointed sets.

Since the pseudo-double category structure $\omega : T\Omega \to \Omega$ was defined as the classifying map of the discrete opfibration $Tu : T\Omega_\bullet \to T\Omega$, we can *compute* what the induced composition of spans of sets must be. Given a composable sequence $S_1, \ldots, S_n$ of spans of sets, $\omega(S_1, \ldots, S_n)$ must send it to the span of sets whose elements are given by the fibers $Tu^{-1}(S_1, \ldots, S_n)$, or, in other words, sequences $s_1, \ldots, s_n$ of elements $s_i \in S_i$. This is the usual pullback composition of spans by another name.

In other words, because double categories are (pseudo-)algebras for a suitably *cartesian* 2-monad on a suitable 2-category of presheaves, we can compute the double category structure on $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ and derive it's special role as *strict* discrete opfibration classifier.

In this paper, we abstract the above story to the general setting of *opfibrantly cartesian* 2-monads $T$ (Definition 4.17) on *enhanced* 2-categories, or $\mathscr{F}$-categories. We work with enhanced 2-categories as defined by Lack and Shulman (2012) because our theorem concerns the interplay between *strict* and *lax* algebra morphisms; the theory of enhanced 2-categories was developed for just this purpose.

We begin by recalling what we need from the theory of enhanced 2-categories in Section 2. In Section 2.3, we give our version of Weber and Mesiti's notions of (good) discrete opfibration classifier to the enhanced setting. Then, in Section 3, we consider *cartesian $T$-morphisms* for an enhanced 2-monad $T$; this notion generalizes the *unique lifting of factorizations* when $T$ is the free category monad on graphs of sets (as first noticed by Leinster, see Remark 3.2), but also appropriate *descent* conditions such as extensivity (see Example 3.4).

In Section 4, we come to our main theorem. In Theorem 4.8, we show that if an enhanced 2-monad $T$ preserves discrete opfibrations and the discrete opfibration classifier $\Omega$ has a strict $T$-algebra structure $\omega : T\Omega \to \Omega$ which makes the 2-classifier $u$ into a (strict) $T$-cartesian morphism, then $\omega$ is the good 2-classifier of $\mathcal{A}\ell g_{\mathrm{lx}}(T)$, the enhanced 2-category of (strict) $T$-algebras and lax $T$-morphisms. We also observe in

Corollary 4.10 that the lax classifying map $\mathrm{d}p : B \to \Omega$ of a strict discrete opfibration $p : E \to B$ is a pseudo-$T$-morphism just when $p$ is $T$-cartesian in the sense of Section 3.

We expect that Theorem 4.8 will be the most useful form of theorem we consider here since it concerns *strict* algebras, which are the more useful sort of algebra in general, and since it is often clear how to produce the strict algebra structure $\omega$ needed. But it is somewhat disappointing that it requires this strict $T$-algebra structure as input. For the rest of Section 4, we consider more general properties of the enhanced 2-monad $T$ alone which will let us deduce the algebra structure $\omega$. In particular, in Lemma 4.22 we show that the classifying map $\omega : T\Omega \to \Omega$ of $Tu$ (where $u$ is the 2-classifier) is a pseudo-$T$-algebra when $T$ is *opfibrantly cartesian*; and, in Section 4.4, we show that under suitable exactness assumptions on the underlying enhanced 2-category, the pseudoalgebra coclassifier $T'$ of $T$ satisfies the strict assumptions of Theorem 4.8. Under these conditions, which hold in all presheaf 2-categories, the induced structure $\omega : T\Omega \to \Omega$ classifies strict discrete opfibrations amongst pseudo-$T$-algebras.

We end the paper in Section 5 with a smattering of examples. We have taken pains to make the main theorem *iterable*, in the sense that when $T : \mathcal{K} \to \mathcal{K}$ is an appropriate enhanced 2-monad on an appropriate enhanced 2-category (which, for lack of a better name, we call a *plumbus* in Definition 2.19), then the enhanced 2-category $\mathcal{Alg}_{\mathrm{lx}}(T)$ of strict algebras and lax morphisms is also appropriate. We may therefore apply the main theorem again to any appropriate enhanced 2-monad on $\mathcal{Alg}_{\mathrm{lx}}(T)$. We use this in Example 5.8 to deduce that the cartesian monoidal structure on $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ classifies strict monoidal, strict double discrete opfibrations via lax monoidal, lax double functors.

## 1.1 Related work

Late during the preparation of this work, we became aware of Koudenburg (2022), where Koudenburg proves a general lifting theorem for Yoneda structures in the setting of (augmented virtual) equipments. The result likely subsumes ours, which should be an instance of his for certain (augmented virtual) equipments of tight two-sided discrete opfibrations in a suitable 2-category—we give a more detailed account of the relationship of the two in Section A, once all the necessary terminology has been introduced.

However, this does not mean *ibid.* makes the present work useless. First of all, ours is a very different approach—perhaps more elementary, if less general. Since we work 2-categorically rather than double-categorically, we provide an alternative point of view, one closer to the way most '2-algebra' has been developed so far Second, our Section 4.4 deals with the question of strictness—which Koudenburg does not touch, since his theorem is stated in the general setting of colax $T$-algebras. Likewise, we give much attention on the question of tightness, i.e. strictness, of representables, and thus identify the notions of perfect (op)fibration and $T$-cartesian morphisms and objects.

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### 1.3 Notation and conventions

In the following, $\mathcal{K}$ will denote a 2-category, often enhanced (see below for a definition). Therein, objects are usually denoted by uppercase Latin letters $A, B, C, \ldots$, 1-cells by lowercase Latin letters $f, g, h, \ldots$ and 2-cells by lowercase Greek letter $\alpha, \beta, \ldots$. Notable exception are algebra maps which are denoted by the Greek equivalent of their carrier, for ease of tracking (so the algebra structure associated to $A$ is usually denote $\alpha$), as well as co/laxator associated to maps of algebras $f$ which are denoted as $\overline{f}$.

We use the convention of calling any object of a 2-category *category*, any 1-cell $a : X \to A$ a *(generalized) object* $a$ of $A$, and any 2-cell $\phi : a \Rightarrow b : X \to A$ a *(generalized) morphism* $\phi : a \to b$ of $A$.

## 2 Preliminaries

We assume the reader is familiar with basic 2-category theory. In the rest of this section, we establish some notation and terminology, as well as introduce some specialized notions. Most of the background required to read this work can be found in Lack (2009).

We recall just one result:

**Theorem 2.1** (Pasting lemma for commas)**.** *Given a diagram in a 2-category $\mathcal{K}$ as below, where the right square is a comma, then the whole diagram is a comma if and only if the left square is a pullback:*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \Swarrow"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd} \tag{2.1}$$

### 2.1 Enhanced 2-categories

*Enhanced 2-categories*, introduced by Lack and Shulman (2012), are 2-categories equipped with a distinguished wide sub-2-category whose 1-cells are called *tight*.

We work will with with such 2-categories for much of the same reasons Lack and Shulman do, that is, to single out the well-behaved *strict* morphisms of 2-algebras amongst (co)lax ones.

**Definition 2.2.** An **enhanced 2-category**, or **$\mathscr{F}$-category**, is a 2-category $\mathcal{K} \equiv: \mathcal{K}_l$ whose 1-cells are called **loose** and a wide and locally full subcategory $J_{\mathcal{K}} : \mathcal{K}_t \hookrightarrow \mathcal{K}$.

**Remark 2.3.** Such an enhanced 2-category is called **chordate** (dually, an enhanced 2-category with only identity tight morphisms is called **inchordate**). Indeed, in what follows, when we say *2-category* we always mean *chordate enhanced 2-category*.

In other words, being tight is a mere property of 1-cells of an enhanced 2-category, and 2-cells between tight 1-cells are the same as the 2-cells between the same 1-cells considered as loose. In particular, $\mathcal{K}_t \hookrightarrow \mathcal{K}_l$ is identity-on-objects, faithful and locally fully faithful. This is equivalent to the definition as a category enriched in the category $\mathscr{F}$ of full embeddings of categories (i.e. 'full subcategories'), as already noted by Lack and Shulman (2012).

**Example 2.4.** A trivial but useful example is taking $\mathcal{C}at$ with all its morphisms considered tight.

From the definition as enriched categories, we also get that *enhanced 2-functors* are 2-functors that preserve tightness, while *enhanced 2-natural transformations*, or *tight natural transformations*, are 2-natural transformations whose components are all tight.

**Definition 2.5** (Enhanced 2-monad)**.** An **enhanced 2-monad** is an $\mathscr{F}$-monad, thus a 2-monad $(T, i, m)$ such that $T$ preserves tightness and where $i$ and $m$ have tight components.

Given an enhanced 2-monad $T$, we can define various enhanced 2-categories of 2-algebras, depending on the weakness of the morphisms we would like to have. By taking suitable Eilenberg–Moore objects in the 2-category of enhanced 2-categories, one obtains enhanced 2-categories of 2-algebras where the tights are tight (in the underlying 2-category) and strict (as $T$-morphisms), while the loose have the desired weakness. See §4.3 in Lack and Shulman (2012), or Blackwell et al. (1989) for definitions of 2-algebras of a 2-monad, and (weak) 1- and 2-morphisms thereof.

**Definition 2.6** (Enhanced 2-category of $T$-algebras and lax morphisms)**.** The **enhanced 2-category of $T$-algebras and lax $T$-morphism** $\mathcal{A}lg_{\mathrm{lx}}(T)$ for an enhanced 2-monad $T$ on the enhanced 2-category $\mathcal{K}$ is the enhanced 2-category so comprised:

1. its objects are strict $T$-algebras whose structure map is tight in $\mathcal{K}$,
2. its loose maps are lax $T$-morphisms,

$$\begin{tikzcd} TA \arrow[r, squiggly, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \arrow[dl, Rightarrow] \\ A \arrow[r, squiggly, "f"'] & B \end{tikzcd} \tag{2.2}$$

3. its tight maps are strict $T$-morphisms whose underlying map is tight in $\mathcal{K}$,

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{2.3}$$

4. its 2-morphisms are $T$-2-morphisms.

In general, we define $\mathcal{A}lg_w(T)$ to be 2-category of 2-algebras with tight carriers and $w$-weak $T$-morphisms, enhanced by choosing the strict morphisms therein.

We now define the three kinds of enhanced 2-limits we will use in this paper, chiefly featuring in the definition of *plumbus* (Definition 2.19), which is the setting in which Section 4 takes place.

We start from a notion of pullback of tight maps along loose maps, which moreover marks the left projection as tight (this is how the $\mathscr{F}$-enrichment intervenes in the definition of enhanced 2-limit, see again Lack and Shulman (2012) for more details on the theory of enhanced 2-limits).

**Definition 2.7** (Left-tight pullback)**.** A **left-tight pullback** is an enhanced 2-limit of shape

$$V^\ell = \left\{ \begin{tikzcd} & b \arrow[d] \\ a \arrow[r, squiggly] & c \end{tikzcd} \right\} \tag{2.4}$$

and weighted as defined below, i.e. where $\Phi_l$ is the standard (conical) weight for a pullback, $\Phi_t$ marks the left projection as tight, and where $\varphi$ is the unique natural transformation of its type.

$$\begin{tikzcd} V^\ell_l \arrow[d, "J_{V^\ell}"'] \arrow[dr, bend left, "\Phi_l"] \arrow[d, phantom, shift left=4, "\overset{\varphi}{\Downarrow}"] & \\ V^\ell_t \arrow[r, bend right, "\Phi_t"'] & \mathcal{C}at \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[d, equal] \arrow[rr, dotted, "\varphi_b"] & & 1 \arrow[d, equal] \\ 1 \arrow[dr, dotted, "\varphi_a"'] & 0 \arrow[rr, dotted, "\varphi_c"] & & 1 \\ & 1 \arrow[urr, phantom] \arrow[rr, equal] & & 1 \end{tikzcd} \tag{2.5}$$

Thus a diagram of shape $V^\ell$ in an enhanced 2-category $\mathcal{K}$ is a cospan whose right leg is tight; and its $\Phi$-weighted limit is given by a pullback square whose left leg is tight and detects tightness, meaning when $p_A h$ as below is tight, so is $h$.

$$\begin{tikzcd} X \arrow[r, squiggly, "h"] \arrow[dr, bend right, "p_A h"'] & f \times_C g \arrow[r, squiggly, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ & A \arrow[r, squiggly, "g"'] & C \end{tikzcd} \tag{2.6}$$

Notice, in particular, that any left-tight pullback is also a pullback.

Next, we have the following 'almost tight' flavour of comma object:

**Definition 2.8** (l-rigged comma)**.** An **l-rigged comma** is an enhanced 2-limit of the same shape as in Diagram 2.4 and weighted as defined below, i.e. where $\Psi_l$ is the standard weight for a comma, $\Psi_t$ marks the left projection as tight, and where $\psi$ is the identity:

$$
\begin{tikzcd}
V_l^\ell \arrow[dd, "J_{V^\ell}"'] \arrow[dr, bend left, "\Psi_l"] \arrow[dd, phantom, shift left=3, "\psi\,\|"] & \\
& \mathcal{C}at \\
V_t^\ell \arrow[ur, bend right, "\Psi_t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[d, "d_0"] \arrow[rr, dotted, dash] & & \\
1 \arrow[r, "d_1"] \arrow[rr, dotted, dash, shift right=2] & 2 \arrow[rr, dotted, dash] & & 1 \arrow[d, "d_0"] \\
& & 1 \arrow[r, "d_1"'] & 2
\end{tikzcd}
\tag{2.7}
$$

Thus a diagram of shape $V^\ell$ in an enhanced 2-category $\mathcal{K}$ is a cospan whose right leg is tight; and its $\Psi$-weighted limit is given by a comma square whose legs are tight and detect tightness, meaning that when $\partial_1 h$ and $\partial_0 h$ as below are tight, so is $h$.

$$
\begin{tikzcd}
X \arrow[dr, squiggly, "h"] \arrow[drr, bend left, "\partial_0 h"] \arrow[ddr, bend right, "\partial_1 h"'] & & \\
& f/g \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\lrcorner\ \swarrow"] & B \arrow[d, "f"] \\
& A \arrow[r, squiggly, "g"'] & C
\end{tikzcd}
\tag{2.8}
$$

The epithet '*l-rigged*' comes from Lack and Shulman (2012)—dually, an *r-rigged* comma is one where the tightness assumption on the legs of Diagram 2.4 is swapped. Rigged limits are precisely the ones created in $\mathcal{A}lg_{\mathrm{lx}}(T)$ and $\mathcal{A}lg_{\mathrm{cx}}(T)$, specifically:

**Proposition 2.9** (Lack (2005), Proposition 4.6)**.** *The forgetful enhanced 2-functor $\mathcal{A}lg_{\mathrm{lx}}(T) \to \mathcal{K}$ creates l-rigged comma objects of the form $s/g$, where $g$ is lax and $s$ is strict:*

$$
\begin{tikzcd}
s/g \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\lrcorner\ \swarrow"] & E \arrow[d, "s"] \\
A \arrow[r, squiggly, "g"'] & B
\end{tikzcd}
\tag{2.9}
$$

*Dually, the forgetful enhanced 2-functor $\mathcal{A}lg_{\mathrm{cx}}(T) \to \mathcal{K}$ creates 'r-rigged' comma objects of the form $f/s$, where $f$ is colax and $s$ is strict:*

$$
\begin{tikzcd}
f/s \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\lrcorner\ \nearrow"] & E \arrow[d, "s"] \\
A \arrow[r, squiggly, "f"'] & B
\end{tikzcd}
\tag{2.10}
$$

**Remark 2.10.** It is the colaxator of $f$ (and $s$—note one could accomodate a laxator there!) which induces the $T$-algebra structure on $f/s$:

$$
\begin{tikzcd}[column sep=small]
 & T(f/s) \arrow[dl, "T\partial_0"'] \arrow[dr, "T\partial_1"] & \\
TA \arrow[d, "\alpha"'] \arrow[dr, squiggly, "Tf"] \arrow[rr, phantom, "\xRightarrow{T\chi}"] & & TE \arrow[dl, "Ts"'] \arrow[d, "\eta"] \\
A \arrow[r, phantom, "\xRightarrow{\overline{f}}"] \arrow[dr, squiggly, "f"'] & TB \arrow[d, dashed, "\beta"] & E \arrow[dl, "s"] \\
 & B &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & T(f/s) \arrow[dl, "T\partial_0"'] \arrow[dr, "T\partial_1"] \arrow[d, dashed, "\exists!\overline{f/s}"] & \\
TA \arrow[d, "\alpha"'] & f/s \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & TE \arrow[d, "\eta"] \\
A \arrow[rr, Rightarrow, "\chi"] \arrow[dr, squiggly, "f"'] & & E \arrow[dl, "s"] \\
 & B &
\end{tikzcd}
\tag{2.11}
$$

We also observe such commas are also created by each of the inclusions $\mathcal{A}\!\ell\!g_{\mathrm{st}}(T) \to \mathcal{A}\!\ell\!g_{\mathrm{ps}}(T) \to \mathcal{A}\!\ell\!g_{\mathrm{cx}}(T)$.

Finally, recall that an enhanced 2-limit is *tight* just when it is weighted 'in a chordate way', i.e. its shape is chordate and its weight has equal tight and loose parts—see §3.5.1 in Lack and Shulman (2012). Thus:

**Definition 2.11** (Tight terminal object)**.** A **tight terminal object** in an enhanced 2-category $\mathcal{K}$ is a terminal object 1 in $\mathcal{K}_l$ such that the unique maps $! : X \to 1$ are tight.

## 2.2 Discrete opfibrations

In this section, we establish some basic facts about discrete opfibrations in enhanced 2-categories.

**Definition 2.12** ((Tight) discrete opfibration)**.** A **(tight) discrete opfibration** in a(n enhanced) 2-category $\mathcal{K}$ is a (tight) map $p : E \to B$ that admits unique (*cocartesian*) lifts:

$$
\begin{tikzcd}
X \arrow[r, squiggly, "e"] \arrow[dr, squiggly, "b"'] & E \arrow[d, two heads, "p"] \arrow[dl, phantom, "\varphi\Downarrow"] \\
 & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, squiggly, "e"] \arrow[r, squiggly, bend right, "\varphi_* e"'] \arrow[r, phantom, "\Downarrow\exists!"] \arrow[dr, squiggly, "b"'] & E \arrow[d, two heads, "p"] \\
 & B
\end{tikzcd}
\tag{2.12}
$$

The chief example of discrete opfibration is given by projections out of a comma:

**Definition 2.13** (Representable discrete opfibration)**.** A discrete opfibration is **representable** if it is equivalent to the projection out of a comma object dashed below:

$$\begin{tikzcd} b/B \arrow[r] \arrow[d, dashed, two heads, "\partial_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, squiggly, "b"] \arrow[dl, Rightarrow] \\ B \arrow[r, equal] & B \end{tikzcd} \tag{2.13}$$

We say $b/B \xrightarrow{\partial_1} B$ is **represented by the object** $b : X \to B$. When $X = 1$, we say it is **globally representable**. When $b = \mathrm{id}_B$, we get the **domain opfibration** associated to $B$.

