

Sharp angle estimates for second order divergence operators

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This article is about the (minimal) sector containing the numerical range of the principal part of a linear second-order elliptic differential operator defined by a form on closed subspaces V of the first-order Sobolev space $W^{1,2}(\Omega)$ incorporating mixed boundary conditions. We collect a comprehensive array of results on the angle of sectoriality and the H^∞ -angle attached to realizations of the elliptic operator. We thereby consider the operator in several scales of Banach spaces: the Lebesgue space, the negative Sobolev space, and their interpolation scale. For the latter two types of spaces, we rely on recent results regarding the Kato square root property. We focus on minimal assumptions on geometry, and we consider both real and complex coefficients. Not all results presented are new, but we strive for a streamlined and comprehensive overall picture from several branches of operator theory, and we complement the existing results with several new ones, in particular aiming at explicit estimates built on readily accessible problem data. This concerns for example a new estimate on the angle of the sector containing the numerical range of a linear, continuous and coercive Hilbert space operator, but also an explicit estimate for the angle of sectoriality for the elliptic operator on $L^p(\Omega)$ with complex coefficients without any assumptions on geometry and a general transfer principle for the Crouzeix-Delyon theorem from bounded operators to sectorial ones, keeping the explicit constant.

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1 Introduction

One of the concepts that mathematicians usually teach already to undergraduate students in Linear Algebra courses is how to understand the precise behavior of linear transformations by looking at their spectrum. This paradigm stays strong even far beyond graduate studies and applies equally well to closed operators B on Banach spaces X that arise when studying partial differential equations through the lens of functional analysis. For example, studying an abstract linear Cauchy problem in X such as

$$u'(t) + Bu(t) = f(t) \quad \text{in } X \quad (0 < t < T), \quad u(0) = u_0,$$

one wants to define the operator exponential $S(t) \simeq e^{-tB}$ of $-B$ to determine the solution by the Duhamel formula $u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)ds$. It is well known that in the relevant case of an unbounded operator, where B is, say, a linear second-order elliptic differential operator, the existence of such an operator exponential is already tied to its spectral behavior—meaning both the location of the spectrum *and* suitable decay of the resolvent operators outside of it; this is the famous Hille-Yosida theorem [75, Ch. 1.3]. But even more, if the spectrum of B is contained in a sector Σ_θ in the complex plane of opening (half) angle $\theta < \pi/2$ and the norms of the resolvents $(B - \lambda)^{-1}$ decay asymptotically like $1/\lambda$, then the semigroup is colloquially called *holomorphic*, and one identifies the abstract Cauchy problem as *parabolic* [75, Ch. 2.5], since one is enabled to derive classically parabolic effects such as instant smoothing for the solution u . (In this work, sectors are closed. We will introduce all objects properly in the main text below.)

A disadvantage of the whole idea is that the spectrum of unbounded operators is rather unwieldy. As a remedy, it is convenient to work with the *numerical range* of B ,

$$N(B) := \left\{ u^*(Bu) : u \in \text{dom}(B), \|u\|_X = 1, u^* \in J(u) \right\}$$

where $J: X \rightrightarrows X^*$ is the duality mapping. (If X is reflexive, which it always will be in this work, then J is single-valued.) It is well known that the numerical range of an operator carries much information on its spectral behavior. In many interesting cases, the (closure of) the numerical range in fact contains the spectrum. It is thus often attractive to work with the numerical range instead of the spectrum. In fact, the numerical range has found countless applications in the most diverse disciplines of mathematics, be that matrix analysis, numerical analysis, or quantum physics. We do not attempt to list them all but refer to [15, 47] for a good overview of basic results and a comprehensive list of further literature.

In the present work, we concentrate on the case where B corresponds to a linear elliptic second-order differential operator of the form $u \mapsto -\text{div}(\mu \nabla u)$ with a coefficient matrix μ and we are interested precisely in the *numerical range* of B being contained in a sector Σ_θ with $\theta < \pi/2$, and in the infimum ω of all such θ —or, at least, a good estimate for the latter. In the basic case of a Hilbert space X , it turns out that since the open left

half plane belongs to the resolvent set $\rho(B)$ of B due to the Lax-Milgram lemma, the property $N(B) \subseteq \Sigma_\theta$ already implies that the spectrum of B is also contained in Σ_θ and that the resolvent $(B - \lambda)^{-1}$ decays like $1/|\lambda|$ —that is, B is a *sectorial* operator and $-B$ is the generator of a bounded holomorphic semigroup S . The minimal angle of such a sector is called the *spectral angle* $\phi(B)$ of B and there is a direct correspondence to the maximal angle $\vartheta(B)$ for the sector of holomorphy of the semigroup S via $\phi + \vartheta = \pi/2$. There is also an intimate relation to B admitting a bounded holomorphic functional calculus (H^∞ -calculus) on the sector Σ_θ° , and to the notion of *spectral sets* [16, 26]. The bounded H^∞ -calculus of angle $< \pi/2$ also implies *maximal parabolic regularity* for B . We refer to Section 3.1 below for all these concepts and results.

Contribution. From the foregoing explanations, it becomes apparent that it is most important to understand well the sectors that contain the numerical range of a prototype linear second-order elliptic operator. Such an operator can be considered in different Banach spaces X . It is also clear that *a priori* it is not at all obvious or expected that there is stability of these sectors when the Banach space X is varied, even not within in a natural scale like the L^p -spaces. Thus we are not only concerned with the sectors and their angles *per se*, but also how they depend on certain other reference angles. The reader has surely already guessed that the optimal angle for the L^2 -realization of the differential operator will play a central role.

We proceed as follows. We consider the realization A of the differential operator $u \mapsto -\operatorname{div}(\mu \nabla u)$ with mixed boundary conditions and the uniformly bounded and elliptic coefficient function $\mu: \Omega \rightarrow \mathcal{L}(\mathbb{C}^d)$ in $L^2(\Omega)$ for an open set Ω via the form method [56, Ch. 6]. The form will thus be the classical Dirichlet form

$$\mathfrak{t}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \ni (u, v) \mapsto \int_{\Omega} (\mu \nabla u, \nabla v) \, dx \quad (2.3)$$

and we also consider the restriction $\mathfrak{t}_V$ to closed subspaces V of $W^{1,2}(\Omega)$ that carry mixed boundary conditions. Thus A is the operator corresponding to the form $\mathfrak{t}_V$. In Section 2, we first give a new estimate for an angle α such that $N(\mathfrak{t}) \subseteq \Sigma_\alpha$, where $N(\mathfrak{t})$ is the numerical range of the form $\mathfrak{t}$. This improves upon the recent result in [44]. The estimate uses only explicit data of the coefficient function μ and transfers to $\mathfrak{t}_V$. It is sharp in several cases. Such an estimate is relevant for what was explained above, because $N(A)$ is a dense subset of $N(\mathfrak{t}_V)$, so both numerical ranges share the same optimal angle ω for a sector containing them. It also shows that we can find uniform sectors for operators with different coefficient functions as long as the data entering the estimate is uniform.

We then collect several concepts such as sectoriality, holomorphic semigroups, the H^∞ -calculus, maximal parabolic regularity, and related results in Section 3.1, before putting them to use for the $L^2(\Omega)$ -operator A in Section 3.2. Most of these results are well known and we merely aim for a clear and streamlined presentation. However, we give a proof of the Crouzeix-Delyon theorem [25, 27, 77] for unbounded, sectorial operators in

which the constant is transferred from the bounded operator case. This seems to be a new result, see Theorem 3.13.

Next, we concern ourselves with the realization A_p of A in $L^p(\Omega)$ spaces, treating first the case of real coefficients in Section 3.3.1 and then the—more involved—case of complex coefficient in Section 3.3.2. In the real case, we do not claim much originality except for an interpolation result for the H^∞ -angle in Theorem 3.20. It turns out that without further assumptions on the geometry of Ω and its boundary, we will not recover the optimal L^2 -angle ω for $N(A_p)$, but at least one that can be shown to be sharp in some cases (Theorem 3.19). A way to recover ω is via Gaussian estimates and a Sobolev embedding assumption for the form domain V , cf. (3.14). In the complex coefficients case, the situation is more dire. Again, with some geometric assumptions on Ω and its boundary, one can recover the L^2 -angle ω for A_p as well. Without these assumptions, mirroring the real case, we can only determine an estimate for a sector containing $N(A_p)$ using the optimal angle of sectoriality for the coefficient function. We do so in Theorem 3.30, thereby (slightly) improving a similar result in [18] with a direct proof. Previously, there were only similar results known for the case where the imaginary part $\text{Im } \mu$ of the coefficient function was symmetric [23]. The new proof follows a particular structure that is also implicitly present in the proofs for the real case that the authors are aware of. It shows that the deciding sector is the one that contains a p -adapted numerical range of the coefficient function. We rely on p -ellipticity [19] and associated results to show these results. Let us point out that there is no assumption on the geometry of Ω and its boundary at all.

Finally, in Section 3.4, we switch to a weak setting and consider A in $W_D^{-1,q}(\Omega)$, a negative (dual) Sobolev space carrying generalized mixed boundary conditions, for certain values of q , and then in interpolation spaces between $L^q(\Omega)$ and $W_D^{-1,q}(\Omega)$. These spaces have become prominent in the treatment of nonlinear evolution equations with nonsmooth data corresponding to real world problems, such as semiconductor dynamics [32, 72] or the thermistor problem [70, 71]. Here, we use the operator square root $(A_p + 1)^{1/2}$ isomorphism ([10, 39]) for $p = q'$ to move between $L^q(\Omega)$ and $W_D^{-1,q}(\Omega)$. The geometric assumptions to achieve this are strong enough to recover the L^2 -angle ω also for the operator in the negative Sobolev spaces and interpolation spaces in between.

We give a streamlined presentation that traces the occurrence of the several angles mentioned before throughout the theory of sectorial operators on the usual function spaces. Thus, not all results are new, and not always the greatest generality in setup is achieved. Nevertheless, we have been careful not to assume anything on geometry when it was not required, but *if* it was required, then we have usually opted for a good compromise of generality and distracting effort needed to introduce concepts.

Notation and conventions. The notation that used in the article will generally be standard or properly introduced at point. We nevertheless mention a few conventions: All Banach spaces under consideration are complex vector spaces. Given a Hilbert space

H , the norm derived from the inner product $(\cdot, \cdot)_H$ is denoted by $|\cdot|_H$. For the particular Hilbert space $H = \mathbb{C}^d$, we do not label norms and inner products. Norms on Banach spaces X will be written as usual $\|\cdot\|_X$. For operator norms we use the notation $\|\cdot\|_{X \rightarrow Y}$ with the obvious meaning. We have tried to be consistent with using z for generic complex numbers, λ for resolvent points, and δ or t for real parameters. There will be various angles of different meaning in the following. The convention to have in mind is as follows: ω signifies an optimal angle of sectoriality of something, ϕ the optimal spectral angle, ψ the optimal H^∞ -angle, and finally α for various estimates or upper bounds on angles. Free variables for angles will be θ and, if necessary, ϑ .

2 A sharp inclusion of the numerical range of the form

We start with a few definitions. Let $0 \leq \theta < \pi$. Then

$$\Sigma_\theta := \left\{ z \in \mathbb{C} : |\arg z| \leq \theta \right\} \cup \{0\}$$

denotes the closed sector of opening angle 2θ in the complex plane. Let further H be a complex Hilbert space and let L be a linear operator in H . We define its *numerical range* or *field of values* by

$$N(L) := \left\{ (Lx, x)_H : x \in \text{dom}(L), |x|_H = 1 \right\} \subseteq \mathbb{C}.$$

The *numerical radius* of L is defined by

$$n(L) := \sup \left\{ |\lambda| : \lambda \in N(L) \right\}.$$

When L is in fact a bounded operator on H , we also employ its decomposition into a sum of a selfadjoint and a skewadjoint operator, using the real and imaginary operator parts of L ,

$$L = \frac{L + L^*}{2} + i \frac{L - L^*}{2i} =: \Re L + i \Im L.$$

Note that both $\Re L$ and $\Im L$ are selfadjoint bounded operators whose numerical ranges are convex subsets of $\mathbb{R}$, and their numerical radii coincide with their respective operator norms. Moreover, $\text{Re}(Lx, x)_H = (\Re Lx, x)_H$ and $\text{Im}(Lx, x)_H = (\Im Lx, x)_H$.

We thus say that L is *coercive* with the constant $m > 0$, or, for short, $\Re L \succeq m$, if $(\Re Lx, x)_H \geq m|x|_H^2$ for all $x \in H$.

Let us proceed with the following elementary yet quite precise lemma, that we were surprised not to find in the common literature.

Lemma 2.1. *Let L be a bounded operator in a Hilbert space H satisfying the coercivity condition $\Re L \succeq m > 0$. Then*

$$N(L) \subseteq \Sigma_\alpha \quad \text{with} \quad \tan(\alpha) = \frac{n(\Im L)}{m}.$$

Proof. Let $x \in H$ with $|x|_H = 1$. We have with the coercivity condition

$$|\operatorname{Im}(Lx, x)_H| = |(\mathfrak{Im} Lx, x)_H| \leq n(\mathfrak{Im} L) \leq \frac{n(\mathfrak{Im} L)}{m} \operatorname{Re}(Lx, x)_H.$$

Hence, $|\arg(Lx, x)_H| \leq \arctan(\frac{n(\mathfrak{Im} L)}{m}) = \alpha$ and $N(L) \subseteq \Sigma_\alpha$ follows. $\square$

Remark 2.2. A few comments on Lemma 2.1:

- (i) Let us recall that since $\mathfrak{Im} L$ is selfadjoint, $n(\mathfrak{Im} L) = \|\mathfrak{Im} L\|_{H \rightarrow H}$. If L itself is selfadjoint, then $N(L) \subseteq \mathbb{R}_+ = \Sigma_0$ which is faithfully reproduced by $N(L) \subseteq \Sigma_\alpha$ in Lemma 2.1, since then $\mathfrak{Im} L = 0$ and so $\alpha = 0$.
- (ii) A variant of Lemma 2.1 is the main result in [44]. There, it is shown that

$$N(L) \subseteq \Sigma_{\bar{\alpha}} \quad \text{with} \quad \bar{\alpha} = \arctan \sqrt{\frac{\|L\|_{H \rightarrow H}^2}{m^2} - 1}.$$

In fact, this follows from the estimate in Lemma 2.1, since

$$|(\mathfrak{Im} Lx, x)_H|^2 = |\operatorname{Im}(Lx, x)_H|^2 \leq |(Lx, x)_H|^2 - m^2,$$

so $n(\mathfrak{Im} L)^2 \leq n(L)^2 - m^2 \leq \|L\|_{H \rightarrow H}^2 - m^2$ and thus $\alpha \leq \bar{\alpha}$. However, there are interesting cases where the angle α is indeed substantially smaller than $\bar{\alpha}$, for example when L is nearly selfadjoint but with large numerical radius. The angles α and $\bar{\alpha}$ coincide whenever the numerical radius of L is attained on $[\operatorname{Re} z = m]$.

Remark 2.3. An alternative, geometric proof of Lemma 2.1 would go as follows: It is easy to see that $N(L) \subseteq N(\Re L) + \imath N(\mathfrak{Im} L)$. Thus, with the coercivity assumption,

$$\overline{N(L)} \subseteq \left\{ z = a + \imath b \in \mathbb{C} : m \leq a \leq n(\Re L), b \in N(\mathfrak{Im} L) \right\} \cap \overline{B(0, n(L))}. \quad (2.1)$$

The rays in the complex plane starting from 0 with opening angle $\pm\alpha$ pass precisely through $m \pm \imath n(\mathfrak{Im} L)$, and at least one of those points is a left corner point of the set on the right in (2.1). Thus Σ_α is the minimal sector which contains that set, which in turn contains $N(L)$.

