TRUSSES, DITRUSSES, WEAK TRUSSES

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Abstract. In this paper we extend to left skew trusses $(T, +, \circ, \sigma)$ previous work on left skew rings. We had presented a left skew ring as a group $(N, +)$ with two binary operations $\circ$ and $\cdot$ with $\circ$ associative, $\cdot$ left distributive over the addition $+$ of the group, and such that the difference of the two operations $\circ$ and $\cdot$ is the binary operation $\pi_1 \colon N \times N \to N$. Here we extend this idea to the left skew trusses introduced in 2019 by Brzeziński, replacing the operation π_1 with the binary operation $\sigma\pi_1 \colon T \times T \to T$. The case where the semigroup morphism $\lambda^T \colon T \to \mathrm{End}_{\mathsf{Gp}}(T, +)$ is constant turns out to be particular interesting. We get several canonical category isomorphisms. For instance, we get a category isomorphism between the category of all left skew trusses $(T, +, \circ, \sigma)$ with $\lambda^T \colon (T, \circ) \to \mathrm{End}_{\mathsf{Gp}}(T, +)$ a constant semigroup morphism and σ, λ^T_0 image-commuting idempotent endomorphisms and the category of all associative interchange near-rings. Interchange near-rings were introduced by Edmunds in 2016. When σ is an idempotent group endomorphism of the group $(T, +)$ and $\lambda^T \colon (T, \circ) \to \mathrm{End}_{\mathsf{Gp}}(T, +)$ is a semigroup morphism constantly equal to a group endomorphism τ, we also get a sort of duality exchanging the mappings σ and τ.

1. Introduction

Left skew braces were introduced by Guarnieri and Vendramin [6] in the context of the study of set-theoretic solutions of the Yang-Baxter equation. Left skew braces—or, more precisely, left skew rings [4, 8]—can be viewed as modified left near-rings. Specifically, a left skew ring is a triple $(G, +, \circ)$ where $(G, +)$ is a (not necessarily abelian) group, $(G, \circ)$ is a semigroup, and the *left skew distributive law* holds, that is, $a \circ (b + c) = (a \circ b) - a + (a \circ c)$ for every $a, b, c \in G$. In the case of left skew braces, we additionally require that $(G, \circ)$ is a group.

Left skew rings can be also equivalently described as quadruples $(G, +, \circ, \cdot)$, where $(G, +)$ is a group, $\circ$ and $\cdot$ are binary operations, $\circ$ is associative, $\cdot$ is left distributive over $+$, and the difference between $\circ$ and $\cdot$ in the additive group of all binary operations on G equals the operation $\pi_1 \colon G \times G \to G$, the projection map onto the first factor [4].

The notion of left skew ring was generalized by Brzeziński to that of a left skew truss in [1]. A *left skew truss* is a quadruple $(T, +, \circ, \sigma)$, where $(T, +)$ is a group, $(T, \circ)$ is a semigroup, $\sigma \colon T \to T$ is a unary operation, and the *left skew σ-distributive law* holds, that is $a \circ (b + c) = (a \circ b) - \sigma(a) + (a \circ c)$ for every $a, b, c \in T$.

In this paper, we extend the framework developed for left skew rings in [4] to the setting of left skew trusses. We show that any left skew truss can be viewed as a quintuple $(G, +, \circ, \cdot, \sigma)$ where $(G, +)$ is a group, $\circ$ and $\cdot$ are binary operations, σ is a unary operation, $\circ$ is associative, $\cdot$ is distributive over $+$, and the difference $\circ - \cdot$ is the operation $\sigma\pi_1 \colon G \times G \to G$ (Subsection 2.2). A particularly nice situation arises when the mapping $\sigma \colon G \to G$ is an idempotent group endomorphism of the additive group $(G, +)$ (Subsection 2.3).

Key words and phrases. Skew brace; Truss; Skew ring; Yang-Baxter equation; Distributive law.
2020 *Mathematics Subject Classification.* Primary 16Y99. Secondary 16T25, 20N99.

For any left skew truss $(T,+,\circ,\sigma)$ and each $a \in T$, let $\lambda_a^T: T \to T$ be the mapping defined by $\lambda_a^T(b) = -\sigma(a) + (a \circ b)$ for every $b \in T$. Then: λ_a^T is a group endomorphism of $(T,+)$; we have $a \circ 0_T = \sigma(a)$ for all $a \in T$; and σ is an idempotent mapping. Moreover, if $\sigma(0_T) = 0_T$, then $\lambda_{0_T}^T$ is necessarily idempotent, and $0_T \circ a = \lambda_0^T(a)$ for all $a \in T$ (Theorems 2.4 and 3.6). We define a *left ditruss* as a quintuple $(D,+,\sigma,\circ,\cdot)$ where $+$, $\circ$, and $\cdot$ are three binary operations, σ is unary, $(D,+)$ is a group, and $\sigma(a) + a \cdot b = a \circ b$ for every $a, b \in D$. Given a left skew truss $(T,+,\circ,\sigma)$, defining the operation $\cdot$ setting $a \cdot b = -\sigma(a) + a \circ b$ for all $a, b \in T$, yields a left ditruss $(T,+,\sigma,\circ,\cdot)$.

Under mild conditions on a ditruss $(D,+,\sigma,\circ,\cdot)$, we show (Theorems 3.4 and 3.6, and Proposition 3.7):
(1) the operation $\circ$ is associative if and only if the operation $\cdot$ is *left weakly σ-associative*, i.e.,

$$(\sigma(a) + a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a, b, c \in D;$$

(2) the operation $\cdot$ is distributive if and only if the operation $\circ$ is *left skew σ-distributive*, that is, $a \circ (b + c) = a \circ b - \sigma(a) + a \circ c$ for all $a, b, c \in D$; and
(3) σ and $\lambda_{0_D}^D$ are commuting idempotent group endomorphisms of the group $(D,+)$.
Moreover, left weak σ-associativity of $\cdot$ is equivalent to requiring that the map

$$\lambda^D: D \to \mathrm{End}_{\mathsf{Gp}}(D,+)$$

is a semigroup homomorphism from $(D,\circ)$ to the semigroup $\mathrm{End}_{\mathsf{Gp}}(D,+)$ (under composition).

Among semigroup morphisms $\varphi: D \to E$ between two semigroups $D \neq \emptyset$ and E, the most natural are the constant morphisms, i.e., the morphisms $\varphi: D \to E$ for which $\varphi(a) = \varphi(b)$ for all $a, b \in D$. These constant morphisms $D \to E$ are in one-to-one correspondence with the idempotent elements of E. Studying the case where the semigroup morphism $\lambda^D: D \to \mathrm{End}_{\mathsf{Gp}}(D,+)$ is constant leads us to a connection with associative interchange near-rings (Section 4). An *interchange near-ring* is a triple $(G,+,\circ)$ where $(G,+)$ is a group and $\circ$ is a binary operation on G satisfying the *interchange law*:

$$(w + x) \circ (y + z) = (w \circ y) + (x \circ z) \text{ for every } w, x, y, z \in G.$$

Interchange near-rings were introduced in [3]. Two endomorphisms ε, η of an additive group G are said to be *image-commuting* if $\varepsilon(x)+\eta(y) = \eta(y)+\varepsilon(x)$ for all $x, y \in G$. By [3, Theorem 3.5], the interchange near-ring structures on any group $(G,+)$ are all of the form $(G,+,\varepsilon\pi_1+\eta\pi_2)$ for a pair (ε,η) of image-commuting group endomorphisms of $(G,+)$. We find natural category isomorphisms. The category of all left ditrusses $(D,+,\sigma,\circ,\cdot)$ with $\circ$ associative, constant semigroup morphism $\lambda^D: (D,\circ) \to \mathrm{End}_{\mathsf{Gp}}(D,+)$, and σ, λ_0^D image-commuting idempotent endomorphisms of $(D,+)$, is isomorphic to the category of associative interchange near-rings (Theorem 4.3). Furthermore, the functor

$$(D,+,\sigma,\circ,\tau\pi_2) \mapsto (D,+,\tau,\circ,\sigma\pi_2)$$

is an involutive categorical automorphism of the category of left ditrusses $(D,+,\sigma,\circ,\cdot)$ with $\circ$ associative and $\lambda^D: (D,\circ) \to \mathrm{End}_{\mathsf{Gp}}(D,+)$ a constant semigroup morphism. We define a *left weak truss* as a quadruple $(W,+,\cdot,\sigma)$ where: $(W,+)$ is a group, σ is a unary operation, $\cdot$ is a binary operation that is left weakly σ-associative, and $\cdot$ is left distributive:

$$(\sigma(a) + a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ and } a(b + c) = ab + ac \text{ for every } a, b, c \in W.$$

The category of left skew trusses $(T,+,\circ,\sigma)$ with σ an idempotent group endomorphism of $(T,+)$ is canonically isomorphic to the category of left weak trusses $(W,+,\cdot,\sigma)$ with σ an idempotent group endomorphism of $(W,+)$ (Theorem 4.4). This isomorphism associates to each left skew truss $(T,+,\circ,\sigma)$ with σ an idempotent group endomorphism the left weak truss $(T,+,\cdot,\sigma)$, where $\cdot$ is the binary operation on T defined by $a\cdot b=-\sigma(a)+a\circ b$ for all $a,b\in T$.

