GROUP EQUIVARIANT RADON-NIKODÝM PROPERTY AND ITS CHARACTERIZATIONS

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Abstract. We introduce and study *equivariant* versions of the Radon–Nikodým property for Banach spaces, together with the closely related notions such as dentability, the Bishop–Phelps and Krein–Milman properties, and Lindenstrauss' property A, all considered in the presence of a continuous group action by linear isometries. While in the classical setting the Radon–Nikodým property, the Bishop–Phelps property and dentability are equivalent, the equivariant situation turns out to depend essentially on the acting group and requires nontrivial tools from abstract harmonic analysis and representation theory. We establish several implications among the equivariant counterparts of these properties. Namely, for a compact group G, the G-Bishop–Phelps property implies strong G-dentability, which in turn implies the G-Krein–Milman property. Moreover, for a locally compact, σ-compact, separable group G, weak G-dentability is equivalent to the G-Radon–Nikodým property.

Contents

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1. Introduction

The Radon-Nikodým property for Banach spaces, which–as the name suggests–concerns Banach spaces for which the Radon-Nikodým theorem for vector-valued measures holds, has been a central topic in Banach space theory for nearly a century. It first appeared in the works of Dunford [19, 20] and played a pivotal role in the development of the theory of vector-valued measures. Although by definition an analytic property involving measures, the Radon-Nikodým property (RNP, for short) admits numerous equivalent characterizations (see, for instance, [17, Chapter VII.6]), some of them purely geometrical, that connect several major branches of Banach space theory. We refer the reader to the classical monographs [33, 10, 17, 43], as well as to more recent references such as [21, 46, 51].

In this paper, we are interested in what can be dubbed an equivariant Radon-Nikodým property; that is, the RNP in the presence of a continuous group action on a Banach space by linear isometries. The motivations are at least twofold. First, and that was the original and foremost motivation for us, we want to extend the already rich and expanding research on norm-attaining operators that are invariant under a group action by linear isometries. We refer to the papers [15, 16, 22, 23] for some of the very recent developments in this direction, as well as to [24, 25] for related work. One of the cornerstones in the norm-attaining theory is the Bishop-Phelps theorem for functionals and its extension to more general operators turns out to be valid exactly for those Banach spaces having the RNP. The equivariant Bishop-Phelps theorem, that is, the version concerning group equivariant operators, is therefore a natural continuation in this line of work and a starting point of our study of the equivariant RNP.

Second, research in this direction belongs to the emerging area of equivariant Banach space theory. From the category-theoretical point of view, it amounts to upgrading the classical category of Banach spaces into an equivariant category–whose objects are Banach spaces endowed with group actions and whose morphisms are equivariant linear operators. This perspective parallels the developments that took place several decades ago in topology and geometry, where equivariant homotopy and co/homology theories have become central in the subject. There are now plenty of tools at our disposal. Indeed, equivariant Banach space theory can draw on representation theory of group actions on Banach spaces, which is particularly well developed for compact groups (see e.g. the monograph [30]), and on analytic group theory studying rigidity of group actions on Banach spaces (see e.g. [2, 6]). Moreover, category-theoretic and homology methods in Banach space theory have recently been developed by Cabello-Sánchez and Castillo in [12] and extended to the equivariant setting in the very recent work [14].

Here we consider equivariant versions of the RNP and several related concepts, including the Riesz representation property, the already mentioned Bishop-Phelps property, dentability of bounded subsets, the Krein-Milman property and the Lindenstrauss' property A, all defined in the presence of a group action. While in the classical case, all these conditions, except for the last two which are implied by the rest, are equivalent, the equivariant setting turns out to be substantially more intricate. Several of the implications require non-trivial tools from abstract harmonic analysis and demand that the acting group be either compact, in which case one employs the Big Peter-Weyl theorem, or locally compact and σ-compact, taking advantage of the existence of a σ-finite Haar measure.

We review the main results below. Note that it is not an exhaustive list of the results but rather a summary of the main implications among the equivariant properties. Throughout the paper, we adopt the convention that equivariant (or sometimes invariant) properties are denoted by the prefix 'G-', where G stands for a generic group and serves as a shorthand for *G-equivariant* or *G-invariant*. For example, we write *G-RNP* in place of the more accurate *group equivariant Radon-Nikodým property*. This notation will be used consistently for the reader's convenience. For the precise definitions that appear in the following, we send the reader to Section 3 below.

Main results.

(1) The G-Bishop–Phelps property implies the strong G-dentability when G is compact.
(2) The strong G-dentability implies the G-Krein–Milman property, again when G is compact.
(3) The weak G-dentability implies the G-RNP if G is separable, and the converse holds if G is locally compact, σ-compact, and separable.

In addition, we present several examples showing that not all these implications can be reversed, together with further phenomena illustrating how the equivariant cases differ from the classical one. We also thoroughly investigate the equivariant analogue of Lindenstrauss' property A.

The paper is organized as follows. Section 2 recalls the Radon-Nikodým property, the central notion of the paper, and sets the notation and basic notions for the sequel. Section 3 rigorously introduces all the new equivariant versions of the properties discussed informally above and shows their fundamental properties. The core of the paper, where, in particular, all the results stated above are proved, lies in Sections 4 to 6. Section 7 deals with norm-attaining equivariant operators. Interestingly, the original motivation for this work has evolved into a natural offshoot of the main part of the paper, yet it naturally follows from the preceding sections and is of independent interest. Finally, Section 8 summarizes the results and suggests open problems. A reader reaching this point will recognize that there is a plethora of new directions to be explored.

To conclude, we believe that developing an equivariant counterpart of the Radon-Nikodým property opens a promising path toward a systematic understanding of Banach space geometry under group actions. In particular, the equivariant viewpoint not only extends the reach of classical Banach space theory but also provides a new language in which linear and group-theoretic aspects coexist naturally. The equivariant Radon–Nikodým property, together with its related notions studied here, thus constitutes a natural starting point for further investigations at the intersection of functional analysis, abstract harmonic analysis, and geometric group theory.

2. Preliminaries

As the main goal of this paper is the study of the group equivariant version of the Radon-Nikodým property for Banach spaces, we begin by briefly recalling its classical definition. For simplicity and as used already in the introduction above, we will be denoting this property as RNP. A Banach space X is said to have the RNP if the classical Radon-Nikodým theorem (see, for instance, [17, Chapter III]) extends to X-valued measures. More precisely, X has the RNP if and only if for every σ-finite measure space $(\Omega, \Sigma, \lambda)$ and for every X-valued measure $\mu : \Sigma \to X$ that has bounded variation and is λ-absolutely continuous, there exists a λ-Bochner integrable function $g : \Omega \to X$ such that

$$\mu(E) = \int_E g\, d\lambda \tag{2.1}$$

for all $E \in \Sigma$. It is enough to consider the case where $\lambda(\Omega) < \infty$ and $\|\mu(E)\| \leq \lambda(E)$ for all $E \in \Sigma$; in this situation, any derivative $g : \Omega \to X$ is necessarily bounded. Equivalently, X satisfies the RNP if and only if for every vector measure $\mu : \Sigma \to X$ with $\|\mu(E)\| \leq \lambda(E)$, there exists a bounded measurable function $g : \Omega \to X$ such that (2.1) holds for all $E \in \Sigma$. This provides an equivalent formulation of the RNP, which will be useful throughout the text. This is in turn equivalent to the statement that, for every continuous linear operator $T : L_1(\Omega, \Sigma, \lambda) \to X$, there exists a bounded measurable function $g : \Omega \to X$ such that

$$T(f) = \int_\Omega fg\, d\lambda, \quad \text{for all } f \in L_1(\Omega, \Sigma, \lambda).$$

It is well known that c_0 and $L_1[0, 1]$ do not satisfy the RNP. On the other hand, reflexive spaces do satisfy the RNP [45], as do separable dual spaces [20]. Moreover, if a Banach space X has the RNP, then every closed subspace of X also has it. Conversely, if every closed separable subspace of X has the RNP [49], then X itself has the RNP (see also [52]). In particular, if every separable subspace of X is isomorphic to a subspace of a separable dual space, then X has the RNP. In particular, $\ell_1(\Gamma)$ has the RNP for any uncountable set Γ.

In what follows, we provide the basic notation and conventions we will be using throughout the paper. We follow standard notation coming, essentially, from the references [17, 21].

Notation. In the present paper, all the Banach spaces X will be considered over the *real* numbers. We denote by B_X its closed unit ball and by S_X its closed unit sphere. We also denote by X^* the topological dual of X. If $x \in X$ and $\varepsilon > 0$, we denote by $B_\varepsilon(x)$ the open ball of radius ε given by $\{y \in X : \|x - y\| < \varepsilon\}$. Given a finite set $\{x_1, \ldots, x_n\} \subseteq X$, the symbol $B_\varepsilon(x_1, \ldots, x_n)$ stands for the union of open balls of radius ε centered at the points $x_1, \ldots, x_n$, i.e., $B_\varepsilon(x_1, \ldots, x_n) = \bigcup_{i=1}^n B_\varepsilon(x_i)$. We denote by diam A the diameter of a subset A. The symbols $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ stand for the convex hull and closed convex hull of A, respectively. We denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X into Y.

Let G be a topological group, which we always assume to be Hausdorff. We denote by $1_G \in G$ the identity element of the group G. On a few occasions, we shall employ the Haar measure, which exists if and only if the group is locally compact, and which is the unique (up to a scalar multiple) non-trivial Radon Borel measure on the group which is invariant under left multiplication. The monograph [30] serves as our reference for topological groups, to which we refer for any unexplained notation. Given a Banach space X, an important example of a topological group is $\mathrm{Iso}(X)$, the group of all linear isometries equipped with the strong operator topology (SOT, for short), which is the topology of pointwise convergence in norm.

We shall denote the actions by the symbol $\alpha : G \curvearrowright X$ or simply by $G \curvearrowright X$, when there is no risk of confusion. For $g \in G$ and $x \in X$, we write $\alpha(g, x) \in X$ for the application of the element g to x under the action α. As before when no confusion can arise, we omit the symbol α and simply write $g \cdot x$, or even gx, instead of $\alpha(g, x)$. For subsets $F \subseteq G$ and $A \subseteq X$ we write $F \cdot A$ to denote the set $\{f \cdot a \colon f \in F, a \in A\}$. All group actions considered in this paper are continuous. If X is a Banach space, which will be the most common case, continuity of the action refers to the continuity of the map $G \times X \to X$, or equivalently, to the continuity of the corresponding homomorphism from G into $\mathrm{Iso}(X)$. In case X is a measure space, the continuity of the action is defined in the appropriate sense for that context.

Throughout the text, when convenient, we refer to a *G-Banach space* as a Banach space endowed with a continuous action of a topological group G by linear isometries.

Let X and Y be Banach spaces, and let G be a topological group acting continuously by linear isometries on X and Y. We say that a linear operator $T : X \to Y$ is *G-equivariant*, or just *equivariant*, if

$$T(g \cdot x) = g \cdot T(x), \text{ for all } g \in G \text{ and } x \in X.$$

We denote by $\mathcal{L}^G(X, Y)$ the Banach space of all G-equivariant bounded linear operators from X into Y. In the language of representation theory, such operators are often referred to as *intertwining operators*. Finally, a subset C of a G-Banach space is said to be *G-invariant* if $g \cdot C \subseteq C$ for every $g \in G$.

3. Definitions, basic properties and examples

In this section, we introduce the central definitions of the paper and show their basic properties. More specifically, we focus on the G-versions of dentability, Radon-Nikodým, Bishop-Phelps, Krein-Milman, and Lindenstrauss A properties.

3.1. G-Bishop-Phelps property.

Recall that a nonempty bounded closed subset C of a Banach space X is said to have the *Bishop-Phelps property* (BP, for short) whenever given any Banach space Y and any bounded linear operator $T \in \mathcal{L}(X, Y)$, there exists $S \in \mathcal{L}(X, Y)$ such that $\|S - T\| \approx 0$ and $\sup\{\|S(x)\| : x \in C\}$ is attained. We will say then that the Banach space X has the BP if every bounded closed absolutely convex subset of X has the BP.

Bourgain's result (see [11, Proposition 1]) states that if X has the Bishop-Phelps property then it is dentable (recall its definition at the beginning of Section 3.2 below). One of our goals here is to obtain an equivariant version of this result (see Theorem 4.2 below) by putting some conditions on the group G such that an analogous conclusion holds true for G-equivariant operators. We consider then the following natural definition.

Definition 3.1 (G-Bishop-Phelps property)**.** Let G be a topological group, and let C be a bounded closed absolutely convex G-invariant subset of a G-Banach space X. We say that C has the *G-Bishop-Phelps property* (G-BP, for short) whenever, for every G-Banach space Y, for any G-equivariant operator $T \in \mathcal{L}^G(X, Y)$ and $\delta > 0$, there exists another G-equivariant operator $T_\delta \in \mathcal{L}^G(X, Y)$ such that $\|T_\delta - T\| < \delta$

and $\sup\{\|T_\delta(x)\| : x \in C\}$ is attained. We will say that X has the G-BP if every bounded closed absolutely convex G-invariant subset of X has the G-BP.

A natural choice is to consider $C = B_X$ for which we obtain the following specialized definition–one of the original motivations for our research. Before introducing this new definition, let us provide some notation. Let X and Y be G-Banach spaces. We denote by $\mathrm{NA}^G(X,Y)$ the set of all G-equivariant norm-attaining operators $T \in \mathcal{L}^G(X,Y)$, that is, those for which there exists $x \in B_X$ satisfying $\|Tx\| = \|T\|$.

Definition 3.2. Let G be a topological group. A G-Banach space X is said to have *G-property A* (in the sense of Lindenstrauss) if for every G-Banach space Y, the set $\mathrm{NA}^G(X,Y)$ is dense in $\mathcal{L}^G(X,Y)$.

Let us remark that it is not at all clear whether the classical Bishop-Phelps property implies the G-Bishop-Phelps property. However, we do know that the converse is not true since L_1 has the G-BP, for $G = \mathrm{Iso}(L_1)$, by Theorem 7.9 below, while L_1 does not have the BP as L_1 is clearly not dentable.

3.2. ***G*-dentability and weaker notions.** In this subsection, we introduce our G-versions of dentability. Recall that a subset $C \subseteq X$ of a Banach space X is said to be dentable if for every $\varepsilon > 0$, there exists a point $x \in C$ such that $x \notin \overline{\mathrm{co}}(C \setminus B(x,\varepsilon))$. The Banach space X is said to be dentable if every nonempty bounded subset of X is dentable.