Remark 2.4. We discuss the quality and sharpness of the $N(L) \subseteq \Sigma_\alpha$ sector inclusion estimate in Lemma 2.1:

- (i) The sector inclusion can still be rather crude. For example, consider the operator L induced on $\mathbb{C}^2$ by a 2×2 diagonal matrix with $m = 1$ and $10 + \imath$ on the diagonal. Then the numerical range $N(L)$ consists of the line between 1 and $10 + \imath$ in the complex plane; accordingly, the optimal sector including $N(L)$ would be of opening half angle $\arctan(\frac{1}{10})$ which is quite small, however, $\alpha = \arctan 1 = \frac{\pi}{4}$.

- (ii) On the other hand, in Remark 2.3 we have seen that one can determine the angle α as the one for which the rays in the complex plane starting from 0 with opening angles $\pm\alpha$ pass through $m \pm in(\Im L)$. We thus see that α is sharp precisely when $z = m + in(\Im L)$ or its complex conjugate $\bar{z}$ is a boundary point of $N(L)$. In fact, there is a direct link between sharpness of α and z or $\bar{z}$ being an eigenvalue of L which becomes an equivalence if $N(L)$ is closed. (In particular, if H is finite-dimensional.) By symmetry, it is sufficient to argue for z only. Indeed, if $z \in N(L)$, then z is also a corner point of that set. So when z is an eigenvalue of L , then α is sharp, because eigenvalues are contained in the numerical range. Vice versa, if the angle is sharp and $z \in N(L)$ instead of merely $z \in \overline{N(L)}$, then z must be a corner of $N(L)$ and from that it follows that it is an eigenvalue of L ([34, Thm. 1]). This situation occurs precisely for Ornstein-Uhlenbeck operators [20, Rem. 2].
- (iii) In fact, let us note that the possible shapes of the numerical range are rather well understood for operators on finite-dimensional spaces $\mathbb{C}^d$, in particular so for $d = 2, 3$. We refer to the classical work [60], see also [59]. For $d = 2$, α can only be sharp if L is normal. Indeed, if L is not normal then $N(L)$ is a nondegenerate ellipse or a ball and the eigenvalues of L are in the interior which contradicts the general necessary condition that z or $\bar{z}$ is an eigenvalue. (The numerical range of a finite-dimensional operator is always closed.) Similarly, for $d = 3$, the angle α can only be sharp when the matrix A associated to L is unitarily equivalent to a matrix of the form $\begin{pmatrix} a & 0 \\ 0 & \mathcal{A} \end{pmatrix}$ with $a \in \mathbb{C}$ and $\mathcal{A} \in \mathbb{C}^{2 \times 2}$, where a is either z or $\bar{z}$.

From Lemma 2.1 we see that a coercive bounded operator on a Hilbert space H is *sectorial-valued* in the sense that its numerical range is contained in a sector. It will be useful to define the *optimal* angle for a sector containing the numerical range of a sectorial-valued (possibly unbounded) operator L :

$$\omega(L) := \inf \left\{ \theta \in [0, \pi) : N(L) \subseteq \Sigma_\theta \right\}.$$

We will stick with the notation ω for optimal angles of various kinds. Let further $(\mathcal{L}_x)$ be a family of bounded, sectorial-valued operators on a Hilbert space indexed by x . Then we say that $(\mathcal{L}_x)$ is *uniformly* sectorial-valued if

$$\omega(\mathcal{L}) := \sup_x \omega(\mathcal{L}_x) < \infty.$$

We next apply Lemma 2.1 to determine a sector which includes the numerical range of a *form*. Therefor, we recall that if $\mathbf{a}$ is a sesquilinear form in H , then its numerical range is given by

$$N(\mathbf{a}) := \left\{ \mathbf{a}(x, x) : x \in \text{dom}(\mathbf{a}), |x|_H = 1 \right\}.$$

We say that $\mathbf{a}$ is a *sectorial form* if there is $\theta \in [0, \pi)$ such that $N(\mathbf{a}) \subseteq \Sigma_\theta$. Of course we also set

$$\omega(\mathbf{a}) := \inf \left\{ \theta \in [0, \pi) : N(\mathbf{a}) \subseteq \Sigma_\theta \right\} \quad (2.2)$$

The particular form that we consider will be that defining an elliptic operator in divergence form.

Theorem 2.5. *Let $\Omega \subseteq \mathbb{R}^d$ be any open set and let $\mu: \Omega \rightarrow \mathcal{L}(\mathbb{C}^d)$ be bounded and Lebesgue measurable. Let*

$$\mathfrak{t}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \ni (u, v) \mapsto \int_{\Omega} (\mu \nabla u, \nabla v) \, dx \quad (2.3)$$

be the Dirichlet form on $L^2(\Omega)$ with $\text{dom}(\mathfrak{t}) = W^{1,2}(\Omega)$.

- (i) *Suppose that μ is uniformly sectorial-valued a.e. on Ω with $\omega(\mu) \leq \pi/2$. Then $\mathfrak{t}$ is sectorial and $N(\mathfrak{t}) \subset \Sigma_{\omega(\mu)}$. In particular, $\omega(\mathfrak{t}) \leq \omega(\mu)$.*
- (ii) *Suppose that $\Re \mu(x) \succeq m(x) > 0$ for almost every $x \in \Omega$, and that*

$$\tan(\alpha) := \text{ess sup}_{x \in \Omega} \frac{n(\Im \mu(x))}{m(x)} < \infty. \quad (2.4)$$

Then μ is uniformly sectorial-valued almost everywhere on Ω with $\omega(\mu) \leq \alpha < \pi/2$.

Proof. For (i), we estimate pointwisely. By assumption, there is a subset $\Lambda \subseteq \Omega$ with $|\Omega \setminus \Lambda| = 0$ such that

$$|\text{Im}(\mu(x)\xi, \xi)| \leq \tan(\omega(\mu)) \text{Re}(\mu(x)\xi, \xi) \quad (x \in \Lambda, \xi \in \mathbb{C}^d).$$

Thus,

$$\begin{aligned} |\text{Im } \mathfrak{t}(u, u)| &\leq \int_{\Omega} |\text{Im}(\mu(x)\nabla u(x), \nabla u(x))| \, dx \\ &\leq \tan(\omega(\mu)) \int_{\Omega} \text{Re}(\mu(x)\nabla u(x), \nabla u(x)) \, dx = \tan(\omega(\mu)) \text{Re } \mathfrak{t}(u, u), \end{aligned}$$

so that $N(\mathfrak{t}) \subseteq \Sigma_{\omega(\mu)}$.

Assertion (ii) follows directly from Lemma 2.1 applied to $L := \mu(x)$. $\square$

Some further comments on Theorem 2.5 are in order.

Remark 2.6. (i) It is worthwhile to observe that the value of α in (2.4) is invariant under a multiplicative scalar perturbation ρ of the coefficient matrix μ , where ρ is a real, bounded and strictly positive function on Ω . Such a uniform sectoriality property is often of particular interest for nonautonomous parabolic or quasilinear problems, see for example *Hypothesis 1* in the seminal work [5], or also [46, Thm. 6.1].

- (ii) Due to the assumption that (2.4) is bounded, no uniform ellipticity constant for μ is required in Theorem 2.5. However, if the pointwise ellipticity constants $m(x)$ degenerate $\downarrow 0$ over Ω , i.e., there is no common positive lower bound, then the form $\mathfrak{t}$ is in general *not* closed when considered as a form in $L^2(\Omega)$ with $\text{dom}(\mathfrak{t}) = W^{1,2}(\Omega)$, but at best closable [62, Rem. 6.4]. See also [56, Ch. VI.§1.3/4]. While it is known that if the numerical range of $\mathfrak{t}$ is contained in a given sector, then so is the numerical range of the closure of the form $\mathfrak{t}$, we concede that dealing with the closure of the form causes several technical issues down the line which obfuscate the matter at hand. For this reason, we will avoid this case and soon require that there is uniform lower bound $m(x) \geq m_\bullet > 0$ a.e. on Ω . Then (2.4) is always satisfied.
- (iii) In the situation of Theorem 2.5, it is clear that for any shift $\delta \in \Sigma_{\omega(\mathfrak{t})}$ of $\mathfrak{t}$, in particular for $\delta \geq 0$, the numerical range of $\mathfrak{t} + \delta$ is still contained in $\Sigma_{\omega(\mathfrak{t})}$.
- (iv) Let ν be a measure on Ω that is absolutely continuous with respect to the Lebesgue measure. Then it is easy to extend and state Theorem 2.5 for the form $\mathfrak{t}$ defined on $W^{1,2}(\Omega; \nu)$ with

$$W^{1,2}(\Omega; \nu) := \left\{ u \in W_{\text{loc}}^{1,2}(\Omega) : u, \nabla u \in L^2(\Omega; \nu) \right\}$$

and the Lebesgue-space $L^2(\Omega; \nu)$ with respect to ν . Such a setting is often considered in the area of Dirichlet forms. It would work in particular for measures with a bounded and uniformly positive density with respect to the Lebesgue measure and allows to consider for example Ornstein-Uhlenbeck operators as the corresponding realizations in $L^2(\Omega; \nu)$. We refer to [20, 21] where similar results to this paper are derived, and the references there, but do not pursue this particular avenue in this work.

In the same vein, the authors are convinced that the foregoing may be generalized to forms which lead to elliptic *systems*—as long as the coefficient function is (essentially) uniformly bounded and pointwise coercive. However, we do not go into details here.

Moreover, clearly, the sectoriality of $\mathfrak{t}$ as in (2.3) is preserved when restricting that form to closed subspaces V of $W^{1,2}(\Omega)$. In particular, this applies to subspaces encoding Dirichlet boundary conditions on (parts of) $\partial\Omega$ and to finite-dimensional subspaces, for example arising in Galerkin methods, most prominently Finite Elements.

Corollary 2.7. *Let V be any closed subspace of $W^{1,2}(\Omega)$ and denote the form $\mathfrak{t}$ restricted to V by $\mathfrak{t}_V$. Then Theorem 2.5 stays true for $\mathfrak{t}_V$. In particular, $\mathfrak{t}_V$ is a sectorial form with $\omega(\mathfrak{t}_V) \leq \omega(\mathfrak{t}) \leq \omega(\mu) \leq \alpha$, where α is as in (2.4).*

In a most useful turn of events, the numerical range inclusion for $\mathfrak{t}_V$ also transfers to the operator A on $L^2(\Omega)$ induced by $\mathfrak{t}_V$, and the associated versions on $L^p(\Omega)$. We define this operator in the next section.

3 Consequences for the operators induced by the form

We proceed to collect several more intricate consequences of the sectoriality of $\mathfrak{t}$ as in Theorem 2.5 and of $\mathfrak{t}_V$ as in Corollary 2.7 for any closed subspace V of $W^{1,2}(\Omega)$. The setting and assertions of these results are thus taken for granted in the following.

Armed with the knowledge that the Dirichlet forms $\mathfrak{t}_V$ are sectorial in the setting of Theorem 2.5 and Corollary 2.7, we recall that

$$\omega(\mathfrak{t}_V) = \inf \left\{ \theta \in [0, \pi) : N(\mathfrak{t}_V) \subseteq \Sigma_\theta \right\}.$$

We will neglect the parameter V in ω_V if the context is clear, which we expect it to always be. Clearly, α as in (2.4) is an upper bound for ω since it is an upper bound for $\omega(\mu)$. In the following, we will derive several results and angles which will depend on these optimal angles $\omega(\mathfrak{t}_V)$ for $\mathfrak{t}_V$ and $\omega(\mu)$ for μ . This will give the most precise and sharpest results, yet of course it is of importance to have a good upper bound like α at our disposal, for example when we are interested in *uniformity* of associated sectors. Such a situation was already mentioned in Remark 2.6. In any of the following, we can readily substitute the upper bound α for $\omega(\mathfrak{t}_V)$ and $\omega(\mu)$ to get a concrete upper bound of the desired quantity.

As already announced above, we will concentrate on the situation where the coefficient $\Re \mu$ is uniformly positive definite a.e. on Ω and the form $\mathfrak{t}_V$ is closed; that is, we make for all what follows, the following supposition:

Assumption 3.1 (Standing assumptions). In the context of Theorem 2.5 and Corollary 2.7, we suppose the following assumptions to be valid from now on:

- (i) The ellipticity constants $m(x)$ for the matrices $\mu(x)$ admit a *uniform* positive lower bound $m(x) \geq m_\bullet > 0$ for almost all $x \in \Omega$, i.e., $\Re \mu \succeq m(x) \geq m_\bullet > 0$ uniformly on Ω .
- (ii) The form domain V under consideration is dense in $L^2(\Omega)$ and it is a *closed* subspace of $W^{1,2}(\Omega)$.

Note that the assumption that V is dense in $L^2(\Omega)$ is needed to make the following operator A induced by $\mathfrak{t}_V$ on $L^2(\Omega)$ well defined, which will be the basis for all further considerations:

$$\begin{aligned} \text{dom}(A) &:= \left\{ u \in V : \exists f \in L^2(\Omega) : \mathfrak{t}(u, v) = (f, v)_{L^2(\Omega)} \text{ for all } v \in V \right\} \\ Au &:= f. \end{aligned} \tag{3.1}$$

It is quickly verified that A is a closed operator when V is a closed subspace of $W^{1,2}(\Omega)$ and μ is uniformly elliptic, as we have assumed. Since by definition, $(Au, u)_{L^2(\Omega)} = \mathfrak{t}(u, u)$

for all $u \in \text{dom}(A) \subseteq V$, it is clear that $N(A + \delta) \subseteq N(\mathbf{t}_V + \delta)$ for all $\delta \geq 0$. In fact, $N(A + \delta)$ is dense in $N(\mathbf{t}_V + \delta)$ ([56, Cor. VI.§2.2.3]), so we have:

$$\omega(A + \delta) = \omega(\mathbf{t}_V + \delta) \leq \omega(\mu) \leq \alpha \quad (\delta \geq 0) \quad (3.2)$$

with α as in (2.4). We will write ω for $\omega(A) = \omega(\mathbf{t}_V)$ in the following.

The rest of this work will be spent collecting and deriving a host of results which are related to (3.2) and $\omega(A)$ in general. We subdivide these by the following settings:

- (i) Considering A in $L^2(\Omega)$, where $\omega(A)$ directly determines the resulting quantities.
- (ii) The realizations of A in $L^p(\Omega)$:
 - (a) The general case, where the results are stated in some (explicit) dependence on $\omega(\mu)$ and p , and
 - (b) peculiar geometric situations which yield p -independent results, so the situation for $p = 2$ is decisive.
- (iii) Functional spaces other than $L^p(\Omega)$, in particular negative (dual) Sobolev spaces.

We will thereby be mostly interested in the case where V corresponds to a subspace of $W^{1,2}(\Omega)$ that carries mixed boundary conditions in the usual sense. Since this will already be an comprehensive endeavor, we will not consider the manifold possible extensions, for example considering Robin boundary conditions or dynamical boundary conditions.

Before we go into details we re-establish some definitions and known abstract results which we employ later. Several of these results can be found in the literature at one place or the other. However, we strive for the simplest possible reasoning by collecting them at all at one place to see how the individual results build upon each other and need very little extra heavy machinery to establish.

3.1 Definitions and preliminary results

As before, we use Σ_θ for the closed sector of semi-angle $\theta \in [0, \pi)$, and we denote its interior by $\Sigma_\theta^\circ := (\Sigma_\theta)^\circ$. It will be useful to have in mind that

$$\mathbb{C} \setminus \Sigma_\theta = -\Sigma_{\pi-\theta}^\circ.$$

Also, recall that the symbol $\rho(B)$ stands for the resolvent set of a linear operator B in X , that is, the set of all $\lambda \in \mathbb{C}$ for which $B - \lambda$ has a continuous inverse $X \rightarrow X$. Accordingly, $\sigma(B) := \mathbb{C} \setminus \rho(B)$ is the spectrum of B .