Throughout this paper, by "algebra" we mean "algebra" in the sense of Universal Algebra, and we sometimes denote algebras in an informal way. For instance, an additive group will be denoted both in the form $(G,+,-,0_G)$ as one usually correctly does in Universal Algebra and in the more common form $(G,+)$. Similarly, a near-ring will be denoted in both forms $(R,+,-,0_R,\cdot)$ and $(R,+,\cdot)$. Also, when we say that a binary operation $\cdot$ on an additive group $(G,+)$ is "left distributive", we mean that $\cdot$ distributes over addition of the group G, i.e., that $a\cdot(b+c)=a\cdot b+a\cdot c$ for every $a,b,c\in G$.

2. Basic terminology on trusses and idempotent endomappings

2.1. **Left skew trusses.** The definition of skew trusses is due to Tomasz Brzeziński [1, Section 2]. A *left skew truss* is an algebra $(T,+,-,0_T,\circ,\sigma)$, where $(T,+,-,0_T)$ is a group, $\circ$ is a binary operation on T, σ is a unary operation, and the following two axioms are satisfied:

(a) (*associativity of the operation* $\circ$) $\quad a\circ(b\circ c)=(a\circ b)\circ c,$

and

(b) (*left skew σ-distributivity*) $$a\circ(b+c)=(a\circ b)-\sigma(a)+(a\circ c) \tag{3.1}$$

for every $a,b,c\in T$.

The first examples of left skew trusses are:
(1) For the unary operation $\sigma\colon T\to T$ defined by $\sigma(a)=0_T$ for all $a\in T$, left skew trusses $(T,+,-,0_T,\circ,\sigma)$ are exactly left near-rings.
(2) For $\sigma=\mathrm{id}_T$ (the identity mapping $\mathrm{id}_T\colon T\to T$), left skew trusses $(T,+,-,0_T,\circ,\mathrm{id}_T)$ are exactly the left skew rings studied in [4].

2.2. **Idempotent endomappings.** Let T be a set. In [4, Subsection 4.1] we saw that a particularly important binary operation on T is the operation $\pi_1\colon T\times T\to T$ defined by $a\,\pi_1\,b:=a$ for every $a,b\in T$. In fact, the set of all binary operations on T turns out to be a monoid with respect to a suitably defined multiplication of binary operations, and π_1 turns out to be the identity of this monoid [7, Theorem 1.2]. For any unary operation $\sigma\colon T\to T$ on the set T, the binary operation on T naturally corresponding to π_1 is the operation $\sigma\pi_1\colon T\times T\to T$ defined by $a(\sigma\pi_1)b:=\sigma(a)$ for every $a,b\in T$. It is now important to notice that:

Lemma 2.1. *Let $\sigma\colon T\to T$ be a unary operation on a set T. The binary operation $\sigma\pi_1$ on T is associative if and only if the mapping σ is idempotent.*

Proof. For every $a,b,c\in T$, we have
$$a\,(\sigma\pi_1)(b\,(\sigma\pi_1)\,c)=\sigma(a)\quad\text{and}\quad(a\,(\sigma\pi_1)\,b)(\sigma\pi_1)\,c=\sigma(a\,(\sigma\pi_1)\,b)=\sigma(\sigma(a)).$$
□

Since we have met with idempotent endomappings σ on T, that is, mappings σ of a set T into itself such that $\sigma^2=\sigma$, let me briefly recall here the notion of semidirect product in Universal Algebra [5].

We can construct an *internal semidirect-product decomposition* of any algebra A making use of any subalgebra B of A and any congruence ω on A such that the intersection of B with any congruence class of A modulo ω is a singleton [5]. Internal semidirect-product decompositions $A = B \ltimes \omega$ of an algebra A are in one-to-one correspondence with the idempotent endomorphisms of the algebra A. If $\sigma\colon A \to A$ is an idempotent endomorphism of an algebra A, the corresponding internal semidirect-product decomposition of A is $A = \sigma(A) \ltimes \sim_\sigma$, where $\sim_\sigma$ is the kernel of the endomorphism σ, that is, the congruence $\sim_\sigma$ on A defined, for every $a, a' \in A$, by $a \sim_\sigma a'$ if $\sigma(a) = \sigma(a')$. For instance, in the case of sets (algebras with no operations), we have that, for a set A, idempotent endomappings of A are in one-to-one correspondence with semidirect-product decompositions $A = B \ltimes \sim$ of A, that is, with the set of all pairs $(B, \sim)$, where $\sim$ is any equivalence relation on A, and B is an irredundant set of representatives of A modulo $\sim$. Given any such pair $(B, \sim)$, the corresponding idempotent endomapping $\sigma\colon A \to A$ maps any element a of A to the unique element of $B \cap [a]_\sim$. Conversely, conversely, given any idempotent endomapping $\sigma\colon A \to A$, the corresponding pair is $(\sigma(A), \sim_\sigma)$.

External semidirect products (or, better, *semidirect-product extensions* of B) are constructed from, and parametrized by, all product-preserving functors $F\colon \mathcal{C}_B \to \mathsf{Set}_*$ from a suitable category $\mathcal{C}_B$ containing B (called the *enveloping category* of B or the *term category* of B) to the category Set_* of pointed sets [5]. The objects of $\mathcal{C}_B$ are the n-tuples $(b_1, \dots, b_n)$ of elements of B, with $n \geq 0$. (That is, the objects of $\mathcal{C}_B$ are all words in the alphabet B.) A morphism $(b_1, \dots, b_n) \to (c_1, \dots, c_m)$ in $\mathcal{C}_B$ is any m-tuple $(p_1, \dots, p_m)$ of n-ary terms [2, Definitions 10.1 and 10.2] such that $p_j(b_1, \dots, b_n) = c_j$ for every $j = 1, 2, \dots, m$. Composition of morphisms is defined by

$$(q_1, \dots, q_r) \circ (p_1, \dots, p_m) = (q_1(p_1, \dots, p_m), q_2(p_1, \dots, p_m), \dots, q_r(p_1, \dots, p_m)).$$

See [5, Section 4]. Two objects $(a_1, \dots, a_n), (b_1, \dots, b_m)$ are isomorphic in $\mathcal{C}_B$ if and only if there is a bijection $\varphi\colon \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$ such that $a_{\varphi(j)} = b_j$ for every $j = 1, \dots, n$.

In the category $\mathcal{C}_B$ the product of two objects $(a_1, \dots, a_n), (b_1, \dots, b_m)$ of $\mathcal{C}_B$ is the $(n+m)$-tuple $(a_1, \dots, a_n, b_1, \dots, b_m)$ (justapposition of words), with projections

$$\pi_a\colon (a_1, \dots, a_n, b_1, \dots, b_m) \to (a_1, \dots, a_n) \quad \text{and} \quad \pi_b\colon (a_1, \dots, a_n, b_1, \dots, b_m) \to (b_1, \dots, b_m),$$

the tuples of terms $\pi_a = (x_1, \dots, x_n)$ and $\pi_b = (x_{n+1}, \dots, x_{n+m})$. In order to check that the Universal Property of Product holds, fix any object $(c_1, \dots, c_r)$ of $\mathcal{C}_B$ and any pair of morphisms $f_a\colon (c_1, \dots, c_r) \to (a_1, \dots, a_n)$ and $f_b\colon (c_1, \dots, c_r) \to (b_1, \dots, b_m)$, where $f_a = (p_1, \dots, p_n)$ and $f_b = (q_1, \dots, q_m)$ are tuples of r-ary terms such that $p_j(c_1, \dots, c_r) = a_j$ for every $j = 1, \dots, n$ and $q_k(c_1, \dots, c_r) = b_k$ for every $k = 1, \dots, m$. It is easily seen that the unique morphism $f\colon (c_1, \dots, c_r) \to (a_1, \dots, a_n, b_1, \dots, b_m)$ such that $\pi_a f = f_a$ and $\pi_b f = f_b$ is the $(n+m)$-tuple of terms $f = (p_1, \dots, p_n, q_1, \dots, q_m)$.

