A SKEW GROUP RING OF $\mathbb{Z}/2\mathbb{Z}$ OVER $U(\mathfrak{sl}_2)$, LEONARD TRIPLES AND ODD GRAPHS

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Abstract. We employ a skew group ring of $\mathbb{Z}/2\mathbb{Z}$ over $U(\mathfrak{sl}_2)$ to construct modules over the universal Bannai–Ito algebra. In addition, we give the conditions under which the defining generators act as Leonard triples on the resulting modules. As a combinatorial realization, we establish an algebra homomorphism from the universal Bannai–Ito algebra onto the Terwilliger algebra of an odd graph. This homomorphism provides a unified description of Leonard triples on all irreducible modules over the Terwilliger algebra.

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1. Preliminaries on skew group rings and Hopf algebras

Throughout this paper, we assume that all algebras are unital and associative, and that all algebra (anti-)homomorphisms are unital. For any two elements x, y of an algebra, we write $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$. Let $\mathbb{N}$ denote the set of nonnegative integers. By convention, a vacuous summation is equal to 0 and a vacuous product is equal to 1.

In analogy with the way semidirect products generalize the direct product of groups, the skew group rings may be regarded as a natural generalization of group rings. Let R be a ring with unit 1. Let G be a finite group. Suppose that there is a group homomorphism $\varphi : G \to \mathrm{Aut}(R)$. The *skew group ring $R \rtimes_\varphi G$ of G over R induced by φ* consists of all formal sums

$$\sum_{g\in G} a_g g \qquad \text{where } a_g \in R \text{ and } g \in G.$$

The addition operation is componentwise. The multiplication is distributively defined by

$$a_1 g_1 \cdot a_2 g_2 = (a_1 \cdot \varphi(g_1)(a_2))\, g_1 g_2 \tag{1}$$

for all $a_1, a_2 \in R$ and all $g_1, g_2 \in G$. Each $a \in R$ can be identified with $a \cdot 1 \in R \rtimes_\varphi G$. Each $g \in G$ can be identified with $1 \cdot g \in R \rtimes_\varphi G$. Let $\mathbb{F}$ denote a field. Suppose that R is an algebra over $\mathbb{F}$. Then $R \rtimes_\varphi G$ is an algebra over $\mathbb{F}$ with scalar multiplication defined by

$$c\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} (c a_g) g$$

where $c \in \mathbb{F}$ and $a_g \in R$ for all $g \in G$.

Let A denote an algebra over $\mathbb{F}$. The multiplication map $m : A \otimes A \to A$ is the $\mathbb{F}$-bilinear map defined by $m(a \otimes b) = ab$ for all $a, b \in A$. The unit map $\iota : \mathbb{F} \to A$ is given by $\iota(c) = c \cdot 1$ for all $c \in \mathbb{F}$. Recall that A is called a *Hopf algebra* if the following conditions hold:

- There is an algebra homomorphism $\Delta : A \to A \otimes A$ such that

$$(1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta.$$

The map Δ is called the *comultiplication* of A.

- There is an algebra homomorphism $\varepsilon : A \to \mathbb{F}$ such that
$$m \circ (1 \otimes (\iota \circ \varepsilon)) \circ \Delta = 1,$$
$$m \circ ((\iota \circ \varepsilon) \otimes 1) \circ \Delta = 1.$$
The map ε is called the *counit* of A.
- There is an algebra anti-homomorphism $S : A \to A$ such that
$$m \circ (1 \otimes S) \circ \Delta = \iota \circ \varepsilon,$$
$$m \circ (S \otimes 1) \circ \Delta = \iota \circ \varepsilon.$$
The map S is called the *antipode* of A.

In what follows, particular emphasis will be placed on two classical Hopf algebras and a Hopf algebra arising from their skew group ring.

Let $\mathfrak{g}$ denote a finite-dimensional Lie algebra over the complex number field $\mathbb{C}$. There exists a unique algebra $U(\mathfrak{g})$ over $\mathbb{C}$, up to isomorphism, together with a Lie algebra homomorphism $h : \mathfrak{g} \to U(\mathfrak{g})$ satisfying the following universal property: For any algebra A over $\mathbb{C}$ and any Lie algebra homomorphism $\phi : \mathfrak{g} \to A$, there exists a unique algebra homomorphism $\widetilde{\phi} : U(\mathfrak{g}) \to A$ such that $\phi = \widetilde{\phi} \circ h$. The algebra $U(\mathfrak{g})$ is called the *universal enveloping algebra* of $\mathfrak{g}$. By the Poincaré–Birkhoff–Witt theorem, the map h is injective. Hence each element of $\mathfrak{g}$ can be naturally identified with an element of $U(\mathfrak{g})$. The algebra $U(\mathfrak{g})$ is generated by all $u \in \mathfrak{g}$. In addition, $U(\mathfrak{g})$ carries the following Hopf algebra structure:

- The comultiplication $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the algebra homomorphism that maps
$$u \quad \mapsto \quad u \otimes 1 + 1 \otimes u \qquad \text{for all } u \in \mathfrak{g}.$$
- The counit $U(\mathfrak{g}) \to \mathbb{C}$ is the algebra homomorphism that maps
$$u \quad \mapsto \quad 0 \qquad \text{for all } u \in \mathfrak{g}.$$
- The antipode $U(\mathfrak{g}) \to U(\mathfrak{g})$ is the algebra anti-homomorphism that maps
$$u \quad \mapsto \quad -u \qquad \text{for all } u \in \mathfrak{g}.$$

Let G be a group. The group ring $\mathbb{C}[G]$ of G over $\mathbb{C}$ carries the following Hopf algebra structure:

- The comultiplication $\mathbb{C}[G] \to \mathbb{C}[G] \otimes \mathbb{C}[G]$ is the algebra homomorphism that maps
$$g \quad \mapsto \quad g \otimes g \qquad \text{for all } g \in G.$$
- The counit $\mathbb{C}[G] \to \mathbb{C}$ is the algebra homomorphism that maps
$$g \quad \mapsto \quad 1 \qquad \text{for all } g \in G.$$
- The antipode $\mathbb{C}[G] \to \mathbb{C}[G]$ is the algebra anti-automorphism that maps
$$g \quad \mapsto \quad g^{-1} \qquad \text{for all } g \in G.$$

We now assume that G is a finite subgroup of $\mathrm{Aut}(\mathfrak{g})$. By the universal property of $U(\mathfrak{g})$, each $g \in G$ extends uniquely to an algebra automorphism of $U(\mathfrak{g})$. Thus G embeds naturally into $\mathrm{Aut}(U(\mathfrak{g}))$. This gives rise to the skew group ring of G over $U(\mathfrak{g})$, denoted by
$$U(\mathfrak{g})_G := U(\mathfrak{g}) \rtimes G$$
for brevity. In this case $U(\mathfrak{g})_G$ naturally inherits the Hopf algebra structure from those of $U(\mathfrak{g})$ and $\mathbb{C}[G]$. Specifically, the Hopf algebra structure of $U(\mathfrak{g})_G$ is as follows:

- The comultiplication $\Delta : U(\mathfrak{g})_G \to U(\mathfrak{g})_G \otimes U(\mathfrak{g})_G$ is the algebra homomorphism defined by

$$\Delta(u) = u \otimes 1 + 1 \otimes u \quad \text{for all } u \in \mathfrak{g}, \tag{2}$$

$$\Delta(g) = g \otimes g \qquad \text{for all } g \in G. \tag{3}$$

- The counit $\varepsilon : U(\mathfrak{g})_G \to \mathbb{C}$ is the algebra homomorphism defined by

$$\varepsilon(u) = 0 \qquad \text{for all } u \in \mathfrak{g},$$
$$\varepsilon(g) = 1 \qquad \text{for all } g \in G.$$

- The antipode $S : U(\mathfrak{g})_G \to U(\mathfrak{g})_G$ is the algebra anti-automorphism defined by

$$S(u) = -u \qquad \text{for all } u \in \mathfrak{g},$$
$$S(g) = g^{-1} \qquad \text{for all } g \in G.$$

This section provides the necessary background on skew group rings and Hopf algebras.

2. Introduction

The Lie algebra $\mathfrak{sl}_2$ over $\mathbb{C}$ consists of all 2×2 complex trace-zero matrices. For a Q-polynomial distance-regular graph, the Terwilliger algebra with respect to a base vertex is the subalgebra of the full matrix algebra generated by the adjacency matrix and the corresponding dual adjacency matrix.

The interplay between $\mathfrak{sl}_2$ and Terwilliger algebras has been extensively investigated since the study of hypercubes [8]. More recently, the Clebsch–Gordan rule for $U(\mathfrak{sl}_2)$ was realized through the decompositions of Terwilliger algebras of Hamming graphs [14]. This line of work continued with a clarification of the subtle link between the Clebsch–Gordan coefficients of $U(\mathfrak{sl}_2)$ and the Terwilliger algebras of Johnson graphs [15]. A q-analogue of this link was then established for Grassmann graphs, relating the Clebsch–Gordan coefficients of $U_q(\mathfrak{sl}_2)$ to their Terwilliger algebras [16]. Related developments include the use of $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{\mathfrak{sl}}_2)$ to study the Terwilliger algebras of dual polar graphs and Grassmann graphs [2, 21, 26].

In parallel, an algebra homomorphism from the universal Hahn algebra into $U(\mathfrak{sl}_2)$ was established to capture the relation between the Terwilliger algebras of a hypercube and its halved graph [17]. This line of work further led to an algebra homomorphism from the universal Racah algebra into $U(\mathfrak{sl}_2)$, elucidating its connection with the Terwilliger algebras of halved cubes [18]. Collectively, these results demonstrate a rich and fruitful set of links between $\mathfrak{sl}_2$ and Terwilliger algebras.

Building on these developments, this paper contributes to the study of the interplay between $\mathfrak{sl}_2$ and Terwilliger algebras by means of skew group rings. Recall that $\mathfrak{sl}_2$ admits the standard basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

There is an involutive Lie algebra automorphism $\rho : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ that sends

$$E \mapsto F, \qquad F \mapsto E, \qquad H \mapsto -H. \tag{5}$$

The subgroup of $\mathrm{Aut}(\mathfrak{sl}_2)$ generated by ρ, which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, gives rise to the skew group ring

$$U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}.$$

Let $d \geq 1$ denote an integer. Recall that the *odd graph* O_{d+1} is the Q-polynomial distance-regular graph whose vertices are the d-element subsets of a $(2d+1)$-element set, with two vertices adjacent if and only if the corresponding subsets are disjoint. Extending the framework developed for hypercubes in [8], we establish a connection between the representations of

$$U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2} := U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \otimes U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$$

and the Terwilliger algebras of odd graphs, mediated by the universal Bannai–Ito algebra.

The Bannai–Ito polynomials were originally introduced in the monograph of Bannai and Ito on association schemes [1]. They were later characterized as the $q \to -1$ limit of the q-Racah polynomials [25]. In parallel, a set of algebraic relations can be formulated, which are known as the Askey–Wilson relations associated with the Bannai–Ito polynomials [23, 24]. The Bannai–Ito algebras are defined as algebras satisfying these relations. To provide a uniform framework for their structural study, the universal Bannai–Ito algebra was defined as follows: The *universal Bannai–Ito algebra* $\mathfrak{BI}$ is an algebra over $\mathbb{C}$ generated by X, Y, Z and the relations assert that each of

$$\{X, Y\} - Z, \qquad \{Y, Z\} - X, \qquad \{Z, X\} - Y \tag{6}$$

is central in $\mathfrak{BI}$ [7, 13]. In other words, the Bannai–Ito algebra is obtained by assigning scalar values to the defining central elements (6) of $\mathfrak{BI}$.

By (2) and (3) the comultiplication $\Delta : U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}} \to U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$ of $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ is an algebra homomorphism given by

$$\Delta(E) = E \otimes 1 + 1 \otimes E, \tag{7}$$

$$\Delta(F) = F \otimes 1 + 1 \otimes F, \tag{8}$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \tag{9}$$

$$\Delta(\rho) = \rho \otimes \rho. \tag{10}$$

For any $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$-module V and any scalar $\theta \in \mathbb{C}$, we define

$$V(\theta) = \{v \in V \mid \Delta(H)v = \theta v\}. \tag{11}$$

Among these, the case $\theta = 1$ is of particular interest. In the first main result of this paper, we establish a $\mathfrak{BI}$-module structure on $V(1)$.