Definition 3.2 (Sectorial operator). Let B be an operator in a Banach space X . We call B *sectorial* of angle $\phi \in [0, \pi)$ if $\sigma(B) \subseteq \Sigma_\phi$ and

$$\sup_{\lambda \in \mathbb{C} \setminus \Sigma_\theta} \|\lambda(B - \lambda)^{-1}\|_{X \rightarrow X} < \infty.$$

Moreover, we denote by

$$\phi(B) := \inf \left\{ \phi \in [0, \pi) : B \text{ is sectorial of angle } \phi \right\}$$

the *spectral angle* for the operator B .

Remark 3.3. A more convenient way of working with sectorial operators as before is by means of real resolvent points only. Indeed, it is well known that an operator B is sectorial *if and only if* $(-\infty, 0) \subset \rho(B)$ and

$$\sup_{t>0} \|t(B + t)^{-1}\|_{X \rightarrow X} < \infty. \quad (3.3)$$

Remark 3.4. Sectorial operators enjoy several useful properties *out of the box*. Indeed, their resolvent set is nonempty, so they are automatically closed. Moreover, if a sectorial operator is injective, then its range is dense. Even more, if the underlying Banach space is *reflexive*, then a sectorial operator is densely defined, and if its range is dense, then it is also injective. See for example [24, Theorem 3.8].

We next relate the sectoriality of an operator to the property of being a generator of an analytic semigroup.

Definition 3.5 (Bounded holomorphic/analytic, holomorphy angle). For a Banach space X , we say that a strongly continuous semigroup $T: [0, \infty) \rightarrow \mathcal{L}(X)$ is an *holomorphic* or *analytic semigroup of angle* $\vartheta > 0$ if T has a holomorphic extension to the sector Σ_ϑ which is bounded on $\Sigma_\theta \cap \{z \in \mathbb{C} : |z| \leq 1\}$ for every $\theta \in (0, \vartheta)$. In addition, T is *bounded holomorphic of sector* ϑ if T is bounded on every sector Σ_θ for $\theta \in (0, \vartheta)$. The angle ϑ is called the *holomorphy angle* for the semigroup T .

Sectorial operators of spectral angle $< \pi/2$ are precisely the (negative) generators of bounded holomorphic semigroups, and there is a direct relation between the spectral angle and the holomorphy angle of the semigroup, see [3, Thm. 3.7.11] or [74, Thm. 1.45]. In fact, the spectral angle $\phi < \pi/2$ of a sectorial operator B is in one-to-one correspondence with the semi-angles ϑ of sectors on which the semigroup T , generated by $-B$, is holomorphic by $\vartheta + \phi = \pi/2$, and one is enabled to convert every statement on the spectral angle below into a statement on the angle of holomorphy for the corresponding semigroup and vice versa. In particular, we have $\tan(\phi) = \cot(\vartheta)$. Without loss of generality, we choose to concentrate on the spectral angle in the following.

We introduce one more property of (generators of) holomorphic semigroups that is one of the most desirable properties to be inferred from the results collected throughout this work.

Definition 3.6 (Maximal parabolic regularity). Let X be a Banach space and let B be the negative generator of a holomorphic semigroup S on X . For $T \in (0, \infty]$ and $q \in (1, \infty)$, we say that B satisfies *maximal parabolic regularity* on X , if for every $f \in L^q(0, T; X)$, the unique mild solution

$$u(t) := \int_0^t S(t-s)f(s) \, ds \quad (0 < t < T)$$

to the Cauchy problem

$$u'(t) + Bu(t) = f(t) \quad (0 < t < T), \quad u(0) = 0$$

satisfies $u(t) \in \text{dom}(B)$ for almost all $t \in (0, T)$, and $u', Bu \in L^q(0, T; X)$.

It is no restriction to suppose that B is the negative operator of a holomorphic semigroup in Definition 3.6, since it is a necessary condition for maximal parabolic regularity. We also mention that maximal parabolic regularity is independent of $q \in (1, \infty)$, and of T , if finite. For finite T , we can also rescale u exponentially to see that B satisfies maximal parabolic regularity if and only if $B + \delta$ does so for any $\delta \in \mathbb{R}$, say. Finally, if X is a Hilbert space, than *any* negative generator of a holomorphic semigroup satisfies maximal parabolic regularity. See [36, 63] and [1, Ch. III]. The latter also offers a great point of view why maximal parabolic regularity is a highly desirable property for the treatment of nonlinear problems, cf. also [76].

Now, the aim of the next considerations is to introduce instruments that allow one to deduce sectoriality and more from properties of the numerical range of the operator in question. For convenience, we state the following directly for Banach spaces for which we recall the definition of the numerical range and, associated, the duality mapping first.

Let X be a Banach space and let $J: X \rightrightarrows X^*$ be its duality mapping, that is, the *a priori* set-valued mapping

$$J(u) := \left\{ u^* \in X^*: u^*(u) = \|u\|_X, \|u^*\|_{X^*} = 1 \right\}.$$

By the Hahn-Banach theorem, $J(u) \neq \emptyset$ for every $u \in X$. If X^* is strictly convex, then $J(u)$ is in fact single valued. We will always be in this situation when we use the numerical range below for the family $X = L^p(\Omega)$ in the reflexive range $1 < p < \infty$.

Let B be a closed operator in the Banach space X . Then the *numerical range* of B is given by

$$N(B) := \left\{ u^*(Bu): u \in \text{dom}(B), \|u\|_X = 1, u^* \in J(u) \right\}.$$

This definition coincides with the one given before if X is a Hilbert space.

Definition 3.7 (Accretive). Let B be a closed operator in the Banach space X . We say that B is *accretive* if $N(B) \subseteq [\text{Re } z \geq 0] = \Sigma_{\frac{\pi}{2}}$. Further, B is *m-accretive* if it is accretive and $-1 \in \rho(B)$. Finally, we call B *m- θ -accretive* for $\theta \in [0, \pi/2]$ if it is m-accretive and in fact $N(B) \subseteq \Sigma_\theta$.

We remark that it is well known that an accretive operator B is m -accretive if and only if there is *any* λ with $\operatorname{Re} \lambda < 0$ and $\lambda \in \rho(B)$. (It can also be deduced from the proof of Corollary 3.9 right below.) The -1 in Definition 3.7 is just for convenience. By the Lumer-Philips theorem, m -accretive operators are precisely the negative generators of semigroups of contractions [3, Theorem 3.4.5], and m - θ -accretive operators correspond to *holomorphic* semigroups of contractions. We say that T is a semigroup of contractions if $\|T(t)\|_X \leq 1$ for all $t \geq 0$.

In the following, we leverage a numerical range sector inclusion using the following fundamental and well-known proposition which gives decay of the resolvent in relation to distance to the numerical range. See [67, Prop. 1.3.1], compare also [75, Thm. 1.3.9]. For context for the following estimate, note that, given a set $\mathcal{O} \subseteq \mathbb{C}$ and $\lambda \in \mathbb{C} \setminus \overline{\mathcal{O}}$, we have $\sup_{z \in \mathcal{O}} \left| \frac{1}{z - \lambda} \right| = \frac{1}{\operatorname{dist}(\lambda, \mathcal{O})}$.

Proposition 3.8. *Let B be an operator in a Banach space X . Let U be a connected component of $\mathbb{C} \setminus \overline{N(B)}$. If $U \cap \rho(B) \neq \emptyset$, then $U \subseteq \rho(B)$, and*

$$\|(B - \lambda)^{-1}\|_{X \rightarrow X} \leq \frac{1}{\operatorname{dist}(\lambda, N(B))} \quad (\lambda \in U).$$

In the present case, we will be able to invest that the numerical range of B is contained in a sector, and indeed Proposition 3.8 then implies the sectoriality of B , and quite a bit more, provided that at least one resolvent point exists in the left half-plane:

Corollary 3.9. *Let B be an operator in a Banach space X and let $\theta \in [0, \pi/2]$. Suppose that B is m - θ -accretive. Then we have the following.*

- (i) *The numerical range sector contains the spectrum: $\sigma(B) \subset \Sigma_\theta$.*
- (ii) *If $\lambda \in \mathbb{C} \setminus \Sigma_\theta$, then $\|(B - \lambda)^{-1}\|_{X \rightarrow X} \leq \frac{1}{\operatorname{dist}(\lambda, \Sigma_\theta)}$.*
- (iii) *For $\theta = \pi/2$, we have*

$$\|(B - \lambda)^{-1}\|_{X \rightarrow X} \leq \frac{1}{|\operatorname{Re} \lambda|} \quad (\operatorname{Re} \lambda < 0).$$

Otherwise, for every $\vartheta \in (\theta, \pi/2]$,

$$\|\lambda(B - \lambda)^{-1}\|_{X \rightarrow X} \leq \frac{1}{\sin(\vartheta - \theta)} \quad (\lambda \in \mathbb{C} \setminus \Sigma_\theta). \quad (3.4)$$

- (iv) *The operator B is sectorial with spectral angle $\phi(B) \leq \theta$.*
- (v) *The operator B is the negative generator of a semigroup of contractions. If $\theta < \pi/2$, then the semigroup is also a holomorphic semigroup of contractions on $\Sigma_{\frac{\pi}{2} - \theta}$.*

Proof. (i) and (ii) are easily deduced from Proposition 3.8 by choosing U to be the connected component of $\mathbb{C} \setminus N(B)$ that contains $\mathbb{C} \setminus \Sigma_\theta$. For (iii), we use (ii). The first case $\theta = \pi/2$ is clear. Otherwise, let $\lambda \in \mathbb{C} \setminus \Sigma_\vartheta$. Suppose first that $\lambda \in \mathbb{C} \setminus \Sigma_{\pi/2+\theta}$. Then $\text{dist}(\lambda, \Sigma_\theta) = |\lambda|$ because 0 is the closest point in Σ_θ to λ . (Make a sketch.) On the other hand, if $\lambda \in \Sigma_{\pi/2+\theta} \setminus \Sigma_\vartheta$, then with a bit of elementary geometry we find

$$\text{dist}(\lambda, \Sigma_\theta) = |\lambda| \sin(|\arg(\lambda)| - \theta) \geq |\lambda| \sin(\vartheta - \theta)$$

and (iii) follows. This in turn implies (iv) according to Definition 3.2 and Remark 3.3. The first part of (v) is the Lumer-Phillips theorem. For the second part, from (iii) we already know that the semigroup S generated by $-B$ is bounded holomorphic with holomorphy angle at least $\pi/2 - \theta$. The fact that $S(z)$ is a contraction for each $z \in \Sigma_{\pi/2-\theta}$ follows from the observation that $e^{i\varphi}A$ is still m -accretive for all angles φ with $|\varphi| \leq \pi/2 - \theta$ and writing $S(z) = S(te^{i\varphi}) := S_\varphi(t)$ with $\varphi = \arg z$ and $t = |z|$. Since $e^{i\varphi}A$ is the generator of S_φ , the latter is a semigroup of contractions by the Lumer-Phillips theorem, and the rest of assertion (v) follows. $\square$

The H^∞ -calculus and the Crouzeix-Delyon theorem for unbounded operators

We review a functional calculus for injective sectorial operators B on a Banach space X . It will be much more convenient and not restrictive for what comes later to consider a *reflexive* Banach space. Then B is automatically densely defined, and its range is dense in X as noted in Remark 3.4.

The notion of interest concerns the Dunford-Riesz functional calculus for realizing $f(B)$ for certain classes of holomorphic functions on a sector Σ_θ with $\theta > \phi(B)$, the spectral angle of B . We will stay concise and refer to [30, Sections 1.4, 2.1 & 2.4] for the details, also [9]. The functional calculus is based on the Dunford integral to define $f(B)$ and yields a bounded algebra homomorphism

$$\Phi_B: H_0^\infty(\Sigma_\theta^\circ) \rightarrow \mathcal{L}(X), \quad f \mapsto f(B),$$

where $\theta > \phi(B)$, and

$$H_0^\infty(\Sigma_\theta^\circ) := \left\{ f: \Sigma_\theta^\circ \rightarrow \mathbb{C} \text{ holomorphic, } : \exists c \geq 0, s > 0: |f(z)| \leq c \min(|z|^s, |z|^{-s}) \right\}.$$

Evidently, functions in $H_0^\infty(\Sigma_\theta^\circ)$ are required to decay in a regular way at zero and infinity. However, often, it is of particular interest to extend the foregoing functional calculus also to the class of *bounded* holomorphic functions on Σ_θ . The following definition captures our purpose.

Definition 3.10 (H^∞ -calculus). Let B be an injective sectorial operator on a reflexive Banach space X and let $\theta > \phi(B)$, where $\phi(B)$ is the spectral angle of B . Then B is said to admit a *bounded* $H^\infty(\Sigma_\theta)$ -calculus if there is a constant C_θ such that

$$\|f(B)\|_{X \rightarrow X} \leq C_\theta \sup_{z \in \Sigma_\theta} |f(z)| \quad (3.5)$$

for all $f \in H_0^\infty(\Sigma_\theta^\circ)$. We shall denote the infimum of all θ for which B admits a bounded $H^\infty(\Sigma_\theta)$ -calculus by $\psi(B)$, and we call it the H^∞ -angle for B . Often, we just say that B admits a bounded H^∞ -calculus (on X).

If B admits a bounded $H^\infty(\Sigma_\theta)$ -calculus as just defined, then, via the McIntosh convergence lemma, the uniform estimate (3.5) and the Dunford-Riesz functional calculus Φ_B extend uniquely to

$$H^\infty(\Sigma_\theta) := \left\{ f: \Sigma_\theta \rightarrow \mathbb{C}: f \text{ holomorphic on } \Sigma_\theta^\circ \text{ and bounded on } \Sigma_\theta \right\},$$

see [24, Lem. 2.1] or [48, Prop. 5.1.4], with the *same constant* C_θ as in (3.5) ([48, Prop. 5.3.4]). Some authors also require the foregoing as the definition of the $H^\infty(\Sigma_\theta)$ -calculus.

Having an operator with a bounded H^∞ -calculus at hand is desirable for a quite diverse zoo of problems, see e.g. [28, 31, 50, 61, 64, 66, 73, 78, 79]. In particular, a bounded H^∞ -calculus with H^∞ -angle $\psi(B) < \pi/2$ implies *maximal parabolic regularity* (Definition 3.6) for the operator B via bounded imaginary powers, at least if the corresponding Banach space is a UMD space. We refer to the seminal Dore-Venni paper [35], see also [65]. In relation to the foregoing, from the bounded H^∞ -calculus we recover sectoriality of B —which is built into the definition anyway—and associated estimates as well as the generator property for a holomorphic semigroup by choosing $f(z) = \frac{1}{z-\lambda}$ for λ outside of a sector, and $f(z) = e^{-tz}$, for $t > 0$, for example. A bounded H^∞ -calculus is thus *the strongest* of the properties considered.

In the special case of a Hilbert space, we have the general result that an m -accretive operator B admits a bounded H^∞ -calculus with the H^∞ -angle $\psi(B) = \phi(B)$, the spectral angle [63, Corollary 10.12]. On the other hand, there is an intriguing general result for bounded operators on arbitrary convex open sets that contain the numerical range of the operator—ingeniously, with a *fully explicit constant* independent of the given open set and operator: the famous Crouzeix-Delyon theorem, for which we refer to [25, 27, 77] and the references there.

Theorem 3.11 (Crouzeix-Delyon). *Let H be a Hilbert space. Then there is a constant $\mathcal{Q} > 0$ such that for every convex open set $\mathcal{O} \subset \mathbb{C}$, every bounded operator $B \in \mathcal{L}(H)$ with $\overline{N(B)} \subset \mathcal{O}$, and every function $f \in H^\infty(\mathcal{O})$, we have*

$$\|f(B)\|_{H \rightarrow H} \leq \mathcal{Q} \sup_{z \in N(B)} |f(z)|. \quad (3.6)$$

Moreover, if $\mathcal{Q}^*$ is the infimum of all such constants $\mathcal{Q}$, then $2 \leq \mathcal{Q}^* \leq 1 + \sqrt{2}$.