Also, $\mathcal{C}_B$ has a terminal object (the 0-tuple, i.e., the word of length 0). Thus the category $\mathcal{C}_B$ has finite products.

Example 2.2. *(The variety of sets)* In the case of a set (an algebra with no operations), we have that, for a set A, the objects of the category $\mathcal{C}_A$ are all n-tuples $(a_1, \dots, a_n)$ of elements of A, for any integer $n \geq 0$. The terms in the variables $x_1, \dots, x_n$ are only $x_1, \dots, x_n$, so that a morphism $(a_1, \dots, a_n) \to (b_1, \dots, b_m)$ in $\mathcal{C}_A$ is any m-tuple $(x_{\varphi(1)}, \dots, x_{\varphi(m)})$ with $\varphi\colon \{1, 2, \dots, m\} \to \{1, 2, \dots, n\}$ a mapping such that $(a_{\varphi(1)}, \dots, a_{\varphi(m)}) = (b_1, \dots, b_m)$. The composite morphism of the morphisms

$$(x_{\varphi(1)}, \dots, x_{\varphi(m)})\colon (a_1, \dots, a_n) \to (b_1, \dots, b_m)$$

and

$$(x_{\psi(1)},\dots,x_{\psi(r)})\colon (b_1,\dots,b_m)\to(c_1,\dots,c_r)$$

is

$$(x_{\psi(1)},\dots,x_{\psi(r)})\circ(x_{\varphi(1)},\dots,x_{\varphi(m)})=(x_{\varphi\psi(1)},\dots,x_{\varphi\psi(r)}).$$

Here $\varphi\colon\{1,2,\dots,m\}\to\{1,2,\dots,n\}$ and $\psi\colon\{1,2,\dots,r\}\to\{1,2,\dots,m\}$ are mappings such that $(a_{\varphi(1)},\dots,a_{\varphi(m)})=(b_1,\dots,b_m)$ and $(b_{\psi(1)},\dots,b_{\psi(r)})=(c_1,\dots,c_r)$.

In the case of 1-tuples, that is, of elements of A, we have that in the category $\mathcal{C}_A$ there is no morphism $a\to a'$ for $a\neq a'$ $(a,a'\in A)$, and there is only the identity morphism $a\to a'$ for $a=a'$. Therefore, in order to describe a product-preserving functor $F\colon\mathcal{C}_A\to\mathsf{Set}_*$ it suffices to associate to any $a\in A$ a pointed set (B_a,b_a). Then to any n-tuple $(a_1,\dots,a_n)$ there corresponds via F the pointed set $(B_{a_1}\times\dots\times B_{a_n},(b_{a_1},\dots,b_{a_n}))$, and to any morphism

$$(x_{\varphi(1)},\dots,x_{\varphi(m)})\colon(a_1,\dots,a_n)\to(a'_1,\dots,a'_m)$$

there corresponds the mapping $(\pi_{\varphi(1)}\times\dots\times\pi_{\varphi(m)})\colon B_{a_1}\times\dots\times B_{a_n}\to B_{a'_1}\times\dots\times B_{a'_m}$. The semidirect-product extension corresponding to this functor $F\colon\mathcal{C}_A\to\mathsf{Set}_*$ is the disjoint union $\dot{\bigcup}_{a\in A}B_a$. Thus product-preserving functors $F\colon\mathcal{C}_A\to\mathsf{Set}_*$ parametrize all semidirect-product extensions of the set A. Notice that the idempotent endomapping corresponding to $\dot{\bigcup}_{a\in A}B_a$ is the mapping $e\colon\dot{\bigcup}_{a\in A}B_a\to\dot{\bigcup}_{a\in A}B_a$ that maps all the elements of B_a to b_a. Hence $\dot{\bigcup}_{a\in A}B_a=A\ltimes\omega$, where ω is the equivalence relation on $\dot{\bigcup}_{a\in A}B_a$ with equivalence classes the sets B_a.

2.3. **First properties of left skew trusses.**

Lemma 2.3. *Let $(T,+)$ be an additive group and $\sigma\colon T\to T$ be a mapping. Then the binary operation $\sigma\pi_1$ is:*
(a) *Right distributive if and only if σ is a group morphism.*
(b) *Always left skew σ-distributive.*
(c) *Left distributive if and only if $\sigma(a)=0_T$ for every $a\in T$.*

The proof is elementary.

In view of Lemmas 2.1 and 2.3(b), if $(T,+)$ is any group and $\sigma\colon T\to T$ is any mapping, then $(T,+,\sigma\pi_1,\sigma)$ is a left skew truss if and only if the mapping $\sigma\colon T\to T$ is idempotent. More generally:

Theorem 2.4. *If $(T,+,-,0_T,\circ,\sigma)$ is any left skew truss, then:*
(a) *For every $a\in T$, the mapping $\lambda^T_a\colon T\to T$, defined by $\lambda^T_a(b)=-\sigma(a)+(a\circ b)$ for every $b\in T$, is a group endomorphism of the group $(T,+)$.*
(b) *$a\circ 0_T=\sigma(a)$ for every $a\in T$.*
(c) *$\sigma\colon T\to T$ is an idempotent mapping.*
(d) *If $\sigma(0_T)=0_T$, then the group endomorphism $\lambda^T_{0_T}$ of $(T,+)$ is idempotent, $0_T\circ a=\lambda^T_0(a)$ for every $a\in T$, and the idempotent mappings σ and $\lambda^T_{0_T}$ commute.*

Proof. (a) is equivalent to $\lambda^T_a(b+c)=\lambda^T_a(b)+\lambda^T_a(c)$ for every $a,b,c\in T$, and this is trivially equivalent to Identity (3.1).

(b) follows from (a), because $\lambda_a(0_T)=0_T$, so that $0_T=-\sigma(a)+(a\circ 0_T)$.

(c) follows from (b) and the associativity of $\circ$.

As far as (d) is concerned, suppose that $\sigma(0_T)=0_T$. Then $\lambda^T_{0_T}(a)=-\sigma(0_T)+(0_T\circ a)=0_T\circ a$ for every $a\in T$. Then $\lambda^T_{0_T}(\lambda^T_{0_T}(a))=0_T\circ 0_T\circ a=\sigma(0_T)\circ a=0_T\circ a=\lambda^T_{0_T}(a)$ for every $a\in T$. Finally, σ and $\lambda^T_{0_T}$ commute because $0_T\circ(a\circ 0_T)=(0_T\circ a)\circ 0_T$. □

As always in Universal Algebra, left skew truss homomorphisms are defined to be the mappings that preserve all operations $+,-,0_T,\circ,\sigma$, but in view of Theorem 2.4, it suffices that they preserve addition $+$ and multiplication $\circ$. Left skew trusses form a variety in the sense of Universal Algebra. It is a variety of Ω-groups in the sense of Higgins, it is a semiabelian category.

For every left skew truss $(T,+,\circ,\sigma)$ with $\sigma(0_T)=0_T$, the group endomorphism $\lambda^T_{0_T}$ of $(T,+)$ is idempotent (Theorem 2.4(d)), so that the additive group $(T,+)$ has a semidirect-sum decomposition $T=T_0\rtimes T_c$, where $T_0:=\ker(\lambda^T_{0_T})=\{\,a\in T\mid 0_T\circ a=0_T\,\}$ (the 0-*symmetric part* of T) is a normal subgroup of $(T,+)$, and $T_c:=\lambda^T_{0_T}(T)=\{\,a\in T\mid 0_T\circ a=a\,\}$ (the *constant part* of T) is a subgroup of the additive group T. By making use of Theorem 2.4, it is easy to check that both T_0 and T_c are left skew subtrusses of $(T,+,\circ,\sigma)$, that is, they are also closed for $\circ$ and σ.

An *ideal* of the left skew truss $(T,+,\circ,\sigma)$ is a normal subgroup I of the additive group $(T,+)$ such that $(i+a)\circ b-a\circ b\in I$ and $\lambda_a(i)\in I$ for every $a,b\in T$ and $i\in I$. (Notice that, for a normal subgroup I of the additive group T, $(a+i)\circ b-a\circ b\in I$ for every $i\in I$ if and only if $(i+a)\circ b-a\circ b\in I$ for every $i\in I$.)