If X is a G-Banach space for some topological group G, then for every $C \subseteq X$, we denote by $\overline{\mathrm{co}}_G(C)$ the smallest closed convex G-invariant subset of X containing the subset C.

We introduce the following G-version of dentability.

Definition 3.3. Let G be a topological group. Given a G-invariant subset C of a G-Banach space X, we say that $C \subseteq X$ is *G-dentable* if for every $\varepsilon > 0$, there exists a finite set $\{x_1, \dots, x_n\} \subseteq C$ such that one can find an element $x \in C$ such that

$$x \notin \overline{\mathrm{co}}_G\big(C \setminus B_\varepsilon(x_1, \dots, x_n)\big).$$

Recall from Huff and Morris [31] that a subset C of X is dentable if and only if it is $\{1_G\}$-dentable in the sense of Theorem 3.3. This geometric characterization of dentability, due to Huff and Morris, is in fact the main motivation for our definition of G-dentability for subsets. Observe that for any topological group G and G-invariant subset $C \subseteq X$, G-dentability of C implies its classical dentability.

Let us collect two more equivalent but formally weaker definitions of G-dentability which will be more convenient to verify in later proofs.

Lemma 3.4. Let G be a topological group and let X be a G-Banach space. Let $C \subseteq X$ be a G-invariant subset. The following conditions are equivalent.

(1) for every $\varepsilon > 0$, there is a finite set $\{x_1, \dots, x_n\} \subseteq C$ such that there exists $x \in C$ with $x \notin \overline{\mathrm{co}}_G(C \setminus B_\varepsilon(x_1, \dots, x_n))$.
(2) for every $\varepsilon > 0$, there is a finite set $\{x_1, \dots, x_n\} \subseteq X$ such that there exists $x \in C$ with $x \notin \overline{\mathrm{co}}_G(C \setminus B_\varepsilon(x_1, \dots, x_n))$.
(3) for every $\varepsilon > 0$, there is a finite family of subsets $A_1, \dots, A_n \subseteq X$ with $\mathrm{diam}\, A_i < \varepsilon$ such that $x \notin \overline{\mathrm{co}}_G(C \setminus \bigcup_{i=1}^n A_i)$.

Proof. The chain of implications (1) $\implies$ (2) $\implies$ (3) is clear. Let (3) hold and choose $\varepsilon > 0$ arbitrarily. Then, there are $A_1, \dots, A_n \subseteq X$ with $\operatorname{diam} A_i < \varepsilon$ and $x \in C$ such that $x \notin \overline{\mathrm{co}}_G(C \setminus \bigcup_{i=1}^n A_i)$. Define

$$I_\cap := \{i \in \{1, \dots, n\} : A_i \cap C \neq \emptyset\},$$
$$I_\emptyset := \{i \in \{1, \dots, n\} : A_i \cap C = \emptyset\}.$$

For each $i \in I_\cap$, pick $x_i \in C \cap A_i$. Clearly, $A_i \subseteq B_\varepsilon(x_i)$. By hypothesis, observe that

$$x \notin \overline{\mathrm{co}}_G \left(C \setminus \bigcup_{i=1}^n A_i \right) = \overline{\mathrm{co}}_G \left(C \setminus \bigcup_{i \in I_\cap} A_i \right),$$

where the equality holds since $C \cap A_i = \emptyset$ for $i \in I_\emptyset$. Finally, since $\bigcup_{i \in I_\cap} A_i \subseteq \bigcup_{i \in I_\cap} B_\varepsilon(x_i)$, we obtain that

$$x \notin \overline{\mathrm{co}}_G \left(C \setminus \bigcup_{i \in I_\cap} B_\varepsilon(x_i) \right).$$

Thus, for the arbitrarily chosen $\varepsilon > 0$, we have found a finite set $\{x_i : i \in I_\cap\} \subseteq C$ and $x \in C$ satisfying the condition in (1). $\square$

Let us now present three natural group-invariant analogues of the notion of dentability for Banach spaces.

Definition 3.5 (G-dentabilities for Banach spaces)**.** Let G be a topological group and let X be a G-Banach space. We say that X is

(1) *strongly G-dentable* if every bounded G-invariant subset C of X is G-dentable in the sense of Theorem 3.3.

(2) *weakly G-dentable* if every bounded G-invariant subset C of X is dentable in the classical sense.

(3) *very weakly G-dentable* if every bounded G-invariant subset C of X satisfies that for every $\varepsilon > 0$, there is $x \in C$ such that $x \notin \overline{\mathrm{co}}(C \setminus G \cdot B_\varepsilon(x))$.

Remark 3.6. We make the following observations concerning Theorem 3.5.

(a) We have immediately that strong G-dentability implies weak G-dentability, which in turn implies very weak G-dentability. Moreover, classical dentability implies weak G-dentability. See Diagram 1 below.

(b) Very weak G-dentability seems to be too weak as even L_1 is very weakly G-dentable with $G = \mathrm{Iso}(L_1)$. See Theorem 3.7.

(c) Strong G-dentability makes sense only for compact groups, or more precisely, precompact groups. See Theorem 3.8.

Let us prove now the facts that give items (b) and (c) from Theorem 3.5.

Fact 3.7. Let X be a Banach space and suppose that a topological group G acts on X almost transitively by linear isometries; that is, for every $x, y \in S_X$ and $\varepsilon > 0$, there is $g \in G$ such that $\|g \cdot x - y\| < \varepsilon$. Then, X is very weakly G-dentable. In particular, L_1 is very weakly G-dentable for $G = \mathrm{Iso}(L_1)$.

Proof. Let $C \subseteq X$ be bounded and G-invariant and let $\varepsilon > 0$ be arbitrary. Denote $M = \sup\{\|y\| : y \in C\}$ and find $x \in C$ such that $\|x\| > M - \varepsilon$. We claim that $\overline{\mathrm{co}}(C \setminus G \cdot B_\varepsilon(x)) \subseteq \overline{B_{M-\varepsilon}(0)}$ and hence $x \notin \overline{\mathrm{co}}(C \setminus G \cdot B_\varepsilon(x))$.

Indeed, let $y \in C$ be such that $\|y\| > M - \varepsilon$. We have that $\|x\|, \|y\| \in (M - \varepsilon, M]$ implying that $0 < \eta := \varepsilon - \big|\|x\| - \|y\|\big|$. Since the action of G on X is almost transitive, there is $g \in G$ such that

$$\left\| g \cdot x - \frac{\|x\|}{\|y\|} y \right\| < \eta.$$

Thus, we also have

$$\|g \cdot x - y\| \leqslant \left\| g \cdot x - \frac{\|x\|}{\|y\|} y \right\| + \left\| \frac{\|x\|}{\|y\|} y - y \right\| < \eta + \big|\|x\| - \|y\|\big| = \varepsilon,$$

i.e. $y \in G \cdot B_\varepsilon(x)$. Now, since $C \setminus G \cdot B_\varepsilon(x) \subseteq \overline{B_{M-\varepsilon}(0)}$, the same inclusion also holds for the closed convex hull. $\square$

Before showing the second result, we need some notions. Recall that a topological group G is *precompact* if for every neighborhood U of $1_G \in G$, there exists a finite set F_U such that

$$G \subseteq \bigcup_{g \in F_U} g \cdot U.$$

Equivalently, the completion by its left uniformity is compact.

The assumption in Theorem 3.8 below that the action is *faithful*, i.e., that for every $g \in G$, there exists $x \in X$ such that $g \cdot x \neq x$, is not restrictive. In fact, if the action is not faithful, then the set $N := \{g \in G \colon g \cdot x = x, \forall x \in X\}$ is a non-trivial closed normal subgroup of G. In this case, the action of G on X factorizes through the natural action of G/N on X given by

$$gH \cdot x = g \cdot x \quad (g \in G,\, x \in X),$$

which is faithful.

Fact 3.8. Let X be a Banach space equipped with a faithful action of a topological group G by linear isometries. Suppose that X is strongly G-dentable. Then, there exists a coarser group Hausdorff topology τ on G such that (G, τ) is precompact and such that the action is still continuous.

Proof. Since the action is faithful, we can identify G with $\phi[G]$, where ϕ is the natural continuous homomorphism $\phi : G \to \mathrm{Iso}(X)$. Identifying G with $\phi[G]$ naturally induces a topology τ_ϕ on G, where a net $g_\alpha \to g$ in (G, τ_ϕ) if and only if $\phi(g_\alpha) \to \phi(g)$ in $(\mathrm{Iso}(X), \mathrm{SOT})$, which is coarser than the originally given topology on G (by continuity of ϕ). We claim that $\phi[G]$ is precompact in $\mathrm{Iso}(X)$.

First notice that every orbit of G is totally bounded. Pick any $x \in X$. Since by the assumption the orbit Gx is G-dentable, it follows that, for every $\varepsilon > 0$, there are $x_1, \ldots, x_n \in Gx$ such that

$$Gx \not\subseteq \overline{\mathrm{co}}_G(Gx \setminus B_\varepsilon(x_1, \ldots, x_n)).$$

However, for every nonempty subset $C \subseteq Gx$, we have $Gx \subseteq \overline{\mathrm{co}}_G\, C$, which implies that $Gx \setminus B_\varepsilon(x_1, \ldots, x_n)$ must be empty. Thus, the orbit Gx has a finite ε-net. Since $\varepsilon > 0$ was arbitrary, it follows that Gx is totally bounded; equivalently, precompact (as the ambient space X is a complete metric space).

Now, we show the claim. To simplify the proof, we shall assume that X is separable. The general case is left to the reader and requires a more technical gymnastics with nets. Pick a countable dense set $(x_m)_m \subseteq X$. Notice that for each fixed $m \in \mathbb{N}$, since Gx_m is precompact, there exists a subsequence of $(g_{k_n})_{k_n}$ of $(g_n)_n$ such that $(g_{k_n}x_m)_{k_n}$ is convergent in X. By a diagonal argument, we can find a subsequence $(g_l)_l$ of $(g_n)_n$ such that $g_l x_m$ converges for every m. Since $(x_m)_m$ is dense in X, it follows that $g_l x$ is convergent for every $x \in X$. This implies that $(\phi(g_l))_l$ is a convergent sequence in $\mathrm{Iso}(X)$, as desired. As we assumed X to be separable, $(\mathrm{Iso}(X), \mathrm{SOT})$ is Polish (in particular, complete metrizable). Thus, what we have proved implies that the closure of $\phi[G]$ is compact. □

The argument of Theorem 3.8 can be used to distinguish between weak and strong G-dentability, and classical dentability and strong G-dentability, as follows.

Corollary 3.9. Weak G-dentability and classical dentability do not imply strong G-dentability. For instance, for $1 < p < \infty$, L_p is not strongly G-dentable with respect to $G = \mathrm{Iso}(L_p)$.

Proof. If L_p were strongly G-dentable with respect to $G = \mathrm{Iso}(L_p)$, then the proof of Theorem 3.8 would imply that $(\mathrm{Iso}(L_p), \mathrm{SOT})$ is compact, which is not the case–a fact well known to experts, and easily deducible from, for instance, [26, Proposition 3.2]. On the other hand, L_p is dentable, thus it is also weakly G-dentable. □

In Section 4, we will also need the following auxiliary result, which will be used in the proof that the G-Bishop–Phelps property implies strong G-dentability.

Lemma 3.10. Let G be a topological group, X be a G-Banach space, and let D be a G-invariant subset of X. If $\overline{\mathrm{co}}(D)$ is G-dentable, then so is D.

Proof. Let $\varepsilon > 0$ and find $z \in \overline{\mathrm{co}}(D)$ and $y_1, \dots y_n \in \overline{\mathrm{co}}(D)$ such that

$$z \notin \overline{\mathrm{co}}_G \left((\overline{\mathrm{co}}(D)) \setminus B_\varepsilon(y_1, \dots, y_n) \right) =: C.$$

Let U be an open neighborhood of z such that $U \cap C = \emptyset$. Since $z \in \overline{\mathrm{co}}(D)$, $U \cap \mathrm{co}(D) \neq \emptyset$. Hence, there exists

$$x = \sum_{j=1}^{m} \lambda_j x_j \in \mathrm{co}(D) \setminus C$$

where $x_j \in D$ for $1 \leqslant j \leqslant m$. For a contradiction, assume that

$$\forall 1 \leqslant j \leqslant m : x_j \in \overline{\mathrm{co}}_G \left(D \setminus B_\varepsilon(y_1, \dots, y_n) \right).$$

Enlarging D to $\overline{\mathrm{co}}(D)$, we get

$$\forall 1 \leqslant j \leqslant m : x_j \in \overline{\mathrm{co}}_G \left(\overline{\mathrm{co}}(D) \setminus B_\varepsilon(y_1, \dots, y_n) \right) = C.$$

But since C is convex, we obtain that $x = \sum_{j=1} \lambda_j x_j \in C$ which is a contradiction with the very choice of x. This means that for some $j \in \{1, \dots, m\}$, we have that

$$x_j \notin \overline{\mathrm{co}}_G \left(D \setminus B_\varepsilon(y_1, \dots, y_n) \right).$$

It follows from Theorem 3.4 (since $y_1, \dots, y_n$ are not necessarily elements of D) that D is G-dentable. □

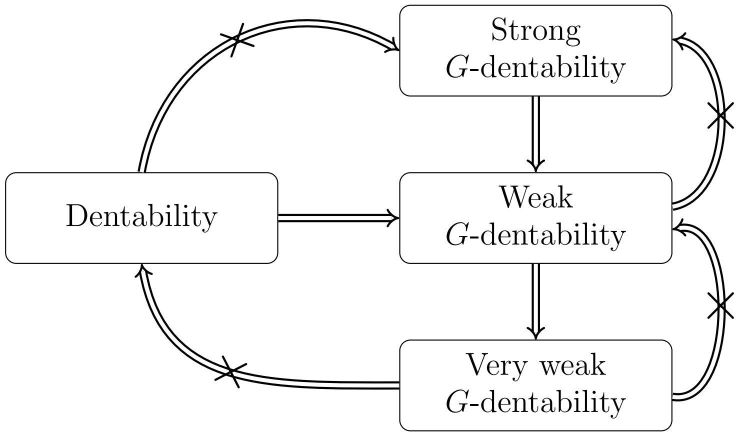

Diagram 1: G-dentability concepts.