While there are versions of the Crouzeix-Delyon theorem for sectorial operators [48, Chapter 7.1.5], unfortunately, there seems to be no rigorous proof in the literature that transfers a Crouzeix-Delyon theorem with a given uniform constant $\mathcal{Q}$ to sectorial

operators when $\mathcal{O}$ is a suitable sector. Since such a result is of independent interest, we continue with a short interlude in which we give a mostly self-contained proof, showing how to transfer Theorem 3.11 to the class of densely defined sectorial operators with dense range, preserving the constant $\mathcal{Q}$.

For this purpose, consider the approximations of a given sectorial operator B on a Hilbert space H given by the operators $B_\varepsilon \in \mathcal{L}(H)$ defined via

$$B_\varepsilon := (B + \varepsilon I)(I + \varepsilon B)^{-1}, \quad \varepsilon > 0. \quad (3.7)$$

By the sectoriality of B , cf. (3.3), these approximations are well defined, invertible, and sectorial themselves [30, Prop. 1.4]. Moreover, if B is densely defined and with dense range, then

$$\lim_{\varepsilon \downarrow 0} f(B_\varepsilon) = f(B) \quad \text{in } \mathcal{L}(H) \quad \text{for all } f \in H_0^\infty(\Sigma_\theta^\circ), \quad (3.8)$$

where $\theta > \phi(B)$ ([30, Thm. 1.7]). It is thus clear how to proceed: Use Theorem 3.11 for B_ε and transfer the $\mathcal{Q}$ -bound in (3.6) to B via (3.8). For this purpose, it will be useful that if the numerical range of B is contained in a convex sector, then so are the numerical ranges of B_ε , and they stay away from zero:

Lemma 3.12. *Let B be an m - θ -accretive operator on a Hilbert space H for some $\theta \in [0, \pi/2]$. Let $\varepsilon > 0$ and let $B_\varepsilon \in \mathcal{L}(H)$ be defined as in (3.7). Then*

$$N(B_\varepsilon) \subset \Sigma_\theta \cap [\operatorname{Re} z \geq \varepsilon].$$

Proof. Due to Corollary 3.9, B is sectorial with $\phi(B) \leq \theta$. Thus, the approximants B_ε are well defined. Let $x \in H$ with $|x|_H = 1$. Then $y = (I + \varepsilon B)^{-1}x \in \operatorname{dom}(B)$ and

$$(B_\varepsilon x, x)_H = ((B + \varepsilon I)(I + \varepsilon B)^{-1}x, x)_H = ((B + \varepsilon I)y, (I + \varepsilon B)y)_H.$$

Expanding, we find

$$(B_\varepsilon x, x)_H = (By, y)_H + \varepsilon^2(y, By)_H + \varepsilon|y|_H^2 + \varepsilon|By|_H^2. \quad (3.9)$$

Now $N(B_\varepsilon) \subset \Sigma_\theta$ follows directly since all summands in (3.9) are in the convex cone Σ_θ by assumption. Further, rewrite

$$B_\varepsilon = \frac{1}{\varepsilon} - \left(\frac{1}{\varepsilon^2} - 1\right)\left(\frac{1}{\varepsilon} + B\right)^{-1}$$

to see that, using Cauchy-Schwarz and sectoriality of B via (3.3),

$$\begin{aligned} \operatorname{Re}(B_\varepsilon x, x) &\geq \frac{1}{\varepsilon}|x|_H^2 - \left(\frac{1}{\varepsilon^2} - 1\right)\left\|\left(\frac{1}{\varepsilon} + B\right)^{-1}\right\|_{H \rightarrow H}|x|_H^2 \\ &\geq \frac{1}{\varepsilon}|x|_H^2 - \varepsilon\left(\frac{1}{\varepsilon^2} - 1\right)|x|_H^2 = \varepsilon|x|_H^2. \end{aligned} \quad \square$$

We now pass to the limit in Theorem 3.11 for the operators B_ε , keeping *the same optimal constant* $\mathcal{Q}^*$. Thanks to (3.8) and Lemma 3.12, this is a straightforward endeavor.

Theorem 3.13. *Let H be a Hilbert space and let $\mathcal{Q}$ be as in Theorem 3.11. Let B be an injective operator on H , let $\theta \in [0, \pi/2)$, and suppose that B is m - θ -accretive. Then B admits a bounded $H^\infty(\Sigma_\vartheta)$ -calculus for every $\vartheta > \theta$ with the constant $\mathcal{Q}$.*

Proof. From the assumptions, B is densely defined (Remark 3.4) and sectorial (Corollary 3.9). Consider first the bounded approximants B_ε of B defined in (3.7). From Lemma 3.12 we have

$$N(B_\varepsilon) \subset \Sigma_\theta \cap [\operatorname{Re} z \geq \varepsilon].$$

Thus, for every $\vartheta > \theta$, the open sector Σ_ϑ° is a neighborhood of $\overline{N(B_\varepsilon)}$. In particular, Theorem 3.11 implies that for every function $f \in H_0^\infty(\Sigma_\vartheta^\circ)$ we have the uniform estimate

$$\|f(B_\varepsilon)\|_{H \rightarrow H} \leq \mathcal{Q} \sup_{z \in \Sigma_\vartheta} |f(z)|.$$

But $f(B_\varepsilon) \rightarrow f(B)$ in $\mathcal{L}(H)$ according to (3.8), so the assertion follows immediately. $\square$

We complement Theorem 3.13 by mentioning also *von Neumann's inequality*. See [48, Thm. 7.1.7/Cor. 7.1.8]. Roughly, and in the present context, it says that Theorem 3.13 also holds true for m -accretive operators B on a Hilbert space H , that is, with $\theta = \pi/2$, and then even with the uniform constant 1. Recall that it was already mentioned before that an m -accretive operator on a Hilbert space admits a bounded H^∞ -calculus and that the H^∞ -angle and the spectral angle coincide.

3.2 The operator on $L^2(\Omega)$

We now apply the foregoing abstract results to the operator A induced by $\mathbf{t}_V$ on $L^2(\Omega)$ as defined in (3.1). Recall that Assumption 3.1 is supposed to hold. The Lax-Milgram lemma implies that the whole (open) left half plane $[\operatorname{Re} z < 0]$ belongs to the resolvent set $\rho(A)$ of A . Also, from Theorem 2.5 and (3.2) we know that $N(A) \subseteq \Sigma_\omega$ and that $\omega \leq \omega(\mu) \leq \alpha < \frac{\pi}{2}$. In particular, A is m - ω -accretive. Recall that we write ω for $\omega(A) = \omega(\mathbf{t}_V)$.

Due to Corollary 3.9, this immediately gives the sectoriality of A with the estimate (3.4) for the generator property for a holomorphic semigroup of contractions for $-A$. Since $L^2(\Omega)$ is a Hilbert space, the latter in fact also already implies maximal parabolic regularity, see [63, Cor. 1.7]. Thus:

Theorem 3.14. *Let A be the operator induced by $\mathbf{t}_V$ on $L^2(\Omega)$. Then A is m - ω -accretive. In particular, it is sectorial with spectral angle ω and for every $\theta \in (\omega, \pi/2]$, we have the explicit estimate*

$$\|\lambda(A - \lambda)^{-1}\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \frac{1}{\sin(\theta - \omega)} \quad (\lambda \in \mathbb{C} \setminus \Sigma_\theta).$$

Moreover, $-A$ is the generator of a holomorphic semigroup of contractions on $L^2(\Omega)$ with the holomorphy angle $\pi/2 - \omega$ and A satisfies maximal parabolic regularity.

Indeed, one can have much more: the Crouzeix-Delyon theorem as in Theorem 3.13 implies a bounded $H^\infty(\Sigma_\theta^\circ)$ -calculus for (injective versions of) A for any $\theta > \omega$.

Theorem 3.15. *Let A be the operator induced by $\mathbf{t}_V$ on $L^2(\Omega)$. Let $\mathcal{Q}$ be as in Theorem 3.11. Then, for every $\delta > 0$ and every $\theta > \omega$, the operator $A + \delta$ admits a bounded $H^\infty(\Sigma_\theta)$ -calculus with the explicit estimate*

$$\|f(A + \delta)\|_{L^2(\Omega) \rightarrow L^2(\Omega)} \leq \mathcal{Q} \sup_{z \in \Sigma_\theta} |f(z)| \quad (f \in H^\infty(\Sigma_\theta^\circ)).$$

In particular, the H^∞ -angle of $A + \delta$ is at most ω . If $\mathbf{t}_V$ is coercive, then the foregoing also holds true for $\delta = 0$.

Proof. It is clear that $N(A + \delta) \subseteq \Sigma_\omega$. Moreover, with Assumption 3.1, the form $\mathbf{t}_V + \delta$ is coercive for any $\delta > 0$, so that $A + \delta$ is injective by Lax-Milgram. So Theorem 3.13 applies. If $\mathbf{t}_V$ is coercive, the same argument also works for $\delta = 0$. $\square$

As mentioned before, we could also have had the bounded H^∞ -calculus on $L^2(\Omega)$ merely from m -accretivity of A by abstract arguments. We prefer to state the explicit version derived from the Crouzeix-Delyon theorem.

Remark 3.16. The form $\mathbf{t}_V$ is coercive whenever one has a Poincaré inequality for V :

$$\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2 \quad (u \in V).$$

The classical case where this fails is the case of only Neumann boundary conditions, i.e., $V = W^{1,2}(\Omega)$, where the constant functions belong to V . For mixed boundary conditions, there is still a Poincaré inequality even for very irregular geometries. See the detailed discussion in [40, Ch. 7].

3.3 The operator on $L^p(\Omega)$

In recent years it has turned out that for the analytic treatment of many real world problems, the $L^2(\Omega)$ -calculus for the corresponding differential operators is not an adequate one, but rather one anchored in an $L^p(\Omega)$ space is so. This usually happens if one is confronted with non-smooth data of the problem (non-smooth, non-convex domain, or discontinuous coefficients, or mixed boundary conditions), because then one cannot rely on elliptic $H^2(\Omega)$ regularity, or if one has to deal with certain nonlinear equations, for example in particular *quasilinear* ones, or semilinear problems without a monotonicity assumption.

Prototypes for such problems could be the classical semiconductor equations (Van Roosbroeck system, [55]) or the Keller-Segel model from mathematical biology ([52], see also [69, Ch. 5]). Here, let us point out that while one might expect from this brief motivation that when going over to an $L^p(\Omega)$ setting, *surely* one would consider $p > 2$. But in fact, it turns out that it indeed can be adequate to consider $p < 2$, as done for example in [52].

In contrast to the $L^2(\Omega)$ setting, for the operator A considered in $L^p(\Omega)$ it makes quite a fundamental difference whether the coefficient function μ is real- or complex valued. We first consider the case of real coefficients where one can derive quite satisfying results under very few structural assumptions. In any case, we specialize on subspaces V of $W^{1,2}(\Omega)$ which incorporate a Dirichlet type boundary condition on a closed subset D of the boundary $\partial\Omega$.

To this end, in addition to Assumption 3.1 we activate *from now on* the following assumption:

Assumption 3.17. Let $D \subseteq \partial\Omega$ be closed, and let V be the closure in $W^{1,2}(\Omega)$ of either

$$C_D^\infty(\Omega) := \left\{ u|_\Omega : u \in C_0^\infty(\mathbb{R}^d) : \text{dist}(\text{supp}(u), D) > 0 \right\} \quad (3.10)$$

or of

$$\underline{W}_D^{1,2}(\Omega) := \left\{ u \in W^{1,2}(\Omega) : \text{dist}(\text{supp}(u), D) > 0 \right\}. \quad (3.11)$$

When $D = \emptyset$, we clearly recover $\underline{W}_\emptyset^{1,2}(\Omega) = W^{1,2}(\Omega)$. If $D \neq \emptyset$, then either space V carries a Dirichlet boundary condition for its elements on D , at least implicitly so. In how far this condition can be made more explicit depends on the regularity of $\partial\Omega$ around D , see e.g. [22, 41]. Often, V built from (3.10) is also said to carry *good Neumann boundary conditions* on $\partial\Omega \setminus D$, e.g. [74].

Note that we have $C_0^\infty(\Omega) \subset V$ in both cases, so both V are indeed dense in $L^2(\Omega)$.

3.3.1 Real coefficients

For this subsection, in addition to Assumption 3.1 and 3.17 we assume that the coefficient function μ takes its values in the *real-valued* matrices only.

Let A be the operator in $L^2(\Omega)$ which is induced by $\mathbf{t}_V$ as in (3.1). The convenient way to define the realizations in $L^p(\Omega)$ works by extrapolation of the semigroup T generated by $-A$ on $L^2(\Omega)$. For this purpose, let us introduce the extrapolation range of T :

$$\mathcal{I}(A) := \left\{ p \in (1, \infty) : \sup_{t>0} \|T(t)\|_{L^p(\Omega) \rightarrow L^p(\Omega)} < \infty \right\}. \quad (3.12)$$

This set will play an important role from now on, at least implicitly so at the beginning, but also in an increasingly prominent manner throughout this work.

For the present case of a real coefficient function, the following strategy is particularly effective for a second-order operator with a scalar shift as we consider in this work, and will not work as well for an operator including first-order or nonconstant zero-order terms. See below for an approach via Gaussian estimates, though, where these can be incorporated. We say that T is *positive* if $T(t)f$ is nonnegative for all nonnegative $f \in L^2(\Omega)$ and all $t > 0$. Clearly, T must necessarily be real—mapping real functions into real functions—in order to be positive; for this we need a real coefficient function as in Assumption 3.17. Further, T is *L^∞ -contractive* if

$$\|T(t)f\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \quad (f \in L^2(\Omega) \cap L^\infty(\Omega), \quad t > 0).$$

A positive and L^∞ -contractive semigroup is also called *sub-Markovian*. Of course, positivity and the contraction property of a semigroup can also be defined on function spaces other than $L^2(\Omega)$, in particular on $L^p(\Omega)$ spaces, with the necessary changes. We refer to [74, Ch. 2.2] for all these notions and many more.

The overall result is then the following.

Proposition 3.18. *The operator A on $L^2(\Omega)$ induced by $\mathbf{t}_V$ is the negative generator of a sub-Markovian, holomorphic semigroup T_2 of contractions on $L^2(\Omega)$, and we have $\mathcal{I}(A) = (1, \infty)$. More precisely, the semigroup extrapolates consistently to a strongly continuous positive contraction semigroup T_p on $L^p(\Omega)$ for every $p \in [1, \infty)$, and the semigroups T_p are also holomorphic for all $p \in (1, \infty) = \mathcal{I}(A)$.*

The individual statements in Proposition 3.18 are scattered in [74] at various places, but collected in detail in [22, Ch. 2.2]. Here we have already used several structural properties of the particular choices of V as in (3.10) and (3.11), and that μ is real. It is worthwhile that with holomorphy of $A = A_2$ and $p \in \mathcal{I}(A)$, holomorphy of the extrapolated semigroup T_p follows from Stein's extrapolation theorem [74, p.96] which also gives a crude estimate on the holomorphy angle. We will derive a more precise estimate below.

For $p \in [1, \infty)$, denote by $-A_p$ the generator of T_p . Then A_p is an m -accretive operator in $L^p(\Omega)$ and it is not hard to see that $-A_p$ is the part of $-A$ in $L^p(\Omega)$. The operators A_p inherit several properties from A_2 . For example, its numerical range is contained in a sector, yet, without further specification of Ω and its boundary, not with exactly the angle of the $L^2(\Omega)$ -operator. Recall that $\omega(\mu)$ is the optimal angle of sectoriality for the coefficient function μ . Then we have the following:

Theorem 3.19 ([22, Thm. 3.4]). *Let $p \in (1, \infty)$. The numerical range of A_p is contained in the sector with semi-angle α_p , that is, $N(A_p) \subseteq \Sigma_{\alpha_p}$ with*

$$\tan(\alpha_p) = \frac{\sqrt{(p-2)^2 + p^2\omega(\mu)^2}}{2\sqrt{p-1}}. \quad (3.13)$$

In particular, A_p is m - α_p -accretive, the spectral angle $\phi(A_p)$ is not larger than α_p , and the holomorphy angle of the semigroup generated by $-A_p$ is at least $\pi/2 - \alpha_p$.