Theorem 2.1. *Suppose that V is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$-module. Then $V(1)$ is a $\mathfrak{BI}$-module given by*

$$X = \Delta(E\rho), \tag{12}$$

$$Y = \frac{1}{2}(H \otimes 1 - 1 \otimes H), \tag{13}$$

$$Z = (E \otimes 1 - 1 \otimes E)\Delta(\rho). \tag{14}$$

Recall that a *Leonard pair* consists of two diagonalizable linear operators on a nonzero finite-dimensional vector space, such that each acts in an irreducible tridiagonal fashion on an eigenbasis of the other. As a natural extension, a *Leonard triple* consists of three diagonalizable linear operators on a nonzero finite-dimensional vector space, such that any two act in an irreducible tridiagonal fashion on an eigenbasis of the third. For the formal definitions, we refer the reader to [4,22]. Leonard pairs and Leonard triples form an important class of operators satisfying the Askey–Wilson relations [6, 9–11, 18, 23, 24]. Consequently, they are naturally connected with $\mathfrak{BI}$-modules. We clarify the position of Theorem 2.1 within the framework of Leonard triples in the second main result:

Theorem 2.2. *Suppose that V is a finite-dimensional irreducible $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module. Then the following conditions are equivalent:*

(i) *$V(1)$ is nonzero.*
(ii) *The $\mathfrak{BI}$-module $V(1)$ is irreducible.*
(iii) *X, Y, Z act on the $\mathfrak{BI}$-module $V(1)$ as a Leonard triple.*

For any set X let $\mathbb{C}^X$ denote the vector space over $\mathbb{C}$ with basis X. Let Ω be a finite set and write 2^Ω for the power set of Ω. For any $x_0 \in 2^\Omega$ we equip $\mathbb{C}^{2^\Omega}$ with a $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module structure, denoted by

$$\mathbb{C}^{2^\Omega}(x_0).$$

By Theorem 2.1 this endows $\mathbb{C}^{2^\Omega}(x_0)(1)$ with a $\mathfrak{BI}$-module structure.

We now assume that Ω is a $(2d+1)$-element set and view $\binom{\Omega}{d}$, the collection of d-element subsets of Ω, as the vertex set of O_{d+1}. Let $\mathbf{A}$ denote the adjacency matrix of O_{d+1}. The action of $\mathbf{A}$ on $\mathbb{C}^{\binom{\Omega}{d}}$ is given by

$$\mathbf{A}x = \sum_{x\cap y=\emptyset} y \qquad \text{for all } x \in \binom{\Omega}{d}, \tag{15}$$

where the sum is over all $y \in \binom{\Omega}{d}$ with $x \cap y = \emptyset$. Pick any $x_0 \in \binom{\Omega}{d}$. Let $\mathbf{A}^*(x_0)$ denote the dual adjacency matrix of O_{d+1} with respect to x_0. Recall from [1,3] that the action of $\mathbf{A}^*(x_0)$ on $\mathbb{C}^{\binom{\Omega}{d}}$ is given by

$$\mathbf{A}^*(x_0)x = \left(2d - \frac{2d+1}{d+1}(|x_0 \setminus x| + |x \setminus x_0|)\right) x \qquad \text{for all } x \in \binom{\Omega}{d}. \tag{16}$$

Let $\mathbf{T}(x_0)$ denote the Terwilliger algebra of O_{d+1} with respect to x_0. As vector spaces, the $\mathfrak{BI}$-module $\mathbb{C}^{2^\Omega}(x_0)(1)$ coincides with the standard $\mathbf{T}(x_0)$-module $\mathbb{C}^{\binom{\Omega}{d}}$. Building on this identification, our third main result gives a combinatorial realization of Theorems 2.1 and 2.2:

Theorem 2.3. *Let $d \geq 1$ be an integer and let Ω be a $(2d+1)$-element set. For any $x_0 \in \binom{\Omega}{d}$ the following statements hold:*

(i) *There exists a unique algebra homomorphism $\mathfrak{BI} \to \mathbf{T}(x_0)$ given by*

$$\begin{aligned} X &\mapsto \mathbf{A}, \\ Y &\mapsto \frac{d+1}{2d+1}\mathbf{A}^*(x_0) + \frac{1}{2(2d+1)}, \\ Z &\mapsto \frac{d+1}{2d+1}\{\mathbf{A}, \mathbf{A}^*(x_0)\} + \frac{1}{2d+1}\mathbf{A}. \end{aligned}$$

In particular the homomorphism is surjective.

(ii) *The elements X, Y, Z act on each irreducible $\mathbf{T}(x_0)$-module as a Leonard triple.*

The paper is organized as follows: Section 3 provides a presentation of the algebra $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$. Section 4 contains the proof of Theorem 2.1. In Section 5, we classify finite-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules and show that every finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module is completely reducible. Section 6 provides the necessary background on finite-dimensional irreducible $\mathfrak{BI}$-modules. Section 7 characterizes when the action of X, Y, Z

on a finite-dimensional irreducible $\mathfrak{BI}$-module forms a Leonard triple. Section 8 provides the proof of Theorem 2.2. The argument for Theorem 2.3 is presented in Section 9. Finally, we present the Clebsch–Gordan rule for $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$, which provides the decomposition formula for finite-dimensional irreducible $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-modules into direct sums of irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules.

3. A presentation for $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$

Recall that $\mathfrak{sl}_2$ is a three-dimensional Lie algebra over $\mathbb{C}$ with basis E, F, H. By (4) the matrices E, F, H satisfy

$$[H,E]=2E, \tag{17}$$

$$[H,F]=-2F, \tag{18}$$

$$[E,F]=H. \tag{19}$$

Hence $U(\mathfrak{sl}_2)$ is an algebra over $\mathbb{C}$ generated by E, F, H subject to the relations (17)–(19).

Lemma 3.1. *The elements*

$$E^i F^j H^k \qquad \text{for all } i,j,k \in \mathbb{N} \tag{20}$$

are a basis for $U(\mathfrak{sl}_2)$.

Proof. By Poincaré–Birkhoff–Witt theorem the lemma follows. □

Lemma 3.2. *The elements*

$$E^i F^j H^k \rho^h \qquad \text{for all } i,j,k \in \mathbb{N} \text{ and } h \in \{0,1\} \tag{21}$$

are a basis for $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$.

Proof. Immediate from Lemma 3.1 and the definition of $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$. □

Theorem 3.3. *The algebra $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ has a presentation generated by E, F, H, ρ subject to the relations* (17)–(19) *and*

$$\rho H = -H\rho, \tag{22}$$

$$\rho E = F\rho, \tag{23}$$

$$\rho^2 = 1. \tag{24}$$

Proof. Since $U(\mathfrak{sl}_2) \subseteq U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ the relations (17)–(19) hold in $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$. By (1) and (5), the relations (22)–(24) hold in $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$. Let $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$ denote the algebra over $\mathbb{C}$ generated by the symbols E, F, H, ρ subject to the relations (17)–(19) and (22)–(24). By construction, there exists a unique algebra homomorphism $\Phi : U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}} \to U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ that sends

$$(E,F,H,\rho) \quad \mapsto \quad (E,F,H,\rho).$$

Combined with Lemma 3.2 the elements (21) of $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$ are linearly independent. To see that Φ is an isomorphism, it remains to verify that the elements (21) span $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$.

Since the relations (17)–(19) hold in $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$, there exists a unique algebra homomorphism $\Psi : U(\mathfrak{sl}_2) \to U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$ that sends

$$(E,F,H) \quad \mapsto \quad (E,F,H).$$

Since the elements (20) of $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$ are linearly independent, it follows from Lemma 3.1 that Ψ is injective. Hence $U(\mathfrak{sl}_2)$ can be considered as a subalgebra of $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$. Let $\mathcal{I}$ denote the subspace of $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$ spanned by (21); equivalently

$$\mathcal{I} = U(\mathfrak{sl}_2) + U(\mathfrak{sl}_2)\rho.$$

Using (22)–(24) yields that $\mathcal{I}$ is closed under right multiplication by E, F, H, ρ. Since the algebra $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$ is generated by these elements, it follows that $\mathcal{I}$ is a right ideal of $U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$. Finally, since $1 \in \mathcal{I}$ we conclude that $\mathcal{I} = U(\mathfrak{sl}_2)'_{\mathbb{Z}/2\mathbb{Z}}$. The result follows. □

4. Proof for Theorem 2.1

Recall $\Delta(H)$ and $\Delta(\rho)$ from (9) and (10).

Lemma 4.1. *The following relations hold in $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$:*

(i) $[\Delta(H), 1 \otimes H] = 0$.
(ii) $[\Delta(H), H \otimes 1] = 0$.
(iii) $[\Delta(H), 1 \otimes E] = 2(1 \otimes E)$.
(iv) $[\Delta(H), E \otimes 1] = 2(E \otimes 1)$.
(v) $\{\Delta(H), \Delta(\rho)\} = 0$.

Proof. (i), (ii): Both $H \otimes 1$ and $1 \otimes H$ commute with $\Delta(H)$.
(iii), (iv): These are obtained by evaluating the commutators using (17).
(v): This follows by evaluating the anti-commutator using (22). □

Recall the notation $V(\theta)$ from (11) for any $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module V and any $\theta \in \mathbb{C}$.

Lemma 4.2. *Let V denote a $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module. For any $\theta \in \mathbb{C}$ the following relations hold:*

(i) $(H \otimes 1)V(\theta) \subseteq V(\theta)$.
(ii) $(1 \otimes H)V(\theta) \subseteq V(\theta)$.
(iii) $(1 \otimes E)V(\theta) \subseteq V(\theta + 2)$.
(iv) $(E \otimes 1)V(\theta) \subseteq V(\theta + 2)$.
(v) $\Delta(\rho)V(\theta) \subseteq V(-\theta)$.

Proof. Immediate from Lemma 4.1. □

The distinguished central element

$$\Lambda = EF + FE + \frac{H^2}{2} \tag{25}$$

is called the *Casimir element* of $U(\mathfrak{sl}_2)$. As in Section 1, the Lie algebra automorphism ρ of $\mathfrak{sl}_2$ can be regarded as an algebra automorphism of $U(\mathfrak{sl}_2)$.

Lemma 4.3. (i) *The algebra automorphism ρ of $U(\mathfrak{sl}_2)$ fixes Λ.*
(ii) *The element Λ is central in $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$.*

Proof. (i): Apply ρ to both sides of (25) and evaluate the resulting equation using (5).
(ii): By (1) the product $\rho \cdot \Lambda$ is equal to $\rho(\Lambda) \cdot \rho$ in $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$. By (i) this implies that Λ commutes with ρ. □

Proof of Theorem 2.1. For convenience, we temporarily set X, Y, Z to denote the right-hand sides of (12)–(14) respectively. Lemma 4.2 implies that $V(1)$ is invariant under X, Y, Z. By Lemma 4.3(ii), the Casimir element Λ of $U(\mathfrak{sl}_2)$ is central in $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$. It therefore suffices to verify that the following relations hold on $V(1)$:

$$\{X, Y\} = Z, \tag{26}$$

$$\{Y, Z\} = X, \tag{27}$$

$$\{Z, X\} = Y + \Lambda \otimes 1 - 1 \otimes \Lambda. \tag{28}$$

Using (10) and (22) a direct calculation shows that

$$\{(E \otimes 1)\Delta(\rho), 1 \otimes H\} = 0,$$
$$\{(1 \otimes E)\Delta(\rho), H \otimes 1\} = 0.$$

Using (10), (17) and (22) a direct calculation shows that

$$\{(E \otimes 1)\Delta(\rho), H \otimes 1\} = 2(E \otimes 1)\Delta(\rho),$$
$$\{(1 \otimes E)\Delta(\rho), 1 \otimes H\} = 2(1 \otimes E)\Delta(\rho).$$

With these four identities, it is straightforward to verify the relations (26) and (27). Applying (7), (10), (23) and (24) a direct computation yields

$$\{(E \otimes 1)\Delta(\rho), \Delta(E\rho)\} = E \otimes F + F \otimes E + 2(EF \otimes 1),$$
$$\{(1 \otimes E)\Delta(\rho), \Delta(E\rho)\} = E \otimes F + F \otimes E + 2(1 \otimes EF).$$

These identities imply that

$$\{Z, X\} - Y = \left(2EF - \frac{H}{2}\right) \otimes 1 - 1 \otimes \left(2EF - \frac{H}{2}\right).$$

By (19) and (25) the element $2EF - \frac{H}{2}$ can be written as $\Lambda - \frac{H(H-1)}{2}$. Hence $\{Z, X\} - Y$ is equal to $\Lambda \otimes 1 - 1 \otimes \Lambda$ plus $\frac{1}{2}$ times

$$1 \otimes H(H-1) - H(H-1) \otimes 1.$$

Using (9) the above element factors as $(1 \otimes H - H \otimes 1)(\Delta(H) - 1)$, which vanishes on $V(1)$ by (11). Therefore (28) holds on $V(1)$. The result follows. $\square$

Let κ, λ, μ denote the defining central elements (6) of $\mathfrak{BI}$. Explicitly,

$$\kappa = \{X, Y\} - Z, \tag{29}$$

$$\lambda = \{Y, Z\} - X, \tag{30}$$

$$\mu = \{Z, X\} - Y. \tag{31}$$

In addition, the distinguished central element

$$X^2 + Y^2 + Z^2$$

is called the *Casimir element* of $\mathfrak{BI}$ [7, Equation (6)].