3.3. ***G*-Krein-Milman property.** In this subsection, we consider a version of the Krein-Milman property that is compatible with a given group action. In Section 5.1, we will show its connection with the G-dentability of a Banach space X (more specifically, see Theorem 5.1).

Recall that if C is a nonempty convex subset of X, a point $x_0 \in C$ is an *extreme point* of C if $x_1 = x_2 = x_0$ whenever $x_1, x_2 \in C$ and x_0 can be written as $x_0 = (1/2)x_1 + (1/2)x_2$. The well-known Krein-Milman theorem says that if C is a nonempty convex compact subset of X, then C contains at least one extreme point and, furthermore, $C = \overline{\text{co}}(\text{ext}(C))$ (see, for instance, [21, Theorem 3.65]).

We say that a Banach space X has the *Krein-Milman property* (KMP, for short) if every closed bounded convex subset of X is the norm closed convex hull of its extreme points. The fact that every Banach space with the RNP has the KMP is due to J. Lindenstrauss (see, for instance, [17, Theorem 7, p. 191]).

The version we introduce below, the so-called G-KMP, is the natural group-compatible analogue of the KMP.

Definition 3.11. Let G be a topological group, X be a G-Banach space, and let $C \subseteq X$ be a closed bounded convex G-invariant subset of X. We say that C has the *G-Krein-Milman property* (G-KMP, for short) whenever $C = \overline{\text{co}}_G(\text{ext}(C))$. We say that the Banach space X satisfies the *G-KMP* if every closed bounded convex G-invariant subset of X has the G-KMP.

Let us notice that $\overline{\text{co}}_G(\text{ext}(C)) = \overline{\text{co}}(\text{ext}(C))$ since the set of extreme points in a G-invariant set is G-invariant. This means we have in fact that C satisfies the G-KMP whenever $C = \overline{\text{co}}(\text{ext}(C))$.

3.4. ***G*-Radon-Nikodým property.** In this subsection, we introduce the group equivariant version of the Radon–Nikodým property for Banach spaces, the main topic of the present paper. As we know from the classical scenario, the dentability of a Banach space is equivalent to the RNP (see, for instance, [46, Theorem 2.9]). Our aim is to prove that, under some hypothesis on the group G, weak G-dentability is equivalent with the G-RNP (see Theorem 3.19 and Theorem 6.4).

Before providing the definition of the G-Radon-Nikodým property, let us recall the relations between linear isometries of L_p spaces for $p \in [1,\infty\}\setminus\{2\}$ and isomorphisms of measure spaces.

Definition 3.12. [27, Definition 3.2.3]. Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces. A set map $T:\Sigma_1\to\Sigma_2$ defined modulo null sets is called a *regular set isomorphism* if

(1) $T(\Omega_1\setminus A) = T\Omega_1\setminus TA$ for all $A\in\Sigma_1$.
(2) $T(\cup_{n=1}^{\infty}A_n) = \cup_{n=1}^{\infty}TA_n$ for disjoint $A_n\in\Sigma_1$.
(3) $\mu_2(TA)=0$ if and only if $\mu_1(A)=0$.

All set equalities and inclusions are understood to be modulo null sets.

Equivalently, denoting by

$$N_i := \{A\in\Sigma_i\colon \mu_i(A)=0\},\quad i\in\{1,2\}$$

the corresponding null ideals, a regular set isomorphism is an isomorphism of the quotient measure algebras Σ_1/N_1 and Σ_2/N_2.

Theorem 3.13 (Banach-Lamperti (see, for instance, [27, Theorem 3.2.5])). Suppose that U is a linear isometry from $L_p(\Omega_1,\Sigma_1,\mu_1)$ into $L_p(\Omega_2,\Sigma_2,\mu_2)$, where $1\leqslant p<\infty$, $p\neq 2$ and we are assuming that the measure spaces are σ-finite. Then, there exists a regular set isomorphism $T:\Sigma_1\to\Sigma_2$ and a function h defined on Ω_2 such that

$$U(f)(t) = h(t)(\widetilde{T}f)(t) \tag{3.1}$$

where $\widetilde{T}:L_p(\Omega_1,\Sigma_1,\mu_1)\to L_p(\Omega_2,\Sigma_2,\mu_2)$ is a linear operator induced from T defined by $\widetilde{T}(\chi_A)=\chi_{TA}$, and h satisfies that

$$\int_{TA}|h|^p\,d\mu_2 = \int_{TA}\frac{d(\mu_1\circ T^{-1})}{d\mu_2}\,d\mu_2 = \mu_1(A),\quad \forall A\in\Sigma_1.$$

Conversely, for any h and T as above, the operator U satisfying (3.1) is an isometry.

Moreover, the linear isometry U is positive if and only if the function h is positive.

Let (Ω,Σ,μ) be a σ-finite measure space. The set of all regular set isomorphisms on (Ω,Σ,μ) forms a group, denoted by $\mathrm{Aut}^*(\Omega,\Sigma,\mu)$, or just $\mathrm{Aut}^*(\mu)$ if there is no risk of confusion. To each $T\in\mathrm{Aut}^*(\mu)$, we can assign the isometry $U_T\in\mathrm{Iso}(L_1(\Omega,\Sigma,\mu))$ given by

$$U_T(f)(t) = \frac{d(\mu\circ T^{-1})}{d\mu}(t)(\widetilde{T}f)(t),$$

where $\widetilde{T}:L_1(\Omega,\Sigma,\mu)\to L_1(\Omega,\Sigma,\mu)$ is defined as above by $\widetilde{T}(\chi_A)=\chi_{TA}$. The map $\Phi:T\mapsto U_T$ is known to be an algebraic isomorphism between $\mathrm{Aut}^*(\mu)$ and the group $\mathrm{Iso}^+(L_1(\Omega,\Sigma,\mu))$ of positive linear isometries on $L_1(\Omega,\Sigma,\mu)$ (see, e.g., [36, Theorem 3.1]). Note that $\frac{d(\mu\circ T^{-1})}{d\mu}(\cdot)$ is a nonnegative measurable function.

We equip $\mathrm{Aut}^*(\mu)$ with the topology obtained from the isomorphism Φ. Moreover, given a topological group G, notice that actions of G on (Ω,Σ,μ) by regular set isomorphisms are in one-to-one correspondence with homomorphisms from G into $\mathrm{Aut}^*(\mu)$. We say that an action of G on (Ω,Σ,μ) by regular set isomorphisms is *continuous* if the corresponding homomorphism from G into $\mathrm{Aut}^*(\mu)$ is continuous.

The Banach–Lamperti theorem (Theorem 3.13) then immediately yields the following result.

Corollary 3.14. Given a topological group G and a finite measure space (Ω, Σ, μ), continuous actions of G on (Ω, Σ, μ) by regular set isomorphisms are in one-to-one correspondence with continuous actions of G on $L_1(\Omega, \Sigma, \mu)$ by positive linear isometries.

Let G be a topological group and X be a G-Banach space. Let (Ω, Σ, μ) be a finite measure space. Given an X-valued vector measure $m : \Sigma \to X$ satisfying $|m| \ll \mu$, define a bounded linear operator $T_m : L_1(\Omega, \Sigma, \mu) \to X$ by

$$T_m(\chi_A) := m(A), \; A \in \Sigma. \tag{3.2}$$

The above well-defines the linear operator. Indeed, the definition of T_m on characteristic functions automatically extends linearly to the definition of T_m on simple functions. If $f := \sum_i \lambda_i \chi_{A_i} \in B_{L_1(\Omega,\Sigma,\mu)}$ with $A_i \cap A_j = \emptyset$ if $i \neq j$, then $T_m(f) = \sum_i \lambda_i \mu(A_i) \frac{m(A_i)}{\mu(A_i)} \in \mu(\Omega) B_X$; thus, T_m is bounded and the definition extends to the whole space $L_1(\Omega, \Sigma, \mu)$.

Definition 3.15. Let G be a topological group and X be a G-Banach space. Suppose that G acts continuously on (Ω, Σ, μ) by regular set isomorphisms. We say that an X-valued vector measure $m : \Sigma \to X$ is *G-quasi-invariant* if for every $g \in G$ and $A \in \Sigma$

$$g \cdot m(A) = T_m\left(\frac{dg_*\mu}{d\mu} \chi_{gA}\right)$$

where $g_*\mu$ is the pushforward measure defined by $g_*\mu(A) := \mu(g^{-1}A)$ and $\frac{dg_*\mu}{d\mu}$ is the Radon-Nikodým derivative of $g_*\mu$ with respect to μ.

Remark 3.16. Notice that if the action of G on (Ω, Σ, μ) preserves the measure, i.e., for every $g \in G$ and $A \in \Sigma$ we have $\mu(gA) = \mu(A)$, then an X-valued measure m is G-quasi-invariant if and only if m is G-equivariant; that is, for all $g \in G$ and $A \in \Sigma$, $m(gA) = g \cdot m(A)$.

To make Theorem 3.15 more transparent we state the following fact which motivated the definition.

Fact 3.17. Let G be a topological group and X be a G-Banach space. Suppose that G acts continuously on (Ω, Σ, μ) by regular set isomorphisms. An X-valued vector measure $m : \Sigma \to X$, satisfying $|m| \ll \mu$, is G-quasi-invariant if and only if the associated operator T_m is G-equivariant.

Proof. Suppose that m is G-quasi-invariant. For any $A \in \Sigma$ and $g \in G$, we then have

$$T_m(g\chi_A) = T_m\left(\frac{dg_*\mu}{d\mu} \chi_{gA}\right) = g \cdot m(A) = g \cdot T_m(\chi_A).$$

By linearity and continuity of T_m and the action of G, we conclude that T_m is G-equivariant.

Conversely, if T_m is G-equivariant, then for every $A \in \Sigma$ and $g \in G$ we have

$$g \cdot m(A) = g \cdot T_m(\chi_A) = T_m(g\chi_A) = T_m\left(\frac{dg_*\mu}{d\mu} \chi_{gA}\right)$$

showing that m is G-quasi-invariant. ∎

While in general we are interested in actions of groups on finite measure spaces (Ω, Σ, μ) by regular set isomorphisms, in many natural examples the group acts on the underlying set Ω itself. We explain the relation between such actions and actions by regular set isomorphisms in the remark below.

Remark 3.18. Suppose that (Ω, Σ, μ) is a finite measure space on which a topological group G acts by a *point action*, that is, G acts on Ω by measurable bijections that preserve the measure class of μ, i.e., $\mu \ll g_*\mu$ and $g_*\mu \ll \mu$, for all $g \in G$. Then, clearly this action induces an action on Σ by regular set isomorphisms.

We note that if G is locally compact and second countable and (Ω, Σ, μ) is a standard probability space, then every continuous action of G on (Ω, Σ, μ) by regular set isomorphisms comes from some point action of G on Ω by measurable bijections that preserve the measure class of μ. This follows from Mackey's theorem (see [47, Theorem 3.3]). Moreover, (Ω, Σ, μ) is, without loss of generality standard probability space if and only if $L_1(\Omega, \Sigma, \mu)$ is separable. One direction is clear. For the other, the separability assumption guarantees that there is a measure algebra isomorphism between Σ/N_μ, where N_μ is the ideal of null sets in Σ, and a measure algebra of a standard probability space (see e.g. [50, Theorem 15.3.4]). The measure algebra isomorphism induces a linear isometry between the corresponding L_1 spaces.

Finally, we are ready to state the main definitions of this subsection and one of the central new notions of the paper.

Definition 3.19. Let G be a topological group and X be a G-Banach space. If for every finite measure space (Ω, Σ, μ) with a continuous action of G by regular set isomorphisms and for every G-quasi-invariant X-valued vector measure m on Σ such that $|m| \ll \mu$, there exists $f \in L_1(\Omega, \Sigma, \mu; X)$ such that

$$m(A) = \int_A f \, d\mu, \quad \text{for all } A \in \Sigma,$$

then we say X has the *G-Radon-Nikodým property* (G-RNP, for short).

By Theorem 3.19, it is immediate that if a G-Banach space X, where G is a topological group, has the RNP, then it has the G-RNP.

It is well-known that the classical RNP is equivalent to the RNP with respect to the measure space $([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra of $[0,1]$ ([17, Corollary 8] or [46, Corollary 2.14]). This is also the case of the G-RNP when G is locally compact, σ-compact and separable (see Theorem 6.5).

Moreover, it is well known that the classical RNP is equivalent to that, for every finite measure space (Ω, Σ, μ), each $T : L_1(\Omega, \Sigma, \mu) \to X$ is *representable*, i.e., there exists $\phi \in L_\infty(\Omega, \Sigma, \mu; X)$ such that

$$Tf = \int f\phi \, d\mu, \quad \forall f \in L_1(\Omega, \Sigma, \mu). \tag{3.3}$$

We shall show that the natural G-analogue of this equivalence still holds for any topological group G. The following is the G-version of the equivalent definition of the RNP in terms of representable operators.

Definition 3.20. Let G be a topological group and X be a G-Banach space. We say that X has the *G-Riesz representation property* if for every finite measure space (Ω, Σ, μ) and continuous action of G on $L_1(\Omega, \Sigma, \mu)$ by positive linear isometries, each G-equivariant bounded linear operator $T : L_1(\Omega, \Sigma, \mu) \to X$ is representable, that is, there exists a function $\phi \in L_\infty(\Omega, \Sigma, \mu; X)$ satisfying (3.3).

Example 3.21. $L_1[0,1]$ does not have the G-Riesz representation property for any action of any group G since the identity on $L_1[0,1]$ (which is, obviously, G-equivariant) does not admit any bounded Radon-Nikodým derivative $\phi : [0,1] \to L_1[0,1]$.

As promised, we present the equivalence between the G-Riesz representation property and the G-RNP.

Proposition 3.22. Let G be a topological group and X be a G-Banach space. The following statements are equivalent.

(1) X has the G-Riesz representation property.
(2) X has the G-RNP.