We note that in the present generality, the angle α_p in Theorem 3.19 is indeed optimal, see [22, Rem. 3.8] and also [20, 21]. Clearly, it also collapses to $\alpha_2 = \omega(\mu)$ which is the natural upper bound for ω , the optimal angle of sectoriality for A . Next, we relate the H^∞ -angle of A with the one of A_p by interpolation.

Theorem 3.20. *Let $p \in (1, \infty)$ and $\delta > 0$. Then $A_p + \delta$ admits a bounded H^∞ -calculus on $L^p(\Omega)$ with H^∞ -angle not larger than $\frac{\pi}{2}|1 - \frac{2}{p}| + \omega(1 - |1 - \frac{2}{p}|)$. In particular, $A_p + \delta$ satisfies maximal parabolic regularity on $L^p(\Omega)$. If A is injective, the assertion is also true for $\delta = 0$.*

Proof. We combine an ingenious interpolation result for H^∞ -angles as in [57, Prop. 4.9] with the result of Duong [37] which says that for $p \in (1, \infty)$, each $A_p + \delta$ has a bounded $H^\infty(\Sigma_\theta)$ -calculus for any $\theta > \pi/2$. For a comprehensive proof, see also [51]. We then interpolate between the angle ω for $p = 2$ and $\pi/2$, which is an upper bound for the H^∞ -angle $\psi(A_r)$ for all the other $r \in (1, \infty) \setminus \{2\}$. Indeed, given $p \in (1, \infty) \setminus \{2\}$, for any $r \in (1, \infty)$ we can pick $\theta \in (0, 1)$ such that

$$L^p(\Omega) = \left[L^r(\Omega), L^2(\Omega) \right]_\theta, \quad \frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{2}.$$

Then, by [57, Prop. 4.9], also $\psi(A_p) = (1-\theta)\psi(A_r) + \theta\omega$. In particular, $\psi(A_p) \leq (1-\theta)\frac{\pi}{2} + \theta\omega$. To get the best possible upper bound, we optimize (maximize) for θ : for $p > 2$, we let $r \uparrow \infty$, for which $\theta \uparrow \frac{2}{p}$, whereas for $p < 2$, we let $r \downarrow 1$, for which $\theta \uparrow 2 - \frac{2}{p}$. Encoding the case distinction in an absolute value, this yields exactly that $\psi(A_p) \leq \frac{\pi}{2}|1 - \frac{2}{p}| + \omega(1 - |1 - \frac{2}{p}|)$ for all $p \in (1, \infty)$, which was the claim.

Maximal parabolic regularity follows from the bounded H^∞ -calculus since $\psi(A_p + \delta) < \pi/2$ and $L^p(\Omega)$ is an UMD space [35, 65]. See also Remark 3.21 below.

It remains to verify the assumptions of the interpolation result [57, Prop. 4.9]. Writing $B_p := A_p + \delta$, these require that there is $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < 0$ such that for any $p, q \in (1, \infty)$, the resolvents in λ of B_p and B_q are consistent, that is,

$$(B_p - \lambda)^{-1}u = (B_q - \lambda)^{-1}u \quad (u \in L^p(\Omega) \cap L^q(\Omega)).$$

But from Proposition 3.18 we already know that the semigroups generated by $-B_p$ and $-B_q$, respectively, are consistent for all $p, q \in (1, \infty)$, and via the Laplace transform of the semigroups this indeed implies consistency of the resolvents. $\square$

Remark 3.21. As observed in the proof, Theorem 3.20 in particular implies that $\psi(A_p + \delta) < \pi/2$ for $p \in (1, \infty)$. We mention that there are also abstract results that imply this relation directly. These exploit intrinsic connections of a bounded H^∞ -calculus to essential features of $\mathcal{R}$ -boundedness. See e.g. [58, Cor. 5.2], where one has to invest that the semigroups T_p generated by $-A_p$ are positive, holomorphic contraction semigroups as stated in Proposition 3.18. While Theorem 3.20 is more explicit, the general statement that indeed $\psi(A_p + \delta) < \pi/2$ is often of foremost interest due to its connection to maximal parabolic regularity via bounded imaginary powers, as mentioned before.

While we were able to transfer the desirable properties of the operator A induced by t_V on $L^2(\Omega)$ to the realizations A_p of A in $L^p(\Omega)$ in Theorems 3.19 and 3.20, so far the corresponding angles were only perturbations or functions of ω and $\omega(\mu)$ instead of ω itself. However, we note that so far, we have not required boundedness of Ω , or *any* kind of boundary regularity.

In fact, investing a bit more in geometry, we find that there are—still very general—situations where non-trivial results allow to deduce that the deciding angle ω for the $L^2(\Omega)$ -operator A is also directly the one for the operators A_p .

One method relies on Gaussian estimates [74, Ch. 6]. These can also be used to extrapolate the operator A to $L^p(\Omega)$ in the case of lower order derivative terms in the differential operator. For these the extrapolation technique based on L^∞ -contractivity may fail, but Gaussian estimates still yield $\mathcal{I}(A) = (1, \infty)$ and even extrapolation to $L^1(\Omega)$. The general assumption posed to get these is the *embedding property*

$$V \hookrightarrow L^{2^*}(\Omega), \tag{3.14}$$

where 2^* is the Sobolev conjugate of 2.

If any general conditions on the geometry of Ω and D such that (3.14) holds true are met (see Remark 3.22 below), then the semigroup operators on $L^2(\Omega)$ are integral operators, the kernels of which satisfy *Gaussian estimates*, see [74, Thm. 6.10].

Having these Gaussian estimates at hand, one can exploit [2, Thms. 5.4/5.7] to obtain that the semigroup T generated by $-A$ extrapolates to $L^p(\Omega)$ for $p \in [1, \infty)$, in particular, $\mathcal{I}(A) = (1, \infty)$, and, defining the operators A_p as the negative generators

of the semigroup T_p on $L^p(\Omega)$ as before, $A_p + \delta$ admit a bounded H^∞ -calculus with H^∞ -angle ω for every $\delta > 0$. So the optimal L^2 -angle is recovered under the minimal assumption (3.14). As before, if t_V is coercive, then the same is true for $\delta = 0$.

Remark 3.22. The convenient way to obtain the embedding property (3.14) is to establish a continuous, linear extension operator $\mathcal{E}: V \rightarrow W^{1,2}(\mathbb{R}^d)$. There are many, very general sufficient conditions for the latter, for example that Ω is a bounded domain that admits bi-Lipschitz boundary charts at any point in $\overline{\partial\Omega \setminus D}$. See Definition 3.23 below. In this case the desired extension operator exists for either choice (3.10) or (3.11) of V . Moreover, whenever $D \neq \emptyset$, a Poincaré inequality as in Remark 3.16 holds true so that t_V is coercive and A is injective. For the latest developments and even more general admissible settings for such an extension operator, at least for V as in (3.10), we refer to [14] and the references there.

3.3.2 Complex coefficients

We now consider the general case of a *complex* coefficient function. In comparison to the case of real coefficients, the situation regarding sectoriality of the parts A_p of A in $L^p(\Omega)$, properties of the associated semigroup and a bounded H^∞ -calculus is (much) more intricate.

In fact, even for the most straightforward case of a second-order operator with complex coefficients and a scalar shift, the technique employed in Proposition 3.18 to extrapolate the semigroup T generated by $-A$ to $L^p(\Omega)$ based on L^∞ -contractivity of T does not work at all: in the case of complex coefficients, there just *is no* L^∞ -contractivity in general, and its failure is quite catastrophic in nature [6, 7, 68]. In fact, for complex coefficients, there are situations where even for self-adjoint differential operators, there is a finite p for which the associated L^2 -semigroup T ceases to exist on $L^p(\Omega)$, see [29].

Nevertheless, in recent years there was considerable progress in understanding and clarifying the situation. We concentrate on contributions involving mixed boundary conditions on domains. We also reverse the order compared to the real coefficients section and begin with abstract results that yield the L^2 -angles also for the parts A_p of A in $L^p(\Omega)$ before deriving an estimate on the angle of sectoriality in $L^p(\Omega)$ in more general situations. For this purpose, let us introduce the following particular geometric setting:

Definition 3.23 (Lipschitz around $\partial\Omega \setminus D$). We say that the open set Ω is *Lipschitz around $\partial\Omega \setminus D$* if it is a bounded domain, $D \subseteq \partial\Omega$ is closed, and there are Lipschitz boundary charts available for $\overline{\partial\Omega \setminus D}$ in the following sense: For every $x \in \overline{\partial\Omega \setminus D}$ there exists a neighborhood U_x and a bi-Lipschitz mapping $\Phi_x: U_x \rightarrow (-1, 1)^d$ such that

$$\Phi_x(U_x \cap \partial\Omega) = (-1, 1)^{d-1} \times \{0\}, \quad \Phi_x(U_x \cap \Omega) = (-1, 1)^{d-1} \times (-1, 0).$$

As before, let A be the operator induced by $\mathbf{t}_V$ on $L^2(\Omega)$ and recall that we denote the semigroup denoted by $-A$ by T . In [39, Thm. 1.3], for V given as the closure of $C_D^\infty(\Omega)$ as in (3.10), and under the assumption that Ω is Lipschitz around $\partial\Omega \setminus D$ as in Definition 3.23, Egert shows the following most intriguing result: *If the semigroup T consistently extrapolates to bounded holomorphic semigroups T_p on $L^p(\Omega)$, that is, if $p \in \mathcal{I}(A)$, where*

$$\mathcal{I}(A) := \left\{ p \in (1, \infty) : \sup_{t>0} \|T(t)\|_{L^p(\Omega) \rightarrow L^p(\Omega)} < \infty \right\}, \quad (3.12)$$

and if we, as usual, denote by A_p the negative generator A_p of T_p in $L^p(\Omega)$, then for any $\delta > 0$, the operator $A_p + \delta$ admits a bounded H^∞ -calculus on $L^p(\Omega)$ with the L^2 -angle H^∞ -angle ω . As before, $\delta = 0$ is allowed if $\mathbf{t}_V$ is coercive. Let us also mention that the result of course also holds for real coefficient functions, and also allows for lower order terms, and thus can be substituted in, say, the Gaussian estimates context to obtain a bounded H^∞ -calculus as soon as one knows that $\mathcal{I}(A) = (1, \infty)$. See also [10] for similar results with less but more involved assumptions on geometry. Choosing the exponentials $f(z) := e^{-tz}$ as particular bounded holomorphic functions on the right half-plane, it also becomes clear that $p \in \mathcal{I}(A)$ is also necessary for the H^∞ -calculus.

The question thus shifts to investigating $\mathcal{I}(A)$. It is well known that $\mathcal{I}(A)$ is an open interval, and that it includes all p such that $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{d}$, in particular, $(1, \infty) \subseteq \mathcal{I}(A)$ for $d = 2$, but also that the condition is sharp in the sense that there is no p satisfying $|\frac{1}{2} - \frac{1}{p}| > \frac{1}{d}$ which is in $\mathcal{I}(A)$ for *every* operator A with a complex coefficient function. We refer to [39, 42, 43] and the references there, and also for the fact that under the given supposition on V (via the embedding property (3.14)) and for a *fixed* complex coefficient function μ and associated operator A , the limit cases for $d \geq 3$ in the foregoing condition $p = 2^* = \frac{2d}{d-2}$ and $p = 2_* = \frac{2d}{d+2}$ indeed belong to $\mathcal{I}(A)$. (And then so does an interval slightly beyond these.) Some of the mentioned works are based on the concept of p -ellipticity that we introduce below in more detail, see also Lemma 3.26.

On the other hand, the extrapolation machinery via Gaussian estimates employed at the end of the foregoing section works also for the complex coefficients case. Indeed, in [43, Thm. 3.8], under slightly stronger assumptions than stated so far, it is shown that A admits Gaussian estimates if the complex coefficient function arises from a real one perturbed by a sufficiently small *imaginary* perturbation. See also [17] for a related result. Having Gaussian estimates at hand, one can again extrapolate the semigroup T to bounded holomorphic semigroups on $L^p(\Omega)$ for all $p \in \mathcal{I}(A) = (1, \infty)$ and the result of Egert yields the bounded H^∞ -calculus automatically.

A direct estimate. In the following, we present a different approach, essentially with the same philosophy as for Theorem 3.19: we *directly* determine an angle for a sector containing the numerical range of the L^p -realization of A , with the optimal angle of sectoriality for the *coefficient function* $\omega(\mu)$ as a reference point, and with no further assumptions on geometry of Ω . In the present case of complex coefficients, this will

be possible provided that the norm of the (componentwise) imaginary part $\text{Im } \mu$ of the coefficient function stands in a certain relation with p and the ellipticity constants. We will only suppose that Assumptions 3.1 and 3.17 are fulfilled, which is the same setting as in the real coefficients case. That is, we work with either form domains given by (3.10) or (3.11), and we assume, as before, that the coefficient function μ is uniformly elliptic with lower bound $m_\bullet > 0$ a.e. on Ω .

A particularly useful concept is the following, popularized by [19]: For $p \in (1, \infty)$, we say that a matrix function $\mu: \Omega \rightarrow \mathbb{C}^{d \times d}$ is *p-elliptic* if $\Delta_p(\mu) > 0$ with

$$\Delta_p(\mu) := \text{ess inf}_{x \in \Omega} \min_{\xi \in \mathbb{C}^d, |\xi|=1} \text{Re}(\mu(x)\xi, \mathcal{J}_p(\xi)) \quad (3.15)$$

where $\mathcal{J}_p$ is the $\mathbb{R}$ -linear map

$$\mathcal{J}_p(\alpha + \imath\beta) := \frac{2\alpha}{p'} + \imath \frac{2\beta}{p}.$$

Of course, 2-ellipticity is exactly the usual uniform ellipticity of μ over Ω . Further, it is easily seen that $\Delta_p(\mu) = \Delta_{p'}(\mu)$, in particular, μ is p -elliptic if and only if it is p' -elliptic [42, Lemma 8]. Moreover, μ is p -elliptic for all $p \in (1, \infty)$ if and only if μ is real [82, Lem. 2.15]. We thus assume that $\text{Im } \mu \neq 0$, thereby excluding $q = \infty$ in the next lemma; otherwise the results from the previous subsections can be consulted.

It will be useful to set $\Psi(s) := \frac{2\sqrt{s-1}}{s-2}$. Then $\Psi: [2, \infty] \rightarrow [0, \infty]$ is continuous and monotonously decreasing with $\lim_{s \downarrow 2} \Psi(s) = \infty$ and $\lim_{s \uparrow \infty} \Psi(s) = 0$. In particular, Ψ is invertible. We point out that although the following lemma is formulated pointwise for x , we still require uniform ellipticity for μ as in Assumption 3.1.

Lemma 3.24. *Suppose that $\Re \mu(x) \succeq m(x) \geq m_\bullet$ for every $x \in \Omega$. Set*

$$\eta := \text{ess sup}_{x \in \Omega} \frac{\|\text{Im } \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{m(x)} \quad \text{and} \quad q := \Psi^{-1}(\eta). \quad (3.16)$$

Then $\Delta_q(\mu) \geq 0$ and μ is p -elliptic if $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{q}|$.