Proposition 2.5. *Let $(T,+,\circ,\sigma)$ be a left skew truss. There is a one-to-one correspondence between the set of all ideals of T and the set of all congruences of the left skew truss T.*

Proof. Clearly, it suffices to prove that, for a normal subgroup I of the additive group of T, the equivalence $\equiv$ on T, defined, for every $a,b\in T$, by $a\equiv b$ if $a-b\in I$, is compatible not only with the addition $+$, but also with the multiplication $\circ$ and the unary operation σ if and only if $(i+a)\circ b-a\circ b\in I$ and $\lambda_a(i)\in I$ for every $a,b\in T$ and $i\in I$.

Suppose both $\circ$ and σ compatible with the equivalence $\equiv$. Then $(i+a)\circ b\equiv(0+a)\circ b=a\circ b$, so that $(i+a)\circ b-a\circ b\in I$. Moreover, for every $a,b\in T$ and every $i\in I$, we have that $a\circ(b+i)\equiv a\circ b$, so that $-(a\circ b)+a\circ(b+i)\in I$. Hence, by left skew σ-distributivity of $\circ$, we get that $-(a\circ b)+a\circ b-\sigma(a)+a\circ i\in I$, that is $\lambda_a(i)=-\sigma(a)+a\circ i\in I$.

Conversely, assume that $(i+a)\circ b-a\circ b\in I$ and $\lambda_a(i)\in I$ for every $i\in I$, $a,b\in T$. Suppose $a,b,c,d\in T$, $a\equiv b$ and $c\equiv d$. Then $b=i+a$ and $d=c+j$ for suitable $i,j\in I$. Therefore $b\circ d=(i+a)\circ(c+j)=(i+a)\circ c-\sigma(i+a)+(i+a)\circ j\equiv a\circ c+\lambda_{i+a}(j)\equiv a\circ c$, so that $\equiv$ is compatible with $\circ$. Finally, $\equiv$ is compatible with σ, that is, $a\equiv b$ implies $\sigma(a)\equiv\sigma(b)$, because $\sigma(a)=a\circ 0_T\equiv b\circ 0_T=\sigma(b)$. □

3. Subtraction of operations and left ditrusses

3.1. **Subtraction of operations and left ditrusses.** Extending the framework developed for left skew rings in [4] to the setting of left skew trusses, we will now write the binary operation $\sigma\pi_1$ on an additive group $(T,+)$ as the difference of an associative operation and a left distributive operation. That is, for a group $(T,+)$, and a mapping $\sigma\colon T\to T$ we will look for the triples of operations $(\sigma,\circ,\cdot)$ on T with σ unary, $\circ$ associative, $\cdot$ left distributive, and $\sigma\pi_1=\circ-\cdot$. This is equivalent to $\sigma(a)=a\circ b-a\cdot b$ for every $a,b\in T$, or to $\sigma(a)+a\cdot b=a\circ b$ for every $a,b\in T$.

Definition 3.1. A *left ditruss* is an algebra $(D,+,-,0,\sigma,\circ,\cdot)$, where $+$, $\circ$ and $\cdot$ are three binary operations, $-$ and σ are unary, and 0 is nullary, satisfying the following conditions:

(1) $(D,+,-,0)$ is a group, not-necessarily abelian; and

(2)
$$\sigma(a)+a\cdot b=a\circ b \tag{3.1}$$
for every $a,b\in D$.

As usual in Universal Algebra, left ditruss morphisms are the mappings compatible with all the operations $+,-,0,\sigma,\circ$ and $\cdot$ of ditrusses, but it is clear that a mapping is a left ditruss morphism if and only if it is compatible with $+$ and with any two of the three operations $\sigma,\circ$ and $\cdot$.

Example 3.2. A particularly interesting full subcategory of the category of left ditrusses is the category $\mathcal{C}$ of the left ditrusses $(D,+,-,0,\sigma,\circ,\cdot)$ for which $\cdot$ depends only on the second factor:

Lemma 3.3. *The following conditions are equivalent for a left ditruss $(D,+,-,0,\sigma,\circ,\cdot)$:*

(a) *The binary operation $\cdot$ depends only on the second factor;*

(b) *$a\cdot b=\tau(b)$ for some mapping $\tau\colon D\to D$;*

(c) *$\cdot=\tau\pi_2$ for some mapping $\tau\colon D\to D$;*

(d) *The mapping $\lambda\colon D\to D^D$ defined by $\lambda\colon a\mapsto\lambda_a$, where $\lambda_a(b)=a\cdot b$ for every $a,b\in D$, is constant.*

The proof of Lemma 3.3 is elementary. Let $\mathcal{C}$ be the category of all left ditrusses satisfying the equivalent conditions of Lemma 3.3. We will show in Subsection 3.2 that the category $\mathcal{C}$ has an involutive automorphism $F\colon\mathcal{C}\to\mathcal{C}$ defined by $F(D,+,-,0,\sigma,\circ,\tau\pi_2)=(D,+,-,0,\tau,\circ',\sigma\pi_2)$. Here the operation $\circ$ is necessarily the operation $\sigma\pi_1+\tau\pi_2$, and $\circ'$ is defined to be $\tau\pi_1+\sigma\pi_2$. The functor F is the identity on morphisms.

Theorem 3.4. *Let $(D,+,-,0,\sigma,\circ,\cdot)$ be a left ditruss. The operation $\cdot$ is left distributive, that is, $a\cdot(b+c)=a\cdot b+a\cdot c$ for all $a,b,c\in D$, if and only if the operation $\circ$ is* left skew σ-distributive*, i.e. $a\circ(b+c)=a\circ b-\sigma(a)+a\circ c$ for all $a,b,c\in D$.*

Proof. One has
$$a\cdot(b+c)=a\cdot b+a\cdot c$$
if and only if
$$-\sigma(a)+a\circ(b+c)=-\sigma(a)+a\circ b-\sigma(a)+a\circ c,$$
that is, if and only if
$$a\circ(b+c)=a\circ b-\sigma(a)+a\circ c.$$
□

Let $(G,+)$ be a group, and $\circ$ and $\cdot$ be two binary operations on the set G with $\cdot$ distributive over $+$. A necessary and sufficient condition for the existence of a mapping $\sigma\colon G\to G$ such that $\sigma\pi_1=\circ-\cdot$, i.e., such that $(G,+,\sigma,\circ,\cdot)$ be a left ditruss, is that the difference $\circ-\cdot$ be a binary operation that does not depend on the second factor, i.e., such that $a\circ b-a\cdot b=a\circ c-a\cdot c$ for all $a,b,c\in G$. Equivalently, such that $a\circ b-a\cdot b=a\circ 0-a\cdot 0$ for all $a,b\in G$. Now left distributivity of $\cdot$ is equivalent to the fact that left multiplication $\lambda_a\colon G\to G$, $\lambda_a\colon b\mapsto a\cdot b$, is a group endomorphism of the group $(G,+)$ for every $a\in G$. In particular, we necessarily have that $a\cdot 0=0$ for every $a\in G$. Therefore, in the particular case of $\cdot$ left distributive, we have that $(G,+,\sigma,\circ,\cdot)$ is a left ditruss for some unary operation σ if and only if $a\circ b-a\cdot b=a\circ 0$

for every $a, b \in G$. Moreover, if this hold, then we must have necessarily that $\sigma(a) = a \circ 0$ for every $a \in G$. This proves that:

Proposition 3.5. *Let $(G, +)$ be a group, and $\circ$ and $\cdot$ be two binary operations on G with $\cdot$ left distributive. Then the following two conditions are equivalent:*
(a) *$a \circ b - a \cdot b = a \circ c - a \cdot c$ for every $a, b, c \in G$, that is, the binary operation $\circ - \cdot$ depends only on the first factor.*
(b) *$a \circ b - a \cdot b = a \circ 0$ for every $a, b, c \in G$.*

Moreover, if the previous two equivalent conditions hold and the mapping $\sigma \colon G \to G$ is defined setting $\sigma(a) = a \circ 0$ for every $a \in G$, then $(G, +, \circ, \cdot, \sigma)$ is a left ditruss and $a \circ (b + c) = a \circ b - \sigma(a) + a \circ c$ for every $a, b, c \in G$.