Proof. Suppose that X satisfies the G-Riesz representation property. Let (Ω, Σ, μ) be a finite measure space such that G acts on it continuously by regular set isomorphisms. By Theorem 3.14, this action induces a continuous action of G on $L_1(\Omega, \Sigma, \mu)$ by positive linear isometries. Now let $m : \Sigma \to X$ be a G-quasi-invariant X-valued vector measure such that $|m| \ll \mu$. Consider the bounded operator $T_m : L_1(\Omega, \Sigma, \mu) \to X$ induced by the vector measure m as in (3.2). By Theorem 3.17, T_m is G-equivariant. Since X has the G-Riesz representation property, there exists $h \in L_\infty(\Omega, \Sigma, \mu; X)$ such that $T_m(f) = \int fh\, d\mu$ for all $f \in L_1(\Omega, \Sigma, \mu)$. In particular, for every $A \in \Sigma$, $m(A) = \int_A h\, d\mu$; so X has the G-RNP with respect to (Ω, Σ, μ) applying the same function $h \in L_\infty(\Omega, \Sigma, \mu; X) \subseteq L_1(\Omega, \Sigma, \mu; X)$.

Conversely, suppose that X has the G-RNP. Let (Ω, Σ, μ) be a finite measure space and suppose G acts continuously by positive linear isometries on $L_1(\Omega, \Sigma, \mu)$. Let $T : L_1(\Omega, \Sigma, \mu) \to X$ be a bounded G-equivariant operator. Define the vector measure $m : \Sigma \to X$ by

$$m(A) := T(\chi_A).$$

Since $\|m(A)\| \leqslant \|T\|\mu(A)$ for every $A \in \Sigma$, we have $|m|(A) \leqslant \|T\|\mu(A)$. Assuming, without loss of generality, that $\|T\| \leqslant 1$, we obtain $|m| \ll \mu$. Moreover, since T is G-equivariant, for every $g \in G$ and $A \in \Sigma$, we have

$$gm(A) = gT(\chi_A) = T(g\chi_A) = T\left(\frac{dg_*\mu}{d\mu}\chi_{gA}\right)$$

verifying that m is G-quasi-invariant. Since X has the G-RNP, there exists $h \in L_1(\Omega, \Sigma, \mu; X)$ such that $m(A) = \int_A h\, d\mu$. Note that $|m|(A) = \int_A \|h\|\, d\mu$ which implies that $\|h\| \leqslant \|T\|$ μ-almost everywhere, thus in fact $h \in L_\infty(\Omega, \Sigma, \mu; X)$. By the linearity and continuity of T, we conclude that $Tf = \int fh\, d\mu$ for every $f \in L_1(\Omega, \Sigma, \mu)$, so T is representable witnessed by the function $h \in L_\infty(\Omega, \Sigma, \mu; X)$. $\square$

The reader may wonder why the functions f and ϕ from Theorem 3.19 and Theorem 3.20, respectively, are not required to have any G-equivariance properties. We show that they do have certain G-equivariance properties and that such properties are, in fact, automatic. Before introducing these properties let us give some motivation for their definition.

Let (Ω, Σ, μ) be a finite measure space and X be a Banach space. Suppose that a topological group G acts on (Ω, Σ, μ) by regular set isomorhisms so that the action comes from a point action on Ω. Then, given $f \in L_1(\Omega, \Sigma, \mu; X)$, we say that f is G-equivariant if, as expected, for every $g \in G$ and almost every $\omega \in \Omega$ we have $f(g\omega) = gf(\omega)$. However, since not every action of G on Σ by regular set isomorphisms comes from a point action, we need a more versatile notion of G-equivariance.

Definition 3.23. Let G be a topological group acting on a finite measure space (Ω, Σ, μ) by regular set isomorphisms and on a Banach space X by linear isometries. We say that a function $f \in L_1(\Omega, \Sigma, \mu; X)$ is *G-equivariant-on-sets* if

$$\forall g \in G,\ \forall A \in \Sigma, \qquad \int_A g \cdot f(\omega) d\mu(\omega) = \int_{gA} f(\omega) \frac{dg_*\mu}{d\mu}(\omega)\, d\mu(\omega).$$

Let us show that this indeed generalizes the standard notion of G-equivariance.

Fact 3.24. Retain the notation from Theorem 3.23. Suppose the action on (Ω, Σ, μ) comes from a point action on Ω. Then, a function $f \in L_1(\Omega, \Sigma, \mu; X)$ is G-equivariant-on-sets if and only if it is G-equivariant.

Proof. Suppose that f is G-equivariant, then by run-of-the-mill substitution we get that for any $A \in \Sigma$

$$\int_A gf(\omega) d\mu(\omega) = \int_A f(g\omega) d\mu(\omega) = \int_{gA} f(\omega) \frac{dg_*\mu}{d\mu}(\omega) d\mu(\omega)$$

showing that f is G-equivariant-on-sets.

Conversely, suppose that f is G-equivariant-on-sets. Fix $g \in G$ and $A \in \Sigma$. We have

$$\int_A gf(\omega) d\mu(\omega) = \int_{gA} f(\omega) \frac{dg_*\mu}{d\mu}(\omega) d\mu(\omega) = \int_{gA} f(\omega) dg_*\mu(\omega) = \int_A f(g\omega) d\mu(\omega),$$

so $\int_A gf(\omega) - f(g\omega) d\mu(\omega) = 0$. Since $A \in \Sigma$ was arbitrary, by [32, Proposition 1.2.14], we obtain that the set $\{\omega \in \Omega \colon gf(\omega) \neq f(g\omega)\}$ has measure zero. $\square$

Now we are ready to show that the functions f and ϕ from Theorem 3.19 and Theorem 3.20, respectively, satisfy this definition of G-equivariance.

Proposition 3.25. Let X be a Banach space and G be a topological group so that $G \curvearrowright X$ linearly and isometrically. Let (Ω, Σ, μ) be a finite measure space and G act on (Ω, Σ, μ) by regular set isomorphisms. Then, the functions $f \in L_1(\Omega, \Sigma, \mu; X)$ from Theorem 3.19 and $\phi \in L_\infty(\Omega, \Sigma, \mu; X)$ from Theorem 3.20 are both G-equivariant-on-sets.

Proof. In view of Theorem 3.22, it is sufficient to prove that f is G-equivariant-on-sets. Let $A \in \Sigma$ and $g \in G$ be given. Observe that if ψ is a simple function in $L_1(\Omega, \Sigma, \mu; X)$, then

$$T_m(\psi\, \chi_{gA}) = \int_{gA} f\psi\, d\mu. \tag{3.4}$$

By continuity of T_m, (3.4) extends to all elements of $L_1(\Omega, \Sigma, \mu; X)$ and by the G-quasi-invariance of μ, we obtain

$$\begin{aligned}\int_A gf(\omega)d\mu(\omega) &= g\left(\int_A f(\omega)d\mu(\omega)\right) = g\cdot m(A)\\ &= T_m\left(\frac{dg_*\mu}{d\mu}\chi_{gA}\right) = \int_{gA} f(\omega)\frac{dg_*\mu}{d\mu}d\mu(\omega).\end{aligned}$$

□

We conclude this section by noting that, throughout the remainder of the paper, we shall use the term "G-RNP", even when referring to arguments involving the G-Riesz representation property, as the two notions are equivalent as we have seen in Theorem 3.22.

4. Consequences of the G-Bishop-Phelps property

The aim of this section is to prove that, as in the classical case (see [11, Proposition 1]), the G-Bishop-Phelps property implies strong G-dentability. However, we prove it under the assumption that G is a compact group (see Theorem 3.6.(c) for a discussion on why compactness of G is required).

The following is the main result of this section.

Theorem 4.1. Let G be a compact group and X be a G-Banach space. If X satisfies the G-BP, then X is strongly G-dentable.

In order to obtain Theorem 4.1, it suffices to prove Theorem 4.2, which deals with *separable* G-invariant subsets, along with the separable reduction argument provided in Theorem 4.4.

Theorem 4.2. Let G be a compact group, X be a G-Banach space, and let $C \subseteq X$ be a separable bounded closed convex G-invariant set. If C has the G-BP, then it is G-dentable.

Before proceeding to the proof of Theorem 4.2, let us first complete the separable reduction argument. To this end, we need the following rather simple auxiliary lemma.

Lemma 4.3. Let G be a compact group, X be a G-Banach space, and let $A \subseteq X$ be a separable subset. Then $G \cdot A$ is separable.

Notice that we *do not* require that the group G is separable.

Proof. Let $A' \subseteq A$ be a countable dense subset. Then, clearly $G \cdot A'$ is dense in $G \cdot A$. Since $G \cdot a$ is compact for every $a \in A'$, it follows that $G \cdot A'$ is σ-compact, therefore separable. Consequently, $G \cdot A$ is separable. □

The following argument generalizes the classical case from [41].

Lemma 4.4. Let G be a compact group and X be a G-Banach space. Let $C \subseteq X$ be a G-invariant subset. If every separable G-invariant subset of C is G-dentable, then C is itself G-dentable.

Proof. Suppose that C is not G-dentable. Then, there exists $\varepsilon > 0$ such that, for every finite tuple $x_1, \dots, x_n \in C$, we have $C = \overline{\mathrm{co}}_G\big(C \setminus B_\varepsilon(x_1, \dots, x_n)\big)$. Therefore, for every $x \in C$ and every finite tuple $x_1, \dots, x_n \in C$, there exists a separable set $E^x_{\{x_1,\dots,x_n\}} \subseteq C \setminus B_\varepsilon(x_1, \dots, x_n\}$ such that

$$x \in \overline{\mathrm{co}}_G(E^x_{\{x_1,\dots,x_n\}}) = \overline{\mathrm{co}}(G \cdot E^x_{\{x_1,\dots,x_n\}}).$$

Notice that by Theorem 4.3, the set $G \cdot E^x_{\{x_1,\dots,x_n\}}$ is separable and it is also G-invariant.

Now given any separable G-invariant subset E of C with a countable dense subset $\{e_n\}_{n\in\mathbb{N}}$, we define a set $\hat{E}$, omitting the countable dense subset from the notation, as follows. First let $(\vec{e}_n)_{n\in\mathbb{N}}$ be an enumeration of all finite tuples of $\{e_n\}_{n\in\mathbb{N}}$. Set

$$\hat{E} := \bigcup_{i\in\mathbb{N},\, j\in\mathbb{N}} G \cdot E^{e_i}_{\vec{e}_j}.$$

Notice that $\hat{E}$ is still separable and G-invariant.

We recursively define G-invariant separable sets D_n increasing with respect to inclusion, $n \geqslant 0$. Pick any $x \in C$ and set $D_0 := \{g \cdot x \colon g \in G\}$. Suppose that D_n has been defined. Choose a countable dense subset $\{d^n_i\}_{i\in\mathbb{N}}$ that contains $\{d^{n-1}_i\}_{i\in\mathbb{N}}$, if $n \geqslant 1$, of D_n and consider the set $\hat{D}_n$ with respect to $\{d^n_i\}_{i\in\mathbb{N}}$. Then, we define

$$D_{n+1} := \hat{D}_n \cup D_n.$$

Proceeding in this way, we obtain $D_0 \subseteq \dots \subseteq D_n \subseteq \dots$.

Finally, we set $\mathsf{D} := \bigcup_n D_n$, which is a G-invariant and separable subset of C as a countable union of G-invariant separable sets. Let us check D is not G-dentable, which will be a contradiction finishing the proof. This is witnessed by $\varepsilon/2$. Suppose there exist a finite tuple $(x_1, \dots, x_n) \in \mathsf{D}$ and $x \in \mathsf{D}$ such that $x \notin \overline{\mathrm{co}}_G\big(\mathsf{D} \setminus B_{\varepsilon/2}(x_1, \dots, x_n)\big)$. Without loss of generality, by approximating $x, x_1, \dots, x_n$ arbirarily well with elements of the form d^m_i, we may assume that there is $m \in \mathbb{N}$ such that $x, x_1, \dots, x_n \in \{d^m_i\}_{i\in\mathbb{N}}$. However, since by definition $E^x_{\{x_1,\dots,x_n\}} \subseteq D_{m+1} \subseteq \mathsf{D}$, we observe that

$$x \in \overline{\mathrm{co}}_G(E^x_{\{x_1,\dots,x_n\}}) \subseteq \overline{\mathrm{co}}_G(\mathsf{D} \setminus B_\varepsilon(x_1, \dots, x_n)),$$

which gives a contradiction. □

Having completed the separable reduction, we now turn to the proof of Theorem 4.2. We need the theorem below implied by the Big Peter-Weyl theorem. Given a topological group G acting by linear isometries on a Banach space X, we say that $x \in X$ is *almost G-invariant* (or *G-finite*) if $\mathrm{span}(G \cdot x)$ is a finite dimensional space. The set of all almost G-invariant elements of X will be denoted by X_{fin}.

Theorem 4.5 (A version of the Big Peter-Weyl Theorem, [30, Theorem 3.51])**.** Let G be a compact group and X be a G-Banach space. Then, we have that $X = \overline{X_{\mathrm{fin}}}$.

Now we are ready to prove Theorem 4.2. The argument is inspired by the proof of [11, Proposition 1]. It is worth emphasizing, however, that our proof diverges substantially from Bourgain's original approach: in our G-equivariant framework the use of the Peter–Weyl theorem is essential, as it provides sufficiently many almost G-invariant elements that serve as the key ingredients for the renorming argument in the proof.

Proof of Theorem 4.2. Let us assume that $C \subseteq B_X$ and that it is not G-dentable. By Theorem 3.4, there exists $\varepsilon > 0$ such that, for every finite set $\{x_1, \ldots, x_n\}$ in X, we have

$$C \subseteq \overline{\mathrm{co}}_G(C \setminus B_\varepsilon(x_1, \ldots, x_n)).$$

Let $Y \subseteq X$ be the separable closed subspace generated by C, which is G-invariant. Thanks to Theorem 4.5, we can choose a countable dense sequence $(y_n)_{n\in\mathbb{N}} \subseteq Y$ such that for every $n \in \mathbb{N}$ the space

$$Y_n := \overline{\mathrm{span}}\{g \cdot y_n \colon g \in G\}$$

is finite-dimensional. We define a new norm $|||\cdot|||$ on X as follows

$$|||x||| := \sqrt{\|x\|^2 + \sum_{n=1}^{\infty} \frac{\mathrm{dist}\,(x, Y_n)^2}{2^n}}, \quad x \in X.$$

It is straightforward to check that $|||\cdot|||$ is an equivalent norm on X which is still G-invariant, i.e., the action of G on $(X, |||\cdot|||)$ is still implemented by linear isometries, since each Y_n is G-invariant.