**Warning 2.14.** We remark that these are not representable in the sense of e.g. Riehl and Verity (2022), in which that means just 'being a discrete opfibration' (as opposed to being in a distinguished subclass).

**Observation 2.15.** For an enhanced 2-category, we are guaranteed that a representable discrete opfibration is tight when the comma is r-rigged (e.g. in $\mathcal{Alg}_{\mathrm{cx}}(T)$ over a 2-category with commas).

Being concerned, as we are in this work, with enhanced 2-categories of 2-algebras and lifting 2-classifiers there, it is important to characterize discrete opfibrations therein:

**Proposition 2.16** (Characterization of tight discrete opfibrations of $T$-algebras)**.** *Let $(T, i, m)$ be an enhanced 2-monad on $\mathcal{K}$, let $p : (E, \eta) \to (B, \beta)$ be a strict $T$-morphism. Then if $p$ is a tight discrete opfibration in $\mathcal{Alg}_{\mathrm{lx}}(T)$, then so it is in $\mathcal{K}$. If, furthermore, $T$ preserves discrete opfibrations, then the converse holds.*

*Proof.* To prove the first claim, we ponder a lifting problem for $p$ in $\mathcal{K}$:

$$\begin{tikzcd} X \arrow[r, squiggly, "e"] \arrow[dr, squiggly, bend right, "b"'] & E \arrow[d, "p"] \arrow[dl, phantom, "\varphi\Downarrow"] \\ & B \end{tikzcd} \tag{2.14}$$

From this, we freely construct the following lifting problem for $p$ in $\mathcal{Alg}_{\mathrm{lx}}(T)$:

$$\begin{tikzcd} TX \arrow[r, squiggly, "Te"] \arrow[dr, squiggly, bend right, "Tb"'] & TE \arrow[r, "\eta"] \arrow[d, "Tp"] \arrow[dl, phantom, "T\varphi\Downarrow"] & E \arrow[d, "p"] \\ & TB \arrow[r, "\beta"'] & B \end{tikzcd} \tag{2.15}$$

By assumption, the latter admits a solution:

$$
\begin{tikzcd}
TX \arrow[r, squiggly, "Te"] \arrow[rr, squiggly, bend right, "(\beta T\varphi)_*(\eta Te)"'] \arrow[dr, squiggly, bend right, "Tb"'] & TE \arrow[r, "\eta"] \arrow[d, phantom, "\exists!\,\lambda\Downarrow"] & E \arrow[d, "p"] \\
& TB \arrow[r, "\beta"'] & B
\end{tikzcd}
\tag{2.16}
$$

Whiskering the above diagram by $i : X \to TX$ yields the desired lift.

We prove uniqueness similarly: suppose we have two solutions $\lambda$, $\lambda'$ for the same lifting problem we started with. The same construction above gives us:

$$
\begin{tikzcd}
TX \arrow[r, squiggly, "Te"] \arrow[r, squiggly, bend right, "T(\varphi_* e)"'] \arrow[r, phantom, "T\lambda\Downarrow\Downarrow T\lambda'"] \arrow[dr, squiggly, bend right, "Tb"'] & TE \arrow[r, "\eta"] & E \arrow[d, "p"] \\
& TB \arrow[r, "\beta"'] & B
\end{tikzcd}
\tag{2.17}
$$

Thus $\eta$ necessarily equalizes $T\lambda$ and $T\lambda'$, since both $\eta(T\lambda)$ and $\eta(T\lambda')$ are $p$-lifts of $\beta T\varphi$, and by assumption $p$ admits unique lifts. But then, whiskering again by $i$ shows that $\lambda = \eta(T\lambda)i = \eta(T\lambda')i = \lambda'$.

Now assume $T$ preserves tight discrete opfibrations. Consider a lifting problem in $\mathcal{A}\ell g_{\mathrm{lx}}(T)$:

$$
\begin{tikzcd}
& & X \arrow[rr, squiggly, "e"] \arrow[drr, squiggly, "b"'] \arrow[rr, phantom, shift right=3, "\varphi\Downarrow"] & & E \arrow[d, "p"] \\
& & & \arrow[l, Leftarrow, "\overline{e}"] & B \\
TX \arrow[uurr, "\xi"] \arrow[rr, squiggly, "Te"] \arrow[drr, squiggly, bend right, "Tb"'] \arrow[rr, phantom, shift right=3, "T\varphi\Downarrow"] & & TE \arrow[uurr, "\eta"] \arrow[d, "Tp"] \arrow[u, phantom, "\overline{b}"] & & \\
& & TB \arrow[uurr, "\beta"'] & &
\end{tikzcd}
\tag{2.18}
$$

We solve it in the back face, getting $\lambda : e \Rightarrow \varphi_* e$:

$$
\begin{tikzcd}
& & X \arrow[rr, squiggly, "e"] \arrow[rr, squiggly, bend right, "\varphi_* e"'] \arrow[rr, phantom, "\exists!\,\lambda\Downarrow"] \arrow[drr, squiggly, "b"'] & & E \arrow[d, "p"] \\
& & & \arrow[l, Leftarrow, "\overline{e}"] & B \\
TX \arrow[uurr, "\xi"] \arrow[rr, squiggly, "Te"] \arrow[rr, squiggly, bend right, "T(\varphi_* e)"'] \arrow[rr, phantom, "T\lambda\Downarrow"] \arrow[drr, squiggly, bend right, "Tb"'] & & TE \arrow[uurr, "\eta"] \arrow[d, "Tp"] \arrow[u, phantom, "\overline{b}"] & & \\
& & TB \arrow[uurr, "\beta"'] & &
\end{tikzcd}
\tag{2.19}
$$

We must now equip $\varphi_* e$ with the structure of a lax $T$-morphism and show that $\lambda$ is a well defined $T$-2-morphism. We define $\overline{\varphi_* e} : \eta T(\varphi_* e) \Rightarrow (\varphi_* e)\xi$ by appealing to the universal property of $\eta T\lambda$ as a cocartesian lift of $p\eta T\lambda = \beta TpT\lambda = \beta T\varphi$: this is legitimate since $Tp$ is a discrete opfibration by assumption on $T$, and any map between discrete opfibrations is cartesian:

$$\begin{tikzcd}
 & e\xi \arrow[rr, "\lambda\xi"] \arrow[dd, mapsto] & & (\varphi_* e)\xi \arrow[dd, mapsto] \\
\eta Te \arrow[ur, "\overline{e}"] \arrow[rr, "\eta T\lambda"] \arrow[dd, mapsto] & & \eta(T\varphi_* e) \arrow[ur, dashed, "\overline{\varphi_* e}"] \arrow[dd, mapsto] & \\
 & pe\xi \arrow[rr, "\varphi\xi"] & & b\xi \\
p\eta Te \arrow[ur, "p\overline{e}"] \arrow[r, "p\eta T\lambda"'] \arrow[d, equals] & p\eta T(\varphi_* e) \arrow[r, equals] & \beta Tb \arrow[ur, "\overline{b}"'] & \\
\beta T(pe) \arrow[urr, bend right, "\beta T\varphi"'] & & &
\end{tikzcd} \tag{2.20}$$

We now embark on a diagram chase to show that $\lambda$ is a well-defined $T$-2-morphism, i.e. that $(\overline{\varphi_* e})\eta T\lambda = (\lambda\xi)\overline{e}$. By uniqueness of lifts along $p$ it suffices to show that these become equal after composing with $p$:

$$\begin{aligned}
p(\overline{\varphi_* e})\eta T\lambda &= \overline{b}(p\eta T\lambda) \\
&= \overline{b}(\beta TpT\lambda) \\
&= \overline{b}(\beta T\varphi) \\
&= (\varphi\xi)(p\overline{e}) \\
&= (p\lambda)\xi(p\overline{e}) \\
&= p((\lambda\xi)\overline{e}).
\end{aligned} \tag{2.21}$$

□

**Remark 2.17.** The converse direction above is significant, since it shows that defining a tight discrete opfibration to be a strict $T$-morphism in $\mathcal{Alg}_{\mathrm{lx}}(T)$ is in fact a natural requirement. When $T$ is sketchable[1], then this is also a consequence of Theorem 7.6 in Arkor et al. (2024).

Note that by Proposition 2.9, for algebras of an enhanced 2-monad $T$ the representable opfibrations are only definable at colax objects $b : X \to B$ rather that at general ones.

In conclusion, we record the following theorem, which substantially amounts to Theorem 5.10 from Capucci and Myers (2024):

**Proposition 2.18.** *Let $\mathcal{K}$ be a 2-category enhanced by a replete and pullback-stable choice of tights, and admitting all l-rigged arrow objects, i.e. l-rigged commas of the form $A/A$[2]. Then, for $(T, i, m)$ a (not necessarily enhanced!) 2-monad over $\mathcal{K}$, in*

[1] $T$ is sketchable when $\mathcal{Alg}_{\mathrm{lx}}(T) \simeq \mathcal{Mod}_{\mathrm{lx}}(\mathbb{T}, \mathcal{K})$ for $\mathbb{T}$ an enhanced 2-sketch (see Arkor et al. (2024) for a definition).

$\mathcal{A}\!\ell\!g_{\mathrm{cx}}(T)$*, tight normal (i.e. lifting identities to identities) fibrations admit pullbacks along arbitrary colax* $T$*-morphisms.*

## 2.3 Classifying discrete opfibrations

We now give the central definitions of the paper, concerning classifiers for discrete opfibrations. We closely follow Mesiti (2024) in doing so, albeit we adapt the theory to the setting of suitably complete enhanced 2-categories we call *plumbuses*:

**Definition 2.19** (Plumbus)**.** A **plumbus** is an enhanced 2-category admitting the following enhanced 2-limits:

1. a tight terminal object (Definition 2.11),
2. all left-tight pullbacks (Definition 2.7) of tight discrete opfibrations.

Specifically, in a plumbus we are guaranteed that the tightness of discrete opfibrations is a pullback-stable property, so that we have a pseudofunctor

$$\begin{tikzcd} \mathcal{K}^{\mathrm{op}} \arrow[rr, "\mathit{tdOpfib}"] & & \mathit{Cat} \\ B \arrow[d, "f"'] & \longmapsto & \mathit{tdOpfib}(B) \\ B' & & \mathit{tdOpfib}(B') \arrow[u, "f^*"'] \end{tikzcd} \tag{2.22}$$

where $\mathit{tdOpfib}(B)$ is the full subcategory of $\mathcal{K}/B$ spanned by tight discrete opfibrations, and $f^*$ is given by (left-tight) pullback (indeed, pullback of discrete opfibrations also induces a functorial action on 2-cells, thanks to their lifting property—see p.296 in Weber (2007b)).

We also note an interesting consequence of being a plumbus:

**Proposition 2.20.** *In a plumbus, tight discrete opfibrations* detect tightness, *i.e. when* $p : E \to B$ *is a tight discrete opfibration and* $pe$ *is tight for some* $e : X \to E$*, then so is* $e$*.*

*Proof.* Apply the tightness detection property of left-tight pullbacks to the pullback of $p$ along $\mathrm{id}_B$. □

In particular, all maps into discrete objects (i.e. those for which ! is a discrete opfibration) are tight, and thus the full subcategory spanned by them is completey contained in $\mathcal{K}_t$.

[2]This is a slightly stronger notion of *paradise* from Capucci and Myers (2024) (Definition 2.3).

**Definition 2.21.** A **pullback-ideal of tight discrete opfibrations** in a plumbus $\mathcal{K}$ is a replete and full subpseudofunctor

$$\mathit{stdOpfib} \subseteq \mathit{tdOpfib} : \mathcal{K}^{\mathrm{op}} \to \mathit{Cat} \tag{2.23}$$

Thus $\mathit{stdOpfib}$ amounts to a pullback-stable choice of tight discrete opfibrations.

The following is essentially Definition 4.1 from Weber (2007b):

**Definition 2.22** (Classifying discrete opfibration)**.** Let $\mathcal{K}$ be a plumbus, and consider a pullback-ideal of tight discrete opfibrations $\mathit{stdOpfib}$ therein, that we call **small**, which contains $\mathrm{id}_1$—and thus, all identity maps.

A **2-classifier for small discrete opfibrations** is a small discrete opfibration:

$$\Omega_\bullet \xrightarrow{u} \Omega \tag{2.24}$$

such that the functor $(-)^*u$ induced by taking pullbacks is fully faithful into tight discrete opfibrations, and corestricts to an equivalence of pseudofunctors $[\mathcal{K}^{\mathrm{op}}, \mathit{Cat}]$ on $\mathit{stdOpfib}$:

$$\begin{tikzcd} \mathit{stdOpfib} \arrow[loop left, "\cong"] \arrow[r, dashed, bend left, "\mathrm{d}(-)"] & \mathcal{K}(-,\Omega) \arrow[l, bend left, "(-)^*u"] \arrow[loop right, "\cong"] \end{tikzcd} \tag{2.25}$$

When $\mathcal{K}$ is equipped with a good 2-classifier, we say it is **representable**.

A map $f : B \to \Omega$ is a **formal copresheaf** and $f^*u$ its **opfibration of elements**, while $\mathrm{d}p : B \to \Omega$ is the **copresheaf of fibers** of an small discrete opfibration $p : E \to B$.

We can unpack the above equivalence informally as saying that, for every small discrete opfibration $p : E \to B$, there is a classifying morphism $\mathrm{d}p : B \to \Omega$ such that $\mathrm{d}p^*u \cong p$ as opfibrations over $B$ and for which the square below is a pullback:

$$\begin{tikzcd} \mathrm{d}p^*u \arrow[r] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, "u"] \\ B \arrow[r, squiggly, "\mathrm{d}p"'] & \Omega \end{tikzcd} \tag{2.26}$$

Note this implies the same square with $p$ *in lieu* of $\partial_1$ is a pullback too.

**Observation 2.23.** Given $u : \Omega_\bullet \to \Omega$ for which $(-)^*u$ is fully faithful, there is a natural notion of small discrete opfibration determined by the replete image of $(-)^*u$. Indeed, one can see smallness as a derived, rather than primary, notion, as done e.g. in Weber (2007b).

**Notation 2.24.** When $p$ is representable by $b$, then we also say the formal copresheaf $\mathrm{d}p$ is **representable** and denote it by $B(b, -)$.

The restriction to a pullback-ideal of small discrete opfibrations is due to a necessity arising in practice, for instance in the archetypal example of the 2-category of large but locally small categories:

**Example 2.25.** The archetypal example of 2-classifier is $\mathrm{Set}_\bullet \xrightarrow{u} \mathrm{Set}$ in $\mathcal{C}at$ (which, for us, is the 2-category of large but locally small categories). As the pullback-ideal of small discrete opfibrations, pick those having small (in the usual sense of 'having the size of a set in the first universe') fibers. This recovers the well-known discrete opfibration of elements construction:

$$\begin{tikzcd} \int F \arrow[r] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Set}_\bullet \arrow[d, "u"] \\ B \arrow[r, "F"'] & \mathrm{Set} \end{tikzcd} \tag{2.27}$$

We can choose naturally $\int \mathrm{d}p \cong p$ and $\mathrm{d}\int F \cong F$ so as to make this into an equivalence.

Notice in Example 2.25 above we could not have chosen *all* discrete opfibrations, since that would have forced $\Omega$ to be the *very large* category of *large sets*, which is not a member of $\mathcal{C}at$.

Moreover, let us also remark that both tightness and smallness are necessary notions: tightness refines arbitrary maps, while smallness refines only tight discrete opfibrations. For instance, in constructing $\mathcal{A}lg_{\mathrm{lx}}(T)$ one uses the notion of tightness to restrict which maps can become algebras, which is unrelated to the notion of smallness governing the classifiability of discrete opfibrations.

Finally, we remark that smallness is related to *admissibility* from , the notion of *attribute* from Weber (2007b), and finally the unnamed 'pullback-stable property $P$' in Mesiti (2024). The *attributes* of Weber are, essentially, what we would call the *small* modules (i.e. two-sided discrete opfibrations), though, unlike Weber, we do not venture in that direction (except briefly in Section A, indeed see Definition A.5). Finally, *admissibility* is the most indirectly related: as shown by Weber, his notion of 2-topos gives rise to a Yoneda structure where the admissible maps are those which represent a small two-sided opfibration. However, we cannot easily express this notion in the setting of representable plumbuses since we lack a duality (as in a Weber 2-topos) which would make two-sided discrete opfibrations instances of one-sided ones.

**Remark 2.26.** Notice $\mathrm{d}p$ is not unique, though any two classifying maps $\mathrm{d}p, \mathrm{d}p'$ must be isomorphic, and then by fully faithfulness of $(-)^*u$, such isomorphism must be unique.

**Remark 2.27** (Perfectness)**.** Crucially for our main result below (Theorem 4.8), $\mathrm{d}p$ does not need to be tight. However, we will need to keep track of those discrete opfibrations whose classifying map is tight: we call them **perfect**.

In fact, we can notice that $\mathcal{K}(-,\Omega)$ naturally lands in $\mathscr{F}$, since $\mathcal{K}$ is $\mathscr{F}$-enriched, and the inclusion of perfect opfibrations into small ones also promotes $\mathit{stdOpfib}$ to be a indexed full subcategory inclusion $\mathcal{K}^{\mathrm{op}} \to \mathscr{F}$. Thus, by definition, the equivalence of indexed categories $(-)^*u$ becomes an equivalence of indexed *full subcategories*.

### 2.3.1 Good 2-classifiers

An important observation, due to Weber (2007b), is that when $\mathrm{id}_1$ is small—and we always assume that in a representable plumbus—then the maps $X \to 1 \xrightarrow{\mathrm{d}(\mathrm{id}_1)} \Omega$ are terminal in each category $\mathcal{K}(X,\Omega)$, and thus $\mathrm{d}(\mathrm{id}_1)$ can be considered 'internally terminal' in $\Omega$, that is:

**Proposition 2.28** (Weber (2007b), Proposition 8.2)**.** *The classifying map* $\mathrm{d}(\mathrm{id}_1)$ *is right adjoint to* $! : \Omega \to 1$.

Moreover, we make the following observation, adapting the analogous one from Shulman (2023):

**Proposition 2.29** (Shulman (2023), Theorem 0.1)**.** *In a representable plumbus where* $\mathrm{id}_1$ *is perfect (Remark 2.27), the following is an l-rigged comma (Definition 2.8):*

$$\begin{tikzcd} \Omega_\bullet \arrow[r] \arrow[d, "u"', two heads] \arrow[dr, phantom, "\lrcorner\ \swarrow"] & 1 \arrow[d, "\mathrm{d}(\mathrm{id}_1)"] \\ \Omega \arrow[r, equal] & \Omega \end{tikzcd} \tag{2.28}$$

*Proof.* We reproduce the argument of Shulman and adapt it to the enhanced setting. Suppose given a lax square:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"', squiggly] \arrow[dr, phantom, "\swarrow"] & 1 \arrow[d, "\mathrm{d}(\mathrm{id}_1)"] \\ \Omega \arrow[r, equal] & \Omega \end{tikzcd} \tag{2.29}$$

We want to show such data is equivalent to a map $h : X \to \Omega_\bullet$. The key observation is that, in turn, a map like $h$ corresponds to a small discrete opfibration (which is the one classified by $uh$) and a section thereof (which picks out the additional 'points' $h$ picks by landing in $\Omega_\bullet$ rather than $\Omega$).