Lemma 4.4. *For any $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module V the following statements hold:*

(i) *κ, λ, μ act on the $\mathfrak{BI}$-module $V(1)$ as 0, 0, $\Lambda \otimes 1 - 1 \otimes \Lambda$ respectively.*
(ii) *$X^2 + Y^2 + Z^2$ acts on the $\mathfrak{BI}$-module $V(1)$ as $\Lambda \otimes 1 + 1 \otimes \Lambda + \frac{3}{4}$.*

Proof. (i): The statement was shown in the proof of Theorem 2.1.

(ii): Recall the actions of X, Y, Z on the $\mathfrak{BI}$-module $V(1)$ from (12)–(14). By the formula for the sum of squares we have

$$Y^2 = \frac{H^2 \otimes 1 + 1 \otimes H^2}{4} - \frac{H \otimes H}{2}.$$

Using (7), (8), (23) and (24) a direct calculation shows that

$$X^2 = EF \otimes 1 + E \otimes F + F \otimes E + 1 \otimes EF,$$
$$Z^2 = EF \otimes 1 - E \otimes F - F \otimes E + 1 \otimes EF.$$

Summing these identities, the Casimir element $X^2 + Y^2 + Z^2$ of $\mathfrak{BI}$ acts on $V(1)$ as

$$\left(2EF + \frac{H^2}{4}\right) \otimes 1 + 1 \otimes \left(2EF + \frac{H^2}{4}\right) - \frac{H \otimes H}{2}. \tag{32}$$

By (19) and (25) the element $2EF + \frac{H^2}{4}$ can be written as $\Lambda + H - \frac{H^2}{4}$. Substituting this into (32) and rewriting in terms of $\Delta(H)$ yields that (32) is equal to $\Lambda \otimes 1 + 1 \otimes \Lambda$ plus

$$\Delta(H)\left(1 - \frac{\Delta(H)}{4}\right).$$

Since $\Delta(H)$ acts as 1 on $V(1)$ by (11), the statement (ii) follows. □

5. Finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules

We begin this section with some standard facts on finite-dimensional $U(\mathfrak{sl}_2)$-modules. For each $n \in \mathbb{N}$ there exists an $(n+1)$-dimensional $U(\mathfrak{sl}_2)$-module L_n with basis $\{v_i^{(n)}\}_{i=0}^n$ satisfying

$$Ev_i^{(n)} = iv_{i-1}^{(n)} \qquad (1 \le i \le n), \qquad Ev_0^{(n)} = 0, \tag{33}$$

$$Fv_i^{(n)} = (n-i)v_{i+1}^{(n)} \qquad (0 \le i \le n-1), \qquad Fv_n^{(n)} = 0, \tag{34}$$

$$Hv_i^{(n)} = (n-2i)v_i^{(n)} \qquad (0 \le i \le n). \tag{35}$$

The Casimir element Λ acts on L_n as scalar multiplication by $\frac{n(n+2)}{2}$. The $U(\mathfrak{sl}_2)$-module L_n is irreducible.

Lemma 5.1. *For each $n \in \mathbb{N}$ every $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)$-module is isomorphic to L_n.*

Proof. See [19, Section V.4] for example. □

Recall that a module over an algebra is called *completely reducible* if it is equal to a (direct) sum of irreducible submodules [20, Section XVII.2].

Lemma 5.2. *Every finite-dimensional $U(\mathfrak{sl}_2)$-module is completely reducible.*

Proof. See [19, Theorem V.4.6] for example. □

Applying Theorem 3.3, the $U(\mathfrak{sl}_2)$-module L_n ($n \in \mathbb{N}$) can be extended to a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module by defining

$$\rho\, v_i^{(n)} = \pm v_{n-i}^{(n)} \qquad (0 \le i \le n). \tag{36}$$

The choice of sign gives the two distinct $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules, denoted by L_n^+ and L_n^-, respectively.

Lemma 5.3. *For any $n \in \mathbb{N}$ the following statements hold:*

(i) *The $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules L_n^+ and L_n^- are irreducible.*
(ii) *The $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules L_n^+ and L_n^- are not isomorphic.*

Proof. (i): By construction, the restriction of the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules $L_n^\pm$ to $U(\mathfrak{sl}_2)$ is the irreducible $U(\mathfrak{sl}_2)$-module L_n. Hence (i) follows.

(ii): Suppose, for contradiction, that there exists a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module isomorphism $f : L_n^+ \to L_n^-$. Then f is a $U(\mathfrak{sl}_2)$-module automorphism of L_n. By Schur's lemma f must be a nonzero scalar multiple of the identity map on L_n. However, by (36) any nonzero scalar multiple of the identity cannot commute with ρ, a contradiction. □

Lemma 5.4. *Suppose that V is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module and W is a $U(\mathfrak{sl}_2)$-submodule of V. Then the following statements hold:*

(i) *ρW is a $U(\mathfrak{sl}_2)$-submodule of V.*
(ii) *$W + \rho W$ is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodule of V.*

Proof. (i): Using (22)–(24) it is routine to verify that ρW is invariant under E, F, H. Since the algebra $U(\mathfrak{sl}_2)$ is generated by these elements, the statement (i) follows.

(ii): By (i) the subspace $W + \rho W$ of V is a $U(\mathfrak{sl}_2)$-module. In view of (24) the $U(\mathfrak{sl}_2)$-module $W + \rho W$ is also invariant under ρ. The statement (ii) follows. □

Lemma 5.5. *Let V denote a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module. Suppose that W is a $U(\mathfrak{sl}_2)$-submodule of V isomorphic to L_n for some $n \in \mathbb{N}$. Then the following statements are true:*

(i) *The $U(\mathfrak{sl}_2)$-module ρW is isomorphic to L_n.*
(ii) *Suppose that $W = \rho W$. Then the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module W is isomorphic to either L_n^+ or L_n^-.*
(iii) *Suppose that $W \neq \rho W$. Then the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module $W + \rho W$ is isomorphic to $L_n^+ \oplus L_n^-$.*

Proof. Since the $U(\mathfrak{sl}_2)$-module W is isomorphic to L_n, there is a $U(\mathfrak{sl}_2)$-module isomorphism $f : L_n \to W$. Define a linear map $g : L_n \to \rho W$ by

$$v_i^{(n)} \quad \mapsto \quad \rho f(v_{n-i}^{(n)}) \qquad (0 \leq i \leq n). \tag{37}$$

(i): By Lemma 5.4(i) the space ρW is a $U(\mathfrak{sl}_2)$-submodule of V. Since f is a $U(\mathfrak{sl}_2)$-module homomorphism, it is routine to verify that g is also a $U(\mathfrak{sl}_2)$-module homomorphism by using (22), (23) and (33)–(35). Since f is a linear isomorphism and ρ is invertible in $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ by (24), the vectors $\{\rho f(v_{n-i}^{(n)})\}_{i=0}^n$ form a basis of ρW. Therefore g is a $U(\mathfrak{sl}_2)$-module isomorphism. The statement (i) follows.

(ii): Since $W = \rho W$ the composition $f^{-1} \circ g$ is a $U(\mathfrak{sl}_2)$-module automorphism of L_n. By Schur's lemma there exists a nonzero scalar $\varepsilon \in \mathbb{C}$ such that $f^{-1} \circ g = \varepsilon \cdot 1$. Consequently $g = \varepsilon \cdot f$. Evaluating both sides at $v_{n-i}^{(n)}$ and applying (37) gives

$$\rho f(v_i^{(n)}) = \varepsilon f(v_{n-i}^{(n)}) \qquad (0 \leq i \leq n). \tag{38}$$

Applying ρ to both sides of (38), the right-hand side is evaluated using (38) and the left-hand side is evaluated by (24). It follows that

$$f(v_i^{(n)}) = \varepsilon^2 f(v_i^{(n)}) \qquad (0 \leq i \leq n).$$

Since the vectors $\{f(v_i^{(n)})\}_{i=0}^n$ are nonzero, it follows that $\varepsilon \in \{\pm 1\}$. Comparing (36) with (38) implies that if $\varepsilon = 1$ then f is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module isomorphism from L_n^+ to W; if $\varepsilon = -1$ then f is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module isomorphism from L_n^- to W. The statement (ii) follows.

(iii): By Lemma 5.4(ii) the space $W + \rho W$ is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodule of V. Define a map $h^+ : L_n^+ \to W + \rho W$ by

$$h^+(v) = f(v) + g(v) \qquad \text{for all } v \in L_n^+.$$

Define a map $h^- : L_n^- \to W + \rho W$ by

$$h^-(v) = f(v) - g(v) \qquad \text{for all } v \in L_n^-.$$

Since f and g are $U(\mathfrak{sl}_2)$-module homomorphisms, the maps h^+ and h^- are also $U(\mathfrak{sl}_2)$-module homomorphisms. Using (24), (36) and (37) yields that

$$\rho\left(h^+(v_i^{(n)})\right) = h^+\left(\rho\, v_i^{(n)}\right) \qquad (0 \leq i \leq n),$$
$$\rho\left(h^-(v_i^{(n)})\right) = h^-\left(\rho\, v_i^{(n)}\right) \qquad (0 \leq i \leq n).$$

Since the vectors $\{v_i^{(n)}\}_{i=0}^n$ span L_n^+ and L_n^-, it follows that h^+ and h^- are $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module homomorphisms.

Since $f : L_n \to W$ and $g : L_n \to \rho W$ are $U(\mathfrak{sl}_2)$-module isomorphisms, the $U(\mathfrak{sl}_2)$-modules W and ρW are irreducible. Since $W \neq \rho W$ by assumption, it follows that $W \cap \rho W = \{0\}$. Combined with the injectivity of f and g, the kernels of h^+ and h^- are trivial. Thus the images of h^+ and h^- are two $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodules of $W + \rho W$, isomorphic to L_n^+ and L_n^-, respectively. Finally, by Lemma 5.3(ii) and since $\dim(W + \rho W)$ and $\dim L_n^+ + \dim L_n^-$ are equal to $2(n + 1)$, the statement (iii) follows. □

We are now in a position to provide a complete classification of finite-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules and establish the complete reducibility of finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules.

Theorem 5.6. *For any $n \in \mathbb{N}$ every $(n + 1)$-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module is isomorphic to either L_n^+ or L_n^-.*

Proof. Let $n \in \mathbb{N}$ be given. Suppose that V is an $(n+1)$-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module. Then V contains a finite-dimensional irreducible $U(\mathfrak{sl}_2)$-submodule W. By Lemma 5.1 the $U(\mathfrak{sl}_2)$-module W is isomorphic to L_m for some $m \in \mathbb{N}$.