We consider the identity operator $I : (X, \|\cdot\|) \to (X, |||\cdot|||)$. Since it is a G-equivariant bounded linear operator and C has the G-Bishop-Phelps property, there exists a G-equivariant bounded linear operator $T : (X, \|\cdot\|) \to (X, |||\cdot|||)$ which attains the maximum on C at an element $x \in C$ and satisfies $\|I - T\| < \varepsilon/4$. Choose $n \in \mathbb{N}$ such that $\mathrm{dist}\,(x, Y_n) < \varepsilon/4$. Let $z_1, \ldots, z_m \in Y_n$ be an $\varepsilon/8$-net in $(1 + \varepsilon)B_{Y_n}$ noting that B_{Y_n} is compact since Y_n is finite-dimensional.

By the choice of $\varepsilon > 0$, we obtain that

$$C \subseteq \overline{\mathrm{co}}_G(C \setminus B_\varepsilon(z_1, \ldots, z_m)).$$

Set $D := B_\varepsilon(z_1, \ldots, z_m)$. We claim that

$$|||2T(x)||| = \sup_{y \in G\cdot(C\setminus D)} |||T(x + y)|||.$$

Indeed, since $C \subseteq \overline{\mathrm{co}}_G(C \setminus D)$, for every $\delta > 0$, there exist $x_1, \ldots, x_k \in C \setminus D$, $g_1, \ldots, g_k \in G$, and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\|x - \sum_{i=1}^k \lambda_i g_i x_i\| < \delta$. Hence, there exists $i \in \{1, \ldots, k\}$ such that

$$\big| |||T(2x)||| - |||T(x + g_i \cdot x_i)||| \big| < \delta \|T\|,$$

which completes the claim, since $\delta > 0$ was arbitrary.

Now if $(y_k)_{k\in\mathbb{N}} \subseteq C \setminus D$ and $(g_k)_{k\in\mathbb{N}} \subseteq G$ are sequences satisfying $|||2x||| = \lim_{k\to\infty} |||T(x + g_k \cdot y_k)|||$ we obtain, since $|||\cdot|||$ is an ℓ_2-sum of seminorms, that

$$\mathrm{dist}\,(T(x), Y_n) = \lim_{k\to\infty} \mathrm{dist}\,(T(g_k \cdot y_k), Y_n).$$

However, notice that for every $y \in C \setminus D$, we have $\mathrm{dist}\,(y, Y_n) \geqslant \frac{7\varepsilon}{8}$. In fact, otherwise there would be $y' \in Y_n$ such that $\|y - y'\| < 7\varepsilon/8$. In particular, this implies that $\|y'\| \leqslant \|y - y'\| + \|y\| < 1 + \varepsilon$. Thus, there exists $i \in \{1, \ldots, m\}$ such that $\|y' - z_i\| < \varepsilon/8$, thus $y \in B_\varepsilon(z_i) \subseteq D$, a contradiction.

Therefore, we obtain that

$$\begin{aligned}\operatorname{dist}(T(x), Y_n) &\leqslant \|T(x)-x\| + \operatorname{dist}(x, Y_n)\\ &\leqslant |||T(x)-x||| + \operatorname{dist}(x, Y_n) \leqslant \frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}\end{aligned}$$

which yields the following contradiction:

$$\begin{aligned}\operatorname{dist}(T(g_k y_k), Y_n) &= \operatorname{dist}(T(y_k), Y_n)\\ &\geqslant \operatorname{dist}(y_k, Y_n) - \|T(y_k)-y_k\|\\ &\geqslant \operatorname{dist}(y_k, Y_n) - |||T(y_k)-y_k||| \geqslant \frac{7\varepsilon}{8}-\frac{\varepsilon}{4}=\frac{5\varepsilon}{8}.\end{aligned}$$

As a consequence, C is G-dentable. □

5. Consequences of G-dentability

In the classical setting, dentability is equivalent to the Radon–Nikodým and Bishop–Phelps properties, and moreover, it implies the Krein–Milman property. In this section, we show that strong G-dentability implies the G-KMP for compact G, and that weak G-dentability implies the G-RNP for separable G, in the two respective subsections. It remains open whether strong or weak G-dentability implies the G-BP for any nontrivial G.

5.1. **Strong G-dentability implies G-KMP.** In this subsection, our main goal is to connect the strong G-dentability and the G-Krein-Milman property of a G-Banach space X. More specifically, we will prove the following result.

Theorem 5.1. Let G be a compact group and X be a G-Banach space. If X is strongly G-dentable, then X has the G-KMP.

Let us emphasize once again that the assumption of compactness of the group G arises from the fact that strong G-dentability is only meaningful when G is (pre)compact (see Theorem 3.8). To prove Theorem 5.1, we will show that if D is a nonempty closed bounded convex subset of a Banach space X such that every closed bounded convex G-invariant subset of D is G-dentable, then D must contain an extreme point (see Theorem 5.2 below). The proof of this last fact draws on ideas from a classical argument due to J. Lindenstrauss establishing that every nonempty closed bounded convex subset of a Banach space with the RNP contains an extreme point (see, for instance, [17, Theorem 5, p.190]). More precisely, we will prove the following result.

Proposition 5.2. Let D be a nonempty convex closed bounded subset of a G-Banach space X. Assume that every convex closed bounded G-invariant subset of $\overline{\mathrm{co}}_G(D)$ is G-dentable. Then, D has an extreme point.

Theorem 5.1 follows as a consequence of Theorem 5.2. Indeed, suppose that X is strongly G-dentable but does not have the G-KMP. That is, there exists a bounded convex closed G-invariant set $D \subseteq X$ such that

$$D \neq \overline{\mathrm{co}}_G(\mathrm{ext}(D)), \tag{5.1}$$

where $\mathrm{ext}(D)$ denotes the subset of extreme points of D.

Find $x_0 \in D \setminus \overline{\mathrm{co}}_G(\mathrm{ext}(D))$. By the (classical) Hahn-Banach separation theorem and the Bishop-Phelps theorem, there exists $x^* \in X^*$ such that

$$x^*(x_0) = \sup x^*(D) > \sup x^*(\overline{\mathrm{co}}_G(\mathrm{ext}(D))). \tag{5.2}$$

Consider the subset $F := \{x \in D : x^*(x) = x^*(x_0)\}$ which is a nonempty convex closed bounded subset of D. Since X is assumed to be strongly G-dentable, in particular, every convex closed bounded G-invariant subset of $\overline{\mathrm{co}}_G(F)$ is G-dentable. Thus, our hypothesis implies that F has an extreme point. Let e be an extreme point of F. We claim that e is an extreme point of D. Indeed, suppose that there are $u, v \in D$ and $\lambda \in [0,1]$ such that $e = \lambda u + (1-\lambda)v$. There we have that

$$x^*(x_0) = x^*(e) = \lambda x^*(u) + (1-\lambda)x^*(v).$$

Since $x^*(x_0) = \sup x^*(D)$, we get that $x^*(u) = x^*(v) = x^*(x_0)$. This implies that $u, v \in F$ and, therefore, $u = v = e$. So, $e \in \mathrm{ext}(D)$. Consequently,

$$x^*(e) = \sup x^*(D) > \sup x^*(\overline{\mathrm{co}}_G(\mathrm{ext}(D))) \geqslant x^*(e),$$

which is a contradiction.

Therefore, in order to prove Theorem 5.1, it remains to prove Theorem 5.2. For this we will use the following well-known result.

Lemma 5.3 (Kuratowski's intersection theorem, [3, (f), p. 6])**.** Let $D_1 \supseteq D_2 \supseteq \cdots$ be a decreasing sequence of closed subsets of a complete metric space M. If there is a sequence $(\varepsilon_n)_{n=1}^\infty \subseteq (0,\infty)$ such that $\varepsilon_n \to 0$ and, for every $n \in \mathbb{N}$, the set D_n has a finite ε_n-net, then $\bigcap_{n\in\mathbb{N}} D_n$ is a nonempty compact set.

Proof of Theorem 5.2. We will inductively construct a decreasing sequence $D_0 \supseteq D_1 \supseteq \cdots$ of nonempty closed convex subsets of D such that D_n has a finite 2^{-n}-net for every $n \in \mathbb{N}$. Put $D_0 = D$.

Assume that $D_n \subseteq D$ has already been constructed. Since D_n is bounded, so is $\overline{\mathrm{co}}_G(D_n)$. Clearly, $\overline{\mathrm{co}}_G(D_n) \subseteq \overline{\mathrm{co}}_G(D)$. Thus, by the assumption, $\overline{\mathrm{co}}_G(D_n) = \overline{\mathrm{co}}(G\cdot D_n)$ is G-dentable. It follows from Theorem 3.10 that $G\cdot D_n$ is also G-dentable. Choosing $\varepsilon_n = 2^{-n-2}$, it follows that there exist $m_n \in \mathbb{N}$, $x_1^n, \ldots, x_{m_n}^n \in G\cdot D_n$, and $x^n \in G\cdot D_n$ such that

$$x^n \notin \overline{\mathrm{co}}_G\left(G\cdot D_n \setminus B_{\varepsilon_n}(x_1^n, \cdots, x_{m_n}^n)\right) =: C_n.$$

If $C_n = \emptyset$, then $D_n \subseteq G\cdot D_n \subseteq B_{\varepsilon_n}(x_1^n, \cdots, x_{m_n}^n)$, which implies that D_n has a finite 2^{-n-1}-net and thus we can set $D_{n+1} = D_n$.

So, assume that $C_n \neq \emptyset$. Since C_n is G-invariant, we may, without loss of generality, assume that $x^n \in D_n$. By the Hahn-Banach theorem, there exists $x_n^* \in X^*$ such that

$$x_n^*(x^n) > \sup x_n^*(C_n).$$

Since D_n is a bounded closed convex set, we may invoke the Bishop-Phelps theorem to find $y_n^* \in X^*$ such that y_n^* attains its maximum on D_n at, say, $y^n \in D_n$ and y_n^* is sufficiently close to x_n^* so that

$$y_n^*(y^n) = \sup y_n^*(D_n) > \sup y_n^*(C_n).$$

Define $M_n := y_n^*(y^n)$ and

$$D_{n+1} := \{x \in D_n : y_n^*(x) = M_n\}.$$

By the definition of M_n, the sets D_{n+1} and C_n must be disjoint. However, $D_{n+1} \subseteq D_n \subseteq G{\cdot}D_n$ and hence, by the definition of C_n, we have that $D_{n+1} \subseteq B_{\varepsilon_n}(x_1^n, \cdots, x_{m_n}^n)$, implying that D_{n+1} has a finite 2^{-n-1}-net. This implies in turn that $D_{n+1} \subseteq D_n$ is a nonempty closed convex set with a finite 2^{-n-1}-net.

Define $K := \bigcap_{n\in\mathbb{N}} D_n$. Using Theorem 5.3, we observe that K is a nonempty compact set. Moreover, since all D_n are convex, so is K. Hence, by the classical Krein-Milman theorem, K has an extreme point, call it $z \in K$.

We claim that z is an extreme point of D. To see this, let $z = \alpha z_1 + (1-\alpha)z_2$ with $\alpha \in [0,1], z_1, z_2 \in D$. We prove by induction that $z_1, z_2 \in D_n$ for all $n \geqslant 0$. The base case $z_1, z_2 \in D_0 = D$ is trivial. Assume that $z_1, z_2 \in D_n$ for some $n \geqslant 0$. Since $z \in K \subseteq D_{n+1}$,

$$M_n = y_n^*(z) = \alpha y_n^*(z_1) + (1-\alpha)y_n^*(z_2).$$

As M_n is the maximal value of y_n^* on D_n, it follows that $y_n^*(z_1) = y_n^*(z_2) = M_n$, implying that $z_1, z_2 \in D_{n+1}$ and completing the induction step. Consequently, $z_1, z_2 \in K$; and since $z \in \mathrm{ext}(K)$, we obtain that $z_1 = z_2 = z$. This proves that z is indeed an extreme point of D. $\square$

5.2. **Weak G-dentability implies the G-RNP.** In this subsection, we establish a connection between weak G-dentability and the G-RNP of a G-Banach space. It turns out that whenever G is a separable group, weak G-dentability implies the G-RNP (see Theorem 5.4 below). We conclude the section by showing that there is no G-equivariant version of Lewis-Stegall's factorization theorem (see Theorem 5.5).

Theorem 5.4. Let G be a separable topological group and X be a G-Banach space. If X is weakly G-dentable, then X has the G-RNP.

Proof. Let (Ω, Σ, μ) be a finite measure space and $T : L_1(\Omega, \Sigma, \mu) \to X$ be a G-equivariant bounded linear operator, where G acts on $L_1(\Omega, \Sigma, \mu)$ by positive linear isometries. We will show that T is G-representable. By the Banach-Lamperti theorem, we can suppose that the action of G on $L_1(\Omega, \Sigma, \mu)$ is induced by an action of G on $\Sigma/N(\mu)$ (see Theorem 3.14).

Given $A \in \Sigma$ with $\mu(A) > 0$, consider

$$C_A := \left\{ T\left(\frac{\chi_B}{\mu(B)}\right) : B \subseteq A,\, \mu(B) > 0 \right\}$$

and

$$D_A = \{Tf : f \geqslant 0,\, \mathrm{supp}(f) \subseteq A,\, f \in S_{L_1(\Omega,\Sigma,\mu)}\}.$$

For simplicity, let us write $x_B := T(\chi_B)/\mu(B)$.

Notice that, for every $A \in \Sigma$, we have

(1) $C_A \subseteq D_A$;
(2) in fact, $D_A = \overline{\mathrm{co}}(C_A)$;
(3) in particular, $\mathrm{diam}\, D_A = \mathrm{diam}\, C_A$;
(4) while for $g \in G$ we do not in general have $gC_A = C_{gA}$, we do have $gD_A = D_{gA}$.

Let now $G' \leqslant G$ be a countable dense subgroup, which exists since G is separable.

Claim 1. Let $A \in \Sigma$, with $\mu(A) > 0$, be G'-invariant. Then, D_A is G-invariant.

To prove this, we first verify that D_A is G'-invariant. Pick $x \in D_A$, which is of the form Tf, where $f \geqslant 0$, $\mathrm{supp}(f) \subseteq A$, $f \in S_{L_1(\Omega,\Sigma,\mu)}$, and $g \in G'$. Then $gTf = Tgf$, so it suffices to check that $gf \geqslant 0$, $gf \in S_{L_1(\Omega,\Sigma,\mu)}$, and $\mathrm{supp}(gf) \subseteq A$. The first two conditions follow since g is a positive isometry. The last condition follows since up to measure zero, $\mathrm{supp}(gf) = g \cdot \mathrm{supp} f$, and A is preserved by g.