Thus we let $q = f^*u$, and then observe that the filler of the square above describes a map of classifying maps $\mathrm{d}(\mathrm{id}_X) \Rightarrow \mathrm{d}q$: by fully faithfulness of $(-)^*u$, this corresponds to a unique map $\mathrm{id}_X \to q$, i.e. a section of $q$.

Clearly, these correspondences are all bijections so we prove (2.28) is a comma. We must then prove the comma above left is l-rigged. First, note that by definition of perfectness, $\mathrm{d}(\mathrm{id}_1)$ is tight. Second, $u$ and $!_{\Omega_\bullet}$ are also tight by assumption, and finally they jointly detect tightness since $u$ does by Proposition 2.20 and postcomposing with $!_{\Omega_\bullet}$ always yields a tight map. $\square$

**Definition 2.30** (Good 2-classifier)**.** Let $\mathcal{K}$ be a representable plumbus, with its chosen class of small discrete opfibrations. We say its 2-classifier $u$ is **good** when $\mathrm{id}_1$ is perfect, and we say $\mathcal{K}$ is **well-representable**.

A good 2-classifier, therefore, substantiates the intuition that $\Omega_\bullet$ is really the object of 'pointed internal sets'.

We can equivalently use $\tau := \mathrm{d}(\mathrm{id}_1)$ to classify discrete opfibrations, now by taking commas: for every small discrete opfibration $p$, a straightforward application of the pasting lemma for commas (Theorem 2.1) shows that $p \cong \tau/\mathrm{d}p$:

$$\begin{tikzcd} E \arrow[r, squiggly] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \tau/\Omega \arrow[r] \arrow[d, "u"] \arrow[dr, phantom, "\Swarrow"] & 1 \arrow[d, "\tau"] \\ B \arrow[r, squiggly, "\mathrm{d}p"'] & \Omega \arrow[r, equal] & \Omega \end{tikzcd} = \begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow"] & 1 \arrow[d, "\tau"] \\ B \arrow[r, squiggly, "\mathrm{d}p"'] & \Omega \end{tikzcd} \tag{2.30}$$

Observe that Theorem 2.1 here extends to a statement about left-tight pullbacks and l-rigged commas: if the pullback above is left-tight, then the comma square on the right is l-rigged, and *vice versa*.

Thus we have again an equivalence of indexed full subcategories

$$\tau/- : \mathcal{K}(-, \Omega) \xrightarrow{\sim} \mathit{stdOpfib}. \tag{2.31}$$

which makes $\tau$ a *good 2-classifier*, in the sense of Definition 2.15 of Mesiti (2024), for the pullback-stable property $P$ of being small.

**Example 2.31.** The archetypal 2-classifier in $\mathcal{C}at$ (Example 2.25) arises from a good 2-classifier, namely the inclusion $1 \xrightarrow{1} \mathrm{Set}$, which is indeed the classifier of $\mathrm{id}_1$.

**Construction 2.32** (Mesiti's construction)**.** As anticipated in the introduction, in Mesiti (2024) it is shown that for any small category $C$, the (chordate enhanced) 2-category $[C, \mathcal{C}at]$ admits a good 2-classifier:

$$\begin{array}{ccc} C & \xrightarrow{\Omega} & \mathcal{C}at \\ c & & [c/C, \mathrm{Set}] \\ {\scriptstyle f}\big\downarrow & \longmapsto & \big\downarrow{\scriptstyle (f^*)^*} \\ c' & & [c'/C, \mathrm{Set}] \end{array} \tag{2.32}$$

with $\tau : 1 \to \Omega$ picking the constant functor $1 \in [c/C, \mathrm{Set}]$ for each $c \in C$.

**Example 2.33** (Good 2-classifier on $\mathrm{Graph}(\mathcal{C}at)$)**.** Following Mesiti's recipe for $\mathrm{Graph}(\mathcal{C}at) = [\{e \rightrightarrows v\}, \mathcal{C}at]$, one finds that:

$$\Omega(e) = \left[ \left\{ \begin{array}{ccc} & \mathrm{id}_e & \\ \swarrow & & \searrow \\ s & & t \end{array} \right\}, \mathrm{Set} \right] = \mathrm{Set}^{\ulcorner} \qquad \Omega(v) = [\{\mathrm{id}_v\}, \mathrm{Set}] \cong \mathrm{Set} \tag{2.33}$$

with $\Omega(s)$ and $\Omega(t)$ given by the evident restrictions. Thus $\Omega$ is the (large) graph of spans of sets, and $\tau : 1 \to \Omega$ picks out the trivial span $1 = 1 = 1$.

**Non-example 2.34.** Not every 2-classifier is good: consider $\mathcal{M}on\mathcal{C}at_{\mathrm{lx}}$, then, as we will prove later in Section 4, $1/\mathrm{Set} \to \mathrm{Set}$ is a 2-classifier which is not good: the culprit is that $\mathrm{d}(\mathrm{id}_1)$ is not tight, i.e. a strict $T$-morphism.

# 3 Cartesianity

We are now going to introduce a general notion of 'cartesianity' for algebras of a 2-monad and morphisms thereof. In the next section we show that the lifting a 2-classifier to the 2-algebras of $T$ happens when such a classifier is a $T$-cartesian morphism. However, the notions of cartesian algebras and morphisms are also of independent interest, and as far as we know, not proposed before, except for Leinster's analogous notion in 1-category theory (see Remark 3.2). They seem to be basic concepts in 2-algebra, and naturally occuring too: cartesian algebras specialize to extensive categories and cartesian monoidal categories.

In the following, assume $T$ is a 2-monad (but note for most of the definitions below, strictness is not necessary) on a 2-category $\mathcal{K}$ admitting the limits we use throughout.

**Definition 3.1** (Cartesian $T$-morphism)**.** Let $f : A \to B$ be a strict $T$-morphism. We call **$T$-cartesianity defect of $f$** the canonical comparison map induced by the pullback below:

$$\begin{tikzcd}
TA \arrow[dr, dashed, "\delta^w_f"] \arrow[drr, bend left, "Tf"] \arrow[ddr, bend right, "\alpha"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TB \arrow[d, "\beta"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd} \tag{3.1}$$

We say $f$ is **$T$-cartesian** when its defect is a split equivalence, and **strictly $T$-cartesian** when it is an isomorphism.

**Remark 3.2.** The notion of (strict) $T$-cartesian morphism, in the 1-dimensional setting, has been singled out before by Leinster who studied them under the name of *pullback-homomorphisms* in Appendix B of Leinster (2012). He shows fc-cartesian morphisms, where fc is the free category monad on graphs of sets, are precisely the functors enjoying the unique lift of factorizations (ULF) property.

**Definition 3.3.** Let $(A, \alpha)$ be a $T$-algebra and $a : X \to A$ a colax $T$-morphism. We say $\alpha$ is **(strictly) cartesian at** $a$ when the representable discrete opfibration at $a$ (Definition 2.13) is a (strictly) $T$-cartesian morphism:

$$\begin{tikzcd} T(a/A) \arrow[r, "T\partial_1"] \arrow[d, "\overline{a}/\alpha"'] \arrow[dr, phantom, "\lrcorner"] & TA \arrow[d, "\alpha"] \\ a/A \arrow[r, "\partial_1"] & A \end{tikzcd} \tag{3.2}$$

An example of this notion we find illuminating is the following:

**Example 3.4.** Let $T = \mathrm{Fam}_{\mathrm{f}}$ be the free finite coproduct completion 2-monad on Cat. Then a category with coproducts $A$, i.e. a $\mathrm{Fam}_{\mathrm{f}}$-algebra, is cartesian at $\mathrm{id}_A$ iff it is extensive.

To prove the claim we made, observe the comma algebra $A/\coprod : \mathrm{Fam}_{\mathrm{f}}(A/A) \to A/A$ is given by

$$(I \in \mathrm{Fin}, (\varphi_i : x_i \to y_i)_{i \in I}) \;\mapsto\; \coprod_{i\in I} \varphi_i : \coprod_{i\in I} x_i \to \coprod_{i \in I} y_i \tag{3.3}$$

Consider

$$\begin{tikzcd} \mathrm{Fam}_{\mathrm{f}}(A/A) \arrow[dr, dashed, "\delta_{\partial_1}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \mathrm{Fam}_{\mathrm{f}}(A) \times_A A/A \arrow[r] \arrow[d, "A/\coprod"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fam}_{\mathrm{f}}(A) \arrow[d, "\coprod"] \\ & A/A \arrow[r, "\partial_1"'] & A \end{tikzcd} \tag{3.4}$$

The $\mathrm{Fam}_{\mathrm{f}}$-cartesianity defect of $\partial_1 : A/A \to A$ sends $(I, (\varphi_i : x_i \to y_i)_i)$ to $((I, (y_i)_i), \coprod_i \varphi_i : \coprod_i x_i \to \coprod_i y_i)$. To say this is a split equivalence means saying that for all $\psi : X \to \coprod_i y_i$ there exists a unique family $(\psi_i : x_i \to y_i)_i$ such that $\coprod_i \psi_i \cong \psi$. This is precisely saying that the canonical map $\prod_i A/y_i \to A/(\coprod_i y_i)$ is an equivalence of categories, i.e. that $A$ is extensive (Carboni et al. (1993)).

**Example 3.5** (Descent)**.** Generalizing the above, suppose $T$ is a free $\Phi$-cocompletion 2-monad on $\mathcal{C}at$, with $\Phi$ a class of weights. Then $A$ is $T$-cartesian at the classifying object, i.e the map $\partial_1 : A/A \to A$ is $T$-cartesian, precisely when $A$ satisfy descent for $\Phi$-colimits, i.e. $\lim_i A/a_i \simeq A/\operatorname{colim}_i a_i$.

**Example 3.6** (Extensive monoidal category)**.** It's easy to see the argument above for extensive categories applies more generally to $T =$ the free symmetric monoidal category 2-monad, thus recovering the notion of *extensive monoidal category* of Example 9.2 in Gálvez-Carrillo et al. (2018). Hence extensive monoidal categories are characterized as those monoidal categories cartesian at the identity.

Indeed, we might define an **extensive $T$-algebra** to be one cartesian at the identity.

**Example 3.7.** In the special case $X = 1$, being cartesian at $a$ means that for every term $t(b_1, \ldots, b_n)$, it is the case that

$$A(a, t(b_1, \ldots, b_n)) \cong t(A(a, b_1), \ldots, A(a, b_n)). \tag{3.5}$$

For $T$ the free symmetric monoidal category 2-monad on $\mathcal{C}at$ is easy to see this corresponds to the characteristic representability of products (we will come back to this example shortly, in Example 3.17).

In general, the above definition is quite literally expressing the fact that the corepresentables at $a$, which normally are just lax $T$-morphisms, be actually *pseudo*. This is Corollary 4.12.

**Example 3.8** (Spanish double categories)**.** A **spanish** double category ('span-like' in Definition 5.3.1.5 of Myers (2021)) is one where any 2-cell out of a unit and into a composite, as below left, factors uniquely as below right:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[dd, equal, "\bullet" marking] & \cdot \arrow[d, "\bullet" marking] & & \cdot \arrow[r] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\Longrightarrow"] & \cdot \arrow[d, "\bullet" marking] \\
\arrow[r, phantom, "\Longrightarrow"] & \cdot \arrow[d, "\bullet" marking] & = & \cdot \arrow[r] \arrow[d, equal, "\bullet" marking] \arrow[dr, phantom, "\Longrightarrow"] & \cdot \arrow[d, "\bullet" marking] \\
\cdot \arrow[r] & \cdot & & \cdot \arrow[r] & \cdot
\end{tikzcd} \tag{3.6}$$

Consider now the free double category 2-monad fc on Graph($\mathcal{C}at$) (see Example 5.5). A strict object $a : 1 \to A$ in a double category $A$ is simply an object of $A$, and $A$ is spanish exactly when it is fc-cartesian at all its objects. Indeed, an object of $\mathrm{fc}(a/A)$ is a composite of triangles such as above right, while $\alpha \times_A \partial_1$ is comprised of single triangles as above left. The cartesian defect of $a$ composes a series of triangles, and asking this map invertible means having unique factorizations as required.

To capture, instead, the cartesianity of cartesian monoidal categories, we must necessarily check at *colax* global elements (i.e. comonoids), though such a definition would be unsatisfying as it would not necessarily exhaust all objects. For instance (we thank Vincent Moreau for this example), in $\mathcal{C}at$, $A = \mathrm{Top}$ with its unique symmetric monoidal closed structure (see Činčura (1979), Pedicchio and Solimini (1986)) is cartesian only at the discrete spaces.

Therefore, for a $T$-algebra $(A, \alpha)$, we must distinguish between **unstructured objects**, which are just objects $a : X \to A$ of its carrier $A$, and $T$**-co/structured objects**, which are co/lax $T$-morphisms $a : X \to A$. When $T$ is the free monoidal category monad, an unstructured object is just an object; a colax structured object is a comonoid, while a lax structured object is a monoid. The goal of the reminder of the section is to define cartesianity at unstructured global objects.

**Proposition 3.9.** *Let $a : X \to A$ be a colax morphism of $T$-algebras. The following are equivalent:*

1. *$A$ is strictly cartesian at $a$,*

*2. the following is a comma square:*

$$
\begin{tikzcd}
T(a/A) \arrow[r, "T\partial_1"] \arrow[d, "T\partial_0"'] & TA \arrow[dd, "\alpha"] \\
TX \arrow[d, "\xi"'] \arrow[ur, phantom, "\overset{\chi}{\nearrow}"] & \\
X \arrow[r, "a"'] & A
\end{tikzcd}
\tag{3.7}
$$

*Proof.* By Proposition 2.9, the square below on the left commutes:

$$
\begin{tikzcd}
T(a/A) \arrow[rr, "T\partial_1"] \arrow[dr, "\overline{a}/\alpha"'] \arrow[dd, "T\partial_0"'] & & TA \arrow[d, "\alpha"] \\
& a/A \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\nearrow"] & A \arrow[d, equal] \\
TX \arrow[r, "\xi"'] & X \arrow[r, "a"'] & A
\end{tikzcd}
\tag{3.8}
$$

Then the claim becomes a direct application of the pasting lemma for commas (Theorem 2.1) to the two squares on the right above, constituting Diagram 3.7. □

**Corollary 3.10.** *The colax morphism $a$ is $T$-cartesian iff the universal comparison map of Diagram 3.7 into the comma square is a split equivalence.*

Observe that Diagram 3.7 above doesn't mention the colax structure of $a$. In fact, when $T$ is 'cartesian at the representable discrete opfibrations', such colax structure is uniquely determined. Here's what we mean:

**Definition 3.11.** A 2-natural transformation $\phi : F \Rightarrow G : \mathcal{K} \to \mathcal{H}$ is **cartesian at a morphism** $f \in \mathcal{K}$ when the 2-naturality square of $\phi$ at $f$ is a strict 2-pullback in $\mathcal{H}$.

**Definition 3.12.** A 2-monad $T$ is **cartesian at a class of maps** $\mathcal{M} \subseteq \mathcal{K}$ when its unit and multiplication, as 2-natural transformations, are cartesian at the maps in question.

**Theorem 3.13** (Costructure for free)**.** *Suppose $T$ has unit and multiplication $i$ and $m$ cartesian at representable discrete opfibrations. Let $(X, \xi)$ and $(A, \alpha)$ be $T$-algebras, and let $a : X \to A$ be an object of $A$. Consider the discrete opfibration represented by $a$ (Definition 2.13) and suppose Diagram 3.7 is a comma. Then $a$ is equipped with a unique $T$-colax structure.*

*Proof.* We construct the colaxator in two steps. First, consider the universal map $\ulcorner \mathrm{id}_a \urcorner : X \to a/A$ induced by factoring the identity square of $a$ through the comma square defining the discrete opfibration represented by $a$:

$$\begin{tikzcd}
X \arrow[drr, bend left, "a"] \arrow[dr, dashed, "\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, equal, bend right] & & \\
& a/A \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\lrcorner\ \Nearrow"] & A \arrow[d, equal] \\
& X \arrow[r, "a"'] & A
\end{tikzcd} \tag{3.9}$$

Then, whisker Diagram 3.7 by $T\ulcorner \mathrm{id}_a \urcorner$ to get the desired 2-cell $\overline{a}$:

$$\begin{tikzcd}
TX \arrow[r, "Ta"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\overline{a}\ \Nearrow"] & TA \arrow[d, "\alpha"] \\
X \arrow[r, "a"'] & A
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
TX \arrow[drr, bend left, "Ta"] \arrow[dr, dashed, "T\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, bend right, "\xi"'] & & \\
& T(a/A) \arrow[r, "T\partial_1"] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \Nearrow"] & TA \arrow[d, "\alpha"] \\
& X \arrow[r, "a"'] & A
\end{tikzcd} \tag{3.10}$$

We now prove unitality and associativity. The first is the equation

$$\begin{tikzcd}
X \arrow[r, "iX"] \arrow[d, "a"'] & TX \arrow[r, "\xi"] \arrow[d, "Ta"'] \arrow[dr, phantom, "\overline{a}\ \Swarrow"] & X \arrow[d, "a"] \\
A \arrow[r, "iA"'] & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "a"'] & X \arrow[d, "a"] \\
A \arrow[r, equal] & A
\end{tikzcd} \tag{3.11}$$

Unraveling the definition of $\overline{a}$, the left-hand side is equal to

$$\begin{tikzcd}
X \arrow[r, "iX"] \arrow[dd, "a"'] \arrow[ddr, phantom, "\lrcorner"] & TX \arrow[drr, bend left, "\xi"] \arrow[dr, dashed, "T\ulcorner \mathrm{id}_a \urcorner"] \arrow[dd, bend right, "Ta"'] & & \\
& & T(a/A) \arrow[r] \arrow[d, "T\partial_1"'] \arrow[dr, phantom, "\lrcorner\ \Swarrow"] & X \arrow[d, "a"] \\
A \arrow[rr, "iA"'] & & TA \arrow[r, "\alpha"'] & A
\end{tikzcd} \tag{3.12}$$

which, using the fact $i$ is cartesian at $a/A$, in turn equals

$$\begin{tikzcd}
X \arrow[rr, "iX"] \arrow[dr, dashed, "\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, bend right, "a"'] & & TX \arrow[drr, bend left, "\xi"] \arrow[dr, dashed, "T\ulcorner \mathrm{id}_a \urcorner"] & & \\
& a/A \arrow[rr, "i(a/A)"] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & T(a/A) \arrow[r] \arrow[d, "T\partial_1"'] \arrow[dr, phantom, "\lrcorner\ \Swarrow"] & X \arrow[d, "a"] \\
& A \arrow[rr, "iA"'] & & TA \arrow[r, "\alpha"'] & A
\end{tikzcd} \tag{3.13}$$

Pasting the comma and pullback (Theorem 2.1), and observing that $iX \circ \xi = \mathrm{id}_X$, we obtain

$$
\begin{tikzcd}
X \arrow[drr, equal, bend left] \arrow[dr, dashed, "\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, bend right, "a"'] & & \\
& a/A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \Swarrow"] & X \arrow[d, "a"] \\
& A \arrow[r, equal] & A
\end{tikzcd}
\tag{3.14}
$$

which is the identity of $a$ by definition. In fact, at this point we note this also constrains $\overline{a}$ to be as defined above, thereby showing uniqueness.