Suppose that $W \neq \rho W$. By Lemma 5.5(iii) the space $W + \rho W$ is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodule of V isomorphic to $L_m^+ \oplus L_m^-$. This contradicts the irreducibility of V as a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module. Hence $W = \rho W$. By Lemma 5.5(ii) the space W is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodule of V isomorphic to either L_m^+ or L_m^-. Since the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module V is irreducible, it follows that $V = W$ and $m = n$. The result follows. □

Theorem 5.7. *Every finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module is completely reducible.*

Proof. Suppose that V is a finite-dimensional $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module. By Lemma 5.2 the $U(\mathfrak{sl}_2)$-module V is a sum of irreducible $U(\mathfrak{sl}_2)$-submodules of V. It follows that

$$V = \sum_{\text{irreducible } U(\mathfrak{sl}_2)\text{-submodules } W \text{ of } V} W + \rho W,$$

where the sum is over all irreducible $U(\mathfrak{sl}_2)$-submodules W of V.

Let W be any irreducible $U(\mathfrak{sl}_2)$-submodule of V. By Lemma 5.1 there exists an $n \in \mathbb{N}$ such that the $U(\mathfrak{sl}_2)$-module W is isomorphic to L_n. By Lemma 5.5 the space $W + \rho W$ is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodule of V isomorphic to L_n^+, L_n^- or $L_n^+ \oplus L_n^-$. Since the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules L_n^+ and L_n^- are irreducible by Lemma 5.3(i), it follows that V is equal to a sum of irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodules of V. The result follows. □

6. Finite-dimensional irreducible $\mathfrak{BI}$-modules

Recall the central elements κ, λ, μ of $\mathfrak{BI}$ defined in (29)–(31).

Proposition 6.1 (Proposition 2.6, [12])**.** *For any $a, b, c \in \mathbb{C}$ and any even integer $n \geq 0$, there exists an $(n+1)$-dimensional $\mathfrak{BI}$-module $O_n(a,b,c)$ satisfying the following conditions:*

(i) *There exists a basis $\{u_i\}_{i=0}^n$ for $O_n(a,b,c)$ such that*

$$(X - \theta_i)u_i = u_{i+1} \quad (0 \leq i \leq n-1), \qquad (X - \theta_n)u_n = 0,$$
$$(Y - \theta_i^*)u_i = \varphi_i u_{i-1} \quad (1 \leq i \leq n), \qquad (Y - \theta_0^*)u_0 = 0,$$

where

$$\theta_i = \frac{(-1)^i(2a - n + 2i)}{2} \qquad (0 \leq i \leq n),$$
$$\theta_i^* = \frac{(-1)^i(2b - n + 2i)}{2} \qquad (0 \leq i \leq n),$$
$$\varphi_i = \begin{cases} \dfrac{i(n+1-2i-2a-2b-2c)}{2} & \text{if } i \text{ is even}, \\ \dfrac{(i-n-1)(n+1-2i-2a-2b+2c)}{2} & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq n).$$

(ii) *The elements κ, λ, μ act on $O_n(a,b,c)$ as scalar multiplication by*

$$2ab - c(n+1),$$
$$2bc - a(n+1),$$
$$2ca - b(n+1),$$

respectively.

Lemma 6.2. *For any $a, b, c \in \mathbb{C}$ and any even integer $n \geq 0$, the traces of X, Y, Z on $O_n(a,b,c)$ are equal to a, b, c respectively.*

Proof. By Proposition 6.1 the traces of X, Y, XY on $O_n(a,b,c)$ are

$$\sum_{i=0}^n \theta_i = a, \qquad \sum_{i=0}^n \theta_i^* = b, \qquad \sum_{i=0}^n \theta_i\theta_i^* + \sum_{i=1}^n \varphi_i = ab(n+1) - \frac{cn(n+2)}{2},$$

respectively. By (29) the element $Z = \{X, Y\} - \kappa$. Taking the trace on $O_n(a,b,c)$ shows that the trace of Z on $O_n(a,b,c)$ is c. The lemma follows. □

Theorem 6.3 (Theorem 2.8, [12])**.** *Let $n \geq 0$ be an even integer. Suppose that V is an $(n+1)$-dimensional irreducible $\mathfrak{BI}$-module. Then there exist unique scalars $a, b, c \in \mathbb{C}$ such that the $\mathfrak{BI}$-module $O_n(a,b,c)$ is isomorphic to V. Moreover, for any $a, b, c \in \mathbb{C}$ the $\mathfrak{BI}$-module $O_n(a,b,c)$ is irreducible if and only if*

$$a+b+c,\, a-b-c,\, -a+b-c,\, -a-b+c \notin \left\{ \frac{n+1}{2} - i \,\middle|\, i = 2, 4, \ldots, n \right\}.$$

Applying Lemma 6.2, we obtain the following criterion for determining the parameters a, b, c of $O_n(a, b, c)$ to which a given odd-dimensional irreducible $\mathfrak{BI}$-module is isomorphic.

Lemma 6.4. *Let $n \geq 0$ be an even integer. Suppose that V is an $(n+1)$-dimensional irreducible $\mathfrak{BI}$-module. For any $a, b, c \in \mathbb{C}$, the $\mathfrak{BI}$-module $O_n(a, b, c)$ is isomorphic to V if and only if the traces of X, Y, Z on V are equal to a, b, c respectively.*

We now turn our attention to even-dimensional irreducible $\mathfrak{BI}$-modules.

Proposition 6.5 (Proposition 2.4, [12])**.** *For any $a, b, c \in \mathbb{C}$ and any odd integer $n \geq 1$, there exists an $(n+1)$-dimensional $\mathfrak{BI}$-module $E_n(a, b, c)$ satisfying the following conditions:*

(i) *There exists a basis $\{u_i\}_{i=0}^n$ for $E_n(a, b, c)$ such that*

$$
\begin{aligned}
&(X - \theta_i)u_i = u_{i+1} \qquad (0 \leq i \leq n-1), \qquad (X - \theta_n)u_n = 0,\\
&(Y - \theta^*_i)u_i = \varphi_i u_{i-1} \qquad (1 \leq i \leq n), \qquad (Y - \theta^*_0)u_0 = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_i &= \frac{(-1)^i(2a - n + 2i)}{2} \qquad (0 \leq i \leq n),\\
\theta^*_i &= \frac{(-1)^i(2b - n + 2i)}{2} \qquad (0 \leq i \leq n),\\
\varphi_i &= \begin{cases} i(n-i+1) & \text{if } i \text{ is even,}\\ c^2 - \left(\dfrac{2a+2b-n+2i-1}{2}\right)^2 & \text{if } i \text{ is odd} \end{cases} \qquad (1 \leq i \leq n).
\end{aligned}
$$

(ii) *The elements κ, λ, μ act on $E_n(a, b, c)$ as scalar multiplication by*

$$
\begin{aligned}
&c^2 - a^2 - b^2 + \left(\frac{n+1}{2}\right)^2,\\
&a^2 - b^2 - c^2 + \left(\frac{n+1}{2}\right)^2,\\
&b^2 - c^2 - a^2 + \left(\frac{n+1}{2}\right)^2,
\end{aligned}
$$

respectively.

Lemma 6.6. *For any $a, b, c \in \mathbb{C}$ and any odd integer $n \geq 1$, the traces of X, Y, Z on $E_n(a, b, c)$ are equal to $-\frac{n+1}{2}$.*

Proof. By Proposition 6.5 the traces of X, Y, XY on $E_n(a, b, c)$ are equal to

$$
\sum_{i=0}^n \theta_i = -\frac{n+1}{2}, \qquad \sum_{i=0}^n \theta^*_i = -\frac{n+1}{2},
$$
$$
\sum_{i=0}^n \theta_i\theta^*_i + \sum_{i=1}^n \varphi_i = \frac{(n+1)(4c^2 - 4a^2 - 4b^2 + n^2 + 2n - 1)}{8},
$$

respectively. By (29) the element $Z = \{X, Y\} - \kappa$. Taking the trace on $E_n(a, b, c)$ shows that the trace of Z on $E_n(a, b, c)$ is $-\frac{n+1}{2}$. The lemma follows. □

The set $\{\pm 1\}$ is a group under multiplication. There is a natural $\{\pm 1\}^3$-action on $\mathbb{C}^3$ given by

$$
\begin{aligned}
(a,b,c)^{(-1,1,1)} &= (-a,b,c),\\
(a,b,c)^{(1,-1,1)} &= (a,-b,c),\\
(a,b,c)^{(1,1,-1)} &= (a,b,-c)
\end{aligned}
$$

for all $(a,b,c)\in\mathbb{C}^3$. There exists a unique $\{\pm 1\}^2$-action on $\mathfrak{BI}$ such that each $g\in\{\pm 1\}^2$ acts as the algebra automorphism of $\mathfrak{BI}$ specified in the following table:

u	X	Y	Z	κ	λ	μ
$(1,1)(u)$	X	Y	Z	κ	λ	μ
$(1,-1)(u)$	X	$-Y$	$-Z$	$-\kappa$	λ	$-\mu$
$(-1,1)(u)$	$-X$	Y	$-Z$	$-\kappa$	$-\lambda$	μ
$(-1,-1)(u)$	$-X$	$-Y$	Z	κ	$-\lambda$	$-\mu$

TABLE 1. The $\{\pm 1\}^2$-action on $\mathfrak{BI}$

Let V be a module over an algebra. For any automorphism ε of the algebra, we denote by V^ε the module obtained from V by twisting the module structure via ε.

Theorem 6.7 (Theorem 2.5, [12]). *Let $n\geq 1$ be an odd integer. Suppose that V is an $(n+1)$-dimensional irreducible $\mathfrak{BI}$-module. Then there exist scalars $a,b,c\in\mathbb{C}$ and a unique element $(\varepsilon,\varepsilon')\in\{\pm 1\}^2$ such that the $\mathfrak{BI}$-module $E_n(a,b,c)^{(\varepsilon,\varepsilon')}$ is isomorphic to V, where the triple (a,b,c) is unique up to the $\{\pm 1\}^3$-action on $\mathbb{C}^3$. Moreover, for any $a,b,c\in\mathbb{C}$ the $\mathfrak{BI}$-module $E_n(a,b,c)$ is irreducible if and only if*

$$
a+b+c,-a+b+c,a-b+c,a+b-c\notin\left\{\frac{n-1}{2}-i\,\middle|\, i=0,2,\ldots,d-1\right\}.
$$

The following lemma gives a criterion for determining the parameters ε,ε' and a,b,c of $E_n(a,b,c)^{(\varepsilon,\varepsilon')}$ to which a given even-dimensional irreducible $\mathfrak{BI}$-module is isomorphic.

Lemma 6.8. *Let $n\geq 1$ be an odd integer. Assume that V is an $(n+1)$-dimensional irreducible $\mathfrak{BI}$-module. For any $a,b,c\in\mathbb{C}$ and any $(\varepsilon,\varepsilon')\in\{\pm 1\}^2$, the $\mathfrak{BI}$-module V is isomorphic to $E_n(a,b,c)^{(\varepsilon,\varepsilon')}$ if and only if the following conditions hold:*

(i) *The traces of X and Y on the $\mathfrak{BI}$-module $V^{(\varepsilon,\varepsilon')}$ are equal to $-\frac{n+1}{2}$.*
(ii) *The elements κ,λ,μ act on the $\mathfrak{BI}$-module $V^{(\varepsilon,\varepsilon')}$ as scalar multiplication by*

$$
c^2-a^2-b^2+\left(\frac{n+1}{2}\right)^2,
$$

$$
a^2-b^2-c^2+\left(\frac{n+1}{2}\right)^2,
$$

$$
b^2-c^2-a^2+\left(\frac{n+1}{2}\right)^2,
$$

respectively.

Proof. (⇒): Condition (i) follows immediately from Lemma 6.6 and Table 1. Condition (ii) follows immediately from Proposition 6.5(ii).

(⇐): By Theorem 6.7 there exist $a', b', c' \in \mathbb{C}$ and $(\epsilon, \epsilon') \in \{\pm 1\}^2$ such that the $\mathfrak{BI}$-module V is isomorphic to $E_n(a', b', c')^{(\epsilon, \epsilon')}$. By Table 1 and Lemma 6.6, the traces of X and Y on V are $-\epsilon \frac{n+1}{2}$ and $-\epsilon' \frac{n+1}{2}$, respectively. Condition (i) then forces $(\epsilon, \epsilon') = (\varepsilon, \varepsilon')$.