Theorem 3.6. *Let $(D, +, -, 0, \sigma, \circ, \cdot)$ be a left ditruss and suppose that the equivalent conditions of* Theorem 3.4 *hold, that is, assume that the operation $\cdot$ is left distributive. Then:*
(a) *For every $a \in D$, the mapping $\lambda^D_a \colon D \to D$, defined by $\lambda^D_a(b) = a \cdot b$ for every $b \in D$, is a group endomorphism of the group $(D, +)$.*
(b) *$a \cdot 0 = 0$, $a \cdot (-b) = -(a \cdot b)$, $a \circ 0 = \sigma(a)$, and $a \circ (-b) = \sigma(a) - a \circ b + \sigma(a)$ for every $a, b \in D$.*
(c) *Assume that the mapping σ is an idempotent group endomorphism of $(D, +)$. Then the operation $\circ$ is associative, that is, $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in D$, if and only if the operation $\cdot$ is* left weakly σ-associative, *i.e.* $(\sigma(a) + a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for all $a, b, c \in D$.*

Moreover, if the previous two equivalent conditions in (c) *hold, then $\sigma(a \cdot b) = a \cdot \sigma(b)$ for every $a, b \in D$.*

Proof. (a) is just a restatement of the hypothesis that the operation $\cdot$ is left distributive.

(b) follows from (a) and Identity (3.1). Moreover, $\sigma(a) = a \circ 0 = a \circ (b + (-b)) = a \circ b - \sigma(a) + a \circ (-b)$.

(c) Assume that σ is an idempotent endomorphism of $(D, +)$. Then $(a \circ b) \circ c = a \circ (b \circ c)$ if and only if $\sigma(\sigma(a) + a \cdot b) + (\sigma(a) + a \cdot b) \cdot c = \sigma(a) + a \cdot (\sigma(b) + b \cdot c)$, that is, if and only if

$$\sigma(a) + \sigma(a \cdot b) + (\sigma(a) + a \cdot b) \cdot c = \sigma(a) + a \cdot \sigma(b) + a \cdot (b \cdot c). \tag{3.2}$$

For $c = 0$, identity (3.2) implies that $\sigma(a \cdot b) = a \cdot \sigma(b)$ for every $a, b \in D$. Therefore identity (3.2) is equivalent to $(\sigma(a) + a \cdot b) \cdot c = a \cdot (b \cdot c)$. □

Notice that left weak σ-associativity, that is $(\sigma(a) + a \cdot b) \cdot c = a \cdot (b \cdot c)$, can be written equivalently as $(a \circ b)c = a(bc)$ (and in this form the name weak σ-associativity is justified), for all $a, b, c \in D$. Equivalently, the two equivalent statements in Theorem 3.6(c) say that the mapping λ is a semigroup morphism of the semigroup $(D, \circ)$ into the semigroup $\mathrm{End}_{\mathsf{Gp}}(D, +)$ (with composition of endomorphisms). We include this fact as statement (a) in the following proposition.

Proposition 3.7. *Let $(D, +, -, 0, \sigma, \circ, \cdot)$ be a left ditruss and suppose that σ is an idempotent additive group endomorphism of $(D, +)$, $\circ$ is associative and $\cdot$ is left distributive. Then:*
(a) *The mapping $\lambda^D \colon (D, \circ) \to \mathrm{End}_{\mathsf{Gp}}(D, +)$, defined by $\lambda^D \colon a \mapsto \lambda^D_a$, is a semigroup morphism. That is, $(a \circ b) \cdot c = a \cdot (b \cdot c)$ for every $a, b, c \in D$.*
(b) *The group endomorphism λ^D_0 of the group $(D, +, -, 0)$ is an idempotent group endomorphism.*

Proof. (a) The position $a \mapsto \lambda^D_a$ defines a mapping of D into $\mathrm{End}_{\mathsf{Gp}}(D, +)$ by Theorem 3.6(a). It is a semigroup morphism $(D, \circ) \to \mathrm{End}_{\mathsf{Gp}}(D, +)$ by Theorem 3.6(c).

(b) We must show that $\lambda_0^D \circ \lambda_0^D = \lambda_0^D$, that is, that $0 \cdot (0 \cdot a) = 0 \cdot a$. But we have already remarked that $0 \cdot a = 0 \circ a$ for all a, so that $0 \cdot (0 \cdot a) = 0 \cdot a$ is equivalent to $0 \circ (0 \circ a) = 0 \circ a$. Now $\circ$ is associative, hence $\lambda_0^D \circ \lambda_0^D = \lambda_0^D$. □

For the next two paragraphs suppose that the left ditruss D satisfies the hypotheses of Proposition 3.7. Then the two idempotent group endomorphisms σ and λ_0^D are right multiplication by 0_D and left multiplication by 0_D respectively (both with respect to the multiplication $\circ$). These two idempotent endomorphisms of the group $(D, +)$ are not necessaily equal, as the following example show. Let σ be a nontrivial idempotent endomorphism of a group $(D, +, -, 0_D)$. Consider the left ditruss $(D, +, -, 0, \sigma, \sigma\pi_1, \circ_0)$, where $\circ_0$ is the operation defined by $a \circ_0 b = 0$ for every $a, b \in D$. Then λ_0^D is the null endomorphism of $(D, +)$, which is different from σ.

Nevertheless, we have already remarked that the two idempotent group endomorphisms σ and λ_0^D always commute, because $0_D \circ (a \circ 0_D) = (0_D \circ a) \circ 0_D$.

3.2. **Properties of left ditrusses with $\cdot = \tau\pi_2$.** In Example 3.2 we have considered the ditrusses in the category $\mathcal{C}$ of all the left ditrusses $(D, +, -, 0, \sigma, \circ, \tau\pi_2)$ for some mapping $\tau \colon D \to D$. There is an involutive category automorphism $F \colon \mathcal{C} \to \mathcal{C}$ defined by

$$F(D, +, -, 0, \sigma, \circ, \tau\pi_2) = (D, +, -, 0, \tau, \circ', \sigma\pi_2).$$

Here the operations are $\circ = \sigma\pi_1 + \tau\pi_2$, and $\circ' = \tau\pi_1 + \sigma\pi_2$. Thus $(D, +, -, 0, \sigma, \circ, \tau\pi_2)$ corresponds to the difference $\sigma\pi_1 = (\sigma\pi_1 + \tau\pi_2) - \tau\pi_2$ and $(D, +, -, 0, \tau, \circ', \sigma\pi_2)$ corresponds to the difference $\tau\pi_1 = (\tau\pi_1 + \sigma\pi_2) - \sigma\pi_2$. Here σ and τ are arbitrary mappings $T \to T$.

In the following proposition we have collected some properties of the left ditrusses in the category $\mathcal{C}$.

Proposition 3.8. *Let $(D, +, -, 0, \sigma, \circ, \tau\pi_2)$ be a left ditruss where σ and τ are arbitrary mappings $D \to D$. Then:*
(a) *The binary operation $\tau\pi_2$ is left distributive if and only if τ is an endomorphism of the group $(D, +)$.*
(b) *Assume $\sigma(0) = \tau(0) = 0$. If the binary operation $\circ = \sigma\pi_1 + \tau\pi_2$ is associative, then σ and τ are idempotent endomappings of D and $\sigma\tau = \tau\sigma$.*
(c) *Let σ and τ be two idempotent endomorphisms of the group $(D, +)$ and suppose $\sigma\tau = \tau\sigma$. Then the binary operation $\circ = \sigma\pi_1 + \tau\pi_2$ is associative.*

Proof. (a) is trivial, because $a(\tau\pi_2)(b+c) = a(\tau\pi_2)b + a(\tau\pi_2)c$ if and only if $\tau(b+c) = \tau(b) + \tau(c)$.

As far as (b) and (c) are concerned, notice that $\circ = \sigma\pi_1 + \tau\pi_2$ is associative if and only if

$$a(\sigma\pi_1 + \tau\pi_2)(b(\sigma\pi_1 + \tau\pi_2)c) = a(\sigma\pi_1 + \tau\pi_2)(\sigma(b) + \tau(c)) = \sigma(a) + \tau((\sigma(b) + \tau(c))$$

is equal to

$$(a(\sigma\pi_1 + \tau\pi_2)b)(\sigma\pi_1 + \tau\pi_2)c = (\sigma(a) + \tau(b))(\sigma\pi_1 + \tau\pi_2)c = \sigma(\sigma(a) + \tau(b)) + \tau(c).$$

Therefore $\circ$ is associative if and only if

$$\sigma(a) + \tau((\sigma(b) + \tau(c)) = \sigma(\sigma(a) + \tau(b)) + \tau(c). \tag{3.3}$$

Now in order to prove (b) suppose $\sigma(0) = \tau(0) = 0$ and $\circ$ associative, so that Identity (3.3) holds for every $a, b, c \in D$. Replacing (a, b, c) in Identity (3.3) with $(a, 0, 0), (0, b, 0), (0, 0, c)$ respectively, we get that $\sigma(a) = \sigma(\sigma(a))$, $\tau(\sigma(b)) = \sigma(\tau(b))$ and $\tau(\tau(c)) = \tau(c)$, respectively. Therefore σ and τ are idempotent and commute. This proves (b).