We move on to associativity:

$$
\begin{tikzcd}
T^2X \arrow[r, "mX"] \arrow[d, "T^2a"'] & TX \arrow[r, "\xi"] \arrow[d, dashed, "Ta"] \arrow[dr, phantom, "\overset{\overline{a}}{\Swarrow}"] & X \arrow[d, "a"] \\
T^2A \arrow[r, "mA"'] & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
=
\begin{tikzcd}
T^2X \arrow[r, "TX"] \arrow[d, "T^2a"'] \arrow[dr, phantom, "\overset{T\overline{a}}{\Swarrow}"] & TX \arrow[r, "\xi"] \arrow[d, dashed, "Ta"] \arrow[dr, phantom, "\overset{\overline{a}}{\Swarrow}"] & X \arrow[d, "a"] \\
T^2A \arrow[r, "T\alpha"'] & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\tag{3.15}
$$

Again, we unravel the definition of $\overline{a}$ and see the left-hand side equals

$$
\begin{tikzcd}
T^2X \arrow[rr, "mX"] \arrow[ddr, bend right, "T^2a"'] & & TX \arrow[drr, bend left, "\xi"] \arrow[dr, dashed, "T\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, bend right, "Ta"'] & & \\
& & & T(a/A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \Swarrow"] & X \arrow[d, "a"] \\
& T^2A \arrow[rr, "mA"'] & & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\tag{3.16}
$$

which, using the fact $m$ is cartesian at $a/A$, in turn equals

$$
\begin{tikzcd}
T^2X \arrow[rr, "mX"] \arrow[dr, dashed, "T^2\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, bend right, "T^2a"'] & & TX \arrow[drr, bend left, "\xi"] \arrow[dr, dashed, "T\ulcorner \mathrm{id}_a \urcorner"] \arrow[ddr, bend right, "Ta"'] & & \\
& T^2(a/A) \arrow[rr, "m(a/A)"] \arrow[d, dashed, "T^2\partial_1"] \arrow[r, phantom, "\lrcorner"] & & T(a/A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \Swarrow"] & X \arrow[d, "a"] \\
& T^2A \arrow[rr, "mA"'] & & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\tag{3.17}
$$

We then recognize the image under $T$ of the square we assumed to be comma in the left part of the diagram:

$$
\begin{tikzcd}
T^2X \arrow[rr, "T\xi"] \arrow[dr, "T^2\ulcorner \mathrm{id}_a \urcorner"] \arrow[dddr, bend right, "T^2a"'] & & TX \arrow[dr, "T\ulcorner \mathrm{id}_a \urcorner"] \arrow[drr, bend left, "\xi"] \arrow[dd, "Ta"] & & \\
& T^2(a/A) \arrow[ur, dashed, "T\xi"] \arrow[rr, dashed, "T(\xi T\partial_0)"] \arrow[dd, "T^2\partial_1"'] \arrow[dr, phantom, "\Swarrow T\chi"] & & T(a/A) \arrow[r] \arrow[dd, "T\partial_1"] & X \arrow[dd, "a"] \arrow[ddl, phantom, "\Swarrow \chi"] \\
& & TA \arrow[dr, equal] & & \\
& T^2A \arrow[ur, "T\alpha"] \arrow[rr, "mA"'] & & TA \arrow[r, "\alpha"'] & A
\end{tikzcd}
\tag{3.18}
$$

Thus whiskering $T\chi$ by $T\ulcorner \mathrm{id}_a \urcorner$ yields $T\overline{a}$, concluding the proof. $\square$

**Corollary 3.14.** *The same holds when the universal comparison map of Diagram 3.7 into the comma square is a split equivalence.*

Thus, while Definition 3.3 defined *$T$-cartesianity at a $T$-costructured object*, Theorem 3.13 defines *$T$-cartesianity at an (unstructured) object*, as desired. We record this definition:

**Definition 3.15.** Let $T$ be a 2-monad cartesian at the representable discrete opfibrations. Given a $T$-algebra $(A, \alpha)$ and a (plain) morphism $a : X \to A$, we say $\alpha$ is **strictly cartesian at** $a$ when Diagram 3.7 is a comma, and **cartesian** when the universal comparison of the latter into the comma square is a split equivalence.

And we have:

**Corollary 3.16.** *$(A, \alpha)$ is (strictly) $T$-cartesian at the (unstructured) object $a$ if and only if it is (strictly) cartesian at the (uniquely) $T$-costructured object $a$.*

**Example 3.17.** Let $T$ be the free symmetric (strict) monoidal category 2-monad. One can verify that indeed this 2-monad is cartesian at all functors, as claimed e.g. in Leinster (2004). Now suppose a $T$-algebra $A$ is cartesian (in the sense of Definition 3.15) at all its *global* objects (i.e. functors $a : 1 \to A$). By Theorem 3.13, this means (1) every object is a commutative comonoid in a unique way and (2) every morphism is a map of comonoids.[3] By Fox's Theorem (Fox (1976)), then $A$ must be a cartesian monoidal category.

**Remark 3.18.** This shows that Theorem 3.13 is a "converse Fox's theorem", i.e. cartesianity at all global objects implies all of them are equipped with a unique $T$-costructure which is furthermore preserved by all maps. To show the latter statement, notice that the above proof also shows that maps of unstructured objects satisfy the conditions for being 2-morphisms of 2-algebras.

[3]Note this is automatic given the way we defined cartesianity.

# 4 Lifting 2-classifiers to 2-algebras

In this section we formulate the main definitions and results of the paper. We start by observing that the completeness properties of plumbuses are stable under forming enhanced 2-categories of 2-algebras and lax morphisms:

**Proposition 4.1.** *When $\mathcal{K}$ is a plumbus (Definition 2.19) and $(T, i, m)$ an enhanced 2-monad, the enhanced 2-category $\mathcal{Alg}_{\mathrm{lx}}(T)$ is a plumbus too.*

*Proof.* The definition of $\mathcal{Alg}_{\mathrm{lx}}(T)$ has been given in Definition 2.6. The tight terminal object is evidently inherited, so let's consider left-tight pullbacks of tight discrete opfibrations. Recall Proposition 2.16, where we have shown that a tight discrete opfibration in $\mathcal{Alg}_{\mathrm{lx}}(T)$ is also a tight discrete opfibration in $\mathcal{K}$. By the dual of Proposition 2.18, we only need to show that, for $p : E \to B$ a discrete opfibration and $b : A \to B$ a lax $T$-morphism $b^*p : E_b \to A$ detects tightness, i.e. for $e : X \to E_b$, if $b^*p(e)$ is tight then so is $e$. In fact, it suffices to check strictness. But $b^*p(e)$ strict means $p(\overline{e}) = \mathrm{id}_{p(e)}$, and since $p$ is discrete we must have had $\overline{e} = \mathrm{id}_e$ to begin with. □

**Remark 4.2.** Observe how the existence of left-tight pullbacks relies crucially on both the discreteness and opfibrancy properties of tight discrete opfibrations.

For the rest of the section, fix a representable plumbus $\mathcal{K}$ (Definition 2.22), having 2-classifier $u : \Omega_\bullet \to \Omega$. The main goal of this section is to give conditions on an enhanced 2-monad $T$ on $\mathcal{K}$ such that $\mathcal{Alg}_{\mathrm{lx}}(T)$, which by Proposition 4.1 above is also a plumbus, is still representable (i.e. admits a 2-classifier) or even well-representable (i.e. admits a good 2-classifier).

We do so in two steps.

First, we prove in Theorem 4.8 that this is true as soon as $\Omega$ supports a sufficiently nice $T$-algebra structure and $T$ preserves small discrete opfibrations—these conditions comprise Assumption 4.4.

We then note that, under that assumption, $T$ is a particularly nice 2-monad we dub *opfibrantly cartesian*. In Section 4.3 we briefly study opfibrantly cartesian 2-monads and conclude by noting being opfibrantly cartesian comes just shy of being equivalent to Assumption 4.4, essentially for strictness reasons. In Section 4.4, we therefore show how a classical strictification technique can be adapted to opfibrantly cartesian 2-monads, so as to close the circle: an opfibrantly cartesian 2-monad may be replaced by one satisfying the stricter Assumption 4.4, at least when the ambient $\mathscr{F}$-category satisfies suitable completeness and exactness conditions.

## 4.1 The lifting theorem

Assume, for the moment, that $T$ preserves small discrete opfibrations. Under this hypothesis, one can ponder what happens to $u$: specifically, it will yield a new small discrete opfibration $Tu$, whose classifying map has the intriguing signature dashed below:

$$\begin{tikzcd} T\Omega_\bullet \arrow[r, squiggly, "\omega_\bullet"] \arrow[d, "Tu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, "u"] \\ T\Omega \arrow[r, squiggly, dashed, "\omega"'] & \Omega \end{tikzcd} \tag{4.1}$$

It would be then reasonable to believe that if $\omega$ were a tight map and a lawful $T$-algebra, we would have lifted $\Omega$ to $\mathcal{Alg}_{\mathrm{lx}}(T)$ (provided we can somehow obtain the same for $\omega_\bullet$) The only remaining obstacle is constructing a laxator for $\mathrm{d}p$: to do that, it turns out one need to know $T$ preserves pullbacks of small discrete opfibrations.

However, note that:

**Lemma 4.3.** *$T$ preserves small discrete opfibrations and their pullbacks if and only if it preserves pullbacks of $u$ and $Tu$ is a small discrete opfibration.*

*Proof.* We only need to prove the converse direction. Thus let $\omega = \mathrm{d}Tu$ be the map classifying $Tu$, as seen in Diagram 4.1. Consider a small discrete opfibration $p : E \to B$, we have:

$$\begin{tikzcd} TE \arrow[r, squiggly] \arrow[d, "Tp"'] & T\Omega_\bullet \arrow[r, squiggly] \arrow[d, "Tu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, "u"] \\ TB \arrow[r, squiggly, "T\mathrm{d}p"'] & T\Omega \arrow[r, squiggly, "\omega"'] & \Omega \end{tikzcd} = \begin{tikzcd} TE \arrow[rr, squiggly] \arrow[d, "Tp"'] & & \Omega_\bullet \arrow[d, "u"] \\ TB \arrow[r, squiggly, "T\mathrm{d}p"'] & T\Omega \arrow[r, squiggly, "\omega"'] & \Omega \end{tikzcd} \tag{4.2}$$

By pasting for pullbacks, the square above right is a pullback if and only if the left half of the square above left is a pullback, and this holds by assumption. Thus $Tp$ is a small discrete opfibration classified by $\omega(T\mathrm{d}p)$. $\square$

We thus arrive to:

**Assumption 4.4.** $\mathcal{K}$ is a representable plumbus with 2-classifier $u : \Omega_\bullet \to \Omega$ and $(T, i, m)$ an enhanced 2-monad, such that

A. $Tu$ is a small discrete opfibration,
B. $T$ preserves pullbacks of $u$,
C. $i$ and $m$ are cartesian at $u$,
D. $Tu$ is perfect,
E. $(\Omega, \omega)$ is a strict $T$-algebra.

**Remark 4.5.** Recall from Remark 2.27 that $Tu$ being perfect means that its classifying map—thus $\omega$—is tight.

**Lemma 4.6.** *Assumption 4.4 entails that (1) $\omega_\bullet : T\Omega_\bullet \to \Omega_\bullet$ is a (tight) strict $T$-algebra structure and (2) $u$ is a strictly cartesian $T$-morphism.*

*Proof.* Once (1) is proven, (2) holds simply by definition (Definition 3.1) since the relevant diagram is precisely Diagram 4.1.

First, note that since tight discrete opfibrations detect tightness (Proposition 2.20), and since $u\omega_\bullet = \omega T u$ is tight, we get $\omega_\bullet$ is tight. To see it is also a lawful strict $T$-algebra, consider the following diagram:

$$\begin{tikzcd}
\Omega_\bullet \arrow[r, "u"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner" very near start] & \Omega \arrow[d, "i"] \arrow[dd, bend left, equal] \\
T\Omega_\bullet \arrow[r, "Tu"] \arrow[d, "\omega_\bullet"'] \arrow[dr, phantom, "\lrcorner" very near start] & T\Omega \arrow[d, "\omega"] \\
\Omega_\bullet \arrow[r, "u"'] & \Omega
\end{tikzcd} \tag{4.3}$$

Since the right triangle commutes and the composite of the two squares in the middle is a pullback, then $\omega_\bullet i = u^* \mathrm{id}_\Omega = \mathrm{id}_{\Omega_\bullet}$. Associativity for $\omega_\bullet$ is proven analogously. □

Combined with Lemma 4.3, we have:

**Corollary 4.7.** *Under Assumption 4.4,* $\mathrm{d}Tp \cong \omega T\mathrm{d}p$.

We are thus ready for:

**Theorem 4.8.** *Suppose Assumption 4.4 is met. Then $\mathcal{Alg}_{\mathrm{lx}}(T)$, where a tight discrete opfibration is* small *when its underlying opfibration is so in $\mathcal{K}$, is a representable plumbus, with the same 2-classifier $u : \Omega_\bullet \to \Omega$.*

*Proof.* By Proposition 4.1, we already know $\mathcal{Alg}_{\mathrm{lx}}(T)$ is a plumbus, and by Assumption 4.4 and Lemma 4.6 $u$ and $(\Omega, \omega)$ are valid members thereof.

Let $p : E \to B$ be a small discrete opfibration in $\mathcal{Alg}_{\mathrm{lx}}(T)$ (recall Proposition 2.16), and denote by $\eta$ and $\beta$, respectively, the algebra structures on $E$ and $B$. Consider the classifying map $\mathrm{d}p : B \to \Omega$ obtained in $\mathcal{K}$. By Proposition 2.9, if $p$ can be classified, it must be by a lax $T$-morphism carried by $\mathrm{d}p$.

Thus we only need to show $\mathrm{d}p$ admits a lax $T$-structure (see (1.1)–(1.3) in Blackwell et al. (1989)), which first of all amounts to a 2-cell:

$$\begin{tikzcd}
TB \arrow[r, squiggly, "T\mathrm{d}p"] \arrow[d, "\beta"'] & T\Omega \arrow[d, "\omega"] \arrow[dl, phantom, "\overline{\mathrm{d}p}"] \\
B \arrow[r, squiggly, "\mathrm{d}p"'] & \Omega
\end{tikzcd} \tag{4.4}$$

To see this, consider the above square as a 2-cell between classifying maps. Such 2-cells, by the structure of a 2-classifier, are in one-to-one correspondence with maps as dashed below, where the (non-dotted) 2-cells are classifying comma squares (squashed beyond recognition):

$$
\begin{tikzcd}
\beta^*E \arrow[ddr] \arrow[drrrr] \arrow[dr, phantom, "\lrcorner", very near start] & & T\mathrm{d}p^*T\Omega_\bullet \arrow[ll, dashed] \arrow[ddl] \arrow[drr, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
& & & & \Omega_\bullet \arrow[dd, "u"] \\
& TB \arrow[rr, squiggly, "T\mathrm{d}p"] \arrow[dr, "\beta"'] & & T\Omega \arrow[dr, "\omega"] \arrow[dl, phantom, "\overline{\mathrm{d}p}"] & \\
& & B \arrow[rr, squiggly, "\mathrm{d}p"'] & & \Omega
\end{tikzcd}
\tag{4.5}
$$

By Corollary 4.7, we have shown that $(T\mathrm{d}p)^*Tu \cong Tp$. Thus we get $\overline{\mathrm{d}p}$ by exhibiting a map $TE \to \beta^*p$ over $TB$, and indeed there is a unique such map, namely the $T$-cartesianity defect of $p$ (Definition 3.1):

$$
\begin{tikzcd}
TE \arrow[dr, dashed, "\delta_p"] \arrow[drr, bend left, "\eta"] \arrow[ddr, bend right, "Tp"'] & & \\
& \beta^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
& TB \arrow[r, "\beta"'] & B
\end{tikzcd}
\tag{4.6}
$$

Therefore we are left with proving $\overline{\mathrm{d}p}$ satisfies the axioms of a laxator.