It follows that the $\mathfrak{BI}$-module V is isomorphic to $E_n(a', b', c')^{(\varepsilon, \varepsilon')}$; equivalently, the $\mathfrak{BI}$-module $V^{(\varepsilon, \varepsilon')}$ is isomorphic to $E_n(a', b', c')$. Using Proposition 6.5 the elements κ, λ, μ act on $V^{(\varepsilon, \varepsilon')}$ as scalar multiplication by

$$c'^2 - a'^2 - b'^2 + \left(\frac{n+1}{2}\right)^2,$$
$$a'^2 - b'^2 - c'^2 + \left(\frac{n+1}{2}\right)^2,$$
$$b'^2 - c'^2 - a'^2 + \left(\frac{n+1}{2}\right)^2,$$

respectively. Comparing with condition (ii), this implies that (a', b', c') lies in the $\{\pm 1\}^3$-orbit of (a, b, c). By Theorem 6.7 the $\mathfrak{BI}$-module $E_n(a, b, c)^{(\varepsilon, \varepsilon')}$ is therefore isomorphic to V. The lemma follows. □

7. Leonard triples on finite-dimensional irreducible $\mathfrak{BI}$-modules

A square matrix is called *tridiagonal* if all nonzero entries lie on the main diagonal, the subdiagonal, or the superdiagonal. A tridiagonal matrix is called *irreducible* if every entry on the subdiagonal and every entry on the superdiagonal is nonzero.

Definition 7.1 (Definition 1.2, [4])**.** Let V be a nonzero finite-dimensional vector space. An ordered triple of linear operators $L : V \to V$, $L^* : V \to V$, $L^\varepsilon : V \to V$ is called a *Leonard triple* if the following conditions hold:

(i) There exists a basis for V with respect to which the matrix representing L is diagonal and the matrices representing L^* and L^ε are irreducible tridiagonal.
(ii) There exists a basis for V with respect to which the matrix representing L^* is diagonal and the matrices representing L^ε and L are irreducible tridiagonal.
(iii) There exists a basis for V with respect to which the matrix representing L^ε is diagonal and the matrices representing L and L^* are irreducible tridiagonal.

In this section, we derive necessary and sufficient conditions for X, Y, Z to form Leonard triples on finite-dimensional irreducible $\mathfrak{BI}$-modules.

Lemma 7.2. *The following relations hold in* $\mathfrak{BI}$*:*

(i) $X^2Y + 2XYX + YX^2 - Y = 2\kappa X + \mu$.
(ii) $X^2Z + 2XZX + ZX^2 - Z = 2\mu X + \kappa$.

Proof. Relation (i) follows by eliminating Z from (31) using (29). Relation (ii) follows by eliminating Y from (29) using (31). □

Lemma 7.3. *Let $a, b, c \in \mathbb{C}$ and $n \geq 0$ be an even integer. Suppose that the $\mathfrak{BI}$-module $O_n(a, b, c)$ is irreducible. Then the following conditions are equivalent:*

(i) *There exists a basis for $O_n(a,b,c)$ with respect to which the matrix representing X is diagonal and the matrices representing Y and Z are irreducible tridiagonal.*
(ii) *X is diagonalizable on $O_n(a,b,c)$.*
(iii) $a \notin \{\frac{n-1}{2} - i \,|\, i = 0, 1, \ldots, n-1\}$.

Proof. We use the notations from Proposition 6.1 in the proof.

(i) $\Rightarrow$ (ii): Trivial.

(ii) $\Leftrightarrow$ (iii): With respect to the basis $\{u_i\}_{i=0}^n$ for $O_n(a,b,c)$, the matrix representing X is lower bidiagonal, with diagonal entries $\{\theta_i\}_{i=0}^n$ and all subdiagonal entries equal to 1. Consequently, the minimal polynomial of X on $O_n(a,b,c)$ is

$$\prod_{i=0}^{n}(x - \theta_i).$$

It follows that X is diagonalizable if and only if the scalars $\{\theta_i\}_{i=0}^n$ are pairwise distinct. The latter condition is equivalent to (iii).

(iii) $\Rightarrow$ (i): By (iii) all eigenvalues $\{\theta_i\}_{i=0}^n$ of X on $O_n(a,b,c)$ are simple. For $0 \leq i \leq n$, let v_i denote a θ_i-eigenvector of X in $O_n(a,b,c)$. Then the vectors $\{v_i\}_{i=0}^n$ are a basis for $O_n(a,b,c)$. Set

$$\theta_{-1} = \frac{n}{2} - a + 1, \qquad \theta_{n+1} = -\frac{n}{2} - a - 1.$$

Applying both sides of Lemma 7.2(i) to v_i $(0 \leq i \leq n)$ shows that

$$(X - \theta_{i-1})(X - \theta_{i+1})Yv_i$$

is a scalar multiple of v_i. Since $\{\theta_i\}_{i=0}^n$ are pairwise distinct by (iii), Yv_i is a linear combination of v_{i-1}, v_i, v_{i+1} for all $i = 1, 2, \ldots, n-1$. By construction, θ_{-1} and θ_{n+1} are not in $\{\theta_1, \theta_3, \ldots, \theta_{n-1}\}$. By (iii), $\theta_{-1} \notin \{\theta_2, \theta_4, \ldots, \theta_n\}$ and $\theta_{n+1} \notin \{\theta_0, \theta_2, \ldots, \theta_{n-2}\}$. It follows that θ_{-1} is not among the scalars $\{\theta_i\}_{i=1}^n$ and θ_{n+1} is not among $\{\theta_i\}_{i=0}^{n-1}$. Consequently, Yv_0 is a linear combination of v_0 and v_1, and Yv_n is a linear combination of v_{n-1} and v_n. Thus, the matrix representing Y with respect to the basis $\{v_i\}_{i=0}^n$ for $O_n(a,b,c)$ is tridiagonal. Similarly, Lemma 7.2(ii), together with condition (iii), implies the matrix representing Z with respect to the same basis for $O_n(a,b,c)$ is tridiagonal. Since the $\mathfrak{BI}$-module $O_n(a,b,c)$ is irreducible by assumption, these tridiagonal matrices are irreducible. The condition (i) follows. □

Lemma 7.4. *Let $a, b, c \in \mathbb{C}$ and $n \geq 1$ be an odd integer. Suppose that the $\mathfrak{BI}$-module $E_n(a,b,c)$ is irreducible. Then the following conditions are equivalent:*

(i) *There exists a basis for $E_n(a,b,c)$ with respect to which the matrix representing X is diagonal and the matrices representing Y and Z are irreducible tridiagonal.*
(ii) *X is diagonalizable on $E_n(a,b,c)$.*
(iii) $a \notin \{\frac{n-1}{2} - i \,|\, i = 0, 1, \ldots, n-1\}$.

Proof. We use the notations from Proposition 6.5. The argument is essentially the same as that in the proof of Lemma 7.3, except for the following difference. In the present case

$$\theta_{-1} = \frac{n}{2} - a + 1, \qquad \theta_{n+1} = \frac{n}{2} + a + 1.$$

The reasons why θ_{-1} is excluded from $\{\theta_i\}_{i=1}^n$ and why θ_{n+1} is excluded from $\{\theta_i\}_{i=0}^{n-1}$ differ slightly. By construction, $\theta_{-1} \notin \{\theta_1, \theta_3, \ldots, \theta_n\}$ and $\theta_{n+1} \notin \{\theta_0, \theta_2, \ldots, \theta_{n-1}\}$. By (iii), $\theta_{-1} \notin \{\theta_2, \theta_4, \ldots, \theta_{n-1}\}$ and $\theta_{n+1} \notin \{\theta_1, \theta_3, \ldots, \theta_{n-2}\}$. □

There exists a unique $\mathbb{Z}/3\mathbb{Z}$-action on $\mathfrak{BI}$ such that each element of $\mathbb{Z}/3\mathbb{Z}$ acts as the algebra automorphism of $\mathfrak{BI}$ described in the following table.

u	X	Y	Z	κ	λ	μ
$(0 \bmod 3)(u)$	X	Y	Z	κ	λ	μ
$(1 \bmod 3)(u)$	Y	Z	X	λ	μ	κ
$(2 \bmod 3)(u)$	Z	X	Y	μ	κ	λ

TABLE 2. The $\mathbb{Z}/3\mathbb{Z}$-action on $\mathfrak{BI}$

Lemma 7.5. *Let $a, b, c \in \mathbb{C}$ and $n \geq 0$ be an even integer. Then the following conditions are equivalent:*

(i) *The $\mathfrak{BI}$-module $O_n(a,b,c)$ is irreducible.*
(ii) *The $\mathfrak{BI}$-module $O_n(c,a,b)$ is irreducible.*
(iii) *The $\mathfrak{BI}$-module $O_n(b,c,a)$ is irreducible.*

Suppose that (i)–(iii) *hold. Then the $\mathfrak{BI}$-modules*

$$O_n(a,b,c), \qquad O_n(c,a,b)^{1 \bmod 3}, \qquad O_n(b,c,a)^{2 \bmod 3}$$

are isomorphic.

Proof. Recall the irreducibility criterion for the $\mathfrak{BI}$-module $O_n(a,b,c)$ from Theorem 6.3. The equivalence of (i)–(iii) is immediate from the irreducibility criterion. By Lemmas 6.2 and 6.4, it is routine to verify the second assertion. □

Lemma 7.6. *Let $a, b, c \in \mathbb{C}$ and $n \geq 1$ be an odd integer. Then the following conditions are equivalent:*

(i) *The $\mathfrak{BI}$-module $E_n(a,b,c)$ is irreducible.*
(ii) *The $\mathfrak{BI}$-module $E_n(c,a,b)$ is irreducible.*
(iii) *The $\mathfrak{BI}$-module $E_n(b,c,a)$ is irreducible.*

Suppose that (i)–(iii) *hold. Then the $\mathfrak{BI}$-modules*

$$E_n(a,b,c), \qquad E_n(c,a,b)^{1 \bmod 3}, \qquad E_n(b,c,a)^{2 \bmod 3}$$

are isomorphic.

Proof. Recall the irreducibility criterion for the $\mathfrak{BI}$-module $E_n(a,b,c)$ from Theorem 6.7. The equivalence of (i)–(iii) is immediate from the irreducibility criterion. By Lemmas 6.6 and 6.8, it is routine to verify the second assertion. □

Theorem 7.7. *Let $a, b, c \in \mathbb{C}$ and $n \geq 0$ be an even integer. Suppose that the $\mathfrak{BI}$-module $O_n(a,b,c)$ is irreducible. Then the following conditions are equivalent:*

(i) *X, Y, Z act on $O_n(a,b,c)$ as a Leonard triple.*
(ii) *X, Y, Z are diagonalizable on $O_n(a,b,c)$.*
(iii) *$a, b, c \notin \{\frac{n-1}{2} - i \mid i = 0, 1, \ldots, n-1\}$.*

Proof. (i) ⇒ (ii): Immediate from Definition 7.1.

(ii) ⇒ (iii) ⇒ (i): By Lemma 7.3 the following conditions are equivalent:

- X is diagonalizable on $O_n(a,b,c)$.

- $a \notin \{\frac{n-1}{2} - i \,|\, i = 0, 1, \ldots, n-1\}$.
- There exists a basis for $O_n(a, b, c)$ with respect to which the matrix representing X is diagonal and those representing Y and Z are irreducible tridiagonal.

By Table 2 the actions of X, Y, Z on the $\mathfrak{BI}$-module $O_n(a, b, c)^{1 \bmod 3}$ correspond respectively to the actions of Y, Z, X on the $\mathfrak{BI}$-module $O_n(a, b, c)$. By Lemma 7.5 the $\mathfrak{BI}$-module $O_n(a, b, c)^{1 \bmod 3}$ is isomorphic to the $\mathfrak{BI}$-module $O_n(b, c, a)$. Hence, combined with Lemma 7.3, the following conditions are equivalent:

- Y is diagonalizable on $O_n(a, b, c)$.
- X is diagonalizable on $O_n(a, b, c)^{1 \bmod 3}$.
- X is diagonalizable on $O_n(b, c, a)$.
- $b \notin \{\frac{n-1}{2} - i \,|\, i = 0, 1, \ldots, n-1\}$.
- There exists a basis for $O_n(a, b, c)$ with respect to which the matrix representing Y is diagonal and those representing Z and X are irreducible tridiagonal.