To prove (c), suppose that σ and τ are two idempotent endomorphisms of $(D,+)$ that commute. In order to prove that the operation $\circ = \sigma\pi_1 + \tau\pi_2$ is associative, we must prove that Identity (3.3) holds for every $a, b, c \in D$. But $\sigma(a)+\tau(\sigma(b)+\tau(c)) = \sigma(a)+\tau(\sigma(b))+\tau(\tau(c)) = \sigma(\sigma(a)) + \sigma(\tau(b)) + \tau(c) = \sigma(\sigma(a)+\tau(b)) + \tau(c)$, as desired. $\square$

Let us go back to left ditrusses $(T, +, -, 0, \sigma, \circ, \cdot)$. In Proposition 3.7(a) we saw that, under suitable hypothesis, the mapping $\lambda^T\colon (T, \circ) \to \mathrm{End}_{\mathsf{Gp}}(T,+)$ is a semigroup morphism. Now, for any two semigroups T and E, the most natural examples of semigroup morphisms $T \to E$ are the *constant morphisms*, that is, the morphisms $\varphi\colon T \to E$ such that $\varphi(t) = \varphi(t')$ for every $t, t' \in T$. Clearly, for T non-empty, constant morphisms $T \to E$ are in one-to-one correspondence with the idempotent elements of E.

Corollary 3.9. *The following conditions are equivalent for an algebra $(D, +, -, 0, \circ, \sigma)$ with $(D, +, -, 0)$ a group, $\circ$ a binary operation on D and σ an idempotent group endomorphism of $(D, +, -, 0)$:*
(a) *$(D, +, -, 0, \circ, \sigma)$ is a left skew truss for which the semigroup endomorphism $\lambda^D\colon (D, \circ) \to \mathrm{End}_{\mathsf{Gp}}(D,+)$ is a constant mapping.*
(b) *There exists an idempotent group endomorphism τ of $(D, +, -, 0)$ that commutes with σ and such that $\circ = \sigma\pi_1 + \tau\pi_2$.*
(c) *There exist an idempotent group endomorphism τ of $(D, +, -, 0)$ that commutes with σ and a binary operation $\circ'$ on D such that $(D, +, -, 0, \circ', \tau)$ is a left skew truss for which the corresponding semigroup endomorphism $\lambda^D\colon (D, \circ') \to \mathrm{End}_{\mathsf{Gp}}(D,+)$ is the mapping constantly equal to σ.*
(d) *There exists an idempotent group endomorphism τ of $(D, +, -, 0)$ that commutes with σ and such that $(D, +, -, 0, \tau\pi_1 + \sigma\pi_2, \tau)$ is a left skew truss.*

Moreover, if these equivalent conditions (a)-(d) *hold, then the operation $\circ$ is not only left skew σ-distributive, but also right skew τ-distributive, and the operation $\circ'$ is not only left skew τ-distributive, but also right σ-distributive.*

Proof. (a) $\Rightarrow$ (b) Suppose that (a) holds, so that $(D, +, -, 0, \circ, \sigma)$ is a left skew truss and there exists a group endomorphisms τ of $(D,+)$ such that $\lambda^D_a = \tau$ for every $a \in D$. Since $\lambda^D\colon (D, \circ) \to \mathrm{End}_{\mathsf{Gp}}(D,+)$ is a semigroup endomorphism, we have that τ is necessarily idempotent. As $(D, +, -, 0, \circ, \sigma)$ is a left skew truss, the operation $\circ$ is associative and $\sigma\pi_1 = \circ - \cdot$, so that $a \circ b = a(\sigma\pi_1)b + a \cdot b = \sigma(a) + \lambda^D_a(b) = \sigma(a) + \tau(b)$. Therefore $\circ = \sigma\pi_1 + \tau\pi_2$. By Proposition 3.8(b), the idempotent endomorphisms σ and τ commute.

(b) $\Rightarrow$ (c) Let τ be an idempotent group endomorphism of $(D, +, -, 0)$ that commutes with σ and such that $\circ = \sigma\pi_1 + \tau\pi_2$. We will prove that the binary operation $\circ' = \tau\pi_1 + \sigma\pi_2$ on D is such that $(D, +, -, 0, \circ', \tau)$ is a left skew truss for which the corresponding semigroup endomorphism $\lambda^D\colon (D, \circ') \to \mathrm{End}_{\mathsf{Gp}}(D,+)$ is the mapping constantly equal to σ. From Example 3.2 we knwo that $(D, +, -, 0, \tau, \circ', \sigma\pi_2)$ is a left ditruss. By Proposition 3.8(c), the operation $\circ'$ is associative. Also $a \circ' (b + c) = \tau(a) + \sigma(b + c) = \tau(a) + \sigma(b) + \sigma(c) = \tau(a) + \sigma(b) - \tau(a) + \tau(a) + \sigma(c) = a \circ' b - \tau(a) + a \circ' c$, so that $(D, +, -, 0, \circ', \tau)$ is a left skew truss. Finally, for the corresponding semigroup morphism λ, we have that $\lambda_a(b) = -\tau(a) + a \circ' b = -\tau(a) + \tau(a) + \sigma(b) = \sigma(b)$.

(c) $\Rightarrow$ (d) Assume that (c) holds. In order to prove (d) it suffices to show that $\circ' = \tau\pi_1 + \sigma\pi_2$. But since $(D, +, -, 0, \circ', \tau)$ is a left skew truss, we know that $a \circ' b = \tau(a) + a \cdot b = \tau(a) + \lambda_a(b) = \tau(a) + \sigma(b) = a(\tau\pi_1 + \sigma\pi_2)b$, as desired.

(d) $\Rightarrow$ (a) Apply the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) to the left skew truss $(D, +, -, 0, \tau\pi_1 + \sigma\pi_2, \tau)$, with the roles of the two commuting idempotent group morphisms σ and τ interchanged. $\square$

4. Interchange near rings, weak right trusses

Recall that for any binary operation $\circ$ on a set D, it is possible to define its opposite $\circ^{\mathrm{op}}$ setting $a \circ^{\mathrm{op}} b = b \circ a$ for all $a, b \in D$.

Proposition 4.1. *Let $(D, +, -, 0, \sigma, \circ, \tau\pi_2)$ be a left ditruss with σ and τ arbitrary mappings $D \to D$, so that $\circ = \sigma\pi_1 + \tau\pi_2$. Assume $\circ' = \tau\pi_1 + \sigma\pi_2$. Then $\circ' = \circ^{\mathrm{op}}$ if and only if σ and τ are* image-commuting*, that is, $\sigma(x) + \tau(y) = \tau(y) + \sigma(x)$ for every $x, y \in D$.*

Proof. $\circ' = \circ^{\mathrm{op}}$ if and only if $a\circ' b = a\circ^{\mathrm{op}} b$ for every $a, b \in D$, that is, if and only if $a(\tau\pi_1+\sigma\pi_2)b = b \circ a$. This is equivalent to $\tau(a) + \sigma(b) = \sigma(b) + \tau(a)$. $\square$

Notice that, as far as the involutive category automorphism described in Corollary 3.9 is concerned (it transforms the operation $\circ = \sigma\pi_1 + \tau\pi_2$ into the operation $\circ' = \tau\pi_1 + \sigma\pi_2$), we necessarily have that $\circ$ must be left skew σ-distributive and right skew τ-distributive.

Example 4.2. *Interchange near-rings, associative interchange near-rings.* An *interchange near-ring* $(G, +, -, 0_G, \circ)$ is a group $(G, +, -, 0_G)$ with a further binary operation $\circ$ for which the *interchange law*

$$(w + x) \circ (y + z) = (w \circ y) + (x \circ z) \tag{4.1}$$

holds, for all $w, x, y, z \in G$ [3]. For an interesting excursus on the history of the interchange law, see the Introduction of [3]. An interchange near-ring $(G, +, -, 0_G, \circ)$ is an *associative* interchange near-ring if the operation $\circ$ is associative.