Finally, to check that D_A is G-invariant, pick $x \in D_A$ and $g \in G$. Let $(g_n)_n \subseteq G'$ be a net converging to g. Then, by the continuity of the action and since D_A is closed as a closed convex hull of C_A, we have $gx = \lim_n g_n x \in D_A$.

Claim 2. For any G'-invariant set $A \in \Sigma$ with $\mu(A) > 0$ and any $\varepsilon > 0$, there exists $B \subseteq A$ with $\mu(B) > 0$ and $\mathrm{diam}\,(C_B) \leqslant \varepsilon$.

Assume to the contrary that there exists a G'-invariant set $A \in \Sigma$ with $\mu(A) > 0$ and $\varepsilon > 0$ such that, for every $B \subseteq A$ with $\mu(B) > 0$, we have $\mathrm{diam}\,(C_B) > \varepsilon$. We then show that the G-invariant set D_A is not dentable, which contradicts the assumptions. Since $D_A = \overline{\mathrm{co}}(C_A)$, note that D_A is dentable if and only if C_A is dentable [44, 48]. So it is enough to show that C_A is not dentable.

To show this, we use an exhaustion argument. Take a maximal disjoint family $\{D_n\}$ of measurable subsets of A such that $\mu(D_n) > 0$ for every $n \in \mathbb{N}$ and that for any $n \in \mathbb{N}$, $\|x_A - x_{D_n}\| > \varepsilon/2$. Put $E = A \setminus (\cup_n D_n)$. We claim that $\mu(E) = 0$. If not, by the assumption, $\mathrm{diam}\,(C_E) > \varepsilon$. Thus, there exists $y \in C_E$ such that $\|x_A - y\| > \varepsilon/2$. Take $E' \subseteq E$ so that $\mu(E') > 0$ and $y = x_{E'}$. The existence of E' contradicts the maximality of $\{D_n\}$. Consequently, we obtain that $A = \cup_n D_n$ (up to null sets). Therefore,

$$x_A = \sum_n \frac{\mu(D_n)}{\mu(A)} x_{D_n}.$$

As the same argument will work for x_B for every $B \subseteq A$ with $\mu(B) > 0$, this shows that C_A is not dentable. This completes the proof of Claim 2.

Thanks to Claim 2, we can consider, for given $\varepsilon > 0$, a maximal disjoint family $\{A_n\}$ of measurable subsets of Ω such that $\mu(A_n) > 0$ and $\mathrm{diam}\,(C_{A_n}) \leqslant \varepsilon$ for every $n \in \mathbb{N}$.

Claim 3. The set Ω is equal to $\bigcup_n A_n$ up to measure zero.

Otherwise, set $E := \Omega \setminus \bigcup_n A_n$ and assume that $\mu(E) > 0$. If there exists a G'-invariant subset $E' \subseteq E$ of positive measure, then by Claim 2, there is $E'' \subseteq E'$ such that $\mu(E'') > 0$ and $\mathrm{diam}\, C_{E''} \leqslant \varepsilon$, which contradicts the maximality of $\{A_n\}$. This implies that the set

$$E \setminus \bigcup_{n \in \mathbb{N},\, g \in G'} g \cdot A_n$$

is μ-null. Since G' is countable, $\mu(E) > 0$, and from the σ-additivity of μ, there exist $n \in \mathbb{N}$ and $g \in G'$ such that $\mu(gA_n \cap E) > 0$. Setting $F := A_n \cap g^{-1}E$,

$$\mathrm{diam}\, C_{gA_n \cap E} = \mathrm{diam}\, D_{gA_n \cap E} \overset{(3),\,(4)}{=\!=} \mathrm{diam}\, C_F \leqslant \mathrm{diam}\, C_{A_n} \leqslant \varepsilon.$$

This again contradicts the maximality of $\{A_n\}$ and completes the proof of Claim 3.

Finally, define $\phi_\varepsilon : \Omega \to X$ by $\phi_\varepsilon(x) = x_{A_n}$ if $x \in A_n$ for some $n \in \mathbb{N}$. Let $f \in S_{L_1(\Omega,\Sigma,\mu)}$ be a positive function and let $\alpha_n = \int_{A_n} f\,d\mu$ for every $n \in \mathbb{N}$. Note that

$$T\left(\frac{f\chi_{A_n}}{\alpha_n}\right) \in D_{A_n} \quad \text{and} \quad x_{A_n} \in C_{A_n}.$$

Thus,

$$\begin{aligned}\left\|Tf - \int f\phi_\varepsilon\,d\mu\right\| &= \left\|\sum_{n=1}^{\infty}\left(T(f\chi_{A_n}) - \int_{A_n} f\phi_\varepsilon\,d\mu\right)\right\| \\ &\leqslant \sum_{n=1}^{\infty}\alpha_n\left\|T\left(\frac{f\chi_{A_n}}{\alpha_n}\right) - x_{A_n}\right\| \leqslant \varepsilon,\end{aligned}$$

where the last inequality holds since $\operatorname{diam}(D_{A_n}) \leqslant \varepsilon$. Therefore, it follows that $\|Tf - \int f\phi_\varepsilon\,d\mu\| \leqslant \varepsilon\|f\|$ for all $f \in L_1(\Omega,\Sigma,\mu)$.

We repeat this procedure with $\varepsilon_k = 1/2^k$, i.e., construct the maximal family $\{A_n^k\}$ as above, and consider the corresponding $\widetilde{\phi}_k := \phi_{\varepsilon_k} \in L_\infty(\Omega,\mu,X)$ satisfying that $\|Tf - \int f\widetilde{\phi}_k\,d\mu\| \leqslant \|f\|/2^k$ for every $f \in L_1(\Omega,\Sigma,\mu)$ and $k \in \mathbb{N}$. Then, we have that $\|\widetilde{\phi}_n - \widetilde{\phi}_m\|_{L_\infty(\Omega,\Sigma,\mu;X)} \leqslant 2^{-n} + 2^{-m}$; thus $(\widetilde{\phi}_k)$ converges in $L_\infty(\Omega,\Sigma,\mu;X)$ to a function ϕ and $Tf = \int f\phi\,d\mu$ for every $f \in L_1(\Omega,\Sigma,\mu)$. This proves that T is G-representable. □

Recall from Lewis-Stegall's result (see, for instance, [17, Theorem 3, page 114]) that a bounded linear operator $T : L_1(\Omega,\Sigma,\mu) \to X$ is representable if and only if T factors through ℓ_1, i.e.,

$$\begin{tikzcd} L_1(\Omega,\Sigma,\mu) \arrow[rr, "T"] \arrow[dr, "S"'] & & X \\ & \ell_1 \arrow[ur, "L"'] & \end{tikzcd}$$

This factorization result leads to a natural question whether every G-equivariant representable operator $T : L_1(\Omega,\Sigma,\mu) \to X$ (see Theorem 3.20) factors through ℓ_1 in a *G-equivariant sense*, i.e., $T = L \circ S$, where L and S both are G-equivariant operators. However, the following example shows that this is not the case.

Proposition 5.5. There is no G-equivariant Lewis-Stegall's factorization theorem.

Proof. Consider $G = \mathrm{Iso}(\mathbb{R}^2)$ and $(B_{\mathbb{R}^2},\Sigma,\mu)$, where μ is the Lebesgue measure on $B_{\mathbb{R}^2}$. Define a bounded linear operator $T : L_1(B_{\mathbb{R}^2},\Sigma,\mu) \to \mathbb{R}^2$ by $T(\chi_A) = \int_A x\,d\mu(x)$ for every $A \in \Sigma$. Notice that T is G-equivariant and the identity map (which is automatically G-equivariant) from $B_{\mathbb{R}^2}$ into $\mathbb{R}^2$ represents T. Notice that G has only two connected components (the orientation preserving isometries forms a connected component of the identity) and that $\mathrm{Iso}(\ell_1)$ is totally disconnected, which can be seen e.g. from the Banach-Lamperti theorem viewing ℓ_1 as an L_1 over a countable atomic measure space. Since continuous actions of G on ℓ_1 can be identified with continuous homomorphisms from G into $\mathrm{Iso}(\ell_1)$, we see that there are no non-trivial actions of G on ℓ_1. Suppose by contradiction that T factors through G-equivariant operators $L : \ell_1 \to \mathbb{R}^2$ and $S : L_1(B_{\mathbb{R}^2},\Sigma,\mu) \to \ell_1$. Since the action of G on ℓ_1 is trivial, while it is transitive on spheres of $\mathbb{R}^2$, we get that L must be trivial. This is a contradiction. □

It is worth mentioning that the space in Theorem 5.5 has the G-RNP since $\mathbb{R}^2$ has the RNP and the RNP implies the G-RNP (see Theorem 3.19 above).

6. Consequences of the G-Radon-Nikodým property

The main goal of this section is to prove that the G-Radon–Nikodým property implies weak G-dentability for groups G satisfying a certain measure condition, which holds, for instance, for locally compact and σ-compact groups. Along the way, we develop the basic theory of G-equivariant-on-sets martingales. We begin with the definition of the class of groups considered in this section.

Definition 6.1. Let G be a topological group. We say that G has *property* $\mathcal{F}$ if there exists a finite measure space (Ω, Σ, μ) such that G admits a point action on Ω (recall Theorem 3.18) satisfying the following conditions:

- the corresponding action on (Ω, Σ, μ) by regular set isomorphisms is continuous;
- the action on Ω is *essentially free*, i.e., for every $g \in G \setminus \{1_G\}$
$$\mu\big(\{\omega \in \Omega\colon g \cdot \omega = \omega\}\big) = 0,$$
- it admits a *measurable fundamental domain*, i.e., there is $F \in \Sigma$ such that $\bigcup_{g\in G} gF = \Omega$ up to measure zero and for every $g \neq h \in G$, $gF \cap hF = \emptyset$.

Example 6.2. Every locally compact σ-compact group G has property $\mathcal{F}$. Indeed, let μ_G be the Haar measure on G (we refer to [30] for Haar measures on locally compact groups). Set $\Omega = G$ and let Σ be the Borel σ-algebra on G. If G is compact, then set $\mu = \mu_G$ and we are done. Otherwise, μ_G is not finite, however by σ-compactness, it is σ-finite. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint measurable sets of strictly positive finite measure such that $G = \bigcup_{n\in\mathbb{N}} K_n$ and let $(\alpha_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence of strictly positive reals such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Then, define
$$\widetilde{\mu_G}(A) := \sum_{i=1}^{\infty} \alpha_i \frac{\mu_G(A \cap K_i)}{\mu_G(K_i)}, \quad A \in \Sigma.$$
It is straightforward to check that $(\Omega, \Sigma, \widetilde{\mu_G})$ is then a probability measure space witnessing that G has property $\mathcal{F}$ (the fundamental domain being the singleton $\{1_G\}$).

In order to show that the G-RNP implies weak G-dentability for a certain group G, we shall need an auxiliary notion of a G-equivariant-on-sets martingale. While it is of independent interest, in the present context it will only play a role of an intermediary between the G-RNP and weak G-dentability.

Definition 6.3. Let G be a topological group and X be a G-Banach space. Let (Ω, Σ, μ) be a finite measure space and suppose G acts continuously on (Ω, Σ, μ) by regular set isomorphisms. A sequence $(M_n)_{n\geqslant 0} \subseteq L_1(\Omega, \Sigma, \mu; X)$ is a *G-equivariant-on-sets martingale* if each M_n, $n \geqslant 0$, individually is G-equivariant-on-sets (recall Theorem 3.23) and $(M_n)_{n\geqslant 0}$ is a martingale with respect to a filtration of G-invariant sub-σ-algebras of Σ. That is, there is a sequence $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \Sigma$, where each $\mathcal{A}_n$, $n \geqslant 0$, is G-invariant and such that each M_n, $n \geqslant 0$, is $\mathcal{A}_n$-measurable and $\mathbb{E}^{\mathcal{A}_n}(M_{n+1}) = M_n$, where $\mathbb{E}^{\mathcal{A}_n}$ is the corresponding conditional expectation. Here, G-invariance of a sub-σ-algebra $\mathcal{A}$ of Σ is understood in the sense that $g \cdot A \in \mathcal{A}$ for all $g \in G$ and $A \in \mathcal{A}$.

We refer to [46, Section 1.3] for details on the classical definition of a Banach space valued martingales.

Theorem 6.4. Let G be a topological group and X be a G-Banach space. Consider the following conditions.

(1) X has the G-RNP.
(2) For every finite measure space (Ω, Σ, μ) equipped with a continuous action of G by regular set isomorphisms, every X-valued G-equivariant-on-sets martingale which is uniformly bounded in L_∞ (i.e. $\sup_{n\in\mathbb{N}} \|M_n\|_{L_\infty} < \infty$) converges a.e. and in $L_1(\Omega, \Sigma, \mu; X)$.
(3) X is weakly G-dentable.

Then, we have that (1) $\implies$ (2) for any G, (2) $\implies$ (3) if G has property $\mathcal{F}$, and (3) $\implies$ (1) if G is separable. In particular, if G is locally compact, σ-compact and separable, then all the conditions are equivalent.

Proof. (3) $\implies$ (1) is already proved in Theorem 5.4.

(1) $\implies$ (2): The proof is the same as the proof of $(i) \implies (ii)$ in [46, Theorem 2.9], to which we refer the reader. Here we only show how the weaker hypothesis of (1) is sufficient to get the weaker conclusion (2).

Let (M_n) be an X-valued uniformly bounded G-equivariant-on-sets martingale. Since (M_n) is uniformly bounded in $L_\infty(\Omega, \Sigma, \mu; X)$, it is clear that (M_n) is uniformly integrable in $L_1(\Omega, \Sigma, \mu; X)$. Thus, the proof of $(i) \implies (ii)$ from [46, Theorem 2.9] shows that the formula

$$m(A) = \lim_{n\to\infty} \int_A M_n d\mu \quad (A \in \Sigma)$$

defines a bounded vector-valued measure. In order to be able to apply the G-RNP instead of the RNP in the proof of $(i) \implies (ii)$ from [46, Theorem 2.9], we only need to check that m is G-quasi-invariant. Then, the rest of the argument is verbatim the same as in [46].