Proof. We note that $\eta < \infty$ since μ is bounded and uniformly elliptic. Thus, also $q > 2$. It is enough to argue for $p > 2$, and useful to set $\sigma_s := \Psi(s)^{-1} = \frac{s-2}{2\sqrt{s-1}}$. Then, the choice of q implies that for almost every $x \in \Omega$ and every $p \in (2, q)$,

$$m(x) \geq \sigma_q \|\text{Im } \mu(x)\| = (\sigma_q - \sigma_p) \|\text{Im } \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} + \sigma_p \|\text{Im } \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}, \quad (3.17)$$

where $\sigma_q - \sigma_p > 0$. Inspecting the proof of [43, Lemma 2.16] (or Lemma 3.28 below), we find that for $\xi \in \mathbb{C}^d$ with $|\xi| = 1$ and for $p \geq 2$,

$$\frac{1}{2} \text{Re}(\mu(x)\xi, \mathcal{J}_p(\xi)) \geq \frac{1}{p} \left(m(x) - \sigma_p \|\text{Im } \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} \right). \quad (3.18)$$

We derive a lower bound for (3.18) that is uniform in $x \in \Omega$. If $2\sigma_p \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} \leq m(x)$, then it follows from (3.18) that

$$\operatorname{Re}(\mu(x)\xi, \mathcal{J}_p(\xi)) \geq \frac{m(x)}{p}.$$

On the other hand, if $2\sigma_p \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} > m(x)$, then using (3.17) in (3.18) shows that

$$\operatorname{Re}(\mu(x)\xi, \mathcal{J}_p(\xi)) \geq \frac{\sigma_q - \sigma_p}{\sigma_p} \cdot \frac{m(x)}{p}.$$

Thus, the claim follows from uniform ellipticity $m(x) > m_\bullet > 0$ and the definition of p -ellipticity (3.15). $\square$

Remark 3.25. We can read off a lower bound for the p -ellipticity constant $\Delta_p(\mu)$ in Lemma 3.24. Indeed, for $p > 2$, the last estimates in the proof show that

$$\Delta_{p'}(\mu) = \Delta_p(\mu) \geq \left(1 \wedge \frac{\sigma_q - \sigma_p}{\sigma_p}\right) \cdot \frac{m_\bullet}{p} > 0.$$

The assertion $\Delta_q(\mu) \geq 0$ with $q = \Psi^{-1}(\eta)$ as in (3.16) implies that the semigroup T on $L^2(\Omega)$ extrapolates to a holomorphic contraction semigroup on $L^p(\Omega)$ if $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{q}|$. This is another result of Egert [42, Thm. 2]. Here we use the particular choices for V as in Assumption 3.17. We mention that the case of V given by the closure of $\underline{W}_D^{1,2}(\Omega)$ as in (3.11) is not treated in [42] explicitly, but it is easily traced—and verified with the author—that the particular structural results established and required for [42, Thm. 2] for V given by the closure of $C_D^\infty(\Omega)$ as in (3.10) stay valid.

Lemma 3.26 ([42, Thm. 2]). *Let q be as in (3.16) and let $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{q}|$. Then $p \in \mathcal{I}(A)$. In particular, T extrapolates to a holomorphic semigroup of contractions T_p on $L^p(\Omega)$.*

Again, we are in the position to conveniently define the $L^p(\Omega)$ versions A_p of A to be the negative generators of the semigroups T_p provided by Lemma 3.26 for $|\frac{1}{2} - \frac{1}{p}| < |\frac{1}{2} - \frac{1}{q}|$. These are already densely defined and sectorial, but the angle of sectoriality is unclear apart from a possibly crude estimate yielded by Stein interpolation.

We thus next present an estimate on the angle of the sector containing the numerical range of the operators A_p with complex coefficients. Then Corollary 3.9 strikes home and delivers resolvent estimates. A similar result is derived in [18, Sect. 8] by elegant and abstract means. We give a slightly more precise result via a different, if more involved route of proof, whose structure will possibly prove useful in the future. For this purpose, let us put $p \in (1, \infty)$ and introduce the p -numerical range of $L \in \mathcal{L}(\mathbb{C}^d)$:

$$N_p(L) := \left\{ (L\xi, \mathcal{J}_p(\xi)) : \xi \in \mathbb{C}^d, |\xi| = 1 \right\}$$

Clearly, one could also define the p -numerical range for an operator L on a complex Hilbert space H with the necessary changes. In any case, sticking to $H = \mathbb{C}^d$, in the above context, $\Delta_p(\mu) \geq 0$ means that $N_p(\mu(x)) \subseteq [\operatorname{Re} z \geq 0] = \Sigma_{\frac{\pi}{2}}$ for almost all $x \in \Omega$. We mention a few more properties of $N_p(L)$.

- (a) Of course, $N_2(L) = N(L)$, the usual numerical range. Further, we have $N_p(L) = N_{p'}(L)$, due to $(L\xi, \mathcal{J}_p(\xi)) = (L(\imath\xi), \mathcal{J}_{p'}(\imath\xi))$ for all $\xi \in \mathbb{C}^d$.
- (b) Moreover,

$$(L\xi, \xi) = \frac{1}{2}(L\xi, \mathcal{J}_p(\xi)) + \frac{1}{2}(L\xi, \mathcal{J}_{p'}(\xi)), \quad (3.19)$$

so $N(L) = N_2(L) \subseteq \frac{1}{2}N_p(L) + \frac{1}{2}N_{p'}(L) = \frac{1}{2}N_p(L) + \frac{1}{2}N_p(L)$. In particular, whenever, $N_p(L)$ is contained in a convex cone—such as a sector—for some $p \in (1, \infty)$, then so is $N(L)$.

Theorem 3.27. *Let $p \in (1, \infty)$ and suppose that $\theta \in [0, \frac{\pi}{2}]$ is such that $N_p(\mu(x)) \subseteq \Sigma_\theta$ for almost all $x \in \Omega$. Then $N(A_p) \subseteq \Sigma_\theta$.*

We postpone the proof of Theorem 3.27. However, we already mention that the θ in Theorem 3.27 necessarily satisfies $\theta \geq \omega(\mu)$, the optimal angle of uniform sectoriality for μ . This follows from (3.19) and its following comment.

Theorem 3.27 allows us to obtain a sector estimate for $N(A_p)$ from a sector estimate for the p -modulated numerical range of the coefficient function. We thus complement Theorem 3.27 with the following estimate, recalling $\sigma_p := \frac{p-2}{2\sqrt{p-1}}$:

Lemma 3.28. *Suppose that $\Re \mu(x) \succeq m(x) \geq m_\bullet$ for every $x \in \Omega$. Let η and $q \geq 2$ be defined as in (3.16). Then, for every $p \in [2, q]$ and almost every $x \in \Omega$, $N_p(\mu(x)) \subseteq \Sigma_{\alpha_p}$ where $\alpha_q := \frac{\pi}{2}$ and, for $p \in [2, q)$,*

$$\tan(\alpha_p) := \operatorname{ess\,sup}_{x \in \Omega} \left[\frac{\tan(\omega_{\mu(x)})m(x) + \sigma_p \|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{m(x) - \sigma_p \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}} \right]. \quad (3.20)$$

Proof. Again, the fundamental calculation is already contained in the proof of [43, Lemma 2.16]. We reproduce it here with a slight modification that allows to obtain the essential supremum in front of the fraction in (3.20). Write $0 \neq \xi = \alpha + \imath\beta$ with $\alpha, \beta \in \mathbb{R}^d$. Then

$$\frac{1}{2}(\mu(x)\xi, \mathcal{J}_p(\xi)) = \frac{1}{p}(\mu(x)\xi, \xi) + \left(1 - \frac{2}{p}\right)(\mu(x)\alpha, \alpha) + \left(1 - \frac{2}{p}\right)\imath(\mu(x)\beta, \alpha). \quad (3.21)$$

It is helpful to note that a suitably chosen weighted Young inequality gives

$$\left(1 - \frac{2}{p}\right)|\alpha||\beta| \leq \frac{p-2}{2\sqrt{p-1}} \left(\frac{1}{p'}|\alpha|^2 + \frac{1}{p}|\beta|^2 \right) = \sigma_p \left(\frac{1}{p'}|\alpha|^2 + \frac{1}{p}|\beta|^2 \right). \quad (3.22)$$

We first estimate (the absolute value of) the imaginary part of (3.21). Set

$$t_\xi(x) := \frac{\frac{1}{p} \operatorname{Re}(\mu(x)v, v) + \left(1 - \frac{2}{p}\right) \operatorname{Re}(\mu(x)\alpha, \alpha)}{\frac{1}{p'}|\alpha|^2 + \frac{1}{p}|\beta|^2}.$$

Thus, using (3.22) and the angle of sectoriality $\omega_{\mu(x)}$ for $\mu(x)$:

$$\begin{aligned} \frac{1}{2} |\operatorname{Im}(\mu(x)\xi, \mathcal{J}_p(\xi))| &\leq \frac{1}{p} \tan(\omega_{\mu(x)}) \operatorname{Re}(\mu(x)\xi, \xi) + \left(1 - \frac{2}{p}\right) \tan(\omega_{\mu(x)}) \operatorname{Re}(\mu(x)\alpha, \alpha) \\ &\quad + \left(1 - \frac{2}{p}\right) \|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} |\alpha| |\beta| \\ &\leq \left(\tan(\omega_{\mu(x)}) t_\xi(x) + \sigma_p \|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} \right) \left(\frac{1}{p'} |\alpha|^2 + \frac{1}{p} |\beta|^2 \right). \end{aligned}$$

Completely analogously we find

$$\frac{1}{2} \operatorname{Re}(\mu(x)\xi, \mathcal{J}_p(\xi)) \geq \left(t_\xi(x) - \sigma_p \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} \right) \left(\frac{1}{p'} |\alpha|^2 + \frac{1}{p} |\beta|^2 \right).$$

Since $t_\xi(x) \geq m(x)$ for all $\xi \in \mathbb{C}^d$, we rediscover Lemma 3.24. But we can also say a bit more. Indeed, for $p \in [2, q]$, the right-hand side is positive, so that

$$\frac{|\operatorname{Im}(\mu(x)\xi, \mathcal{J}_p(\xi))|}{\operatorname{Re}(\mu(x)\xi, \mathcal{J}_p(\xi))} \leq \frac{\tan(\omega_{\mu(x)}) t_\xi(x) + \sigma_p \|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{t_\xi(x) - \sigma_p \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}.$$

The right-hand side in the foregoing inequality is *increasing* in $t_\xi(x)$ for every $\xi \in \mathbb{C}^d$. Thus, we minimize it simultaneously with respect to ξ by replacing $t_\xi(x)$ by the uniform lower bound $m(x)$, and then take the essential supremum over $x \in \Omega$ to complete the proof. Note that from the construction of η and q in (3.16) and the choice of p , the denominator in (3.20) is greater than zero, cf. (3.17). $\square$

Remark 3.29. We obtain a pointwise essential supremum in (3.20). We could also obtain a *uniform-data-estimate* in the following sense: Put

$$\eta_\bullet := \frac{\operatorname{ess\,sup}_{x \in \Omega} \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{m_\bullet} \quad \text{and} \quad q_\bullet := \Psi^{-1}(\eta_\bullet).$$

Then $\eta_\bullet \geq \eta$, so that $q_\bullet \leq q$ with η and q as in (3.16). By simple modifications in the foregoing proof, we obtain $N_p(\mu(x)) \subseteq \Sigma_{\alpha_p^\bullet}$ with

$$\tan(\alpha_p^\bullet) := \frac{\tan(\omega(\mu)) m_\bullet + \sigma_p \operatorname{ess\,sup}_{x \in \Omega} \|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{m_\bullet - \sigma_p \operatorname{ess\,sup}_{x \in \Omega} \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}.$$

This is the angle derived in [18, Sect. 8]—in a more general situation with dynamic boundary conditions—which is in general larger than α_p .

We combine Theorem 3.27 and Lemma 3.28 to obtain:

Theorem 3.30. Suppose that $\Re \mu(x) \succeq m(x) \geq m_\bullet$ for every $x \in \Omega$. Set

$$\eta := \operatorname{ess\,sup}_{x \in \Omega} \frac{\|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{m(x)} \quad \text{and} \quad q := \Psi^{-1}(\eta). \quad (3.16)$$

Then $q > 2$, the operators A_q and $A_{q'}$ are accretive, and for $p \in (q', q)$, we have $N(A_p) \subseteq \Sigma_{\alpha_p}$ with

$$\tan(\alpha_p) := \operatorname{ess\,sup}_{x \in \Omega} \left[\frac{\tan(\omega_{\mu(x)})m(x) + \sigma_p \|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}}{m(x) - \sigma_p \|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}} \right]. \quad (3.20)$$

Let us mention that the way of splitting Theorem 3.30 into the general numerical range principle in Theorem 3.27 and an estimate for the coefficient function as in Lemma 3.28 in fact mimics the structure of the many proofs in this area. In particular, also the sharp estimate (3.13) is derived by following the same logic.

Remark 3.31. While we recover the good estimate $\alpha_2 = \omega(\mu)$ for $p = 2$, the foregoing theorem can essentially never be sharp since it incorporates the imaginary part of μ and there can be several matrices μ with different imaginary parts that nevertheless represent the same operator A through the form $\mathfrak{t}$. We also remark that (3.20) does not reduce to the estimate (3.13) for real matrices if $\operatorname{Im} \mu \equiv 0$, and that there are precise results for the case when $\operatorname{Im} \mu$ is symmetric from [23], see also [33], which obviously cannot be recovered by (3.20). See the brief discussion in [18, Sect. 8]. Nevertheless, we expect that the foregoing theorem can be of particular use in the situation where one can *a priori* assume that $\|\operatorname{Im} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d} \ll m(x)$. Then p can be large and $\tan(\alpha_p)$ becomes a perturbation of $\tan(\omega(\mu))$ in terms of the supremum of $\|\operatorname{Re} \mu(x)\|_{\mathbb{R}^d \rightarrow \mathbb{R}^d}/m(x)$. Such a situation may in particular occur when the imaginary parts of the coefficient in $\mu(x)$ are only small perturbations of 0, for example in measurement errors. Of course, in such a scenario one is generically not in the position to know whether $\operatorname{Im} \mu$ is symmetric or not.

It follows the proof of Theorem 3.27. We proceed in several steps with the big picture being as follows: With the single-valued duality mapping $J(u) = u^* = |u|^{p-2}\bar{u}/\|u\|_{L^p(\Omega)}^{p-1}$ on the reflexive Banach space $L^p(\Omega)$, we would be interested to write, for all $u \in D(A_p)$ with $\|u\|_{L^p(\Omega)} = 1$ to get rid of the normalizing factor,

$$u^*(A_p u) \approx \int_{\Omega} (A_p u) |u|^{p-2} \bar{u} = \mathfrak{t}_V(u, |u|^{p-2} u)$$

and then estimate the form, which will lead to the p -ellipticity expression. However, it is in general not clear whether $|u|^{p-2} u \in V$, the form domain of $\mathfrak{t}_V$. We thus go along a few detours, essentially working backwards. In **Step 1**, we show that (figuratively)

$$\mathfrak{t}(u, |u|^{p-2} u) = \int_{\Omega} (\mu(\alpha + \imath\beta), \mathcal{J}_p(\alpha + \imath\beta))$$

for some functions α, β , if u is smooth and bounded from above and away from zero. This follows essentially the ingenious calculations of Maz'ya and Cialdea [23] and already implies that $\mathbf{t}(u, |u|^{p-2}u) \in \Sigma_\theta$ in view of the given assumption. In order to leverage this result in **Step 2**, we introduce a cut-off $|\cdot|_K$ at levels $1/K$ and K and show that also $\mathbf{t}(u, |u|_K^{p-2}u) \in \Sigma_\theta$ for all $u \in W^{1,2}(\Omega)$. This will allow to recover $N(A_p) \subseteq \Sigma_\theta$ in **Step 3**.