By a similar argument the following conditions are equivalent:

- Z is diagonalizable on $O_n(a, b, c)$.
- X is diagonalizable on $O_n(a, b, c)^{2 \bmod 3}$.
- X is diagonalizable on $O_n(c, a, b)$.
- $c \notin \{\frac{n-1}{2} - i \,|\, i = 0, 1, \ldots, n-1\}$.
- There exists a basis for $O_n(a, b, c)$ with respect to which the matrix representing Z is diagonal and those representing X and Y are irreducible tridiagonal.

Therefore the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) follow. □

Theorem 7.8. *Let $a, b, c \in \mathbb{C}$ and $n \geq 1$ be an odd integer. Suppose that the $\mathfrak{BI}$-module $E_n(a, b, c)$ is irreducible. Then the following conditions are equivalent:*

(i) *X, Y, Z act on $E_n(a, b, c)$ as a Leonard triple.*
(ii) *X, Y, Z are diagonalizable on $E_n(a, b, c)$.*
(iii) *$a, b, c \notin \{\frac{n-1}{2} - i \,|\, i = 0, 1, \ldots, n-1\}$.*

Proof. Relying on Lemmas 7.4 and 7.6, the proof is analogous to that of Theorem 7.7. □

8. Proof for Theorem 2.2

Recall from Section 5 the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules $L_n^{\pm}$ for all $n \in \mathbb{N}$. Let V be a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}^{\otimes 2}$-module. For any $\theta \in \mathbb{C}$, the subspace $V(\theta)$ of V is defined in (11). By Theorem 2.1 the space $V(1)$ acquires a $\mathfrak{BI}$-module structure.

Lemma 8.1. *For any $m, n \in \mathbb{N}$ the vectors*

$$v_i^{(m)} \otimes v_j^{(n)} \qquad (0 \leq i \leq m,\ 0 \leq j \leq n,\ 2(i+j) = m+n-1)$$

form a basis for $(L_m^+ \otimes L_n^+)(1)$.

Proof. Immediate from (9) and (35). □

Lemma 8.2. *For any $m, n \in \mathbb{N}$ the following conditions are equivalent:*

(i) *$(L_m^+ \otimes L_n^+)(1)$ is nonzero.*
(ii) *$m + n$ is odd.*

Proof. Immediate from Lemma 8.1. □

Lemma 8.3. *For any $m, n \in \mathbb{N}$ with $m \leq n$ and $m + n$ odd, the following statements hold:*

(i) *The* $\mathfrak{B}\mathfrak{I}$*-module* $(L_m^+ \otimes L_n^+)(1)$ *is irreducible.*
(ii) X, Y, Z *act on* $(L_m^+ \otimes L_n^+)(1)$ *as a Leonard triple.*

Proof. Since $m \le n$ and $m+n$ is odd, it follows from Lemma 8.1 that the vectors

$$w_i = v_{m-i}^{(m)} \otimes v_{\frac{n-m-1}{2}+i}^{(n)} \qquad (0 \le i \le m)$$

are a basis for $(L_m^+ \otimes L_n^+)(1)$. For notational convenience, let w_{-1} and w_{m+1} denote the zero vector of $(L_m^+ \otimes L_n^+)(1)$ in this proof. Recall from (12) and (13) the actions of X and Y on $(L_m^+ \otimes L_n^+)(1)$. Using (7), (10), (33), (35) and (36) a direct computation yields

$$Xw_i = \left(\frac{m+n+1}{2} - i\right) w_{m-i} + i\, w_{m-i+1} \quad (0 \le i \le m), \tag{39}$$

$$Yw_i = \left(2i - m - \frac{1}{2}\right) w_i \quad (0 \le i \le m). \tag{40}$$

Let W be a $\mathfrak{B}\mathfrak{I}$-submodule of $(L_m^+ \otimes L_n^+)(1)$. We claim that if there is an integer j with $0 \le j \le m$ such that $w_j \in W$, then $w_{j-1} \in W$ and $w_{j+1} \in W$. Since $w_j \in W$ and W is a $\mathfrak{B}\mathfrak{I}$-module, the vector $Xw_j \in W$ and hence the right-hand side of (39) lies in W. By (40) the vectors $\{w_i\}_{i=0}^m$ are eigenvectors of Y corresponding to distinct eigenvalues. It follows that

$$\left(\frac{m+n+1}{2} - j\right) w_{m-j} \in W; \qquad j\, w_{m-j+1} \in W.$$

If $j \ne 0$ then $j\, w_{m-j+1} \in W$ implies $w_{m-j+1} \in W$; if $j = 0$ then $w_{m-j+1} = w_{m+1} = 0 \in W$. Hence $w_{m-j+1} \in W$ in either case. Since $m \le n$ and $0 \le j \le m$, the scalar $\frac{m+n+1}{2} - j \ne 0$ and hence $w_{m-j} \in W$. So far we have seen that if $w_j \in W$ then $w_{m-j+1} \in W$ and $w_{m-j} \in W$. Applying this fact again, the claim follows. To see (i), we assume that W is nonzero and show that $W = (L_m^+ \otimes L_n^+)(1)$. Since the eigenvalues of Y on $(L_m^+ \otimes L_n^+)(1)$ are pairwise distinct, there exists an integer j with $0 \le j \le m$ such that $w_j \in W$. Repeatedly applying the above claim, it follows that all vectors $\{w_i\}_{i=0}^m$ lie in W. The statement (i) follows.

By (39) the trace of X on $(L_m^+ \otimes L_n^+)(1)$ is equal to

$$\begin{cases} \dfrac{n+1}{2} & \text{if } m \text{ is even,} \\ \dfrac{m+1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

By (40) the trace of Y on $(L_m^+ \otimes L_n^+)(1)$ is equal to $-\frac{m+1}{2}$. By (39) and (40) the trace of XY on $(L_m^+ \otimes L_n^+)(1)$ is equal to

$$\begin{cases} -\dfrac{n+1}{4} & \text{if } m \text{ is even,} \\ \dfrac{m+1}{4} & \text{if } m \text{ is odd.} \end{cases}$$

Recall from Section 5 that Λ acts on L_m^+ and L_n^+ as scalar multiplication by $\frac{m(m+2)}{2}$ and $\frac{n(n+2)}{2}$, respectively. By Lemma 4.4(i) the elements κ, λ, μ act on $(L_m^+ \otimes L_n^+)(1)$ as scalar multiplication by $0, 0, \frac{m(m+2)}{2} - \frac{n(n+2)}{2}$, respectively. Using (29) the trace of Z on $(L_m^+ \otimes L_n^+)(1)$

equals

$$\begin{cases} -\dfrac{n+1}{2} & \text{if } m \text{ is even,} \\ \dfrac{m+1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

By Lemmas 6.4 and 6.8 the irreducible $\mathfrak{BI}$-module $(L_m^+ \otimes L_n^+)(1)$ is isomorphic to

$$\begin{cases} O_m\left(\dfrac{n+1}{2}, -\dfrac{m+1}{2}, -\dfrac{n+1}{2}\right) & \text{if } m \text{ is even,} \\ E_m\left(\dfrac{n+1}{2}, \dfrac{m+1}{2}, \dfrac{n+1}{2}\right)^{(-1,1)} & \text{if } m \text{ is odd.} \end{cases}$$

The statement (ii) now follows from Theorems 7.7 and 7.8. □

Let D_4 denote the dihedral group of symmetries of a square. Recall that D_4 admits a presentation with generators σ, τ and the relations

$$\sigma^2 = 1, \qquad \tau^4 = 1, \qquad (\sigma\tau)^2 = 1.$$

Recall the $\{\pm 1\}^2$-action on $\mathfrak{BI}$ from Section 6. There exists a unique group homomorphism $\phi : D_4 \to \{\pm 1\}^2$ that sends

$$\sigma \mapsto (1, -1), \qquad \tau \mapsto (-1, -1). \tag{41}$$

By Theorem 3.3 there exists a unique D_4-action on $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$ such that σ and τ act as the algebra automorphisms of $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$ described in the following table:

u	$E \otimes 1$	$F \otimes 1$	$H \otimes 1$	$\rho \otimes 1$	$1 \otimes E$	$1 \otimes F$	$1 \otimes H$	$1 \otimes \rho$
$\sigma(u)$	$1 \otimes E$	$1 \otimes F$	$1 \otimes H$	$1 \otimes \rho$	$E \otimes 1$	$F \otimes 1$	$H \otimes 1$	$\rho \otimes 1$
$\tau(u)$	$1 \otimes E$	$1 \otimes F$	$1 \otimes H$	$1 \otimes \rho$	$E \otimes 1$	$F \otimes 1$	$H \otimes 1$	$-\rho \otimes 1$

TABLE 3. The D_4-action on $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$

Lemma 8.4. *For any $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module V and any $g \in D_4$, the $\mathfrak{BI}$-module $V^g(1)$ coincides with the $\mathfrak{BI}$-module $V(1)^{\phi(g)}$.*

Proof. Applying Table 3 to (12)–(14), we observe that the actions of X, Y, Z on $(V^\sigma)(1)$ correspond respectively to the actions of $X, -Y, -Z$ on $V(1)$. Likewise, the actions of X, Y, Z on $(V^\tau)(1)$ correspond respectively to the actions of $-X, -Y, Z$ on $V(1)$. Thus, by Table 1 and (41), the lemma is true when $g = \sigma$ and $g = \tau$. Since the group D_4 is generated by σ and τ, the lemma follows. □

We now proceed to the proof for Theorem 2.2:

Proof of Theorem 2.2. The implications (ii) ⇒ (i) and (iii) ⇒ (i) are trivial. It remains to show (i) ⇒ (ii) and (i) ⇒ (iii).

Let V be a finite-dimensional irreducible $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module. By Theorem 5.6 the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules $L_n^{\pm}$ for all $n \in \mathbb{N}$ are all finite-dimensional irreducible $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules up to isomorphism. According to [5, Theorem 3.10.2(ii)], the $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module V is of one of the following forms up to isomorphism:

$$\begin{gathered} L_m^+\otimes L_n^+, \quad L_m^+\otimes L_n^-, \quad L_m^-\otimes L_n^+, \quad L_m^-\otimes L_n^-, \\ L_n^+\otimes L_m^+, \quad L_n^+\otimes L_m^-, \quad L_n^-\otimes L_m^+, \quad L_n^-\otimes L_m^- \end{gathered}$$

for some $m,n\in\mathbb{N}$ with $m\leq n$. By (36) and Table 3 these eight $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-modules are isomorphic to

$$\begin{gathered} (L_m^+\otimes L_n^+)^1, \quad (L_m^+\otimes L_n^+)^{\sigma\tau}, \quad (L_m^+\otimes L_n^+)^{\sigma\tau^{-1}}, \quad (L_m^+\otimes L_n^+)^{\tau^2}, \\ (L_m^+\otimes L_n^+)^{\sigma}, \quad (L_m^+\otimes L_n^+)^{\tau}, \quad (L_m^+\otimes L_n^+)^{\tau^{-1}}, \quad (L_m^+\otimes L_n^+)^{\sigma\tau^2}, \end{gathered}$$

respectively.

Consequently, there exist $m,n\in\mathbb{N}$ with $m\leq n$ and an element $g\in D_4$ such that the $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module V is isomorphic to $(L_m^+\otimes L_n^+)^g$. By Lemma 8.4 the $\mathfrak{BI}$-module $V(1)$ is isomorphic to $(L_m^+\otimes L_n^+)(1)^{\phi(g)}$. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) then follow from Lemmas 8.2 and 8.3 as well as Table 1. $\square$

9. Proof for Theorem 2.3

Recall some notations from Section 2. For a set X, let $\mathbb{C}^X$ be the vector space with basis X. Let Ω be a finite set and 2^Ω be the power set of Ω. In this section we adopt the symbol $\subset\!\!\cdot$ to denote the covering relation in $(2^\Omega,\subseteq)$.