To do so, fix $g \in G$ and $A \in \Sigma$. Observe that

$$g \cdot m(A) = \lim_{n\to\infty} \int_A g \cdot M_n(\omega)\, d\mu(\omega) = \lim_{n\to\infty} \int_{gA} M_n(\omega) \frac{dg_*\mu}{d\mu}(\omega)\, d\mu(\omega)$$

since M_n is G-equivariant-on-sets. For ease of notation, denote

$$h = \frac{dg_*\mu}{d\mu}.$$

To prove that m is G-quasi-invariant, we claim that $g \cdot m(A) = T_m(h\chi_{gA})$.

Note that h is nonnegative; thus

$$\|h\|_1 = \int_\Omega \frac{dg_*\mu}{d\mu}\, d\mu = \mu(g^{-1}\Omega) = \mu(\Omega) < \infty,$$

i.e., $h \in L_1(\Omega, \Sigma, \mu)$. Since simple functions are dense in $L_1(\Omega, \Sigma, \mu)$, we can find a monotone sequence $(h_k)_{k\in\mathbb{N}}$ of nonnegative simple functions such that $h_k \to h$ in $L_1(\Omega, \Sigma, \mu)$ and $h_k \nearrow h$ a.e. By definition, there are a sequence $(n_k)_{k\in\mathbb{N}}$, a sequence of families $(D_i^k)_{i\leqslant n_k}$ of disjoint subsets of Ω, and families of nonnegative scalars $(\alpha_i^k)_{i\in\mathbb{N}}$ such that

$$h_k = \sum_{i=1}^{n_k} \alpha_i^k \chi_{D_i^k}.$$

For each $k \in \mathbb{N}$, denote by

$$I_k = \lim_{n\to\infty} \int_{gA} h_k M_n d\mu. \tag{6.1}$$

First, we need to show that the limit in (6.1) exists. In fact, we have that

$$\begin{aligned} \int_{gA} h_k M_n d\mu = \int_{gA} \sum_{i=1}^{n_k} \alpha_i^k \chi_{D_i^k} M_n d\mu = \sum_{i=1}^{n_k} \alpha_i^k \int_{gA\cap D_i^k} M_n d\mu \\ \xrightarrow[n\to\infty]{} \sum_{i=1}^{n_k} \alpha_i^k m(gA\cap D_i^k) \in X \end{aligned} \tag{6.2}$$

by definition of the vector measure m.

Our goal now is to show that $\lim_{k\to\infty} I_k = g\cdot m(A)$. We start by choosing $\varepsilon > 0$ arbitrarily. Let $K > 0$ be such that for all $n \in \mathbb{N}$ and almost every $\omega \in \Omega$, we have $\|M_n(\omega)\| < K$. Next, choose $k_0 \in \mathbb{N}$ such that $\|h - h_k\| < \varepsilon/K$ for all $k \geqslant k_0$. Then, for all such k, we have

$$\begin{aligned} \left\| \int_{gA} h M_n d\mu - \int_{gA} h_k M_n d\mu \right\| &\leqslant \int_{gA} \|(h-h_k)M_n\| \, d\mu \\ &\leqslant \int_{gA} K\,|h-h_k|\, d\mu < \varepsilon. \end{aligned}$$

Letting $n \to \infty$, we obtain that $\|I_k - g\cdot m(A)\| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it follows that $I_k \to g\cdot m(A)$.

Having this convergence, it is time to start working on I_k's. Note from (6.2) that

$$I_k = \sum_{i=1}^{n_k} \alpha_i^k m(gA\cap D_i^k) = \sum_{i=1}^{n_k} \alpha_i^k T_m(\chi_{gA\cap D_i^k}) = T_m(h_k\chi_{gA}).$$

Since we assume that h_k converge a.e. to h from below, the dominated convergence theorem yields that $h_k\chi_{gA} \to h\chi_{gA}$ in $L_1(\Omega,\Sigma,\mu)$. Since T_m is a bounded operator on $L_1(\Omega,\Sigma,\mu)$, it follows that $T_m(h_k\chi_{gA}) \to T_m(h\chi_{gA})$.

Now we have all the pieces ready; it only remains to put them together to finish the proof of the G-quasi-invariance of m. Indeed,

$$g\cdot m(A) = \lim_{k\to\infty} I_k = \lim_{k\to\infty} T_m(h_k\chi_{gA}) = T_m(h\chi_{gA}) = T_m\left(\frac{dg_*\mu}{d\mu}\chi_{gA}\right).$$

(2) $\implies$ (3): Suppose that G has property $\mathcal{F}$ which is witnessed by an appropriate action on a finite measure space (Ω', Σ', μ'). Let $F' \subseteq \Omega'$ be the corresponding measurable fundamental domain. We define the finite measure space (Ω,Σ,μ) as follows:

- $\Omega := \Omega' \times [0,1]$;
- Σ is the product σ-algebra on Ω generated by Σ' on Ω' and the Borel σ-algebra on $[0,1]$;
- $\mu := \mu' \times \lambda$, where λ denotes the Lebesgue measure on $[0,1]$.

Moreover, we define a point action α_G of G on Ω by

$$\alpha_G(\omega', t) := \big(\alpha'_G(\omega'), t\big), \quad (\omega', t) \in \Omega' \times [0,1],$$

where $\alpha'_G : G \curvearrowright \Omega'$ is the group action witnessing property $\mathcal{F}$ of G. In other words, G acts on the first coordinate as the original action and trivially on the second. Clearly, the corresponding action on (Ω, Σ, μ) by regular set isomorphisms is continuous. It is also clear that $F := F' \times [0,1]$ is a measurable fundamental domain for the new action. As before, we will omit the notation α_G or $\alpha_{G'}$ whenever the context makes the meaning clear.

To reach a contradiction, assume that X is not weakly G-dentable. Then, there is a G-invariant bounded set $D \subseteq X$ and $\delta > 0$ such that for every $x \in D$, we have $x \in \overline{\mathrm{co}}(D \setminus B_\delta(x))$. By a standard argument (see e.g. [46, Lemma 2.7]), we may in fact assume that, for all $x \in D$, we have $x \in \mathrm{co}(D \setminus B_\delta(x))$.

Fix an arbitrary $x_0 \in D$ and define $M_0 : \Omega \to D$ by

$$M_0(\omega) := g \cdot x_0 \quad \text{if } \omega \in g \cdot F \text{ for some } g \in G.$$

Clearly, M_0 is well-defined on a conull set. Moreover, it is G-equivariant and measurable with respect to the sub-σ-algebra $\mathcal{A}_0 := \sigma\big(\{B \times [0,1] \colon B \in \Sigma'\}\big)$.

Since $x_0 \in \mathrm{co}(D \setminus B_\delta(x_0))$, there exist $n \in \mathbb{N}$, $t_1, \ldots, t_n \in (0,1]$, and $x_1, \ldots, x_n \in D$ such that

$$x_0 = \sum_{i=1}^n t_i x_i, \quad \sum_{i=1}^n t_i = 1, \quad \text{and} \quad \|x_i - x_0\| \geqslant \delta, \ \forall i = 1, \ldots, n.$$

We can then find disjoint measurable subsets $A_1, \ldots, A_n \subseteq [0,1]$ such that $\lambda(A_i) = t_i$ and $\bigcup_{i=1}^n A_i = [0,1]$ and define

$$M_1(\omega) := g \cdot x_i \quad \text{if } \omega \in g \cdot F' \times A_i \text{ for some } g \in G \text{ and } i = 1, \ldots, n.$$

It is straightforward that M_1 is well-defined up to a null set and also G-equivariant. Moreover, it is measurable with respect to the sub-σ-algebra $\mathcal{A}_0 \subseteq \mathcal{A}_1$, where $\mathcal{A}_1 := \sigma\big(\{B \times A_i \colon B \in \Sigma', 1 \leqslant i \leqslant n\}\big)$.

Furthermore, observe that, for a.e. $\omega \in \Omega$, we have

$$\|M_0(\omega) - M_1(\omega)\| > \delta.$$

Indeed, let $g \in G$ be such that $\omega \in gF$. Then, there exists $i \in \{1, \ldots, n\}$ such that $\omega \in g \cdot F' \times A_i$, so

$$\|M_0(\omega) - M_1(\omega)\| = \|gx_0 - gx_i\| = \|x_0 - x_i\| > \delta.$$

In the next step, for each $1 \leqslant i \leqslant n$, since $x_i \in \mathrm{co}(D \setminus B_\delta(x_i))$, there exist $n_i \in \mathbb{N}$, $t_1^i, \ldots, t_{n_i}^i \in (0,1]$ and $x_1^i, \ldots, x_{n_i}^i \in D$ such that

$$x_i = \sum_{j=1}^{n_i} t_j^i x_j^i, \quad \sum_{j=1}^{n_i} t_j^i = 1, \quad \text{and} \quad \left\|x_j^i - x_i\right\| \geqslant \delta \text{ for all } j = 1, \ldots, n_i.$$

Then we can find disjoint measurable sets $A_1^i, \ldots, A_{n_i}^i \subseteq A_i$ such that $\lambda(A_j^i) = \alpha_j^i \lambda(A_i)$ and $\bigcup_{j=1}^{n_i} A_j^i = A_i$. Define

$$M_2(\omega) := gx_j^i \quad \text{if } \omega \in g \cdot F' \times A_j^i \text{ for some } g \in G,\ 1 \leqslant i \leqslant n, \text{ and } 1 \leqslant j \leqslant n_i$$

which, as before, we have that it is well-defined up to a null set, G-equivariant and measurable with respect to the sub-σ-algebra $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2$, where $\mathcal{A}_2 :=$

$\sigma\big(\{B \times A_j^i \colon B \in \Sigma', 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n_i\}\big)$. Again as above, we check that, for a.e. $\omega \in \Omega$, we have

$$\|M_1(\omega) - M_2(\omega)\| > \delta.$$

Continuing in the same way, we produce a G-equivariant-on-sets martingale $(M_n)_{n\in\mathbb{N}}$ such that

- $\|M_n(\omega) - M_{n+1}(\omega)\| > \delta$ for all $n \geqslant 0$ and
- $\sup_{n\in\mathbb{N}} \|M_n\|_{L_\infty} \leqslant \sup_{x\in D} \|x\|$, so the martingale is uniformly bounded.

However, the first condition certainly prevents the martingale from converging a.e., which yields a contradiction.

The 'in particular' statement follows since if G is locally compact, σ-compact, then (2) $\implies$ (3) $\implies$ (1) holds, where the first implication holds because of property $\mathcal{F}$ of G (see Theorem 6.2). □

Corollary 6.5. Let G be a locally compact, σ-compact and separable group. Then a G-Banach space X has the G-RNP if and only if it has the G-RNP only with respect to the standard probability space $([0,1], \mathcal{B}, \lambda)$.

Proof. One implication is trivial. For the reverse implication, observe that if G is locally compact, σ-compact and separable, then the finite (probability) measure space $(G, \Sigma_G, \widetilde{\mu_G})$ witnesses the property $\mathcal{F}$ of G (see Theorem 6.2). Moreover, the argument in the proof of (2) $\implies$ (3) of Theorem 6.4 can be adapted, since in this case it actually relies only on the G-RNP with respect to the measure space (Ω, Σ, μ), where $\Omega = G \times [0,1]$, Σ is the Borel σ-algebra of $G \times [0,1]$, and $\mu = \widetilde{\mu_G} \times \lambda$ with $\widetilde{\mu_G}$ a measure equivalent to the Haar measure on G and λ the Lebesgue measure on $[0,1]$.

Now, suppose that X has the G-RNP with respect to $([0,1], \mathcal{B}, \lambda)$. Notice that $\mu = \widetilde{\mu_G} \times \lambda$ is a probability measure that is continuous in the sense that $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$. Since (Ω, Σ) is standard Borel, (Ω, Σ, μ) and $([0,1], \mathcal{B}, \lambda)$ are isomorphic as measure spaces by [34, Theorem 17.41]. It follows that X has the G-RNP with respect to $(G \times [0,1], \Sigma, \mu)$, and by applying (2) $\implies$ (3) of Theorem 6.4, we conclude that X is weakly G-dentable. □

7. Norm-attaining G-equivariant operators

As a way of providing positive results on the denseness of the set of norm-attaining operators, Lindenstrauss introduced what is nowadays known as the *Lindenstrauss property A* (see [37]). A Banach space X is said to satisfy this property if, for every Banach space Y, the set of norm-attaining operators from X into Y is dense in $\mathcal{L}(X,Y)$. In Theorem 3.2, we introduced *G-property A* as a natural equivariant version of this notion which, as mentioned in the introduction, was one of the main motivations for the present work.

There is a clear connection between the Bishop-Phelps property and Lindenstrauss property A: a Banach space has the former if and only if it has the latter under every equivalent renorming. The reader can check that the same relationship holds between the G-BP and G-property A when one considers G-invariant renormings, i.e., those for which the G-action remains isometric with respect to the new norm. In other words, the corresponding linear isomorphism is G-equivariant (see e.g. [1, 18] for recent research on G-invariant renormings).

Since the G-property A is weaker than the G-BP, there exist more examples of G-Banach spaces satisfying it, as we demonstrate in this section (see Theorem 7.1, Theorems 7.2, 7.5 and 7.8, and Theorem 7.9 for positive results on the G-property A, and Theorem 7.3 for a negative one).

We begin by showing that ℓ_1 has the G-property A for every topological group G and any action. In fact, we shall prove something stronger. Recall that the extreme points of the unit ball of ℓ_1 are precisely the canonical basis elements $\{\pm e_i\}_{i\in\mathbb{N}}$; hence every linear isometry on ℓ_1 acts by permuting the canonical basis and changing their signs. In particular, every group acting linearly and isometrically on ℓ_1 induces an action on $\mathbb{N}$, the index set of the basis elements.

Theorem 7.1. Let G be a topological group so that $G \curvearrowright \ell_1$. Then ℓ_1 with this action of G has the G-property A. In fact, if the induced action of G on $\mathbb{N}$ has finitely many orbits, then every G-equivariant operator is norm-attaining.