According to one of the main results of [3] (Theorem 3.5), the interchange near-ring structures on any group $(G, +, -, 0_G)$ are all of the form $(G, +, -, 0_G, \varepsilon\pi_1 + \eta\pi_2)$ for a pair (ε, η) of image-commuting group endomorphisms of $(G, +, -, 0_G)$. The interchange near-ring $(G, +, -, 0_G, \varepsilon\pi_1+ \eta\pi_2)$ is associative if, moreover, ε and η are commuting idempotent group endomorphisms of $(G, +, -, 0_G)$ (Proposition 3.8((b) and (c)) and [3, Theorem 4.3]).

Let $(T, +, -, 0_T)$ be a group and σ, τ be two image-commuting idempotent endomorphisms of the additive group T, and consider a left ditruss $(T, +, -, 0, \sigma, \circ, \cdot)$ with $\circ$ associative and $\lambda^T : (T, \circ) \to \mathrm{End}_{\mathsf{Gp}}(T, +)$ the constant semigroup morphism constantly equal to τ. This simply means that the multiplication $\cdot$ is the operation $\tau\pi_2$, which is always associative, left weakly σ-associative, and left distributive. As a consequence, one has that $a\circ b = \sigma(a)+\tau(b)$. In particular $0_T \circ a = \tau(a)$ and $a \circ 0_T = \sigma(a)$. From associativity we know that $0_T \circ (a \circ 0_T) = (0_T \circ a) \circ 0_T$, that is $\sigma(\tau(a)) = \tau(\sigma(a))$, so that the idempotent endomorphisms σ and τ commute. From [3, Theorem 4.3], we get that $(T, +, -, 0_T, \circ)$ is an associative interchange near-ring.

Conversely, if $(T, +, -, 0_T, \circ)$ is an associative interchange near-ring, there exist two image-commuting idempotent commuting endomorphisms σ and τ of the group $(T, +, -, 0_T, \circ)$ such that $a \circ b = \sigma(a) + \tau(b)$ for every $a, b \in T$. It follows that:

Theorem 4.3. *The following two categories are isomorphic:*
(a) *The category of all left ditrusses $(T, +, -, 0, \sigma, \circ, \cdot)$ with $\circ$ associative,*

$$\lambda^T : (T, \circ) \to \mathrm{End}_{\mathsf{Gp}}(T, +)$$

a constant semigroup morphism, and σ and λ_0^D image-commuting idempotent endomorphisms of the group $(T,+,-,0)$.
(b) *The category of all associative interchange near-rings.*

Now if $(T,+,-,0_T,\circ)$ is an associative interchange near-ring and we replace the multiplication $\circ$ with its opposite operation $\circ^{\mathrm{op}}$, defined by $a\circ^{\mathrm{op}}b := b\circ a$ for all $a,b\in T$, Identity (4.1) becomes $(y+z)\circ^{\mathrm{op}}(w+x) = (y\circ^{\mathrm{op}} w)+(z\circ^{\mathrm{op}} x)$. It follows that if $(T,+,-,0_T,\circ)$ is an associative interchange near-ring, then $(T,+,-,0_T,\circ^{\mathrm{op}})$ is also an associative interchange near-ring. That is, the functor $(T,+,-,0_T,\circ)\mapsto(T,+,-,0_T,\circ^{\mathrm{op}})$ is an involutive categorical isomorphism of the category of all associative interchange near-rings into itself.

From Theorem 4.3 we get that the functor $(T,+,-,0,\sigma,\circ,\tau\pi_2)\mapsto(T,+,-,0,\tau,\circ,\sigma\pi_2)$ is an involutive categorical isomorphism of the category of all left ditrusses $(T,+,-,0,\sigma,\circ,\cdot)$ with $\circ$ associative, $\lambda^T\colon(T,\circ)\to\mathrm{End}_{\mathsf{Gp}}(T,+)$ a constant semigroup morphism constantly equal to τ, and σ,τ image-commuting.

Let $(T,+,-,0_T,\sigma,\circ,\cdot)$ be a left ditruss with σ an idempotent additive group endomorphism of $(T,+)$, $\circ$ associative and $\cdot$ left distributive. Since λ_0^T is an idempotent group endomorphism of the additive group of T, we know that λ_0^T corresponds to a semidirect-sum decomposition of the group $(T,+)$. That is,

$$T_0 := \ker(\lambda_0^T) = \{\, a\in T \mid 0\cdot a = 0 \,\} = \{\, a\in T \mid 0\circ a = 0 \,\}$$

is a normal subgroup of $(T,+)$,

$$T_c := \lambda_0^T(T) = \{\, a\in T \mid 0\cdot a = a \,\} = \{\, a\in T \mid 0\circ a = a \,\}$$

is a subgroup of $(T,+)$, and T is the semidirect sum $T = T_0\rtimes T_c$ as an additive group. We say that T_0 is the 0-*symmetric part* of the left ditruss $(T,+,-,0,\sigma,\circ,\cdot)$, and T_c is its *constant part*. Both T_0 and T_c are subditrusss of $(T,+,-,0,\sigma,\circ,\cdot)$, in the sense that they are additive subgroups of $(T,+)$ and are closed for the three operations σ, $\circ$ and $\cdot$.

A *left weak truss* is an algebra $(W,+,-,0,\cdot,\sigma)$ in which
(1) $(W,+,-,0)$ is a (not-necessarily abelian) group;
(2) σ is a unary operation;
(3) the binary operation $\cdot$ is *left weakly σ-associative*, that is, $(\sigma(a)+a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all $a,b,c\in W$; and
(4) *left distributivity* holds, that is, $a(b+c)=ab+ac$ for every $a,b,c\in W$.

Theorem 4.4. *The category of the left skew trusses $(T,+,-,0_T,\circ,\sigma)$ with σ an idempotent group endomorphism of $(T,+,-,0_T)$ and the category of the left weak trusses $(W,+,-,0,\cdot,\sigma)$ with σ an idempotent group endomorphism of $(W,+,-,0_W)$ are canonically isomorphic. The canonical isomorphism associates to every left skew truss $(T,+,-,0_T,\circ,\sigma)$ with σ an idempotent group endomorphism the left weak truss $(T,+,-,0_T,\cdot,\sigma)$, where $\cdot$ is the binary operation on T defined setting $a\cdot b = -\sigma(a)+a\circ b$ for all $a,b\in T$, and associates to each left skew truss morphism $f\colon T\to T'$ the same mapping f.*

Proof. Let $(T,+,\circ,\sigma)$ be a left skew truss with σ an idempotent group endomorphism of $(T,+)$. Let $\cdot$ be the binary operation on T defined setting $a\cdot b = -\sigma(a)+a\circ b = \lambda_a(b)$ for all $a,b\in T$, so that $(T,+,\sigma,\circ,\cdot)$ is a left ditruss in which $\circ$ is associative and $\cdot$ is left distributive. Then $(T,+,\cdot,\sigma)$ is a left weak truss by Theorems 3.4 and 3.6(c). Notice that a mapping $f\colon T\to T'$, T' a left skew truss, respects $+$, $\circ$ and σ if and only if it respects any two of the three operations

$+$, $\circ$ and σ. Thus we have a canonical functor $(T,+,\circ,\sigma)\mapsto(T',+,\cdot,\sigma)$ of the category of the left skew trusses $(T,+,-,0_T,\circ,\sigma)$ with σ an idempotent group endomorphism into the category of the left weak trusses $(W,+,-,0,\cdot,\sigma)$ with σ an idempotent group endomorphism of the group $(W,+,-,0)$.

Now let $(W,+,\cdot,\sigma)$ be a left weak truss with σ an idempotent group endomorphism, and let $\circ$ be the binary operation on W defined setting $a\circ b=\sigma(a)+a\cdot b$ for every $a,b\in W$. The operation $\circ$ is associative by Theorem 3.6(c) and left skew σ-distributive by Theorem 3.4. Therefore $(W,+,\circ,\sigma)$ is a left skew truss, and we get a canonical functor $(W,+,\cdot,\sigma)\mapsto(W,+,\circ,\sigma)$ of the category of the left weak trusses $(W,+,-,0_W,\circ,\sigma)$ with σ an idempotent group endomorphism into the category of the left skew trusses $(T,+,-,0,\cdot,\sigma)$ with σ an idempotent group endomorphism of the group $(T,+,-,0)$. This functor is clearly an inverse of the functor described in the previous paragraph. $\square$

Examples 4.5. As far as the ditrusses considered in Corollary 3.9 are concered, let us give three examples with three special cases.