We start with the unit axiom, which amounts to the following equation:

$$
\begin{tikzcd}
& & T\Omega \arrow[dr, "\omega"] & \\
B \arrow[r, "iB"] & TB \arrow[ur, squiggly, "T\mathrm{d}p"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "\overline{\mathrm{d}p}\Downarrow"] & & \Omega \\
& & B \arrow[ur, squiggly, "\mathrm{d}p"'] &
\end{tikzcd}
=
\begin{tikzcd}
B \arrow[r, squiggly, bend left, "\mathrm{d}p"] \arrow[r, squiggly, bend right, "\mathrm{d}p"'] \arrow[r, phantom, "\|"] & \Omega
\end{tikzcd}
\tag{4.7}
$$

By pasting a naturality square for $i$, the left hand side reduces to

$$
\begin{tikzcd}
& \Omega \arrow[r, "i\Omega"] & T\Omega \arrow[dr, "\omega"] & \\
B \arrow[ur, squiggly, "\mathrm{d}p"] \arrow[r, "iB"'] & TB \arrow[ur, squiggly, "T\mathrm{d}p"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "\overline{\mathrm{d}p}\Downarrow"] & & \Omega \\
& & B \arrow[ur, squiggly, "\mathrm{d}p"'] &
\end{tikzcd}
=
\begin{tikzcd}
& \Omega \arrow[dr, equal] & \\
B \arrow[ur, squiggly, "\mathrm{d}p"] \arrow[dr, equal] \arrow[rr, phantom, "="] & & \Omega \\
& B \arrow[ur, squiggly, "\mathrm{d}p"'] &
\end{tikzcd}
\tag{4.8}
$$

So we are left with the task of showing that $\overline{\mathrm{d}p}iB$ is an identity. We use the classifier to turn this 2-cell into the map of discrete opfibrations $iB^*\delta_p$:

$$
\begin{tikzcd}
iB^*TE \arrow[rr, "p_2"] \arrow[dd, "p"'] \arrow[dr, "iB^*\delta_p"] \arrow[drr, phantom, "\lrcorner", very near start] & & TE \arrow[dd, "Tp" near end] \arrow[dr, "\delta_p"] & \\
& iB^*\beta^*E \arrow[rr] \arrow[dl, "p"] \arrow[drr, phantom, "\lrcorner", very near start] & & \beta^*E \arrow[dl, "\beta^*p"] \\
B \arrow[rr, "iB"'] & & TB &
\end{tikzcd}
\tag{4.9}
$$

We simplify this by noting $iB^*\beta^*p \cong (\beta iB)^*p \cong \mathrm{id}^*p = p$, so that $iB^*\delta_p$ is really a map $p \to p$. We also observe that in fact $p_2 \cong iE$ since $i$ is cartesian at $Tp$ and thus the pullback of $iB$ along $Tp$ is a naturality square for $i$. We are left with

$$\begin{tikzcd}
E \arrow[rr, "iE"] \arrow[dr, dashed, "iB^*\delta_p"] \arrow[dd, "p"'] & & TE \arrow[dd, "Tp" near end] \arrow[dr, dashed, "\delta_p"] & \\
& E \arrow[rr] \arrow[dl, "p"] \arrow[dr, phantom, "\lrcorner" very near start] & & \beta^*E \arrow[dl, "\beta^*p"] \\
B \arrow[rr, "iB"] & & TB &
\end{tikzcd} \tag{4.10}$$

We perform the following pasting

$$\begin{tikzcd}
E \arrow[rr, "iE"] \arrow[dr, dashed, "iB^*\delta_p"] \arrow[dd, "p"'] & & TE \arrow[dd, "Tp" near end] \arrow[dr, dashed, "\delta_p"] \arrow[rr, "\eta"] & & E \arrow[dd, "p"] \\
& E \arrow[rr] \arrow[dl, "p"] \arrow[dr, phantom, "\lrcorner" very near start] & & \beta^*E \arrow[dl, "\beta^*p"] \arrow[ur] & \\
B \arrow[rr, "iB"] & & TB \arrow[rr, "\beta"] & & B
\end{tikzcd} \tag{4.11}$$

and observe the front face is the identity square of $p$, thus getting $iB^*\delta_p = \eta iE = \mathrm{id}_E$.
Now on to the multiplication axiom:

$$\begin{tikzcd}
& & T\Omega \arrow[dr, "\omega"] & \\
T^2B \arrow[r, "mB"] & TB \arrow[ur, squiggly, "T\mathrm{dp}"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "\overset{\|}{\overline{\mathrm{d}p}}\,\Downarrow"] & & \Omega \\
& & B \arrow[ur, squiggly, "\mathrm{d}p"'] &
\end{tikzcd}
=
\begin{tikzcd}
& TB \arrow[r, "T\omega"] \arrow[d, phantom, "\Downarrow T\overline{\mathrm{d}p}"] & T\Omega \arrow[dr, "\omega"] & \\
T^2B \arrow[ur, squiggly, "T^2\mathrm{d}p"] \arrow[r, "T\beta"'] & TB \arrow[ur, squiggly, "T\mathrm{d}p"] \arrow[dr, "\beta"'] \arrow[rr, phantom, "\overset{\|}{\overline{\mathrm{d}p}}\,\Downarrow"] & & \Omega \\
& & B \arrow[ur, squiggly, "\mathrm{d}p"'] &
\end{tikzcd} \tag{4.12}$$

Again, we prove this equation of 2-cells by proving the corresponding equation between the map of discrete opfibrations they classify.
The first of such maps is

$$\begin{tikzcd}
T^2E \arrow[rr, "mE"] \arrow[dr, dashed, "mB^*\delta_p"] \arrow[dd, "p"'] \arrow[drr, phantom, "\lrcorner" very near start] & & TE \arrow[dd, "Tp" near end] \arrow[dr, dashed, "\delta_p"] & \\
& mB^*\beta^*E \arrow[rr] \arrow[dl, "p"] \arrow[dr, phantom, "\lrcorner" very near start] & & \beta^*E \arrow[dl, "\beta^*p"] \\
T^2B \arrow[rr, "mB"] & & TB &
\end{tikzcd} \tag{4.13}$$

where we used the fact $m$ is cartesian at $p$ to replace the back pullback square with the naturality square of $m$ at $p$.
The second such map is (note the front and back squares below coincide with the ones above):

$$
\begin{tikzcd}
T^2E \arrow[dr, "T\delta_p"] \arrow[ddd, bend right, "T^2p"'] \arrow[rrr, bend left, "mE"] & & & & \\
& T\beta^*TE \arrow[rr] \arrow[dr, "T\beta^*\delta_p"] \arrow[dd, "T\beta^*Tp"] & & TE \arrow[dr, "\delta_p"] \arrow[dd, "Tp"] & \\
& & T\beta^*\beta^*E \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \beta^*E \arrow[dl, "\beta^*p"] \\
& T^2B \arrow[rr, "T\beta"] & & TB &
\end{tikzcd}
\tag{4.14}
$$

By Corollary 4.7, $\omega(T\overline{\mathrm{d}p})$ classifies $T\delta_p$. Then by unicity of the comparison map induced by a pullback (which defines $mB^*\delta_p$), we get the desired equation. This is better shown in the diagram below:

$$
\begin{tikzcd}
T^2E \arrow[dr, "T\delta_p"] \arrow[ddr, bend left, "mB^*\delta_p"] \arrow[rrr, bend left, "mE"] \arrow[ddd, bend right, "T^2p"'] & & & & & \\
& T\beta^*TE \arrow[dr, "T\beta^*\delta_p"] & & TE \arrow[drr, "\eta"] & & \\
& & mB^*\beta^*E \arrow[rr] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & & \beta^*E \arrow[r] & E \arrow[dl, "p"] \\
& T^2B \arrow[rrr, "\beta(T\beta)=\beta(mB)"] & & & B &
\end{tikzcd}
\tag{4.15}
$$

$\square$

We record some immediate consequences of this result, starting from a converse which shows the hypotheses we work with are rather sharp:

**Corollary 4.9** (Converse of Theorem 4.8)**.** *Suppose $T$ preserves small discrete opfibrations, and that $\mathcal{Alg}_{\mathrm{lx}}(T)$ is a representable plumbus, then $T$ satisfies Assumption 4.4.*

*Proof.* Since $T$ preserves small discrete opfibrations, by Proposition 2.16 we know $u : \Omega_\bullet \to \Omega$ must be isomorphic to the 2-classifier of $\mathcal{K}$. We may therefore use this $u$ in Assumption 4.4. $\square$

**Corollary 4.10.** *Let $p : E \to B$ be a small discrete opfibration in $\mathcal{Alg}_{\mathrm{lx}}(T)$. Its classifying map $\mathrm{d}p : B \to \Omega$ is pseudo if and only if $p$ is $T$-cartesian (Definition 3.1).*

*Proof.* It follows by inspection of the proof of the main theorem: the laxator of $\mathrm{d}p$ classifies the strict $T$-cartesianity defect of $p$. $\square$

**Remark 4.11.** Note that to have $\mathrm{d}p$ be *strict* instead is much harder: it should be the case that the composite

$$(T\mathrm{d}p)^*Tu \xrightarrow{\sim} TE \xrightarrow{\delta_p} \beta^*E \tag{4.16}$$

is the identity, which can only happen for $p = u$. As a corollary, we get that $\mathcal{A}lg_{\mathrm{lx}}(T)$ is never well-representable, since $\mathrm{id}_1$ could never be classified by a strict $T$-morphism.

In particular, a representable small discrete opfibration (Definition 2.13) $b/B \xrightarrow{\partial_1} B$ is $T$-cartesian precisely when $B$ is $T$-cartesian at its representing object $b : X \to B$, by definition (Definition 3.3). Thus:

**Corollary 4.12.** *The representable copresheaf $B(b, -) : B \to \Omega$ (Notation 2.24) associated to a representable small discrete opfibration $b/B \xrightarrow{\partial_1} B$ is always a lax $T$-morphism, and it is pseudo precisely when $B$ is $T$-cartesian at $b$.*

**Example 4.13.** A most notable example is any (Parè) representable (§2.1 in (Paré 2011)) for a spanish double category (Example 3.8). This observation plays a major role in the compositionality theorem of (Myers 2021) (Theorem 5.3.3.1 there).

**Remark 4.14** (Decategorification)**.** Theorem 4.8 should decategorify to a theorem about 1-topoi, at least informally. For instance, it is easy to see that a monad for which $Tu$ is a mono (where $u$ is the subobject classifier) induces a $T$-algebra structure on $\Omega$, namely the one given by

$$t(\varphi_1, \ldots, \varphi_n) \ \mapsto\ \top \Leftrightarrow \left(\bigwedge_i \varphi_i\right). \tag{4.17}$$

A result in this direction, involving classifiers for a restricted class of subobjects of $T$-algebras, appears in (Aristote 2024).

## 4.2 Lifting good 2-classifiers to 2-algebras

Suppose now $\mathcal{K}$ is well-representable. Under Assumption 4.4, and by the equivalence of classification explained in Diagram 2.30, we get that there is a comma square:

$$\begin{tikzcd} T\Omega_\bullet \arrow[r] \arrow[d, "Tu"'] \arrow[dr, phantom, "\lrcorner\ \swarrow"] & 1 \arrow[d, "\tau"] \\ T\Omega \arrow[r, "\omega"'] & \Omega \end{tikzcd} \tag{4.18}$$

Speaking informally for the moment, note that the above means that the algebra structure $\omega : T\Omega \to \Omega$ must be given by a sort of limit construction: the 'elements' of $\omega(t(\vec{X}))$ (i.e. maps $\tau \Rightarrow \omega(t(\vec{X}))$) for a $T$-term $t(\vec{X})$ are given by terms $t(\vec{x})$ of 'elements' $x_i : \tau \Rightarrow X_i$.

Moreover, we have:

**Lemma 4.15.** *$\tau$ is a pseudo-$T$-morphism and $(\Omega, \omega)$ is cartesian at $\tau$.*

*Proof.* By a direct application of Theorem 3.13 we get that $\tau$ is colax: indeed $T$ satisfies the necessary assumptions and Diagram 3.7 instantiates to the comma square classifying $Tu$, i.e. Diagram 4.1. Cartesianity is precisely Corollary 3.16.

Moreover, since $\tau$ is right adjoint to $\Omega \to 1$ (Proposition 2.28), and the latter is a strong $T$-morphism (as it is evidently strict), then by doctrinal adjunction (Kelly 1974) $\tau$ must be strong as well. □

Thus the only obstruction to have $u$ be a good 2-classifier in $\mathcal{Alg}_{\mathrm{lx}}(T)$ too is the strictness, rather than the invertibility, of $\overline{\tau}$. For that to be true, we would need $\tau^* u$, to *equal* $\mathrm{id}_1$, which is almost never the case.

Thus we conclude the property of being 'good' does not easily lift to $\mathcal{Alg}_{\mathrm{lx}}(T)$.

## 4.3 Opfibrantly cartesian 2-monads

One might ask whether Assumption 4.4 is *implied* by any other properties of $T$ which don't explicitly refer to $\Omega$. This is essentially the case, except for some subtleties around strictification. In the remainder of this section, we will investigate the relationship between *opfibrantly cartesian* 2-monads and Assumption 4.4.

**Definition 4.16** (Opfibrantly cartesian transformation)**.** An **opfibrantly cartesian transformation** $\alpha : F \Rightarrow G$ between enhanced 2-functors is one cartesian (Definition 3.11) at all small discrete opfibrations.

**Definition 4.17** (Opfibrantly cartesian enhanced 2-monad)**.** An enhanced 2-monad $(T, i, m)$ is **opfibrantly cartesian** when

A'. $T$ preserves small discrete opfibrations,
B'. $T$ preserves pullbacks of small discrete opfibrations,
C'. $i$ and $m$ are opfibrantly cartesian.
D. $Tu$ is perfect.

**Lemma 4.18.** *If $(T, i, m)$ is opfibrantly cartesian, then*

*A. $Tu$ is a small discrete opfibration,*
*B. $T$ preserves pullbacks of $u$,*
*C. $i$ and $m$ are cartesian at $u$,*

Vice versa*, if (A)–(C) above are met, so are (A')–(C') from Assumption 4.4.*

*Proof.* For (A) and (B), this was proven in Lemma 4.3. As for (C), this is an application of the pasting lemma for pullbacks, as illustrated below for $i$ (and $m$ is analogous):

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, two heads, "p"'] & TE \arrow[r, squiggly] \arrow[d, two heads, "Tp"] \arrow[dr, phantom, "\lrcorner", very near start] & T\Omega_\bullet \arrow[r, "\omega_\bullet"] \arrow[d, two heads, "Tu"] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, two heads, "u"] \\ B \arrow[r, "i"'] & TB \arrow[r, squiggly, "T\mathrm{d}p"'] & T\Omega \arrow[r, "\omega"'] & \Omega \end{tikzcd} \quad = \quad \begin{tikzcd} E \arrow[r] \arrow[d, two heads, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, two heads, "u"] \\ B \arrow[r, "\mathrm{d}p"'] & \Omega \end{tikzcd} \tag{4.19}$$

□

**Corollary 4.19.** *When Assumption 4.4 is met, $T$ is opfibrantly cartesian.*

**Remark 4.20** (Relationship between opfibrantly cartesian and cartesian 2-monads)**.** Opfibrantly cartesian 2-monads are both slightly weaker and slightly stronger than the naïve generalization of cartesian monads to enhanced 2-categories. Indeed, any enhanced 2-monad which preserves left-tight pullbacks *and arrow objects*, and for which the unit and multiplication transformations are cartesian is opfibrantly cartesian. This is because discrete opfibrations $p : E \to B$ may be characterized by the following square being a pullback:

$$\begin{tikzcd} E^{\downarrow} \arrow[r, "\partial_0"] \arrow[d, "p^{\downarrow}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ B^{\downarrow} \arrow[r, "\partial_0"'] & B \end{tikzcd} \tag{4.20}$$

where the horizontal maps project out the source of an arrow.

**Remark 4.21.** Note property (D), the fact $Tu$ is perfect (Remark 2.27), does not imply that $T$ preserve perfect small discrete opfibrations, i.e. that if $\mathrm{d}p$ is tight so it $\mathrm{d}Tp$: in fact, we only know $\mathrm{d}Tp \cong \omega(T\mathrm{d}p)$, and tightness is not a replete property (e.g. in $\mathscr{Alg}_{\mathrm{lx}}(T)$, if a 1-cell is isomorphic to a strict $T$-morphism we can only conclude it is pseudo).

One might wonder about a converse to Corollary 4.19: does an opfibrantly cartesian monad satisfy Assumption 4.4? The answer is *almost yes*, except for subtleties concerning strictness:

**Lemma 4.22.** *For $(T, i, m)$ an opfibrantly cartesian 2-monad, the maps $\omega : T\Omega \to \Omega$ and $\omega_\bullet : T\Omega_\bullet \to \Omega_\bullet$ defined in Diagram 4.1 are canonically pseudo-$T$-algebras. Moreover, the 2-classifier $u : \Omega_\bullet \to \Omega$ is a strictly cartesian $T$-morphism.*

*Proof.* We apply the pasting property of pullbacks:

$$\begin{tikzcd} \Omega_\bullet \arrow[r, "i"] \arrow[d, two heads, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & T\Omega_\bullet \arrow[r, "\omega_\bullet"] \arrow[d, two heads, "Tu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, two heads, "u"] \\ \Omega \arrow[r, "i"'] & T\Omega \arrow[r, "\omega"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} \Omega_\bullet \arrow[r, equal] \arrow[d, two heads, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, two heads, "u"] \\ \Omega \arrow[r, equal] & \Omega \end{tikzcd} \tag{4.21}$$

$$\begin{tikzcd} T^2\Omega_\bullet \arrow[r, "m"] \arrow[d, two heads, "T^2u"'] \arrow[dr, phantom, "\lrcorner", very near start] & T\Omega_\bullet \arrow[r, "\omega_\bullet"] \arrow[d, two heads, "Tu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, two heads, "u"] \\ T^2\Omega \arrow[r, "m"'] & T\Omega \arrow[r, "\omega"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} T^2\Omega_\bullet \arrow[r, "m"] \arrow[d, two heads, "T^2u"'] \arrow[dr, phantom, "\lrcorner", very near start] & T\Omega_\bullet \arrow[r, "\omega_\bullet"] \arrow[d, two heads, "Tu"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, two heads, "u"] \\ T^2\Omega \arrow[r, "m"'] & T\Omega \arrow[r, "\omega"'] & \Omega \end{tikzcd} \tag{4.22}$$

Both sides of each line exhibit a classification of the same small discrete opfibration, and thus we can conclude that their bottom sides are uniquely isomorphic. Then, by lifting such an isomorphism along $u$, we get the pseudo-$T$-algebra structure on $(\Omega_\bullet, \omega_\bullet)$.

Finally, $u$ is cartesian by definition of $\omega$ and $\omega_\bullet$ (Diagram 4.1). □

Thus Lemma 4.22 fails short of being a converse to Corollary 4.19 since, in general, $(\Omega, \omega)$ is only proven to be a *pseudo*-$T$-algebra. We close this gap by appealing to general strictification results (specifically, the replacement of $T$ by its pseudoalgebra coclassifier), and show they are compatible with the opfibrantly cartesian properties on $T$ when $\mathcal{K}$ is sufficiently exact: this is the topic of the next section, Section 4.4. Moreover, we conjecture this unfortunate mismatch could disappear if we were to work with pseudoalgebras (even of pseudomonads) throughout, Koudenburg's result below (Theorem A.6) gives us hope.

## 4.4 Replacing an opfibrantly cartesian 2-monad by its pseudoalgebra coclassifier

We have almost shown that an opfibrantly cartesian 2-monad $T$ satisfies Assumption 4.4; but we ended up with a pseudo-algebra, not a strict algebra. In this section, we investigate a way to rectify this mismatch.

A central theme in 2-algebra is that of strictification, and a central technique to that end is replacing 2-monads by their *pseudoalgebra coclassifiers* (see Lack, and §5 and §7 in Lack (2009)). Explicitly, one replaces a 2-monad $T$ with another 2-monad $T'$ which is universal with the property that strict $T'$-algebras are equivalent to pseudo-$T$-algebras.

This is indeed a sort of strictification, since it allows to replace a pseudoalgebra with a strict one, albeit of a different 2-monad. This is useful since strict algebras are easier to work with, but many structures of interest (pseudomonoids, pseudocategories, etc.) often arise as pseudoalgebras. We are no stranger to this phenomenon—as we have seen in Lemma 4.22, opfibrantly cartesian 2-monads naturally induce a *pseudoalgebra* structure on the 2-classifier of a representable plumbus, rather than a strict algebra as our setup would require.

In this section, we show that if $T'$ is a (normal) pseudoalgebra coclassifier for an opfibrantly cartesian (Definition 4.17) enhanced 2-monad $T$, then $T'$ is also opfibrantly cartesian—at least, if the underlying plumbus $\mathcal{K}$ has enough exactness properties. We will then be able to conclude that $(\Omega, \omega)$, which is a pseudo-$T$-algebra by Lemma 4.22, is a strict-$T'$-algebra, bridging the gap with Assumption 4.4.

We begin by showing a straightforward closure property of opfibrantly cartesian 2-monads. In the remainder of this section, we will take for granted that all mentioned discrete opfibrations are small.

Let us fix enhanced 2-monads $(T', i', m')$, $(T, i, m)$. We note that natural transformations of enhanced functors are componentwise tight.