So, from now on, assume that $N_p(\mu(x)) \subseteq \Sigma_\theta$ for some $\theta \in [0, \frac{\pi}{2}]$ and uniformly for almost all $x \in \Omega$. Here is **Step 1**, in sufficient generality for all purposes later:

Lemma 3.32. *Let $\Lambda \subseteq \Omega$ be open and let $u \in C^1(\Lambda) \cap W^{1,2}(\Lambda)$ such that $|u|$ is bounded from above and below by strictly positive constants. Then*

$$\int_{\Lambda} (\mu \nabla u, \nabla(|u|^{p-2}u)) \in \Sigma_\theta.$$

Proof. Observe that due to the supposed properties of u , all the subsequent formal manipulations, essentially following Cialdea/Maz'ya [23], are strictly justified. Putting $v := |u|^{\frac{p-2}{2}}u$, we calculate

$$\begin{aligned} \int_{\Lambda} (\mu \nabla u, \nabla(|u|^{p-2}u)) &= \int_{\Lambda} (\mu \nabla(|v|^{\frac{2-p}{p}}v), \nabla(|v|^{\frac{p-2}{p}}v)) \\ &= \int_{\Lambda} (\mu \nabla v, \nabla v) - \left(1 - \frac{2}{p}\right)^2 \int_{\Lambda} (\mu \nabla |v|, \nabla |v|) \\ &\quad + \left(1 - \frac{2}{p}\right) \left(\int_{\Lambda} \left(\mu \frac{\bar{v}}{|v|} \nabla v, \nabla |v| \right) - \int_{\Lambda} \left(\mu \nabla |v|, \frac{\bar{v}}{|v|} \nabla v \right) \right). \end{aligned}$$

Further following [23, Corollary 1], we put $\phi := \operatorname{Re}(\frac{\bar{v}}{|v|} \nabla v)$ and $\psi := \operatorname{Im}(\frac{\bar{v}}{|v|} \nabla v)$. So:

$$\nabla |v| = \nabla(v\bar{v})^{\frac{1}{2}} = \frac{1}{2} \frac{1}{|v|} (v \nabla \bar{v} + \bar{v} \nabla v) = \operatorname{Re}\left(\frac{\bar{v}}{|v|} \nabla v\right) = \phi.$$

It follows that

$$\int_{\Lambda} (\mu \nabla v, \nabla v) = \int_{\Lambda} \left(\mu \frac{\bar{v}}{|v|} \nabla v, \frac{\bar{v}}{|v|} \nabla v \right) = \int_{\Lambda} (\mu(\phi + \imath\psi), \phi + \imath\psi)$$

and overall

$$\begin{aligned}
\int_{\Lambda} (\mu \nabla u, \nabla(|u|^{p-2}u)) &= \int_{\Lambda} (\mu(\phi + \psi), \phi + \psi) - \left(1 - \frac{2}{p}\right)^2 \int_{\Lambda} (\mu\phi, \phi) \\
&\quad + \left(1 - \frac{2}{p}\right) \left(\int_{\Lambda} (\mu(\phi + \psi), \phi) - \int_{\Lambda} (\mu\phi, \phi + \psi) \right) \\
&= \left(1 - \left(1 - \frac{2}{p}\right)^2\right) \int_{\Lambda} (\mu\phi, \phi) + \int_{\Lambda} (\mu\psi, \psi) \\
&\quad + \frac{2}{p'} \int_{\Lambda} (\mu\psi, \phi) - \frac{2}{p} \int_{\Lambda} (\mu\phi, \psi)
\end{aligned}$$

Staring at the last right-hand side for a minute and setting $\varphi := \frac{2\phi}{p}$, we find

$$\int_{\Lambda} (\mu \nabla u, \nabla(|u|^{p-2}u)) = \int_{\Lambda} (\mu(\varphi + \psi), \mathcal{J}_p(\varphi + \psi))$$

and the claim follows. $\square$

We proceed with **Step 2** to remove the restriction on u in Lemma 3.32. It will cost us a cut-off, but one that we can deal with later.

Lemma 3.33. *Let $u \in C^1(\Omega) \cap W^{1,2}(\Omega)$. Define, for every $K > 0$ and $z \in \mathbb{C}$,*

$$|z|_K := \left(\frac{1}{K} \vee (|z| \wedge K) \right)$$

Then

$$\int_{\Omega} (\mu \nabla u, \nabla(|u|_K^{p-2}u)) \in \Sigma_{\theta}.$$

Proof. We have by the usual (distributional) rules of differentiation

$$\frac{\partial}{\partial x_j} (|u|_K^{p-2}u) = \begin{cases} \frac{\partial}{\partial x_j} (|u|^{p-2}u) & \text{if } \frac{1}{K} < |u| < K, \\ K^{p-2} \frac{\partial u}{\partial x_j} & \text{if } |u| \geq K, \\ \frac{1}{K^{p-2}} \frac{\partial u}{\partial x_j} & \text{if } |u| \leq \frac{1}{K}. \end{cases} \quad (3.23)$$

This gives

$$\begin{aligned}
\int_{\Omega} (\mu \nabla u, \nabla(|u|_K^{p-2}u)) &= \int_{\frac{1}{K} < |u| < K} (\mu \nabla u, \nabla(|u|^{p-2}u)) \\
&\quad + K^{p-2} \int_{|u| \geq K} (\mu \nabla u, \nabla u) + \frac{1}{K^{p-2}} \int_{|u| \leq \frac{1}{K}} (\mu \nabla u, \nabla u).
\end{aligned} \quad (3.24)$$

We put $\Lambda := [\frac{1}{K} < |u| < K]$, which is an open subset of Ω . By the foregoing Lemma 3.33, for the first term in (3.24) we have

$$\int_{\Lambda} (\mu \nabla u, \nabla(|u|^{p-2}u)) \in \Sigma_{\theta}$$

The second and third term in (3.24) are contained in $\Sigma_{\omega(\mu)} \subseteq \Sigma_{\theta}$, recall (3.19). With Σ_{θ} being a convex cone, it follows that

$$\int_{\Omega} (\mu \nabla u, \nabla(|u|_K^{p-2}u)) \in \Sigma_{\theta}. \quad \square$$

Lemma 3.34. *Let $u \in W^{1,2}(\Omega)$ and $K > 0$. Then $|u|_K^{p-2}u \in W^{1,2}(\Omega)$ and $\mathfrak{t}(u, |u|_K^{p-2}u) \in \Sigma_{\theta}$.*

Proof. Let $K > 0$ be arbitrary but fixed. We note that the chain rule (3.23) remains valid for $u \in W^{1,2}(\Omega)$. Thus, using that

$$\frac{\partial}{\partial x_j} |v| = \frac{1}{2} \frac{\bar{v}}{|v|} \frac{\partial v}{\partial x_j} + \frac{1}{2} \frac{v}{|v|} \frac{\partial \bar{v}}{\partial x_j} = \frac{\operatorname{Re}(v \frac{\partial \bar{v}}{\partial x_j})}{|v|},$$

for $v \in W^{1,2}(\Omega)$ bounded away from zero ([74, Prop. 4.4]), we find that

$$\frac{\partial}{\partial x_j} (|u|_K^{p-2}u) = |u|_K^{p-2} \frac{\partial u}{\partial x_j} + \chi_{[\frac{1}{K} < |u| < K]} (p-2)u|u|^{p-4} \operatorname{Re}\left(u \frac{\partial \bar{u}}{\partial x_j}\right). \quad (3.25)$$

It is now easily seen that $|u|_K^{p-2}u \in W^{1,2}(\Omega)$ for every $u \in W^{1,2}(\Omega)$. This is the first assertion. In order to remove the smoothness assumption from Lemma 3.33, let $u \in W^{1,2}(\Omega)$ be arbitrary. Then, by the Meyers-Serrin theorem, there is a sequence (u_n) in $C^1(\Omega) \cap W^{1,2}(\Omega)$ such that $u_n \rightarrow u$ in $W^{1,2}(\Omega)$. From (3.25), we see that the sequence $(|u_n|_K^{p-2}u_n)$ is uniformly bounded in $W^{1,2}(\Omega)$ and thus admits a weakly convergent subsequence that converges to some $v \in W^{1,2}(\Omega)$. Passing to another subsequence, we also know that (u_n) converges pointwise almost everywhere to u . It follows that $v = |u|_K^{p-2}u$ and so, from Lemma 3.33,

$$\mathfrak{t}(u, |u|_K^{p-2}u) = \lim_{n \rightarrow \infty} \mathfrak{t}(u_n, |u_n|_K^{p-2}u_n) \in \Sigma_{\theta}. \quad \square$$

We collect the overall conclusion in **Step 3**:

Proof of Theorem 3.27. Let $u \in \operatorname{dom}(A_p)$ with $\|u\|_{L^p(\Omega)} = 1$. We show that $u^*(A_p u) \in \Sigma_{\theta}$. In fact, it will be enough to show the latter for u from a core of $\operatorname{dom}(A_p)$, because if $v \in \operatorname{dom}(A_p)$ with $\|v\|_{L^p(\Omega)} = 1$ is approximated by a sequence (u_n) in a core of $\operatorname{dom}(A_p)$, then $A_p u_n \rightarrow A_p v$ and $u_n^* := |u_n|^{p-2}u_n / \|u_n\|_{L^p(\Omega)}^{p-1}$ converges weakly to $|v|^{p-2}v = v^*$ in $L^p(\Omega)^*$. Thus, $u_n^*(A_p u_n) \rightarrow v^*(A_p v)$ and if $u_n^*(A_p u_n) \in \Sigma_{\theta}$, then so is $v^*(A_p v)$. Since the

semigroups T_2 and T_p are consistent by construction, $\text{dom}(A_p) \cap \text{dom}(A_2)$ is a core for $\text{dom}(A_p)$, see [22, Lemma 3.1], and that one will work for us. So we show the assertion for $u \in \text{dom}(A_p) \cap \text{dom}(A_2)$ with $\|u\|_{L^p(\Omega)} = 1$. Indeed, with $|u|_K^{p-2}u \in L^p(\Omega) \cap L^2(\Omega)$ and Dominated Convergence we find

$$u^*(A_p u) = \int_{\Omega} (A_p u) \bar{u} |u|^{p-2} = \lim_{K \rightarrow \infty} \int_{\Omega} (A_p u) \bar{u} |u|_K^{p-2} = \lim_{K \rightarrow \infty} \int_{\Omega} (A_2 u) \bar{u} |u|_K^{p-2}. \quad (3.26)$$

In order to relate the last integral in (3.26) with $\mathbf{t}_V(u, |u|_K^{p-2}u)$, we need to show that $|u|_K^{p-2}u \in V$. We do already have $u \in \text{dom}(A_2) \subseteq V$.

- Let V be defined as the closure of $\underline{W}_D^{1,2}(\Omega)$ as in (3.11). If $D = \emptyset$, then $V = W^{1,2}(\Omega)$, and by Lemma 3.34 $|u|_K^{p-2}u \in W^{1,2}(\Omega)$, so we are done. Otherwise, there is a sequence $(u_n) \subseteq \underline{W}_D^{1,2}(\Omega)$ such that $u_n \rightarrow u$ in $W^{1,2}(\Omega)$. By definition, $\text{dist}(\text{supp}(u_n), D) > 0$ for each n , so we recover the same for each $|u_n|_K^{p-2}u_n$. In particular, $(|u_n|_K^{p-2}u_n) \subseteq \underline{W}_D^{1,2}(\Omega) \subseteq V$. As in the proof of Lemma 3.34, there is a subsequence of $(|u_n|_K^{p-2}u_n)$ that converges weakly to $|u|_K^{p-2}u$ in $W^{1,2}(\Omega)$, and that implies that the latter is an element of V .
- If V carries good Neumann boundary conditions, that is, it is defined via (3.10), then there is a sequence $(u_n) \subseteq C_D^\infty(\Omega)$ such that $u_n \rightarrow u$ in the $W^{1,2}(\Omega)$ norm. Again, there is a subsequence of $(|u_n|_K^{p-2}u_n)$ that converges weakly to $|u|_K^{p-2}u$ in $W^{1,2}(\Omega)$. It is thus enough to show that $(|u_n|_K^{p-2}u_n) \subseteq V = W_D^{1,2}(\Omega)$. For this purpose, let n be fixed. As before, it is clear that since $u_n \in C_D^\infty(\Omega)$, we still have $\text{dist}(\text{supp}(|u_n|_K^{p-2}u_n), D) > 0$. From the definition of $C_D^\infty(\Omega)$, there is a $v \in C_c^\infty(\mathbb{R}^d)$ such that $u_n = v|_{\Omega}$. Let (ξ_k) be a usual mollifier family on $\mathbb{R}^d$. Let $v_k := |v|_K^{p-2}v * \xi_k \in C_c^\infty(\mathbb{R}^d)$. Then, for k large enough, we retain $\text{dist}(\text{supp}(v_k), D) > 0$ and $v_k|_{\Omega} \rightarrow |u_n|_K^{p-2}u_n$ in $W^{1,2}(\Omega)$ as $k \rightarrow \infty$. But then the latter is an element of $W_D^{1,2}(\Omega) = V$ as desired.

Now, with the knowledge that $|u|_K^{p-2}u \in V$, we can continue in (3.26), relating the last integral with $\mathbf{t}_V$, to obtain from Lemma 3.33

$$u^*(A_p u) = \lim_{K \rightarrow \infty} \mathbf{t}_V(u, |u|_K^{p-2}u) \in \Sigma_\theta. \quad \square$$

3.4 The operator in negative Sobolev scales

In the previous results we have compiled many results of sector inclusions for the numerical range of second-order elliptic differential operators with real or complex coefficient functions on the function spaces $L^p(\Omega)$, for $1 < p < \infty$. Yet, nowadays it is understood that for the treatment of many real world problems, spaces from the Lebesgue scale are not always sufficient for a satisfactory analytical problem setup, but rather larger function spaces should be considered. This concerns in particular nonlinear, say, quasilinear problems via permanence principles for the dependence of the differential

operator on the solutions, but also structural features of the problem setting. For example, inhomogeneous Neumann boundary data cannot be incorporated in an $L^p(\Omega)$ problem formulation directly, whereas it can be represented easily in a weak formulation in dual spaces of Sobolev spaces such as $W_D^{-1,2}(\Omega)$, the antidual of $W_D^{1,2}(\Omega)$. We define the latter to be the V with good Neumann boundary conditions, that is, given by the closure of $C_D^\infty(\Omega)$ as in (3.10), and do not consider V defined via (3.11) from now on. Yet also $W_D^{-1,2}(\Omega)$, while a Hilbert space and thus quite convenient, is not always adequate to provide a suitable functional-analytical framework to analyze the problem in. For example, the three-dimensional thermistor problem was shown to be wellposed in the ambient (spatial) function space $W_D^{-1,q}(\Omega)$ with $q > d = 3$ (see [70, 71]), which is smaller than $W_D^{-1,2}(\Omega)$ and which we define as follows:

Definition 3.35. Let $q \in (1, \infty)$. Define $W_D^{1,q}(\Omega)$ to be the completion of $C_D^\infty(\Omega)$ as in (3.10) with respect to the $W^{1,q}(\Omega)$ norm. Put further $W_D^{-1,q}(\Omega) := W_D^{1,q'}(\Omega)^*$, the antidual of $W_D^{1,q'}(\Omega)$.

So, we are interested in obtaining also similar results as for the $L^p(\Omega)$ scale also for the operators in $W_D^{-1,q}(\Omega)$ associated with $\mathbf{t}_V$. In order to make use of the good results for $L^p(\Omega)$, we propose to use square root operators to transfer the $L^p(\Omega)$ results to the $W_D^{-1,q}(\Omega)$ scale. Let us make this precise with a few definitions and notations. As before, we denote by A the operator on $L^2(\Omega)$ induced by the form $\mathbf{t}_V$ with $V = W_D^{1,2}(\Omega)$, corresponding to (3.10).