(1) The case $\sigma=\tau$. In this case our difference of operations $\sigma\pi_1=\circ-\cdot$ becomes $\sigma\pi_1=(\sigma\pi_1+\sigma\pi_2)-\sigma\pi_2$, with σ an idempotent group endomorphism (so that $\circ$ is associative). Then we have the left skew truss $(G,+,\circ,\sigma)$, where $\circ$ is defined by $a\circ b=\sigma(a+b)$. Its corresponding left weak truss is $(G,+,\sigma\pi_2,\sigma)$.

(2) The case $\sigma=0$ and $\tau=\mathrm{id}_G$. Recall that, for $\sigma=0$, left skew trusses are exactly left near-rings (Example (1) in Subsection 2.1). In this case our difference of operations $\sigma\pi_1=\circ-\cdot$ is $0=\pi_2-\pi_2$, that is $\circ=\cdot=\pi_2$. In this case the left skew truss (=left near-ring) $(G,+,\pi_2,0)$ corresponds to the left weak truss $(G,+,\pi_2,\mathrm{id}_G)$.

(3) The case $\sigma=\mathrm{id}_G$ and $\tau=0$. Recall that, for $\sigma=\mathrm{id}_G$, left skew trusses are exactly left skew rings. In this case the difference of operations $\sigma\pi_1=\circ-\cdot$ is $\pi_1=\pi_1-\circ_0$, so that $\circ=\pi_1$ and $\cdot=\circ_0$. Thus the left skew truss (=left skew ring) $(G,+,\pi_1)$ corresponds to the left weak truss $(G,+,\circ_0)$.

Examples 4.6. Let $(G,+)$ be a group. Let us go back to the triples of operations $(\sigma,\circ,\cdot)$ on G with σ an idempotent group endomorphism of $(G,+)$, $\circ$ binary and associative, $\cdot$ binary and left distributive, and $\sigma\pi_1=\circ-\cdot$ in the group of all binary operations on G. We already know that this implies that $\circ$ must be necessarily left skew σ-distributive and $\cdot$ must be left weakly σ-associative (Theorems 3.4 and 3.6(c)). Here are two examples.

(1) Let $(G,+)$ be a group and σ,τ be two commuting idempotent group endomorphisms of $(G,+)$. For instance τ could be the identity $\mathrm{id}_G\colon G\to G$, or the zero mapping $G\to G$, or the idempotent endomorphism σ itself, like in Examples 4.5. Consider the difference $\sigma\pi_1=(\sigma\pi_1+\tau\pi_2)-\tau\pi_2$. The operation $\tau\pi_2$ is associative, left weakly σ-associative and left distributive. Thus $(G,+,\tau\pi_2)$ is a left near-ring and $(G,+,\tau\pi_2,\sigma)$ is a left weak truss. The left skew truss corresponding to the left weak truss $(G,+,\tau\pi_2,\sigma)$ is $(G,+,\sigma\pi_1+\tau\pi_2,\sigma)$ (cf. Lemma 2.3). The semigroup morphism λ for this left skew truss is the constant morphism equal to τ. Therefore, for the ditruss $(G,+,\sigma,\sigma\pi_1+\tau\pi_2,\tau\pi_2)$, we have that $G_0=\ker(\tau)$ and $G_c=\tau(G)$.

(2) Let $(G,+)$ be a group and σ,τ be two commuting idempotent group endomorphisms of $(G,+)$. Consider the equality $\sigma\pi_1=(\tau\pi_2+\sigma\pi_1)-(-\sigma\pi_1+\tau\pi_2+\sigma\pi_1)$. Notice that $\tau\pi_2+\sigma\pi_1=(\tau\pi_1+\sigma\pi_2)^{\mathrm{op}}$, so that $\tau\pi_2+\sigma\pi_1$ is associative if and only if $\tau\pi_1+\sigma\pi_2$ is associative, which is true by Proposition 3.8(c). Also $a(-\sigma\pi_1+\tau\pi_2+\sigma\pi_1)b$ is the conjugate $\tau(b)^{\sigma(a)}$ in the group $(G,+)$. Correspondingly, we have the left skew truss $(G,+,(\tau\pi_2+\sigma\pi_1),\sigma)$ and the left weak truss $(G,+,\cdot,\sigma)$ with $a\cdot b=-\sigma(a)+\tau(b)+\sigma(a)$ for every $a,b\in G$ (the conjugate of $\tau(b)$ via

$\sigma(a))$. In this case, $\lambda_0 = \tau$. The semigroup morphism λ is constant if and only if $\lambda_a = \tau$ for every a, that is, if and only if $\tau(b)^{\sigma(a)} = \tau(b)$ for every $a, b \in G$, that is, if and only if σ and τ are image-commuting, that is, in the case of the associative interchange near-rings considered in Theorem 4.3. Notice that:

(a) σ and id_G are image-commuting if and only if $\sigma(G) \subseteq Z(G)$, the center of G. Let us prove that $\sigma(G) \subseteq Z(G)$ if and only if $(G, +)$ has a direct-sum decomposition $G = A \oplus B$, where A is an abelian normal subgroup of G, B is a normal subgroup of G, and $\sigma\colon G \to G$ is the group endomorphism defined by $\sigma(a + b) = a$ for every $a \in A$ and $b \in B$. In fact, assume that $\sigma(G) \subseteq Z(G)$. To the idempotent endomorphism σ, there corresponds a semidirect-product decomposition $G = \sigma(G) \ltimes \ker(\sigma)$, hence an action $\alpha\colon \sigma(G) \to \mathrm{Aut}(\ker(\sigma))$, $\alpha\colon a \in \sigma(G) \mapsto$ conjugation by a. Since $\sigma(G) \subseteq Z(G)$, conjugation by an element $a \in \sigma(G)$ is the identity mapping of $\ker(\sigma)$. Therefore the elements of $\sigma(G)$ and those of $\ker(\sigma)$ commute, so that we have a direct-sum decomposition $G = \sigma(G) \oplus \ker(\sigma)$ of G. Moreover $\sigma(G) \subseteq Z(G)$ implies that $\sigma(G)$ is an abelian group. Set $A := \sigma(A)$ and $B := \ker(\sigma)$, so that A and B are normal subgroups of G, A is abelian and, clearly, $\sigma\colon G \to G$ is the group endomorphism defined by $\sigma(a + b) = a$ for every $a \in A$ and $b \in B$.

(b) σ and σ are image-commuting if and only if $\sigma(G)$ is abelian.

(c) σ and the zero endomorphism of G are always image-commuting.

We conclude with a characterization of 0-symmetric left skew trusses.

Proposition 4.7. *Let $(T, +, \circ, \sigma)$ be a left skew truss with $\sigma\colon T \to T$ any mapping such that $\sigma(0_T) = 0_T$. The following conditions are equivalent:*

(a) *The left skew truss $(T, +, \circ, \sigma)$ is 0-symmetric.*

(b) *$\sigma(a) \circ b = \sigma(a)$ for every $a, b \in T$.*

Proof. (a) $\Rightarrow$ (b) Let $(T, +, \circ, \sigma)$ be a 0-symmetric left skew truss, and let $(T, +, \sigma, \circ, \cdot)$ be the corresponding left ditruss. Then $\circ$ is associative and $0 \cdot b = 0$ for every $b \in T$, that is $\lambda_0 = 0$ (0 is a two-sided zero for the magma $(T, \cdot)$). Since $\cdot$ is weakly σ-associative, we have that $(\sigma(a) + a \cdot 0) \cdot b = a \cdot (0 \cdot b)$ for every $a, b \in T$. Therefore $\sigma(a) \cdot b = 0$. Now σ is an idempotent mapping by Theorem 2.4(c), so the equality $\sigma(x) = x \circ y - x \cdot y$ implies that $\sigma(a) = \sigma(a) \circ b$.

(b) $\Rightarrow$ (a) We know that σ is an idempotent mapping by Theorem 2.4(c). Thus the identity $\sigma(x) = x \circ y - x \cdot y$ implies that $\sigma(a) = \sigma(a) \circ b - \sigma(a) \cdot b$ for every $a, b \in T$. Hence (b) yields that $\sigma(a) \cdot b = 0$. For $a = 0$, we get that $0 \cdot b = 0$, so T is 0-symmetric. □

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