**Lemma 4.23** (Closure of opfibrantly cartesian 2-monads)**.** *Let $q : T' \to T$ be a opfibrantly cartesian transformation ([Definition 4.16]) between enhanced 2-monads. If $T$ is opfibrantly cartesian ([Definition 4.17]), then so is $T'$.*

*Proof.* This follows straightforwardly by pullback pasting.

First, we note that $T'$ preserves pullbacks of tight discrete opfibrations by pullback pasting:

$$\begin{tikzcd} T'E' \arrow[r, "T'\overline{f}"] \arrow[d, "T'p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & T'E \arrow[r, "q"] \arrow[d, "T'p"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tp"] \\ T'B' \arrow[r, "T'f"'] & T'B \arrow[r, "q"'] & TB \end{tikzcd} = \begin{tikzcd} T'E' \arrow[r, "q"] \arrow[d, "T'p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE' \arrow[r, "T\overline{f}"] \arrow[d, "Tp'"] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tp"] \\ T'B' \arrow[r, "q"'] & TB' \arrow[r, "Tf"'] & TB \end{tikzcd} \tag{4.23}$$

Furthermore, $T'$ preserves discrete fibrations because pullbacks of discrete fibrations are discrete fibrations and $T'p$ is the pullback of $Tp$ along $q : T'B \to TB$ by the above lemma.

It therefore remains to show that $i'$ and $m'$ are cartesian along tight discrete opfibrations. These follow by the fact that $q$ is a strict 2-monad morphism and by the above lemma:

$$\begin{tikzcd} E \arrow[r, "i'"] \arrow[d, "p"'] & T'X \arrow[r, "q"] \arrow[d, "T'p"'] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "Tp"] \\ B \arrow[r, "i'"'] & T'Y \arrow[r, "q"'] & TY \end{tikzcd} = \begin{tikzcd} E \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, "Tp"] \\ B \arrow[r, "i"'] & TY \end{tikzcd} \tag{4.24}$$

$$\begin{tikzcd} T'T'E \arrow[r, "m'"] \arrow[d, "T'T'p"'] & T'E \arrow[r, "q"] \arrow[d, "T'p"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tp"] \\ T'T'B \arrow[r, "m'"'] & T'B \arrow[r, "q"'] & TB \end{tikzcd} = \begin{tikzcd} T'T'E \arrow[r, "qq"] \arrow[d, "T'T'p"'] \arrow[dr, phantom, "\lrcorner", very near start] & TTE \arrow[r, "m"] \arrow[d, "TTp"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tp"] \\ T'T'B \arrow[r, "qq"'] & TTB \arrow[r, "m"'] & TB \end{tikzcd} \tag{4.25}$$

Finally, we note that $T'u$ is classified by $\mathrm{d}(Tu) \circ q$ which is a composite of tight maps and therefore tight; so $T'u$ is perfect. $\square$

**Corollary 4.24.** *Let $T : \mathcal{K} \to \mathcal{K}$ be a opfibrantly cartesian monad. Suppose that $q : T' \to T$ is a strict monad morphism and $s : T \to T'$ is a pseudomonad morphism with $qs = \mathrm{id}$ and $\varepsilon : sq \cong \mathrm{id}$ as 2-monad morphisms. If furthermore $T'$ preserves tight discrete opfibrations, then $T'$ is also opfibrantly cartesian.*

*Proof.* By [Lemma 4.23], it suffices to show that $q : T' \to T$ is opfibrantly cartesian. To that end, suppose that $p : E \to B$ is a tight discrete opfibration; then so is $Tp$ since $T$ is opfibrantly cartesian. We will directly show that the following square is a pullback:

$$\begin{tikzcd} T'E \arrow[r, "q"] \arrow[d, "T'p"'] \arrow[dr, phantom, "\lrcorner", very near start] & TE \arrow[d, "Tp"] \\ T'B \arrow[r, "q"'] & TB \end{tikzcd} \tag{4.26}$$

Suppose we have $b : Z \to T'B$ and $e : Z \to TE$ with $qb = (Tp)e$. We then have $se : Z \to T'E$ with $qse = e : Z \to TE$ and $(T'p)se = s(Tp)e = sqb \xrightarrow{\varepsilon} b$. Since by assumption $T'p$ is a discrete opfibration, we have a lift $\overline{\varepsilon} : se \cong b' : Z \to T'E$ with $(T'p)\overline{\varepsilon} = \varepsilon$. In particular, $(T'p)b' = b$; furthermore, $qb' = e$ since $(Tp)q\overline{\varepsilon} = q(T'p)\overline{\varepsilon} = q\varepsilon$ is an identity and $Tp$ is discrete.

It remains to show that any $e' : Z \to T'E'$ with $qe' = e$ and $(T'p)e' = b$ must equal $b'$. Since $qe' = e$, we have that $e' \xrightarrow{\varepsilon^{-1}e'} sqe' = se$ and therefore $\overline{\varepsilon}(\varepsilon^{-1}e') : e' \cong b'$. Since $(T'p)(\overline{\varepsilon}(\varepsilon^{-1}e')) = \mathrm{id}_b$, by discreteness of $T'p$ we conclude that $e' = b'$.

The 2-categorical aspect of the universal property may be checked similarly. □

It therefore suffices to show that the normal pseudoalgebra coclassifier $T'$ preserves tight discrete opfibrations whenever $T$ does. This will not always be true, but it will be true when $\mathcal{K}$ has sufficient exactness properties. Suppose now that $\mathcal{K}$ is locally finitely presentable. Recall that if $T$ is finitary, then $T'$ is the (iso-)codescent object of the truncated simplicial diagram

$$\begin{tikzcd} F^3T \arrow[r, shift left=2, "F^2\alpha"] \arrow[r, "FmT" description] \arrow[r, shift right=2, "mFT"'] & F^2T \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "mT"'] & FT \end{tikzcd} \tag{4.27}$$

where $F$ is the free finitary monad functor on pointed finitary endofunctors of $\mathcal{K}$ and $\alpha : FT \to T$ the algebra map pertaining $T$. It is therefore clear that $T'$ will preserve discrete opfibrations under the following conditions:

1. Whenever $X$ is an opfibrantly cartesian endofunctor (meaning that it preserves discrete opfibrations and their pullbacks) with an opfibrantly cartesian point $\gamma : \mathrm{id} \Rightarrow X$, then $FX$ is an opfibrantly cartesian endofunctor.
2. Whenever $\gamma : X \Rightarrow Y$ is an opfibrantly cartesian natural transformation (meaning that it's naturality squares along discrete opfibrations are pullbacks) between pointed opfibrantly cartesian endofunctors (preserving the point), then $F\gamma$ is opfibrantly cartesian.
3. If $T$ is an opfibrantly cartesian 2-monad, then its algebra map $\alpha$ is opfibrantly cartesian.
4. Codescent objects of truncated simplicial diagrams of discrete opfibrations in $\mathcal{K}$ are discrete opfibrations when the top maps in the diagrams induce pullback squares between the discrete opfibrations.

The first three facts can be seen as showing that $F$ descends to the category of opfibrantly cartesian pointed finitary endofunctors and opfibrantly cartesian transformations. Applying these facts in order, we find that for any discrete opfibration $p$ in $\mathcal{K}$, applying the above truncated simplicial diagram $F^{(-)}T$ of functors to it yields a truncated simplicial diagram of discrete opfibrations whose top squares are pullbacks. The last condition then guarantees that their codescent object, which is $T'p$ is a discrete opfibration.

We will prove these conditions using more primitive exactness conditions in $\mathcal{K}$. In the prescence of a tight discrete opfibration classifier, the exactness conditions we need for (1)–(3) will hold for free; we therefore only need to specially assume the fourth.

**Definition 4.25** (Descendable plumbus)**.** A (well-)representable plumbus (Definition 2.19) $\mathcal{K}$ is **descendable** when

1. $\mathcal{K}$ is locally finitely presentable as an $\mathscr{F}$-category.
2. Filtered colimits of *small* discrete opfibrations are small.
3. In a truncated simplicial diagram of small discrete opfibrations

$$
\begin{tikzcd} p_2 \arrow[r, squiggly, shift left=2, "d_2"] \arrow[r, squiggly] \arrow[r, squiggly, shift right=2] & p_1 \arrow[r, squiggly, shift left, "d_1"] \arrow[r, squiggly, shift right] & p_0 \end{tikzcd} \tag{4.28}
$$

if $d_1$ and $d_2$ are pullback squares, then the (pseudo-)codescent object is a tight discrete opfibration.

$\mathcal{C}at$ is a descendable plumbus (by the lemmas in Section B); therefore, so are all presheaf 2-categories (considered as chordate $\mathscr{F}$-categories). We begin by proving two descent lemmas for descendable plumbuses.

**Lemma 4.26** (Descent in a descendable plumbus)**.** *Let $\mathcal{K}$ be a descendable plumbus. Then tight discrete opfibrations in $\mathcal{K}$ satisfy* descent *along filtered (conical) colimits. Explicitly, if $p : E \Rightarrow B : \mathcal{D} \to \mathcal{K}$ is a cartesian transformation of filtered diagrams which is componentwise a tight discrete opfibration, and $p^{\triangleright} : E^{\triangleright} \Rightarrow B^{\triangleright}$ is an extension of $\gamma$ to cones under the diagrams $E$ and $B$ for which $B^{\triangleright}$ is a colimit cone, then the following are equivalent:*

1. *The cone $E^{\triangleright}$ is a colimit cone.*
2. *$p^{\triangleright}$ is cartesian and its apex is a tight discrete opfibration.*

*If, on the other hand, $B^{\triangleright}$ is not a colimit cone but $\gamma^{\triangleright}$ is cartesian and componentwise a tight discrete opfibration, then the induced square from the colimit of $\gamma$ to the apex of $\gamma^{\triangleright}$ is a pullback.*

*Proof.* First, we show that (1) implies (2). We have a (solid) commuting diagram of (2-)categories

$$
\begin{tikzcd} \mathcal{K} \downarrow \operatorname{colim} B \arrow[r, "\mathrm{cone}^*"'] & \lim(\mathcal{K} \downarrow B) \arrow[l, dashed, bend right, "\mathrm{colim}"'] \\ \mathit{tdOpfib}(\operatorname{colim} B) \arrow[u, hook] \arrow[r, "\mathrm{cone}^*"] & \lim \mathit{tdOpfib}(B) \arrow[u, hook] \\ \mathcal{K}(\operatorname{colim} B, \Omega) \arrow[u] \arrow[r, "\mathrm{cone}^*"'] & \lim \mathcal{K}(B, \Omega) \arrow[u] \end{tikzcd} \tag{4.29}
$$

in which the bottom vertical functors are, by hypothesis, equivalences. The bottom map is an isomorphism by the universal property of the colimit, and therefore the middle horizontal map is an equivalence. The top horizontal map, given by pullback along the colimiting cone under $B$, has a left adjoint given by colimit. The mate of the top square therefore gives a canonical comparison map between the $\operatorname{colim} E$ and the discrete opfibration over $\operatorname{colim} B$

corresponding to the family of discrete fibrations $\gamma : E \Rightarrow B$. Since the inclusion of discrete opfibrations into the slice is fully faithful, this comparison is an isomorphism if and only if $\operatorname{colim}\gamma : \operatorname{colim} E \to \operatorname{colim} B$ is also a discrete opfibration. By fully-faithfulness, it then follows that the entire cone under $\gamma$ is cartesian.

Since being a tight discrete opfibration is a finite limit condition, and since filtered colimits commute with finite limits in locally finitely presentable (enriched) categories (), filtered colimits of tight discrete opfibrations are tight discrete opfibrations. Therefore the apex $\operatorname{colim}\gamma : \operatorname{colim} E \to \operatorname{colim} B$ of $\gamma^{\triangleright}$ is a tight discrete opfibration.

Conversely, suppose if (2) holds, then the apex of $\gamma^{\triangleright}$ is the discrete opfibration corresponding to $\gamma$ under the equivalence cone*. Since by assumption this apex is a tight discrete opfibration, the comparison map described above is an isomorphism, showing that the the cone $E^{\triangleright}$ is a colimit.

As for the second claim, it suffices to show that the pullback of $p^{\text{apex}}$ along the comparison $\text{univ} : \operatorname{colim} B \to B^{\text{apex}}$ is $\operatorname{colim} E$. We have a commuting diagram

$$\begin{tikzcd} & \mathit{tdOpfib}(B^{\text{apex}}) \arrow[dl, "\text{univ}^*"'] \arrow[dr, "\text{cone}^*"] & \\ \mathit{tdOpfib}(\operatorname{colim} B) \arrow[rr, "\text{cone}^*"'] & & \lim \mathit{tdOpfib}(B) \end{tikzcd} \tag{4.30}$$

and therefore $\text{univ}^* p^{\text{apex}}$ sits at the apex of a cone satisfying condition (2) above; we therefore find that $\text{univ}^* p^{\text{apex}} \cong \operatorname{colim} E$ is the colimit, and therefore that the induced square

$$\begin{tikzcd} \operatorname{colim} E \arrow[r] \arrow[d, "\operatorname{colim}\gamma"'] & E^{\text{apex}} \arrow[d, "\gamma^{\text{apex}}"] \\ \operatorname{colim} B \arrow[r] & B^{\text{apex}} \end{tikzcd} \tag{4.31}$$

□

**Lemma 4.27.** *Let $\mathcal{K}$ be a descendable plumbus. Let $p : E \Rightarrow B : \mathcal{D} \to \mathcal{K}$ be a cartesian transformation of filtered diagrams which is componentwise a tight discrete opfibration, and $p^{\triangleright} : E^{\triangleright} \Rightarrow B^{\triangleright}$ be an extension of $\gamma$ to cones under the diagrams $E$ and $B$.*

*If $\gamma^{\triangleright}$ is cartesian and componentwise a tight discrete opfibration, then the induced square from the colimit of $\gamma$ to the apex of $\gamma^{\triangleright}$ is a pullback.*

*Proof.* It suffices to show that the pullback of $p^{\text{apex}}$ along the comparison $\text{univ} : \operatorname{colim} B \to B^{\text{apex}}$ is $\operatorname{colim} E$. We have a commuting diagram

$$\begin{tikzcd} & \mathit{tdOpfib}(B^{\text{apex}}) \arrow[dl, "\text{univ}^*"'] \arrow[dr, "\text{cone}^*"] & \\ \mathit{tdOpfib}(\operatorname{colim} B) \arrow[rr, "\text{cone}^*"'] & & \lim \mathit{tdOpfib}(B) \end{tikzcd} \tag{4.32}$$

and therefore $\text{univ}^* p^{\text{apex}}$ sits at the apex of a cone satisfying condition (2) above; we therefore find that $\text{univ}^* p^{\text{apex}} \cong \operatorname{colim} E$ is the colimit, and therefore that the induced square

$$\begin{tikzcd}
\operatorname{colim} E \arrow[r] \arrow[d, "\operatorname{colim} \gamma"'] & E^{\text{apex}} \arrow[d, "\gamma^{\text{apex}}"] \\
\operatorname{colim} B \arrow[r] & B^{\text{apex}}
\end{tikzcd} \tag{4.33}$$

□

If we do not, for some reason, want to assume that $\mathcal{K}$ has a tight discrete opfibration classifier, then the following lemmas go through with the assumption of descent for tight discrete opfibrations along filtered colimits.

**Remark 4.28** (Descendable plumbuses of algebras)**.** Unfortunately, it is unlikely that $\mathcal{Alg}_{\mathrm{lx}}(T)$ will be descendable even when $\mathcal{K}$ is a descendable plumbus and $T : \mathcal{K} \to \mathcal{K}$ is accessible, opfibrantly cartesian, and preserves (split) codescent objects. This is simply because it is unlikely to be locally finitely presentable. We do not know quite which colimits $\mathcal{Alg}_{\mathrm{lx}}(T)$ will have.

However, we do note that $\mathcal{Alg}_{\mathrm{st}}(T)$ will be locally finitely presentable in the above situation, and that our descent arguments can be adapted to *perfect* discrete opfibrations (Remark 2.27) in this case (since they are precisely those tight discrete opfibrations classified by strict algebra morphisms). It may be that careful attention to relationship between colimits in $\mathcal{Alg}_{\mathrm{st}}(T)$ and $\mathcal{Alg}_{\mathrm{lx}}(T)$ could reveal a way to formulate the results here so that they may apply to $\mathcal{Alg}_{\mathrm{lx}}(T)$.

For this reason, though we formulate the results here in terms of enhanced 2-categories, we encourage the reader to think of $\mathcal{K}$ as being chordate.

We now prove the above lemmas. In the following, $\mathcal{K}$ is always a descendable plumbus.

**Lemma 4.29.** *Suppose that $X : \mathcal{K} \to \mathcal{K}$ is a finitary, opfibrantly cartesian endofunctor with an opfibrantly cartesian point $p : \mathrm{id} \Rightarrow X$. If $F$ is the free monad monad on pointed, finitary endofunctors of $\mathcal{K}$, then $FX$ is opfibrantly cartesian and its point is opfibrantly cartesian.*

*Proof.* We note that $F$ may be constructed as a filtered colimit Lack (2008). Therefore, if $X$ preserves tight discrete opfibrations, so does $FX$. Because finite limits commute with filtered colimits, $FX$ also preserves pullbacks of tight discrete opfibrations, meaning that $FX$ is opfibrantly cartesian.

The point $p : \mathrm{id} \Rightarrow FX$ factors via a cone $p' : \mathrm{id} \Rightarrow D$ over the filtered diagram $D$ of which $FX$ is the colimit via $\gamma : D \to FX$, consisting of whiskers of the point of $X$ by $X$, and perhaps also the injections of filtered colimit cones. Each morphism in $D$ becomes a pullback when applied to a tight discrete opfibration by descent and the assumption that $X$ and its point were opfibrantly cartesian. Also by descent, $\gamma$ consists of pullback square when applied to a tight discrete opfibration. Therefore, the composite $p$ consists of pullback squares when applied to a tight discrete opfibration; it is opfibrantly cartesian. □

**Lemma 4.30.** *Suppose that $\gamma : X \Rightarrow Y : \mathcal{K} \to \mathcal{K}$ is a pointed opfibrantly cartesian transformation between pointed, $\kappa$-accessible, opfibrantly cartesian endofunctors. Then $F\gamma : FX \Rightarrow FY$ is opfibrantly cartesian.*

*Proof.* When applied to a tight discrete opfibration, $F\gamma$ becomes a filtered colimit of pullback squares; it is therefore a pullback square. □

**Lemma 4.31.** *Let $T$ be a finitary, opfibrantly cartesian 2-monad on $\mathcal{K}$. Then its algebra structure $\alpha : FT \Rightarrow T$ is opfibrantly cartesian.*

*Proof.* This follows by the second descent lemma (Lemma 4.27). We note that $FT$ is the colimit of a filtered diagram $D$ built from composites of $T$ and its unit $\eta$. Let's $D^{\triangleright}$ be the cone over $D$ with apex is $T$ which induces $\alpha$ by the universal property of $FT$ as $\operatorname{colim} D$. Let $\pi : E \to B$ be a tight discrete fibration. Then $D^{\triangleright}(\pi)$ is a cartesian transformation $D^{\triangleright}(E) \Rightarrow D^{\triangleright}(B)$ whose components are discrete opfibrations. By the descent lemma, the universal factorization of $D^{\triangleright}(\pi)$ through $\operatorname{colim} D(\pi) = (FT)(\pi)$ is a pullback; but this is the component of $\alpha$ at $\pi$. Therefore $\alpha$ is opfibrantly cartesian. □

Putting these together, we prove the main theorem of this section.