Theorem 9.1 (Theorem 13.2, [8]). *There exists a unique $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}$ given by*

$$Ex=\sum_{y\subset\!\cdot\, x} y, \tag{42}$$

$$Fx=\sum_{x\subset\!\cdot\, y} y, \tag{43}$$

$$Hx=(|\Omega|-2|x|)x \tag{44}$$

for all $x\in 2^\Omega$, where the two sums are taken over all $y\in 2^\Omega$ with $y\subset\!\cdot\, x$ and $x\subset\!\cdot\, y$, respectively.

Let $x_0\in 2^\Omega$ be given. By Theorem 9.1 the spaces $\mathbb{C}^{2^{x_0}}$ and $\mathbb{C}^{2^{\Omega\setminus x_0}}$ are $U(\mathfrak{sl}_2)$-modules. Hence $\mathbb{C}^{2^{\Omega\setminus x_0}}\otimes\mathbb{C}^{2^{x_0}}$ is a $U(\mathfrak{sl}_2)^{\otimes 2}$-module. Define a linear map $\iota(x_0):\mathbb{C}^{2^\Omega}\to\mathbb{C}^{2^{\Omega\setminus x_0}}\otimes\mathbb{C}^{2^{x_0}}$ by

$$x \quad\mapsto\quad (x\setminus x_0)\otimes(x\cap x_0) \qquad \text{for all } x\in 2^\Omega. \tag{45}$$

Note that $\iota(x_0)$ is a linear isomorphism.

Lemma 9.2 (Lemma 5.4, [15]). *For any $x_0\in 2^\Omega$ and $u\in U(\mathfrak{sl}_2)$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{C}^{2^\Omega} & \xrightarrow{\iota(x_0)} & \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \Delta(u)} \\ \mathbb{C}^{2^\Omega} & \xrightarrow[\iota(x_0)]{} & \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}} \end{array}$$

By pulling back via the comultiplication of $U(\mathfrak{sl}_2)$, every $U(\mathfrak{sl}_2) \otimes U(\mathfrak{sl}_2)$-module can be regarded as a $U(\mathfrak{sl}_2)$-module. Thus, Lemma 9.2 can also be interpreted as asserting that $\iota(x_0)$ realizes a $U(\mathfrak{sl}_2)$-module isomorphism from $\mathbb{C}^{2^\Omega}$ to $\mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$.

Theorem 9.3. (i) *There exists a unique $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module $\mathbb{C}^{2^\Omega}$ satisfying* (42)–(44) *and*

$$\rho\, x = \Omega \setminus x \qquad \text{for all } x \in 2^\Omega. \tag{46}$$

(ii) *For any $x_0 \in 2^\Omega$ and $u \in U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{C}^{2^\Omega} & \xrightarrow{\iota(x_0)} & \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \Delta(u)} \\ \mathbb{C}^{2^\Omega} & \xrightarrow[\iota(x_0)]{} & \mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}} \end{array}$$

Proof. (i): By Theorem 9.1 the relations (17)–(19) hold on $\mathbb{C}^{2^\Omega}$. According to Theorem 3.3, it remains to verify the relations (22)–(24) on $\mathbb{C}^{2^\Omega}$.

Let $x \in 2^\Omega$ be given. By (46) the left-hand side of (24) fixes x, so the relation (24) holds on $\mathbb{C}^{2^\Omega}$. Evaluating both sides of (22) at x using (44) and (46), both sides are equal to $(|\Omega| - 2|x|) \cdot (\Omega \setminus x)$. Hence the relation (22) holds on $\mathbb{C}^{2^\Omega}$. Evaluating the left-hand side of (23) at x using (42) and (46) gives $\sum_{y \subset x} \Omega \setminus y$. Evaluating the right-hand side at x using (43) and (46) gives $\sum_{\Omega\setminus x \subset y} y$. Both expressions coincide with

$$\sum_{\Omega \setminus y \subset x} y.$$

Therefore the relation (23) holds on $\mathbb{C}^{2^\Omega}$. The statement (i) follows.

(ii): Let $x \in 2^\Omega$ be given. By (45) and (46) the composition $\iota(x_0) \circ \rho$ maps x to

$$\big((\Omega \setminus x) \setminus x_0\big) \otimes \big((\Omega \setminus x) \cap x_0\big).$$

By (10), (45) and (46) the composition $\Delta(\rho) \circ \iota(x_0)$ maps x to

$$\big((\Omega \setminus x_0) \setminus (x \setminus x_0)\big) \otimes \big(x_0 \setminus (x \cap x_0)\big).$$

The first factors of both tensor products are equal to $\Omega \setminus (x \cup x_0)$, and the second factors are equal to $x_0 \setminus x$. Therefore the two tensor products coincide. Combined with Lemma 9.2, the statement (ii) follows. □

We now regard $\mathbb{C}^{2^\Omega}$ as a $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module by identifying it with $\mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$ via the linear isomorphism $\iota(x_0)$. The resulting $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module structure depends on the choice of x_0, and we denote this module by

$$\mathbb{C}^{2^\Omega}(x_0).$$

Lemma 9.4. *For any $x_0 \in 2^\Omega$, the following equations hold for all $x \in 2^\Omega$ in the $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module $\mathbb{C}^{2^\Omega}(x_0)$:*

(i) $(H \otimes 1)x = (|\Omega \setminus x_0| - 2|x \setminus x_0|)x$.
(ii) $(1 \otimes H)x = (|x_0| - 2|x \cap x_0|)x$.
(iii) $\Delta(H)x = (|\Omega| - 2|x|)x$.
(iv) $\Delta(E\rho)x = \sum_{y \Subset \Omega\setminus x} y$, *where the sum is over all $y \in 2^\Omega$ with $y \Subset \Omega \setminus x$.*

Proof. (i), (ii): These are immediate from (44) and the construction of $\mathbb{C}^{2^\Omega}(x_0)$.

(iii): This is immediate from (i), (ii) and (9). Alternatively, the equation (iii) can be deduced directly from (44) by applying Theorem 9.3(ii) with $u = H$.

(iv): Using (42) and (46) shows that $(E\rho)x$ equals the right-hand side of (iv) for all $x \in 2^\Omega$. Applying Theorem 9.3(ii) with $u = E\rho$, the statement (iv) follows. $\square$

We next turn to the proof for Theorem 2.3:

Proof of Theorem 2.3. Assume that $d \geq 1$ is an integer and $|\Omega| = 2d + 1$, which are the hypotheses of Theorem 2.3. By Theorem 2.1, the space $\mathbb{C}^{2^\Omega}(x_0)(1)$ carries a $\mathfrak{B}\mathfrak{I}$-module structure on which X and Y act as $\Delta(E\rho)$ and $\frac{H\otimes 1 - 1\otimes H}{2}$, respectively. It follows from Lemma 9.4(iii) that

$$\mathbb{C}^{2^\Omega}(x_0)(1) = \mathbb{C}^{\binom{\Omega}{d}} \qquad \text{for any } x_0 \in 2^\Omega.$$

Now let $x_0 \in \binom{\Omega}{d}$.

(i): Let $x \in \binom{\Omega}{d}$. For any $y \in 2^\Omega$, the condition $y \Subset \Omega \setminus x$ is equivalent to $x \cap y = \emptyset$ and $y \in \binom{\Omega}{d}$. Hence, comparing (15) with Lemma 9.4(iv), the operator X on $\mathbb{C}^{2^\Omega}(x_0)(1)$ coincides with $\mathbf{A}$.

Since $|\Omega| = 2d + 1$ and $|x_0| = d$, it follows from Lemma 9.4(i), (ii) that Yx is a scalar multiple of x with scalar

$$\frac{1}{2} - |x \setminus x_0| + |x \cap x_0|. \tag{47}$$

Since $x \cap x_0 = x_0 \setminus (x_0 \setminus x)$, we have $|x \cap x_0| = d - |x_0 \setminus x|$. Hence (47) can be rewritten as

$$d + \frac{1}{2} - (|x \setminus x_0| + |x_0 \setminus x|).$$

By comparing this with (16), the operator Y on $\mathbb{C}^{2^\Omega}(x_0)(1)$ coincides with

$$\frac{d+1}{2d+1}\mathbf{A}^*(x_0) + \frac{1}{2(2d+1)}.$$

By Lemma 4.4(i) the element κ vanishes on $\mathbb{C}^{2^\Omega}(x_0)(1)$. Using (29) the operator Z on $\mathbb{C}^{2^\Omega}(x_0)(1)$ coincides with

$$\frac{d+1}{2d+1}\{\mathbf{A}, \mathbf{A}^*(x_0)\} + \frac{1}{2d+1}\mathbf{A}.$$

By the preceding arguments, the existence of the homomorphism $\mathfrak{BI} \to \mathbf{T}(x_0)$ follows. The uniqueness is immediate since $\mathfrak{BI}$ is generated by X, Y, Z. The homomorphism is surjective since $\mathbf{T}(x_0)$ is generated by $\mathbf{A}, \mathbf{A}^*(x_0)$. The statement (i) follows.

(ii): By Theorem 5.7 the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules $\mathbb{C}^{2^{\Omega\setminus x_0}}$ and $\mathbb{C}^{2^{x_0}}$ are completely reducible. Hence the $U(\mathfrak{sl}_2)^{\otimes 2}_{\mathbb{Z}/2\mathbb{Z}}$-module $\mathbb{C}^{2^{\Omega\setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$, and consequently $\mathbb{C}^{2^{\Omega}}(x_0)$, is completely reducible. The statement (ii) now follows from Theorem 2.2. □

10. The Clebsch–Gordan rule for $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$

Recall the $U(\mathfrak{sl}_2)$-module L_n $(n \in \mathbb{N})$ from (33)–(35). The $U(\mathfrak{sl}_2)$-module L_n satisfies the following universal property:

Lemma 10.1. *Let V denote a finite-dimensional $U(\mathfrak{sl}_2)$-module. Suppose that there exist an integer $n \in \mathbb{N}$ and a vector $v \in V$ such that $Hv = nv$ and $Ev = 0$. Then there exists a unique $U(\mathfrak{sl}_2)$-module homomorphism $L_n \to V$ that maps $v_0^{(n)}$ to v.*

Proof. See [19, Theorem V.4.4] for example. □

For any integers i and n with $0 \leq i \leq n$, the symbol

$$\binom{n}{i} = \frac{n(n-1)\cdots(n-i+1)}{i!}$$

denotes the binomial coefficient.

Lemma 10.2. *Let $m, n \in \mathbb{N}$. For any integer p with $0 \leq p \leq \min\{m, n\}$, the following statements hold:*

(i) *There exists a unique $U(\mathfrak{sl}_2)$-module homomorphism $f : L_{m+n-2p} \to L_m \otimes L_n$ such that*

$$v_0^{(m+n-2p)} \quad \mapsto \quad \sum_{i=0}^{p} (-1)^i \binom{p}{i} v_i^{(m)} \otimes v_{p-i}^{(n)}.$$

In particular f is injective.

(ii) *$f\big(v_{m-p}^{(m+n-2p)}\big)$ is a linear combination of $\{v_{m-i}^{(m)} \otimes v_i^{(n)}\}_{i=1}^{\min\{m,n\}}$ plus*

$$(-1)^p \binom{m+n-2p}{m-p}^{-1} v_m^{(m)} \otimes v_0^{(n)}.$$

(iii) *$f\big(v_{n-p}^{(m+n-2p)}\big)$ is a linear combination of $\{v_i^{(m)} \otimes v_{n-i}^{(n)}\}_{i=1}^{\min\{m,n\}}$ plus*

$$\binom{m+n-2p}{n-p}^{-1} v_0^{(m)} \otimes v_n^{(n)}.$$

Proof. Using Lemma 10.1 the existence and uniqueness of f is straightforward to verify. Since the $U(\mathfrak{sl}_2)$-module L_{m+n-2p} is irreducible, the homomorphism f is injective. The statements (ii) and (iii) follow from direct computation using (i), (8) and (34). □

Let $m, n \in \mathbb{N}$ be given. By Lemma 10.2(i) the $U(\mathfrak{sl}_2)$-module $L_m \otimes L_n$ contains the $U(\mathfrak{sl}_2)$-submodules isomorphic to L_{m+n-2p} for all integers p with $0 \leq p \leq \min\{m, n\}$. A dimension argument then shows that the $U(\mathfrak{sl}_2)$-module $L_m \otimes L_n$ is isomorphic to

$$\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}. \tag{48}$$

This decomposition is known as the *Clebsch–Gordan rule* for $U(\mathfrak{sl}_2)$. As a consequence, for each integer p with $0 \le p \le \min\{m,n\}$, there is a unique $U(\mathfrak{sl}_2)$-submodule of $L_m \otimes L_n$ that is isomorphic to L_{m+n-2p}.