For this subsection, we suppose that Ω is a bounded domain in $\mathbb{R}^d$ which is Lipschitz around $\partial\Omega \setminus D$ as in Definition 3.23. Moreover, D is *Ahlfors-David-regular*, that is, there is a constant $C > 0$ such that, with the $(d-1)$ -dimensional Hausdorff measure $\mathcal{H}_{d-1}$,

$$Cr^{d-1} \leq \mathcal{H}_{d-1}(D \cap B_r(x)) \leq C^{-1}r^{d-1} \quad (x \in D, r \leq 1).$$

This is not the most general geometric setup that we could use (cf. [10, 13]) but we consider it a fair trade-off between generality vs. technical effort at this point. As mentioned in Remark 3.22, the Lipschitz property of $\partial\Omega \setminus D$ provides us with a Sobolev extension operator. In particular, we have the embedding property $W_D^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ as in (3.14) and the results laid out in Sections 3.3.1 and 3.3.2 apply. Indeed, as explained there, for $p \in \mathcal{I}(A)$ with, as the reader recalls,

$$\mathcal{I}(A) := \left\{ p \in (1, \infty) : \sup_{t>0} \|T(t)\|_{L^p(\Omega) \rightarrow L^p(\Omega)} < \infty \right\}, \quad (3.12)$$

we obtain well defined operators A_p on $L^p(\Omega)$ that are consistent with $A = A_2$, whose negatives generate analytic semigroups, and whose angle of sectoriality $\phi(A_p)$ and associated ones coincide with the corresponding angles of A , so for $p = 2$. For a general complex coefficient function μ , the set $\mathcal{I}(A)$ contains at least all p satisfying $|\frac{1}{2} - \frac{1}{p}| \leq \frac{1}{d}$, in particular, for $d = 2$, we have $\mathcal{I}(A) = (1, \infty)$, and equally so for a real coefficient function (in any dimension).

As announced before, we would like to work with the square root of operators, in particular the ones of A_p or rather—for injectivity reasons— $A_p + 1$. We thus note that with sectoriality of A_p , the operators $A_p + 1$ are *positive* operators in the sense of Triebel [83, Ch. 1.15.1]:

Definition 3.36. We say that a densely defined closed linear operator B on a Banach space X is *positive* if $(-\infty, 0] \subset \rho(B)$ and there is a constant $C \geq 0$ such that

$$\|(B + t)^{-1}\|_{X \rightarrow X} \leq \frac{C}{1 + t} \quad (t \in [0, \infty)). \quad (3.27)$$

Compare also [3, Sect. 3.8]. Note that a positive operator in the foregoing sense is sectorial as in Definition 3.2; the main difference is found in the requirement that $0 \in \rho(B)$ is invertible and the thus-adapted growth condition (3.27). A useful feature of positive operators is that one can define their fractional powers B^β by direct means—the Balakrishnan formula—without passing through a full functional calculus.

The following abstract result, specialized to the present use case, shows that precisely those fractional powers are a convenient means to transfer operator properties.

Proposition 3.37 ([8, Lem. 11.4]). *Let X, Y be Banach spaces such that $X \hookrightarrow Y$ densely. Suppose that B is a positive operator on X and that there is a $\beta \in (0, 1)$ such that $B^{-\beta}$ admits an extension to a topological isomorphism $Y \rightarrow X$. Then we have the following:*

- (i) *There is a unique extension $\overline{B}$ of B to Y which is also a positive operator on Y .*
- (ii) *If B admits a bounded H^∞ -calculus on X , then $\overline{B}$ admits a bounded H^∞ -calculus on Y with the same H^∞ -angle.*

In fact, in the setting of Proposition 3.37, the operators $\overline{B}$ and B are similar via the extension of $B^{-\beta}$. Note that one can frame Proposition 3.37 in the context of *extrapolation* scales, cf. [1, Ch. V.I], where Y is defined as the completion of X with respect to the graph norm of B^β . Then the isomorphism property required in Proposition 3.37 is true by design, yet there is only an implicit understanding of the space Y in general. By contrast, we fix a particular space Y from the beginning—here it will, of course, be a negative Sobolev space—and rely on sophisticated results that provide the required isomorphism property for the inverse fractional power, which will be a square root.

Indeed, we know that the operators $A_p + 1$ have the desirable properties that can be transferred via Proposition 3.37, so, specializing to $\beta = 1/2$, all that remains—yet that is far from obvious—is the isomorphism property for the square root $(A_p + 1)^{-1/2}$. Fortunately, we are standing on the shoulders of giants, and from the main results in [10, 39] (see also [11]) and taking adjoints as in the proof of [8, Thm. 11.5], we obtain:

Proposition 3.38. *There exists $\varepsilon > 0$ such that if $p \in \mathcal{I}(A) \cap (1, 2 + \varepsilon)$ and $q = p'$, then $(A_q + 1)^{-\frac{1}{2}}: L^q(\Omega) \rightarrow L^q(\Omega)$ extends to a topological isomorphism*

$$\widetilde{(A_q + 1)^{-\frac{1}{2}}}: W_D^{-1,q}(\Omega) \rightarrow L^q(\Omega). \quad (3.28)$$

The lower bound on q in Proposition 3.38 comes from ε , which depends on the coefficient function μ and the geometry of Ω and D . Recall from the above explanations regarding $\mathcal{I}(A)$ that if the coefficient function μ is real or if $d = 2$, then (3.28) is certainly true for all $q \in [2, \infty)$, whereas for a complex coefficient function with $d \geq 3$, it is at least always true for all $q \in [2, 2^*]$, with the Sobolev conjugate 2^* of 2. Of course, there are more precise statements on the range of admissible q for (3.28), linked to optimal elliptic regularity results, but we do not go into details here and refer to [10, 39] and the references there. We can afford to do so because the most crucial case for us $q > d$ is already admissible for $d = 3$ without further ado due to $2^* = 6$.

Now, in order to leverage Proposition 3.38, let $\mathcal{A}: W_D^{1,2}(\Omega) \rightarrow W_D^{-1,2}(\Omega)$ be the operator associated to $\mathbf{t}_V$ on $V = W_D^{1,2}(\Omega)$, that is, such that

$$\mathbf{t}_V(u, v) = \langle \mathcal{A}u, v \rangle \quad (u, v \in W_D^{1,2}(\Omega))$$

Now let q be as in Proposition 3.38. If $q > 2$, consider the part $\mathcal{A}_q$ of $\mathcal{A}$ in $W_D^{-1,q}(\Omega)$, whereas for $q < 2$, we define $\mathcal{A}_q$ to be the *closure* of $\mathcal{A}$ in $W_D^{-1,q}(\Omega)$. Then $\mathcal{A}_q + 1$ is the natural candidate for the extension of $A_q + 1$ to $W_D^{-1,q}(\Omega)$ in Proposition 3.37 via Proposition 3.38, and we indeed obtain:

Corollary 3.39. *Let q be as in Proposition 3.38. Then $\mathcal{A}_q + 1$ is a positive operator on $W_D^{-1,q}(\Omega)$ and admits a bounded H^∞ -calculus on $W_D^{-1,q}(\Omega)$ with the H^∞ -angle ω . In particular, $\mathcal{A}_q$ satisfies maximal parabolic regularity on $W_D^{-1,q}(\Omega)$.*

In particular, the extended square root $\widetilde{(A_q + 1)^{-1/2}}$ in (3.28) coincides with the inverse square root $(\mathcal{A}_q + 1)^{-1/2}$ of $\mathcal{A}_q + 1$. Moreover, $\mathcal{A}_q$ is the negative generator of an holomorphic, strongly continuous semigroup $\mathcal{T}_q$ on $W_D^{-1,q}(\Omega)$ whose angle of holomorphy again coincides with ω .

We mention also that via similarity, we can factorize the resolvents of $\mathcal{A}_q$ and the semigroup $\mathcal{T}_q$ generated by $-\mathcal{A}_q$ by means of A_q and T_q :

$$(\mathcal{A}_q + \lambda)^{-1} = (A_q + 1)^{1/2} (A_q + \lambda)^{-1} (\mathcal{A}_q + 1)^{-1/2} = (\mathcal{A}_q + 1)^{1/2} (A_q + \lambda)^{-1} (\mathcal{A}_q + 1)^{-1/2} \quad (3.29)$$

and, with T_q being the semigroup generated by $-A_q$ on $L^q(\Omega)$,

$$\mathcal{T}_q(t) = (A_q + 1)^{1/2} T_q(t) (\mathcal{A}_q + 1)^{-1/2} = (\mathcal{A}_q + 1)^{1/2} T_q(t) (\mathcal{A}_q + 1)^{-1/2}.$$

From (3.29), it is immediate to see how resolvent estimates and in particular sectoriality of A_q explicitly transfer to $\mathcal{A}_q$.

3.4.1 The operators in the negative Bessel scale

Let us now consider the realization of the operator $\mathcal{A}_q$ on interpolation spaces between $L^q(\Omega)$ and $W_D^{-1,q}(\Omega)$. The motivation for doing so is as follows: In many cases, if one is confronted with nonlinear evolution equations whose reaction terms do not depend only on the physical quantities themselves but also on their gradients, then ordinary negative Sobolev spaces as introduced in the subsection before do not allow to adequately capture the corresponding nonlinearities. For example, this is the case for the VanRoosbroeck system ([54, 80]), describing the flow of electrons and holes in semiconductors, when the well-known Avalanche generation ([38, 49, 81]) is in power. See [32, 72] for more details. Yet, other aspects of the problem, such as inhomogeneous Neumann boundary data or discontinuous coefficients, do not allow to revert to the $L^p(\Omega)$ scale.

It turns out that in such cases certain interpolation spaces $\mathbf{X}_s := [L^q(\Omega), W_D^{-1,q}(\Omega)]_s$ with $s \in (0, 1)$ are adequate for the functional-analytic treatment of the corresponding equations [32, 72]. The very general geometric framework for Ω and D considered so far does not quite allow to identify $\mathbf{X}_s$ with a well known brand of function spaces. It is clear that the default candidate would be of type $H_D^{-s,q}(\Omega)$, that is, from the scale of (duals of) Bessel potential spaces, and indeed, under not much stronger geometric assumptions than required so far, it is established in the seminal paper [12] that $\mathbf{X}_s$ can be identified as the (anti-) dual of a Bessel potential space $H_D^{s,q'}(\Omega)$ whose elements are of fractional smoothness s and incorporate a homogeneous Dirichlet boundary condition on D , at least so if $s > 1 - 1/q$. While the precise identification of $\mathbf{X}_s$ with a classical type of function space is useful in many scenarios, it is not required for the following, so we do not elaborate further.

Now, let q be as in Proposition 3.38, and let $\mathbf{A}_q$ be the part of $\mathcal{A}_q$ in $\mathbf{X}_s$. Then it is by definition consistent with A_q on $L^q(\Omega)$ and with $\mathcal{A}_q$ on $\mathbf{X}_s$, and we obtain quite directly:

Proposition 3.40. *Let q be as in Proposition 3.38 and let $s \in (0, 1)$. Then $\mathbf{A}_q + 1$ is a positive operator on $\mathbf{X}_s$. Moreover, it admits a bounded H^∞ -calculus on $\mathbf{X}_s$ with the H^∞ -angle ω . In particular, $\mathbf{A}_q$ satisfies maximal parabolic regularity on $\mathbf{X}_s$.*

Proof. Let $\lambda \geq 1$. Since the resolvents $(A_q + \lambda)^{-1}$ on $L^q(\Omega)$ and $(\mathcal{A}_q + \lambda)^{-1}$ on $W_D^{-1,q}(\Omega)$ are consistent, that is, $(\mathcal{A}_q + \lambda)^{-1}$ coincides with $(A_q + \lambda)^{-1}$ on $L^q(\Omega)$, complex interpolation shows that the positive operator estimates (3.27) for these resolvents carry over to $\mathbf{X}_s$, recall also (3.29). This implies that $\mathbf{A}_q + 1$ is a positive operator on $\mathbf{X}_s$, so in particular sectorial and with the same spectral angle as $A_q + 1$ and $\mathcal{A}_q + 1$, cf. Corollary 3.39. Again by interpolation, it follows that $\mathbf{A}_q + 1$ in fact admits a bounded H^∞ -calculus on $\mathbf{X}_s$ with the H^∞ -angle ω since $A_q + 1$ and $\mathcal{A}_q + 1$ do so on $L^q(\Omega)$ and $W_D^{-1,q}(\Omega)$. $\square$

We remark that in the proof we see nicely the remarkable value of the square root isomorphism property (3.28): It allowed to transfer the desirable properties of the operators

from $L^q(\Omega)$ to $W_D^{-1,q}(\Omega)$, with explicit estimates due to the similarity relation (3.29). By jumping over a whole order of differentiability, all the spaces $\mathbf{X}_s$ *in between* are covered automatically.

Comments and open questions

We close with some concluding remarks.

- (i) The form method used to construct the operators A on $L^2(\Omega)$ does not require the particular space combination $W^{1,2}(\Omega)$ (or subspaces thereof) and $L^2(\Omega)$ but works in a very general setup in general Hilbert spaces V and H linked by a transference map $j \in \mathcal{L}(V \rightarrow H)$ with dense range [4]. It was already mentioned in Remark 2.6 that it is also of interest to consider weighted spaces of the same type, that is, with a measure different from the Lebesgue measure. This then also transfers to the corresponding L^p -realizations. The crucial point remains that the decisive sector inclusion is that of the *form*.
- (ii) Continuing the previous point, one can also incorporate dynamic boundary conditions in the abstract operator A by constructing the operator in hybrid L^2 -spaces, so in a product $L^2(\Omega) \otimes L^2(\Gamma)$ with a subset Γ of the boundary $\partial\Omega$. The corresponding measure is a sum of the Lebesgue measure on the domain and the $(d-1)$ -dimensional Hausdorff measure $\mathcal{H}_{d-1}$ on Γ . We refer to [22, 45, 53] and the recent contribution [18].
- (iii) In the context of Theorem 2.5 and Corollary 2.7, we have

$$\omega(\mathbf{t}_V) \leq \omega(\mu) \leq \operatorname{ess\,sup}_{x \in \Omega} \frac{n(\Im \mu(x))}{m(x)}.$$

We have already discussed sharpness of the second inequality in Remark 2.4, yet, of course, it is clearly also very interesting to understand whether or when the first one is sharp as well. In fact, there is a generic way to generate examples for which $\omega(\mathbf{t}_V) < \omega(\mu)$: choose V and a coefficient matrix μ such that μ is not selfadjoint, but the operator A induced by the form $\mathbf{t}_V$ is so. Such a construction seems to be hinted at in [21, Rem. 4]. Then $N(A)$ is real, thus so is $N(\mathbf{t}_V)$ and we obtain $\omega(\mathbf{t}_V) = 0$. But $N(\mu)$ cannot be a subset of $\mathbb{R}$, otherwise μ would be selfadjoint, so that $\omega(\mu) > 0$. Let for example $V = W_{\partial\Omega}^{1,2}(\Omega) = W_0^{1,2}(\Omega)$ with an open set $\Omega \subseteq \mathbb{R}^2$ and pick, say, $\mu = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$ with $\alpha \in \mathbb{R}$. Then A becomes the Dirichlet Laplacian. As mentioned before, such examples depend on both the coefficient matrix and the subspace V . Indeed, for $V = W^{1,2}(\Omega)$ with Ω bounded (in any space dimension), the example does not work and one can show that $\omega(\mathbf{t}) = \omega(\mu)$, see [2, Rem. 5.12]. It seems however quite hard to give a precise and useful characterization on when the angles coincide or not. In one space dimension it is not hard to see that there is always equality, but this relies crucially on the particular dimension 1. We leave further investigations and partial results in this direction for future work.

- (iv) We consider it frustrating that in the complex coefficients case considered in Section 3.3.2, in particular Theorem 3.30, we were not able to derive an estimate that collapses to the corresponding one for the real case considered in Theorem 3.19 if $\operatorname{Im} \mu \equiv 0$. It is our hope that some improvement will be achieved here in the future. Following our strategy of proof via Theorem 3.27, it would be sufficient to work on a sector inclusion for $N_p(\mu)$, that is, the question becomes one of matrix analysis.

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