**Theorem 4.32** (Normal pseudoalgebra coclassifier of opfibrantly cartesian 2-monad is opfibrantly cartesian)**.** *Let $\mathcal{K}$ be a descendable plumbus and $T : \mathcal{K} \to \mathcal{K}$ be a finitary, opfibrantly cartesian 2-monad. Then the normal pseudoalgebra coclassifier $T'$ of $T$ is also opfibrantly cartesian.*

Since the free monad on an (unpointed) endofunctor $X$ is $F(\mathrm{id} + X)$, we may deduce an analogue of the above result for (non-normal) pseudo-algebras when binary coproducts of tight discrete opfibrations are tight in $\mathcal{K}$.

**Theorem 4.33** (Pseudoalgebra coclassifier of opfibrantly cartesian 2-monad is opfibrantly cartesian)**.** *Let $\mathcal{K}$ be a descendable plumbus in which binary coproducts of tight discrete opfibrations are tight discrete opfibrations and $T : \mathcal{K} \to \mathcal{K}$ be a finitary, opfibrantly cartesian 2-monad. Then the pseudoalgebra coclassifier $T'$ of $T$ is also opfibrantly cartesian.*

Therefore in this situation we can prove a converse of Corollary 4.19:

**Proposition 4.34.** *Let $\mathcal{K}$ be a well-representable descendable plumbus. Suppose, moreover, that $T$ is opfibrantly cartesian and admits a (normal) pseudoalgebra coclassifier $T'$. Then Assumption 4.4 is met for $T'$, specifically $\Omega$ admits a canonical strict-$T'$-algebra structure $\omega' : T'\Omega \to \Omega$, $u$ is a strict $T'$-morphism, and $(\Omega, \omega')$ is strictly cartesian at $u$.*

*Proof.* Most of Assumption 4.4 is handled by Theorem 4.32 or Theorem 4.33. By Lemma 4.22, $\omega : T\Omega \to \Omega$ and $\omega_\bullet : T\Omega_\bullet \to \Omega_\bullet$ are pseudo-algebras and $u : \Omega_\bullet \to \Omega$ is a strict cartesian map between them. Therefore, $u$ is a strict cartesian map between the associated *strict* algebras $\omega' : T'\Omega \to \Omega$ and $\omega'_\bullet : T'\Omega_\bullet \to \Omega_\bullet$ corresponding to $\omega$ and $\omega_\bullet$ respectively. Cartesianness follows by appealing to the fact that $q : T' \to T$ is opfibrantly cartesian, as we showed in the proof of Corollary 4.24. □

We note that if $T$ is furthermore *flexible* (i.e. $q : T' \to T$ is a split equivalence with *strict* inverse), then we may simply replace the pseudo-algebra structure $\omega : T\Omega \to \Omega$ by a strict $\omega' : T\Omega \to \Omega$ by transporting across this split equivalence. In other words, for flexible $T$ we may choose the algebra structure on $\Omega$ so that it is strict, causing $T$ to satisfy Assumption 4.4.

# 5 Examples and applications

In this section, we will deploy our main theorem (Theorem 4.8) to construct a number of examples of strict discrete opfibration classifiers in 2-categories of algebras and lax morphisms. We will begin with an example that emphasizes the difference between strict discrete opfibration classifiers amongst lax morphisms and discrete opfibration classifiers amongst strict morphisms.

**Example 5.1** (Presheaves with lax transformations)**.** Consider a category $C$, with set of objects $i : C_0 \hookrightarrow C$. It is a classical observation (see e.g. Example 6.6 Blackwell et al. (1989)) that $\mathcal{Alg}_{\mathrm{lx}}(T)$, where $T$ is the 'free indexed category' 2-monad $T : X \mapsto \sum_{c \in C} C(c, -) \times X(c)$ induced by the 2-adjunction $\mathrm{lan}_i \dashv i^*$, is the 2-category $[C, \mathcal{Cat}]_{\mathrm{lx}}$ of $C$-indexed categories, lax natural transformation between them, and modifications.

Notice that $T$ satisfies the conditions of Assumption 4.4, since everything is checked pointwise. So our Theorem 4.8 equips $[C, \mathcal{Cat}]_{\mathrm{lx}}$ with a 2-classifier, namely $1 \xrightarrow{1} \underline{\mathrm{Set}}$ which picks a (fixed) singleton $1 \in \mathrm{Set}$ at each $c \in C_0$ (we are denoting with $\underline{X}$ the indexed category constant at $X \in \mathcal{Cat}$). This is very different from the 2-classifier described by Mesiti in Mesiti (2024), which we described in Construction 2.32!

There is no contradiction though, since Mesiti's 2-classifier is for the 2-category $[C, \mathcal{Cat}]_{\mathrm{st}}$ of indexed categories with *strict* natural transformations between them. Classification therein is significantly different, since, as we have seen above, classifying maps are in general not strict $T$-morphisms. Indeed, one can see that a strict $T$-morphism $B \to \underline{\mathrm{Set}}$ necessarily classifies a trivial opfibration, i.e. ones of the form $\pi_X : \underline{E} \times B \to B$ for a fixed category $E$.

**Example 5.2** (Symmetric monoidal categories)**.** Rather famously, symmetric monoidal categories are the *strict* algebras for a 2-monad $T$ on the 2-category $\mathcal{Cat}$ of categories. Explicitly, if $\mathrm{Fin}^{\cong}$ is the (skeletal) groupoid of (standard) finite sets and bijections and $i : \mathrm{Fin}^{\cong} \hookrightarrow \mathcal{Cat}$ is the inclusion of the finite discrete categories, then $TC$ may be defined as the bijective-on-objects/fully-faithful factor of the inclusion $\mathsf{Tree}_{0,2}(C) \to \mathrm{Fin}^{\cong}/^{\mathrm{cx}}C$ from the free pointed magma of nullary-or-binary trees with leaves labeled in C to the colax slice of the finite discrete categories and bijections over $C$, sending a tree to its set of leaves.

As a corollary of the *monoidal Grothendieck construction* Moeller (2020), lax symmetric monoidal functors $C \to \mathrm{Set}$ correspond to *strict* monoidal discrete opfibrations $E \to C$. Let's see how this follows from Theorem 4.8.

It is straightforward to show that $T$ is a cartesian 2-monad and that it preserves arrow objects. (This also follows because it is familial, see Example 5.3) As a result, it is opfibrantly cartesian. We may now proceed in one of two ways:

1. We may note that Set admits a symmetric monoidal structure of cartesian product and therefore an algebra structure $\omega : T\mathrm{Set} \to \mathrm{Set}$. We can then check that $\omega$ is cartesian at $\tau : 1 \to \mathrm{Set}$ (which is the inclusion of the terminal object), i.e. that the following square is a pullback:

$$\begin{tikzcd} T(\tau/\mathrm{Set}) \arrow[r] \arrow[d, "\tau/\omega"'] \arrow[dr, phantom, "\lrcorner", very near start] & T\mathrm{Set} \arrow[d, "\omega"] \\ \tau/\mathrm{Set} \arrow[r] & \mathrm{Set} \end{tikzcd} \tag{5.1}$$

   This square being a pullback simply means that a point in the cartesian product $p \in \prod_{i\in I} X_i$ corresponds to a family of points $(p_i \in X_i)_{i\in I}$; this is of course the universal property of the product.
2. By Lemma 4.22, we may *compute* a $T$-algebra structure $\omega : T\mathrm{Set} \to \mathrm{Set}$ on the discrete opfibration classifier $\mathrm{Set} \in \mathcal{C}at$ as the classifier of $Tu : T(\tau/\mathrm{Set}) \to T(\mathrm{Set})$ where $u : \tau/\mathrm{Set} \to \mathrm{Set}$ is the 2-classifier. The fiber of $Tu$ over $X : I \to \mathrm{Set}$ in $T(\mathrm{Set})$ is the lifts of $X$ to $X' : I \to \tau/\mathrm{Set}$, or in other words the families $(x_i \in X_i)_{i\in I}$ of elements. We therefore see that $\omega(X) = \prod_{i\in I} X_i$ computes the cartesian product.

The first approach is useful when we have a good guess at what the appropriate algebra structure on the discrete opfibration classifier should be; the second is useful when we don't.

In any case, Theorem 4.8 now confirms to us that the category of sets, together with its symmetric monoidal structure of cartesian product, classifies strict discrete opfibrations of symmetric monoidal categories $p : E \to C$ via lax symmetric monoidal functors $\mathrm{d}p : C \to \mathrm{Set}$.

**Example 5.3** (Familial 2-monads)**.** We may extend Example 5.2 by noting that the free symmetric monoidal category monad $T : \mathcal{C}at \to \mathcal{C}at$ is a *polynomial 2-monad* in the sense of Weber (2007a), represented by the polynomial

$$* \longleftarrow \mathrm{Fin}_*^{\cong} \longrightarrow \mathrm{Fin}^{\cong} \longrightarrow * \tag{5.2}$$

Let $T : \mathcal{C}at/I \to \mathcal{C}at/I$ be a polynomial 2-monad represented by a polynomial

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I \tag{5.3}$$

Any polynomial 2-monad is cartesian. If furthermore $I$ is discrete and $p$ is a split opfibration, then $T$ is familial by Theorem 4.4.5 of Weber (2015) and therefore preserves discrete opfibrations by Theorem 6.2 of Weber (2007a). Any such polynomial monad is therefore opfibrantly cartesian.

For any $I$, $\mathcal{C}at/I$ has discrete opfibration classifier given by $I \times \mathrm{Set} \to I$ (by Theorem 4.6 of Weber (2007b)). We may therefore conclude by Theorem 4.8 that $I \times \mathrm{Set}$ admits the structure of a $T$-algebra which then classifies strict discrete opfibrations in $\mathcal{A}lg_{\mathrm{lx}}(T)$. Explicitly, $\omega : T(I \times \mathrm{Set}) \to I \times \mathrm{Set}$ classifies $Tu$ where $u : I \times \mathrm{Set}_* \to I \times \mathrm{Set}$ is the 2-classifier.

**Example 5.4** (Strict factorization systems)**.** The walking arrow $\Delta[1]$ is, like any category, a comonoid with respect to the cartesian product. Therefore, the 2-functor $X \mapsto X^{\Delta[1]} : \mathcal{C}at \to \mathcal{C}at$ is a 2-monad; as observed in Korostenski and Tholen (1993), its strict algebras are the strict factorization systems. This is a fun example of a 2-monad which is opfibrantly cartesian but not cartesian (despite preserving pullbacks).

It is easy to see that $(-)^{\Delta[1]}$ is opfibrantly cartesian. As a right adjoint $(-)^{\Delta[1]}$ preserves discrete opfibrations and pullbacks. For a map $p : E \to B$, a naturality square for $i$ is a pullback just when $p$ reflects identities and the a naturality square for $m$ is a pullback when $p$ has unique lifting of factorizations—discrete opfibrations satisfy both these properties.

The algebra structure on Set classifies the discrete opfibration $u^{\Delta[1]} : \mathrm{Set}_*^{\Delta[1]} \to \mathrm{Set}^{\Delta[1]}$ whose fibers are (isomorphically) given by the elements of the domain of an arrow in Set. That is, the algebra structure is $\mathrm{dom} : \mathrm{Set}^{\Delta[1]} \to \mathrm{Set}$, or in other words the strict factorization system $(\mathrm{id}, \mathrm{all})$.

Therefore, $(\mathrm{Set}, (\mathrm{id}, \mathrm{all}))$ classifies strict discrete opfibrations of strict factorization systems in the 2-category of strict factorization systems and *lax* morphisms.

We now treat the case of double categories by using the results from Section 4.4:

**Example 5.5** (Double categories)**.** Let $\mathrm{Graph} = \mathrm{Graph}(\mathcal{C}at)$ be the 2-category $2\mathcal{C}at(\bullet \rightrightarrows \bullet, \mathcal{C}at)$ of graphs of categories. In a graph $\mathbb{D} = \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$ we refer to $\mathbb{D}_0$ as the category of objects and $\mathbb{D}_1$ as the category of edges.

The free (pseudo-)double category monad $T : \mathrm{Graph} \to \mathrm{Graph}$ may be defined by sending a graph $\mathbb{D} = \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$ to the graph $T\mathbb{D} = T_1\mathbb{D}_1 \rightrightarrows \mathbb{D}_0$ where $T_1\mathbb{D}_1$ is the category whose objects are bracketed paths

$$(\to (\to\to)(\to (\to\to\to) \to (\to\to)) \to) \tag{5.4}$$

in the graph $\mathbb{D}$ where a morphism between the *unbracketed* paths $P_1 \to P_2$; that is, a path of morphisms in $\mathbb{D}_1$. The unit includes each edge as a singleton path, and the composition flattens a bracketing-of-bracketings. Since the underlying edges are not touched in this construction, the resulting 2-monad $T$ is cartesian and preserves arrow objects; it is therefore opfibrantly cartesian.

We computed the discrete opfibration classifier of Graph in Example 2.33—it is the graph of spans of sets $\mathbb{S}\mathrm{pan}(\mathrm{Set}) = \mathrm{Set}^{\ulcorner} \rightrightarrows \mathrm{Set}$. This has a $T$-algebra structure given by the usual composition of spans; more explicitly, we send a bracketed path to the iterated pullback described by that particular bracketing. This algebra structure is cartesian at the inclusion of the terminal object $\tau : 1 \to \mathbb{S}\mathrm{pan}(\mathrm{Set})$ by the universal property of these iterated pullbacks. We therefore conclude that $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ is the tight discrete opfibration classifier in the 2-category of double categories and lax functors.

We could also deduce that double category structure of $\mathbb{S}\mathrm{pan}(\mathrm{Set})$ *must* be induced by pullback from the arguments of Lemma 4.22: since the algebra structure of $\Omega$ must classify $Tu$, we have to send a bracketed path of spans to the span whose elements are bracketed paths of elements of the apexes of those spans.

We will now exemplify the *iterability* property of Theorem 4.8. Indeed, once established $\mathcal{Alg}_{\mathrm{lx}}(T)$ is a representable plumbus again, given a new $S$ opfibrantly cartesian over it—or equivalently, given $S$ distributing over $T$ and still opfibrantly cartesian *qua* 2-monad over $\mathcal{Alg}_{\mathrm{lx}}(T)$—we can conclude $\mathcal{Alg}_{\mathrm{lx}}(ST)$ is a representable plumbus too.

Note that, given a distributive law $\lambda : TS \Rightarrow ST$, where $T$ and $S$ are both opfibrantly cartesian 2-monads, then the resulting composite $ST$ is also opfibrantly cartesian as soon as $\lambda$ is an opfibrantly cartesian transformation. Equivalently, for the lift $S^T$ to still be opfibrantly cartesian on $\mathcal{Alg}_{\mathrm{lx}}(T)$ it suffices for $S$ to be opfibrantly cartesian on $\mathcal{K}$ and *preserving strictness for $T$-morphisms*, since all the other conditions depend on pullbacks, which are computed in $\mathcal{Alg}_{\mathrm{lx}}(T)$ just as in $\mathcal{K}$.

**Example 5.6** (Monoidal factorization systems)**.** Consider the 2-monad $S = (-)^{\Delta[1]}$ described in Example 5.4, and observe that $S$ lifts to $\mathcal{SymMon}_{\mathrm{lx}}(\mathcal{Cat})$, which is a representable plumbus by Example 5.2. Observe $S$ is still opfibrantly cartesian, since $S$ preserve the monoidal strictness of the functors it is applied to. Therefore we conclude monoidal factorization systems also admit a 2-classifier given by $1/\mathrm{Set} \xrightarrow{u} \mathrm{Set}$ equipped with the two, compatible, algebra structures we exhibited in Example 5.4 and Example 5.2.

**Example 5.7** (Duoidal categories)**.** A duoidal category is a pseudomonoid in $\mathcal{Mon}_{\mathrm{lx}}(\mathcal{Cat})$. The latter is strictly 2-monadic over $\mathcal{Cat}$ so subject to our theorem. It is thus a representable plumbus. To conclude duoidal categories and lax duoidal functors form a representable plumbus, we thus need to prove the (replacement of the) free monoidal category 2-monad $T$ preserves strict monoidality of monoidal functors, and this is again immediate to verify. The resulting 2-classifier is $1/\mathrm{Set} \xrightarrow{u} \mathrm{Set}$ where Set is equipped with the duoidal structure $(\times, \times)$.

**Example 5.8** (Symmetric monoidal structures)**.** Suppose that $\mathbb{T}$ is an $\mathscr{F}$-sketch (Definition 5.8 of Arkor et al. (2024)) which only marks pullback squares. In this case, which includes for example the sketch for pseudo-categories and other structures determined by Segal conditions, we may post-compose by the free symmetric monoidal category monad $S : \mathcal{Cat} \to \mathcal{Cat}$ to get a monad on the $\mathscr{F}$-category $\mathcal{Mod}_{\mathrm{lx}}(\mathbb{T}, \mathcal{Cat})$ of $\mathbb{T}$-models and lax morphisms, since $S$ is cartesian.

If furthermore $\mathcal{Mod}_{\mathrm{lx}}(\mathbb{T}, \mathcal{Cat})$ is equivalent to $\mathcal{Alg}_{\mathrm{lx}}(T)$ for an opfibrantly cartesian 2-monad $T : \mathcal{Mod}_{\mathrm{lx}}(\mathbb{T}_t, \mathcal{Cat}) \to \mathcal{Mod}_{\mathrm{lx}}(\mathbb{T}_t, \mathcal{Cat})$ on the tights, then $S$ gives a $\mathscr{F}$-monad on $\mathcal{Alg}_{\mathrm{lx}}(T)$.

If we are in a position to compute a tight discrete opfibration classifier of $\mathcal{Mod}_{\mathrm{lx}}(\mathbb{T}_t, \mathcal{C}at) = \mathcal{Mod}_s(\mathbb{T}_t, \mathcal{C}at)$ — for example, if it is merely a presheaf 2-category — then this tight discrete opfibration classifier lifts to models of $T$ and lax morphisms.

Double categories provide an example of this. In particular, the cartesian monoidal structure on $\mathbb{S}$pan(Set) classifies strict monoidal, strict double discrete opfibrations via lax monoidal, lax double functors.

**Remark 5.9** (Monadicity of $\mathscr{F}$-sketches)**.** In forthcoming work, Jason Brown and the second will provide conditions on a $\mathscr{F}$-sketch for its models to be monadic over the models of its tights; all these monads will be opfibrantly cartesian. For example, it suffices for the sketch to arise as the 2-category of corners of a crossed double category over its category of tights, where the crossed double category structure must interact well enough with the structure of the sketch. This occurs, for example, for $\mathscr{F}$-sketches constructed from *extendable algebraic patterns*.