In order to formulate the Clebsch–Gordan rule for $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$, the signs $+$ and $-$ in the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-modules L_n^+ and L_n^- $(n \in \mathbb{N})$ are identified with the real scalars $+1$ and -1, respectively.

Theorem 10.3. *Suppose that $m,n \in \mathbb{N}$ and $\delta, \varepsilon \in \{\pm 1\}$. Then the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module $L_m^\delta \otimes L_n^\varepsilon$ is isomorphic to*

$$\bigoplus_{p=0}^{\min\{m,n\}} L_{m+n-2p}^{(-1)^p \cdot \delta \cdot \varepsilon}.$$

Proof. As a $U(\mathfrak{sl}_2)$-module, $L_m^\delta \otimes L_n^\varepsilon$ is isomorphic to $L_m \otimes L_n$. By the Clebsch–Gordan rule for $U(\mathfrak{sl}_2)$, the $U(\mathfrak{sl}_2)$-module $L_m^\delta \otimes L_n^\varepsilon$ is isomorphic to (48). Fix an integer p with $0 \le p \le \min\{m,n\}$. Let

$$f : L_{m+n-2p} \to L_m^\delta \otimes L_n^\varepsilon$$

be the injective $U(\mathfrak{sl}_2)$-module homomorphism given by Lemma 10.2(i).

Let W denote the image of f. Then the $U(\mathfrak{sl}_2)$-module W is isomorphic to L_{m+n-2p}. By Lemma 5.4(i), the space ρW is a $U(\mathfrak{sl}_2)$-submodule of $L_m^\delta \otimes L_n^\varepsilon$. Moreover, Lemma 5.5(i) implies that ρW is isomorphic to L_{m+n-2p}. Thus

$$W = \rho W.$$

By Lemma 5.4(ii) the space W is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-submodule of $L_m^\delta \otimes L_n^\varepsilon$. It then follows from Lemmas 5.3(ii) and 5.5(ii) that there exists a unique $\epsilon \in \{\pm 1\}$ for which W is isomorphic to L_{m+n-2p}^ϵ. It suffices to prove that

$$\epsilon = (-1)^p \delta \varepsilon. \tag{49}$$

Since the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module W is isomorphic to L_{m+n-2p}^ϵ, there exists a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module isomorphism from L_{m+n-2p}^ϵ to W. As this map is also a $U(\mathfrak{sl}_2)$-module isomorphism, by Schur's lemma it must be a nonzero scalar multiple of f. Hence the map

$$f : L_{m+n-2p}^\epsilon \to W$$

is a $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module isomorphism. In particular f commutes with ρ. Consider the relation

$$(\rho f)\big(v_{m-p}^{(m+n-2p)}\big) = (f\rho)\big(v_{m-p}^{(m+n-2p)}\big). \tag{50}$$

By (36) and Lemma 10.2(iii), the right-hand side of (50) is a linear combination of $\{v_i^{(m)} \otimes v_{n-i}^{(n)}\}_{i=1}^{\min\{m,n\}}$ plus

$$\epsilon \cdot \binom{m+n-2p}{n-p}^{-1} v_0^{(m)} \otimes v_n^{(n)}. \tag{51}$$

Similarly, by (36) and Lemma 10.2(ii), the left-hand side of (50) is a linear combination of $\{v_i^{(m)} \otimes v_{n-i}^{(n)}\}_{i=1}^{\min\{m,n\}}$ plus

$$(-1)^p \delta \varepsilon \cdot \binom{m+n-2p}{m-p}^{-1} v_0^{(m)} \otimes v_n^{(n)}. \tag{52}$$

Since $\binom{m+n-2p}{n-p} = \binom{m+n-2p}{m-p}$, comparing (51) and (52) yields (49). The result follows. $\square$

Finally, we apply the Clebsch–Gordan rule for $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$ (Theorem 10.3) to decompose the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module $\mathbb{C}^{2^\Omega}$ as described in Theorem 9.3(i). For any vector space V and any positive integer n, we write $n \cdot V$ for the direct sum of n copies of V.

Corollary 10.4. *The $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module $\mathbb{C}^{2^\Omega}$ is isomorphic to*

$$\bigoplus_{i=0}^{\lfloor \frac{|\Omega|}{2} \rfloor} \frac{|\Omega|-2i+1}{|\Omega|-i+1}\binom{|\Omega|}{i} \cdot L^{(-1)^i}_{|\Omega|-2i}. \tag{53}$$

Proof. We prove the decomposition by induction on $|\Omega|$. By Theorem 9.3(i), the decomposition holds for $|\Omega| = 0$ and $|\Omega| = 1$, since in these cases $\mathbb{C}^{2^\Omega}$ is isomorphic to L_0^+ and L_1^+ respectively.

Assume that $|\Omega| \geq 2$. For notational convenience, set $D = |\Omega|$ and define

$$m_i(k) = \frac{k-2i+1}{k-i+1}\binom{k}{i}$$

for any integers i and k with $0 \leq i \leq k$. Let $x_0 \in 2^\Omega$ with $|x_0| = 1$. By Theorem 9.3(ii) the $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^\Omega}$ is isomorphic to $\mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$. As noted earlier, the $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^{x_0}}$ is isomorphic to L_1^+. By the induction hypothesis, the $U(\mathfrak{sl}_2)$-module $\mathbb{C}^{2^{\Omega \setminus x_0}}$ is isomorphic to

$$\bigoplus_{i=0}^{\lfloor \frac{D-1}{2} \rfloor} m_i(D-1) \cdot L^{(-1)^i}_{D-2i-1}.$$

The distributive law of $\otimes$ over $\oplus$ yields the following decomposition of $\mathbb{C}^{2^{\Omega \setminus x_0}} \otimes \mathbb{C}^{2^{x_0}}$:

$$\bigoplus_{i=0}^{\lfloor \frac{D-1}{2} \rfloor} m_i(D-1) \cdot (L^{(-1)^i}_{D-2i-1} \otimes L_1^+). \tag{54}$$

Applying Theorem 10.3 the $U(\mathfrak{sl}_2)$-module $L^{(-1)^i}_{D-2i-1} \otimes L_1^+$ is isomorphic to

$$\begin{cases} L^{(-1)^i}_{D-2i} \oplus L^{(-1)^{i+1}}_{D-2i-2} & \text{if } i \leq \frac{D}{2}-1, \\ L_1^{(-1)^i} & \text{otherwise} \end{cases} \qquad \left(0 \leq i \leq \left\lfloor \tfrac{D-1}{2} \right\rfloor\right).$$

If D is even then the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module (54) is isomorphic to

$$m_0(D-1) \cdot L_D^+ \oplus m_{\frac{D}{2}-1}(D-1) \cdot L_0^{(-1)^{\frac{D}{2}}} \oplus \bigoplus_{i=1}^{\frac{D}{2}-1} (m_{i-1}(D-1) + m_i(D-1)) \cdot L^{(-1)^i}_{D-2i}. \tag{55}$$

If D is odd then the $U(\mathfrak{sl}_2)_{\mathbb{Z}/2\mathbb{Z}}$-module (54) is isomorphic to

$$m_0(D-1) \cdot L_D^+ \oplus \bigoplus_{i=1}^{\frac{D-1}{2}} (m_{i-1}(D-1) + m_i(D-1)) \cdot L^{(-1)^i}_{D-2i}. \tag{56}$$

Observe that $m_0(D) = m_0(D-1) = 1$ and

$$m_{i-1}(D-1) + m_i(D-1) = m_i(D) \qquad (1 \leq i \leq D-1).$$

Note that if D is even and $i = \frac{D}{2}$, then $m_i(D-1) = 0$. Substituting these equalities into (55) and (56) yields the decomposition (53). The result follows. $\square$

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References

1. E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin-Cummings, Menlo Park, 1984.
2. P.-A. Bernard, N. Crampé, and L. Vinet, *The Terwilliger algebra of symplectic dual polar graphs, the subspace lattices and $U_q(\mathfrak{sl}_2)$*, Discrete Mathematics **345** (2022), 113169.
3. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.
4. B. Curtin, *Modular Leonard triples*, Linear Algebra and its Applications **424** (2007), 510–539.
5. P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, *Introduction to representation theory*, Student Mathematical Library 59, American Mathematical Society, Providence, RI, 2011.
6. S. Gao, Y. Wang, and B. Hou, *The classification of Leonard triples of Racah type*, Linear Algebra and its Applications **439** (2013), 1834–1861.
7. V. X. Genest, L. Vinet, and A. Zhedanov, *Embeddings of the Racah algebra into the Bannai–Ito algebra*, SIGMA **11** (2015), 050, 11 pages.
8. J.T. Go, *The Terwilliger algebra of the hypercube*, European Journal of Combinatorics **23** (2002), 399–429.
9. B. Hou and S. Gao L. Wang, *The classification of Leonard triples that have Bannai/Ito type and odd diameter*, Linear Algebra and its Applications **439** (2013), 2667–2691.
10. B. Hou, Y. Xu, and S. Gao, *The Classification of Leonard Triples of Bannai/Ito Type with Even Diameters*, Communications in Algebra **43** (2015), 3776–3798.
11. H.-W. Huang, *The classification of Leonard triples of QRacah type*, Linear Algebra and its Applications **436** (2012), 1442–1472.
12. ———, *Finite-dimensional irreducible modules of the Bannai–Ito algebra at characteristic zero*, Letters in Mathematical Physics **110** (2020), 2519–2541.
13. ———, *The Racah algebra as a subalgebra of the Bannai–Ito algebra*, SIGMA **16** (2020), 075, 15 pages.
14. ———, *The Clebsch–Gordan rule for $U(\mathfrak{sl}_2)$, the Krawtchouk algebras and the Hamming graphs*, SIGMA **19** (2023), 017, 19 pages.
15. ———, *The Clebsch–Gordan coefficients of $U(\mathfrak{sl}_2)$ and the Terwilliger algebras of Johnson graphs*, Journal of Combinatorial Theory, Series A **203** (2024), 105833.
16. ———, *An imperceptible connection between the Clebsch–Gordan coefficients of $U_q(\mathfrak{sl}_2)$ and the Terwilliger algebras of Grassmann graphs*, Journal of Combinatorial Theory, Series A **214** (2025), 106028.
17. H.-W. Huang and C.-Y. Wen, *A connection behind the Terwilliger algebras of $H(D,2)$ and $\frac{1}{2}H(D,2)$*, Journal of Algebra **634** (2023), 456–479.
18. ———, *Johnson graphs as slices of a hypercube and an algebra homomorphism from the universal Racah algebra into $U(\mathfrak{sl}_2)$*, Journal of Algebraic Combinatroics **61** (2025), article number 56.
19. C. Kassel, *Quantum groups*, Graduate Texts in Mathematics 155, Springer-Verlag, New York, 1995.
20. S. Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, Springer, New York, 2002.
21. X. Liang, T. Ito, and Y. Watanabe, *The Terwilliger algebra of the Grassmann scheme $J_q(N,D)$ revisited from the viewpoint of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$*, Linear Algebra and its Applications **596** (2020), 117–144.
22. P. Terwilliger, *Two linear transformations each tridiagonal with respect to an eigenbasis of the other*, Linear Algebra and Its Applications **330** (2001), 149–203.
23. P. Terwilliger and R. Vidūnas, *Leonard pairs and the Askey–Wilson relations*, Journal of Algebra and Its Applications **3** (2004), 411–426.

24. R. Vidūnas, *Normalized Leonard pairs and Askey–Wilson relations*, Linear Algebra and its Applications **422** (2007), 39–57.
25. L. Vinet and A. Zhedanov, *A limit $q = -1$ for the big q-Jacobi polynomials*, Transactions of the American Mathematical Society **364** (2012), 5491–5507.
26. C. Worawannotai, *Dual polar graphs, the quantum algebra $U_q(\mathfrak{sl}_2)$, and Leonard systems of dual q-Krawtchouk type*, Linear Algebra and its Applications **438** (2013), 443–497.

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