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# Appendix A Comparison with Koudenburg's theorem

As anticipated in the introduction, Koudenburg's Theorem 8.1 in Koudenburg (2022) generalizes Theorem 4.8 in many ways, chiefly by abstracting away small discrete opfibrations in plumbuses to arbitrary loose arrows in an augmented virtual equipment.

In this section we compare our setting and Koudenburg's, and observe the second agrees with the first, albeit its generality makes it challenging to compare precisely.

**Warning A.1.** The theory of *augmented* virtual double categories and *Yoneda morphisms* has been developed in Koudenburg (2022) and later expounded in Koudenburg (2024). We invite the reader interested in the following to familiarize themselves with those notions there, since it would take too much space to re-introduce them here. We give pointers to Koudenburg (2022) where necessary.

However, knowledge of virtual double categories and equipments (in the sense of Cruttwell and Shulman (2010)) suffices to read most of what follows.

**Warning A.2.** In the setting of double categories *et similia* we adopt the tight/loose convention to refer to 1-cells. The reader is warned not to confuse this with the terminology for enhanced 2-categories we used in the above—we promise to avoid ambiguity.

The definition below instantiates Definition 4.2 in Koudenburg (2024) for $A = 1$:

**Definition A.3.** In an augmented virtual double category with tight terminal object 1, a **Yoneda morphism** (for said object 1) is a tight morphism $1 \xrightarrow{\tau} \Omega$ such that for every loose arrow $p : 1 \nrightarrow B$ there exists a unique $\mathrm{d}p : B \to \Omega$ making the conullary square below cartesian:

$$
\begin{tikzcd}
1 \arrow[rr, "p", "|" marking] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Downarrow\mathrm{cart}"] & B \arrow[dl, dashed, "\exists!\mathrm{d}p"] \\
& \Omega &
\end{tikzcd}
\tag{A1}
$$

A Yoneda morphism is **good** when $\tau$ is companiable.

Intuitively, the cartesianity of such a square amounts to a hom-wise isomorphism $p(b) \cong \Omega(\tau, \mathrm{d}p(b))$.

**Remark A.4.** The conullary 'square' appearing in Definition A.3 above is exactly what the *augmented* in *augmented virtual double category* affords. Note every virtual double category with units is augmented by taking conullary squares to be counary squares with bottom a unit. Augmentation is essentially a way to deal with issues of sizes, alternative to the more common one (employed, for instance, here) of keeping track of a class of 'small' (our *small*) arrows and objects. A exemplified discussion of this approach can be found in Koudenburg (2020).

In this setting, the augmented virtual double category of $w$-weak algebras and $w'$-weak morphisms (for $w, w' \in \{\text{lax, colax, pseudo}\}$) of an augmented virtual monad $T$, denoted $\mathbb{A}\mathrm{lg}^{w}_{w'}(T)$, has been thoroughly described by Koudenburg in §6 of Koudenburg (2022), along with numerous examples including monoidal and double categories with their respective notion of profunctor.

Given any well-representable plumbus $\mathcal{K}$, we would like to define an associated augmented virtual equipment $\mathbb{M}\mathrm{od}(\mathcal{K})$ of tight two-sided discrete fibrations—also called *modules*—as defined e.g. in Corollary 8.1.14 in Riehl and Verity (2022). However, we need to address size.

**Definition A.5.** Call **small** a discrete object $X \in \mathcal{K}$ for which $X \to 1$ is a small discrete opfibration. Now say $A \xleftarrow{q} E \xrightarrow{p} B$, a module in $\mathcal{K}$, is **small** when for each $a : X \to A$ with $X$ small, the leg $p_a$ in the simultaneous pullback below is a small discrete opfibration:

$$\begin{tikzcd} X \arrow[d, "a"'] & E_a \arrow[l, two heads, "q_a"'] \arrow[r, two heads, "p_a"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, equal] \\ A & E \arrow[l, two heads, "q"] \arrow[r, two heads, "p"'] & B \end{tikzcd} \tag{A2}$$

We define thus the **augmented virtual equipment of small modules** $\mathrm{p}\mathbb{M}\mathrm{od}(\mathcal{K})$ to be the evident restriction of $\mathbb{M}\mathrm{od}(\mathcal{K})$.

Clearly, $\mathrm{p}\mathbb{M}\mathrm{od}(\mathcal{K})$ contains the small discrete opfibrations over $B$ as the loose arrows $1 \nrightarrow B$, and moreover their pullback correspond to restriction (see Riehl and Verity (2022), Proposition 8.2.1):

$$\begin{tikzcd} E_b \arrow[r] \arrow[d, two heads, "p_b"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ X \arrow[r, "b"'] & B \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} 1 \arrow[r, equal] \arrow[d, "p_b"', "\shortmid" marking] \arrow[dr, phantom, "\overset{\mathrm{cart}}{\Longrightarrow}"] & 1 \arrow[d, "p", "\shortmid" marking] \\ X \arrow[r, "b"'] & B \end{tikzcd} \tag{A3}$$

Thus, by working in arbitrary augmented virtual double category with suitable restrictions, we generalize this situation.

Note the *2-classifier* is replaced by the loose $u = \Omega(\tau, 1)$, i.e. the companion of $\tau$, i.e. the loose point classified by $\mathrm{id}_\Omega$—if it exists, that is when $\tau$ is good. In turn, $u$ classifies loose points by cartesian squares:

$$\begin{tikzcd} 1 \arrow[rr, dashed, "u", "\shortmid" marking] \arrow[dr, "\tau"'] & {} \arrow[d, phantom, "\Downarrow\mathrm{cart}"] & \Omega \arrow[dl, equal] \\ & \Omega & \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "p", "\shortmid" marking] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathrm{cart}"] & B \arrow[d, dashed, "\exists!\mathrm{d}p"] \\ 1 \arrow[r, "u"', "\shortmid" marking] & \Omega \end{tikzcd} \tag{A4}$$

Moreover, the choice of tights giving the enhancement of $\mathcal{K}$ is roughly subsumed by companiability in $\mathrm{p}\mathbb{M}\mathrm{od}(\mathcal{K})$. Indeed, the companion of a tight morphism $f : A \to B$ is given by the representable module (see Riehl and Verity (2022), Example 8.2.3)

$$\begin{tikzcd} & f/B \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\ A & & B \end{tikzcd} \tag{A5}$$

Thus, as long as this module is small, $f$ will be companiable. Moreover, if we now have a monad $T$ on $\mathrm{p}\mathbb{M}\mathrm{od}(\mathcal{K})$, Lemma 7.4 in Koudenburg (2022) characterizes the companiable maps in $\mathbb{A}\mathrm{lg}^{w}_{w'}(T)$ as those w'-$T$-morphisms which are (1) companiable in $\mathrm{p}\mathbb{M}\mathrm{od}(\mathcal{K})$ (and thus tight in the plumbus) and (2) are in fact pseudo.

This makes the analogue of opfibrantly cartesian 2-monads (Definition 4.17) simply monads $(T, i, m)$ in augmented virtual double categories such that

1. the naturality squares of its structure maps are cartesian at all loose arrows $1 \to B$,
2. if $p : 1 \nrightarrow B$ is classified to a companiable map, so is $1 \overset{!^*}{\nrightarrow} T1 \overset{Tp}{\nrightarrow} TB$.

Indeed, the preservation properties of an opfibrantly cartesian 2-monad are subsumed by the fact that (1) by definition, an augmented virtual double monad sends loose arrows to loose arrows and (2) companions are preserved by any functor of augmented virtual double categories (this is Proposition 6.4 in Shulman (2008) and Corollary 5.5 in Koudenburg (2020)).

Cartesian $T$-morphisms (in the sense of Definition 3.1) are those whose conjoint exists in the equipment of algebras and it's given by a cartesian (in the sense of equipments) square (this makes more sense if you read the definition of loose arrow of algebras from Koudenburg (2022)), i.e.

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & TB \arrow[d, "\beta"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
TB \arrow[r, "TB(1{,}f)", "|"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\Downarrow \mathrm{cart}"] & TA \arrow[d, "\alpha"] \\
B \arrow[r, "B(1{,}f)"', "|"] & A
\end{tikzcd}
\tag{A6}
$$

Let's now state Koudenburg's theorem (Theorem 8.1 in Koudenburg (2022)) specialized for the case $A = 1$ and to target pseudo-$T$-algebra rather than colax ones.

**Theorem A.6** (Koudenburg's Yoneda Lifting Theorem)**.** *Let $T = (T, i, m)$ be a monad on an augmented virtual double category $\mathbb{K}$. Consider a good Yoneda morphism $\tau : 1 \to \Omega$ admitting the companion $u = \tau_*$ in $\mathbb{K}$. Letting $! : T1 \to 1$, assume that*

1. *the conjoint $!^*$ exists,*
2. *the right pointwise composite (Definition 3.17 in Koudenburg (2022)) $(!^* \odot Tu)$ exists,*
3. *the cell $m_J$ is pointwise left $(\tau!)$-exact (Definition 5.15 in Koudenburg (2022)) for each $J : 1 \nrightarrow B$,*
4. *the cell $m_{Tu}$ is pointwise left $(\tau! \circ m_1)$-exact, while $i_u$ is pointwise left $(\tau!)$-exact,*
5. *$T$ preserves any unary cartesian cell with $u$ as (loose) target,*
6. *the $T$-image of the cocartesian cell defining $(!^* \odot Tu)$ is left $(\tau!)$-exact.*

*Then the morphism $\omega := (!^* \odot Tu)^\lambda : T\Omega \to \Omega$ dashed below extends to a pseudo-$T$-algebra structure $(\Omega, \omega)$ on $\Omega$:*

$$\begin{tikzcd}
1 \arrow[r, "!^*", "|" marking] \arrow[dr, "\tau"'] & T1 \arrow[r, "Tu", "|" marking] \arrow[d, phantom, "\Downarrow cart"] & T\Omega \arrow[dl, dashed, "\omega"] \\
& \Omega &
\end{tikzcd} \tag{A7}$$

*With respect to this structure $\tau\colon 1 \to \Omega$ admits a pseudo-$T$-morphism structure that makes it into a good Yoneda embedding both in $\mathbb{A}\mathrm{lg}^{\mathrm{ps}}_{\mathrm{lx}}(T)$.*

Observe that, when instantiated in the augmented virtual equipment $\mathrm{p}\mathbb{M}\mathrm{od}\,(\mathcal{K})$ of small modules in a plumbus introduced above, the above yields a definition of $\omega$ which is substantially the same, i.e. the map classifying $Tu$.

As for the assumptions, finding an exhaustive mapping is challenging, but we believe there is a rough correspondence as follows.[4]

Assumption (1) is met since the augmented virtual equipment of modules admits such conjoints: $!^*$ is the terminal module:

$$\begin{tikzcd}
& T1 \arrow[dl, "!"'] \arrow[dr, equal] & \\
1 & & T1
\end{tikzcd} \tag{A8}$$

In the definition of representable plumbus (Definition 2.22), we ask for $\mathrm{id}_1$ to be small and since $T$ preserves smallness, the above is a valid loose arrow in $\mathrm{p}\mathbb{M}\mathrm{od}\,(\mathcal{K})$. Likewise, (2) is trivially met since the required composite exists by triviality of $!^*$, As for (3) and (4), they assert that $i$ and $m$ satisfy certain pasting properties that correspond to the pasting properties afforded by our cruder hypothesis that $i$ and $m$ be cartesian at tight discrete opfibrations. Then, in light of the above observation regarding restrictions in $\mathrm{p}\mathbb{M}\mathrm{od}\,(\mathcal{K})$, (5) implies $T$ preserves pullbacks of the 2-classifier $u$. Similarly, (6) can be taken to correspond to the requirement $T$ preserves pullbacks of the 2-classifier, which is a consequence of Assumption 4.4 as proven in Lemma 4.3.

## Appendix B  Lemmata on descent in $\mathcal{C}at$

By *(1-)fork* in a 2-category we mean a pair of parallel 1-morphisms therein, and by a *2-fork* a parallel pair of 2-morphisms. These are respectively given by diagram of the shapes below, where we marked a 'back' 1-cell in each case:

$$\begin{tikzcd}
1 \arrow[r, bend left, "\text{back}"] \arrow[r, bend right] & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, bend left, "\text{back}"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\,\Downarrow"] & 2
\end{tikzcd} \tag{B1}$$

A *morphism* of 1- or 2-forks is just a natural transformation of diagrams of that shape.

Coinserters and coequifiers are, respectively, weighted limits of diagrams of the first and second shape (see Kelly (1989)). Obviously, any map of forks $p : E \Rightarrow B$ induces a map between the weighted colimits $\mathrm{colim}^W p : \mathrm{colim}^W E \to \mathrm{colim}^W B$.

[4] We are indebted to Seerp Roald Koudenburg for his help in unraveling such correspondence.

From now on, we work in the 2-category $\mathcal{C}at$.

**Lemma B.1.** *Suppose given a morphism $p : E \Rightarrow B$ of 1-forks. Then the induced map $\operatorname{coins} p : \operatorname{coins} E \to \operatorname{coins} B$ is a discrete opfibration if and only if (1) each component is a discrete opfibration and (2) the naturality square for the back map is a pullback:*

$$\begin{tikzcd}
E_1 \arrow[rr, bend left=10, "f'"] \arrow[dr, "g'"'] \arrow[dd, two heads, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & E_2 \arrow[dr, bend left=10, "k_E"] \arrow[dd, two heads, "p_2" near start] \arrow[dl, phantom, "\Swarrow \nu'"] & \\
& E_2 \arrow[rr, bend right=10, "k_E"' near start] \arrow[dd, two heads, "p_2" near start] & & \operatorname{coins} E \arrow[dd, dashed, two heads, "\operatorname{coins} p"] \\
B_1 \arrow[rr, bend left=10, "f" near start] \arrow[dr, "g"'] & & B_2 \arrow[dr, bend left=10, "k_B"] \arrow[dl, phantom, "\Swarrow \nu"] & \\
& B_2 \arrow[rr, bend right=10, "k_B"'] & & \operatorname{coins} B
\end{tikzcd} \tag{B2}$$

*Proof.* We first describe how coinserters can be constructed in $\mathcal{C}at$. Let $\operatorname{coins} B$ be the category with precisely the same objects as $B_2$, and as morphisms the same as $B_2$ plus formal ones $\nu_{b_1} : fb_1 \to gb_1$ for each $b_1 \in B_1$, thus every morphism in $\operatorname{coins} B$ is a formal composite of morphisms of $B_2$ and $\nu$s, quotiented by naturality equations (i.e. $g\nu = f\nu$). This moreover shows $k_B : B_2 \to \operatorname{coins} B$ is just the evident embedding, and that $p_2 = k_B^* \operatorname{coins} p$. The same applies to $\operatorname{coins} E$, and similarly, $\operatorname{coins} p$ is defined as $p_2$ on objects and $E_2$-morphisms, and as $\operatorname{coins} p(\nu') = \nu_{p_1}$ on $\nu$. Note this implies that $p_2 = k^* \operatorname{coins} p$.

Given this description, we can therefore conclude that when $p$ satisifies the cartesianity assumptions, $\operatorname{coins} p$ is still a discrete 1-cell, and we can check the claimed unique lifting property separately on the $B_2$-morphisms and $\nu$.

So consider a morphism $\varphi : \operatorname{coins} p(e) \to b$ in $\operatorname{coins} B$.

Suppose, first, that $\varphi$ is a $B_2$-morphism. Then we can use $p_2$ to lift it to an $E_2$-morphism in $\operatorname{coins} E$, and this lift is still unique since when $\operatorname{coins} \psi = \varphi$, necessarily $\psi$ must be a $E_2$-morphism and thus is equal to $\operatorname{lift}_{p_2}\varphi$ by assumption on $p_2$.

Suppose instead $\varphi = \nu_{b_1} : fb_1 \to gb_1$. Thus, specifically, $fb_1 = \operatorname{coins} p(e_2) = pe_2$ for some $e_2 \in E_2$, and since $E_1 \cong p_2 \times_{B_2} f$, there must be a unique $e_1 \in E_1$ such that $p_1 e_1 = b_1$. Thus, $\operatorname{coins} p(\nu'_{e_1}) = \nu_{p_1 e_1} = \nu_{b_1}$, showing $\nu_{e_1}$ is the sought lift. Uniqueness follows from uniqueness of $e_1$.

Conversely, we quickly see that $\operatorname{coins} p$ makes $p_2$ a discrete opfibration. Now the same argument we just deployed shows that uniqueness of the lifts of $\nu$s translates to $p_1$ be a discrete opfibration given by pulling back $p_2$ along $f$. ☐

**Lemma B.2.** *Suppose given a morphism $p : E \Rightarrow B$ of 2-forks. Then the induced map $\operatorname{coef} p : \operatorname{coef} E \to \operatorname{coef} B$ is a discrete opfibration if (1) each component is a discrete opfibration, (2) the naturality square for the back map is a pullback, and (3) the 2-cells in $E$ are cartesian over the 2-cells in $B$:*

$$
\begin{tikzcd}
E_1 \arrow[rr, bend left, "f'"] \arrow[rr, bend right, "g'"'] \arrow[rr, phantom, "\alpha'\Downarrow\ \Downarrow\beta'"] \arrow[d, two heads, "p_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & & E_2 \arrow[r, "k_E"] \arrow[d, two heads, "p_2"'] & \operatorname{coef} E \arrow[d, dashed, two heads, "\operatorname{coef} p"] \\
B_1 \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\alpha\Downarrow\ \Downarrow\beta"] & & B_2 \arrow[r, "k_B"] & \operatorname{coef} B
\end{tikzcd}
\tag{B3}
$$

*Proof.* A coequifier in $\operatorname{coef} B$ is given by imposing extra equations between the morphisms of $B$. This means $\operatorname{coef} p$ works just like $p_2$, and its lifting property is still in effect. Thus to show $\operatorname{coef} p$ is a discrete opfibration we only need to check that if $\varphi = \varphi'$ in $B_2$ then $\operatorname{lift}_{p_2}\varphi = \operatorname{lift}_{p_2}\varphi'$ in $E_2$ too. Note it suffices to prove this for $\alpha = \beta$, since all the other newly introduced equations are 'whiskerings' thereof, i.e. $\varphi\alpha\psi = \varphi\beta\psi$.

Now assume given $b_1 \in B_1$ and an $e_2 \in E_2$ such that $p_2 e_2 = f b_1$, so that we have lifting problems for $\alpha_{b_1}, \beta_{b_1}$. By assumption, there is a unique $e_1 \in E_2$ such that $p_1 e_1 = b_1$ and $f' e_1 = e_2$, so that $\alpha'_{e_1}$ and $\beta'_{e_1}$ solve the two lifting problems uniquely—but by construction of $\operatorname{coef} E$, these are equal, proving the claim. $\square$

We do not know if the converse direction of Lemma B.2 holds.