Proof. Let $I_1 \subseteq \mathbb{N}$ be the set consisting of all $k \in \mathbb{N}$ for which there exists $g \in G$ such that $g \cdot e_k = \pm e_1$. If $I_1 \neq \mathbb{N}$, then let $n_2 \in \mathbb{N}$ be the smallest natural number such that $n_2 \notin I_1$, and I_2 be the finite set of all $k \in \mathbb{N}$ for which there is $g \in G$ with $g \cdot e_{n_2} = \pm e_k$. Inductively, we can consider the subsets $\{I_n\}_{n\in\Gamma}$ of $\mathbb{N}$. Under the corresponding group action on $\mathbb{N}$, each set I_n is an orbit, $\cup_{n\in\Gamma} I_n = \mathbb{N}$ and $I_n \cap I_m = \emptyset$ whenever $n \neq m$ in Γ. Let Y be any Banach space and $T \in \mathcal{L}^G(\ell_1, Y)$. We split the proof into two cases.

Case 1: The corresponding group action on $\mathbb{N}$ has only finitely many orbits. For simplicity, suppose that $I_1 \cup \cdots \cup I_k = \mathbb{N}$. In this case, since $\|T\| = \sup_{n\in\mathbb{N}} \|Te_n\|$, we conclude that $T \in \mathrm{NA}^G(\ell_1, Y)$.

Case 2: The corresponding group action on $\mathbb{N}$ has infinitely many orbits.

Given $\varepsilon \in (0, 1)$, let $n \in \mathbb{N}$ be such that $\|Te_n\| > \|T\| - \varepsilon\|T\|/2$. For simplicity, say $\|T\| = 1$ and $\|Te_1\| > 1 - \varepsilon/2$. Let $P_1 : \ell_1 \to \ell_1$ be the canonical projection on $\ell_1(I_1)$, that is, $(P_1(x))(i) = x(i)$ when $i \in I_1$ and 0 otherwise (noting that $I_1 \neq \mathbb{N}$). Consider $S : \ell_1 \to Y$ defined by

$$Sx := Tx + \varepsilon T \circ P(x)$$

for every $x \in \ell_1$. Notice that S is G-equivariant and $\|S - T\| < \varepsilon$. Moreover,

$$\|Se_i\| = \|Te_i + \varepsilon Te_i\| > (1 + \varepsilon)\left(1 - \frac{\varepsilon}{2}\right) > 1$$

for every $i \in I_1$, while $\|Se_j\| = \|Te_j\| \leqslant 1$ for all $j \in \mathbb{N} \setminus I_1$. This implies that $\|S\| = \sup_{i\in I_1} \|Se_i\| = \|Se_1\|$. This shows once again that $S \in \mathrm{NA}(\ell_1, Y)$. □

An application of Theorem 7.1 yields the following result.

Proposition 7.2. Let G be a topological group and X be a reflexive G-Banach space. Suppose that G also acts on c_0 linearly isometrically. Then $\mathrm{NA}^G(X, c_0)$ is dense in $\mathcal{L}^G(X, c_0)$.

Proof. Let $T \in \mathcal{L}^G(X, c_0)$ and $\varepsilon > 0$. If we equip X^* with the standard dual action, then the property of being G-equivariant passes to adjoint operators, so $T^* \in \mathcal{L}^G(\ell_1, X^*)$. By Theorem 7.1, there exists $S \in \mathrm{NA}^G(\ell_1, X^*)$ with $\|T^* - S\| < \varepsilon$.

Taking adjoints again, we get that $S^* \in \mathrm{NA}^G(X^{**}, \ell_\infty)$ and $\|T^{**} - S^*\| < \varepsilon$. Since X is reflexive, we have that $T = T^{**}$ can be approximated arbitrarily well by operators from $\mathrm{NA}^G(X, \ell_\infty)$. All that remains is to show that actually $S^* \in \mathrm{NA}^G(X, c_0)$.

By inspecting the proof of Theorem 7.1, we see that $S = T^* + \varepsilon T^* \circ P$, where $P : \ell_1 \to \ell_1$ is the canonical projection onto $\operatorname{span}\{e_i : i \in I\}$, where $I \subseteq \mathbb{N}$ is the set of indices i such that either e_i or $-e_i$ is an element of $G \cdot e_j$ for some fixed $j \in \mathbb{N}$. Looking at the adjoint of S, we get $S^* = T + \varepsilon P^* \circ T$, where $P^* : \ell_\infty \to \ell_\infty$ is the canonical projection onto the subspace $\ell_\infty(I) = \{x \in \ell_\infty : (\forall i \notin I)\ x(i) = 0\}$. However, since the range of restricted map $P^*\restriction_{c_0} : c_0 \to \ell_\infty$ is contained in c_0, we conclude that the image of S^* is contained in c_0; hence, $S^* \in \mathrm{NA}^G(X, c_0)$. □

In contrast to Theorem 7.2, when we consider c_0 in the domain we get the following negative result.

Proposition 7.3. For every compact group G acting on c_0 by linear isometries, there is a G-Banach space Y such that $\overline{\mathrm{NA}^G(c_0, Y)} \neq \mathcal{L}^G(c_0, Y)$.

Proof. Let $(X, \|\cdot\|)$ be a separable Banach space and G be a compact group such that $\alpha : G \curvearrowright X$ by linear isometries. By [1, Proposition 2.1] or [18, Corollary 4.4 (1)], there exists a strictly convex renorming $Y = (X, |||\cdot|||)$ such that the norm is invariant under the group action G, that is, $|||x||| = |||\alpha(g, x)|||$ for every $x \in X$ and $g \in G$. Notice that, for such a renorming Y, the identity operator $I : (X, \|\cdot\|) \to Y = (X, |||\cdot|||)$ is G-equivariant.

As a concrete case, take $X = c_0$ with the supremum norm. Let $Y = (c_0, |||\cdot|||)$ be a strictly convex space such that $|||\cdot|||$ is G-invariant. It is well known that if a bounded linear operator from c_0 to Y is norm-attaining, then it must be of finite rank (see, for instance, [40, Lemma 2]); so $I \in \mathcal{L}^G(c_0, Y)$ cannot be approximated by norm-attaining operators. □

In order to obtain further examples of G-Banach spaces with G-property A, we recall the following definitions that will be useful in this context.

Let M be a metric space and G be a topological group acting continuously by isometries on M. By $M \mathbin{/\!/} G$ we shall denote the space of closures of orbits of G equipped with the infimum distance. That is,

$$M \mathbin{/\!/} G := \{\overline{G \cdot x} : x \in M\}$$

and for $\overline{G \cdot x}, \overline{G \cdot y} \in M \mathbin{/\!/} G$, we set

$$D(\overline{G \cdot x}, \overline{G \cdot y}) := \inf_{g \in G} d_M(x, gy).$$

It is straightforward to verify that this is a well-defined metric. We can now introduce the following definition.

Definition 7.4. Let G be a topological group, X be a G-Banach space, and let $n \in \mathbb{N}$. We say that the action $G \curvearrowright X$ is *n-cocompact* whenever the metric space $B_X^n \mathbin{/\!/} G$ is compact, where B_X^n is endowed with the sum metric.

A separable Banach space X is said to be *ω-categorical* if $\mathrm{Iso}(X) \curvearrowright X$ is n-cocompact for every $n \in \mathbb{N}$.

We remark that the notion of an ω-categorical Banach space comes from model theory of Banach spaces where it forms one of the most interesting classes of separable Banach spaces fully determined by their first order theory. The above definition comes from (the continuous version of) the Ryll-Nardzewski theorem which reformulates the original definition of ω-categoricity through the actions of linear isometry groups (see [9]). This, in turn, connects these Banach spaces with the group equivariant properties we are after. For model theory of Banach spaces, and more generally metric structures, we refer the interested reader to [7].

The following is a short list of examples of such Banach spaces (also called $\aleph_0$-*categorical Banach spaces*). For more information about them, we send the reader to the references provided below along with the examples as well as to [35].

(1) $\ell_2(\mathbb{N})$ -folklore, see also the example below;
(2) L_p, for all $1 \leqslant p < \infty$ - see [7];
(3) $C(2^{\mathbb{N}})$ - by Henson (unpublished);
(4) the Gurarii space - see [8];
(5) $L_p(L_q)$, for all $1 \leqslant p, q < \infty$ - see [29];
(6) if $1 \leqslant p < \infty$ and X is ω-categorical, then so is $L_p(X)$ (V. Ferenczi, private communication).

Proposition 7.5. Let X be a G-Banach space, for some topological group G, and suppose $G \curvearrowright X$ is 1-cocompact. Then, every G-equivariant operator is norm-attaining. In other words, $\mathrm{NA}^G(X,Y) = \mathcal{L}^G(X,Y)$ for every G-Banach space Y. In particular, X has the G-property A.

Proof. Let $T \in \mathcal{L}^G(X,Y)$ be given. Let us consider the map $T' : B_X \,/\!\!/\, G \to Y \,/\!\!/\, G$ given by $T'(\overline{G \cdot x}) = \overline{G \cdot Tx}$ for each $x \in X$. We claim that the map is well-defined. In fact, if $\overline{G \cdot x} = \overline{G \cdot y}$, then there exists a sequence $(g_n) \subseteq G$ such that $\|x - g_n y\| \to 0$. Then

$$\|Tx - g_n Ty\| = \|Tx - T(g_n y)\| \leqslant \|T\|\|x - g_n y\| \to 0.$$

Thus $\overline{G \cdot Tx} = \overline{G \cdot Ty}$. Moreover, notice that

$$\|Tx\| = \inf_{g \in G} \|Tx - g \cdot 0\| = D(\overline{G \cdot Tx}, 0) = D(T'(\overline{G \cdot x}), 0)$$

for every $x \in B_X$; $\|T\| = \sup\{D(T'(\overline{G \cdot x}), 0) : \overline{G \cdot x} \in B_X \,/\!\!/\, G\}$. If we prove that T' is continuous, then the preceding supremum is indeed achieved since $B_X \,/\!\!/\, G$ is compact. To see that T' is continuous, pick $\overline{G \cdot x} \in B_X \,/\!\!/\, G$ and $\varepsilon > 0$. Then there is $\delta > 0$ such that for every $y \in B_X$ with $\|x - y\| < \delta$ we have $\|T(x) - T(y)\| < \varepsilon$. Then if $\overline{G \cdot z} \in B_X \,/\!\!/\, G$ is such that $D(\overline{G \cdot x}, \overline{G \cdot z}) < \delta$ there is $g \in G$ so that $\|x - gz\| < \delta$ thus

$$D(T'(\overline{G \cdot x}), T'(\overline{G \cdot z})) = \inf_{h \in G} \|Tx - hTz\| \leqslant \|Tx - T(gz)\| < \varepsilon.$$

□

As an immediate consequence of Theorem 7.5 we have that ω-categorial Banach spaces satisfy the G-property A.

Corollary 7.6. Every ω-categorical Banach space X has the G-property A with respect to $G = \mathrm{Iso}(X)$.

We will prove another positive result in Theorem 7.8 for Banach spaces that contain big points. Let us recall the notions of big points, convex transitivity, and almost transitivity.

Definition 7.7. Let X be a Banach space. We say that

(a) $x \in S_X$ is a *big point* if $\overline{\mathrm{co}}\big(\mathrm{Iso}(X) \cdot x\big) = B_X$.
(b) X is *convex-transitive* if every element $x \in S_X$ is a big point.
(c) X is *almost-transitive* if there is/for every $x \in S_X$ the orbit $\mathrm{Iso}(X) \cdot x$ is dense in S_X.

Clearly, every almost-transitive Banach space is convex-transitive, whereas $C(K)$, where K denotes the Cantor set, is a convex-transitive Banach space that is not almost-transitive [42]. For more information on convex-transitive Banach spaces and big points as well as almost-transitive Banach spaces we refer the reader to [4, 5]. We note that these properties developed as a weakening of transitive Banach spaces related to the famous Mazur rotation problem and we refer to the survey [13] as a background for these connections. Now we can provide the promised result.

Proposition 7.8. Let X be a Banach space that contains a big point. Then, every G-equivariant operator is norm-attaining at that point, where $G = \mathrm{Iso}(X)$. In particular, X has the G-property A.

Proof. Let Y be a G-Banach space, $T \in \mathcal{L}^G(X, Y)$, and let $x \in X$ be a big point. We claim that $\|T\| = \|T(x)\|$. Indeed, for any $\varepsilon > 0$, there exist $(\lambda_i)_{i\leqslant k} \subseteq \mathbb{R}$ satisfying $\sum_{i=1}^k \lambda_i = 1$ and $(g_i)_{i\leqslant k} \subseteq G = \mathrm{Iso}(X)$ such that

$$\left\| T\left(\sum_{i=1}^k \lambda_i g_i x\right)\right\| > \|T\| - \varepsilon.$$

It follows that

$$\|T(x)\| = \sum_{i=1}^k \lambda_i \|T(x)\| = \sum_{i=1}^k \lambda_i \|g_i T(x)\| > \|T\| - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $\|T(x)\| \geqslant \|T\|$, so $\|T(x)\| = \|T\|$. $\square$

For Banach spaces that are moreover almost-transitive, one can get something stronger. In particular, it applies to every L_p space with $p \in [1, \infty)$, more generally, to $L_p(\Omega, \Sigma, \mu; E)$ whenever μ is a non-atomic positive measure and E is almost transitive (see [28]).

Corollary 7.9. Let X be an almost-transitive Banach space and $G = \mathrm{Iso}(X)$. Then, X has the G-BP.

Proof. By Theorem 7.8, we get that B_X has the G-BP. So, it is enough to show that, up to scaling, there are no other convex closed bounded G-invariant subsets. Fix $C \subseteq X$ to be a nonempty bounded closed convex G-invariant subset. We will show that $C = rB_X$ for some $r \geqslant 0$. Denote $r := \sup_{x\in C} \|x\|$. Then, clearly $C \subseteq rB_X$ and if $r = 0$, then $\{0\} = C = rB_X$. This means we can assume $r > 0$. Let $\varepsilon \in (0, r)$ and find $x \in C$ such that $\|x\| > r - \varepsilon$. By the almost-transitivity and closedness of C, we have that $\|x\| S_X \subseteq C$. By the convexity of C, we even have that $\|x\| B_X \subseteq C$. Since $\varepsilon \in (0, r)$ was arbitrary, using that C is closed once again, we conclude that

$$rB_X = \overline{\bigcup_{\varepsilon \in (0,r)} (r-\varepsilon)B_X} \subseteq \overline{C} = C.$$

$\square$

8. Summary of the results

For the convenience of the reader, we review all the implications we have proved between the various properties that we have considered throughout the text, as well as the classical counterparts, in Diagram 2 below.

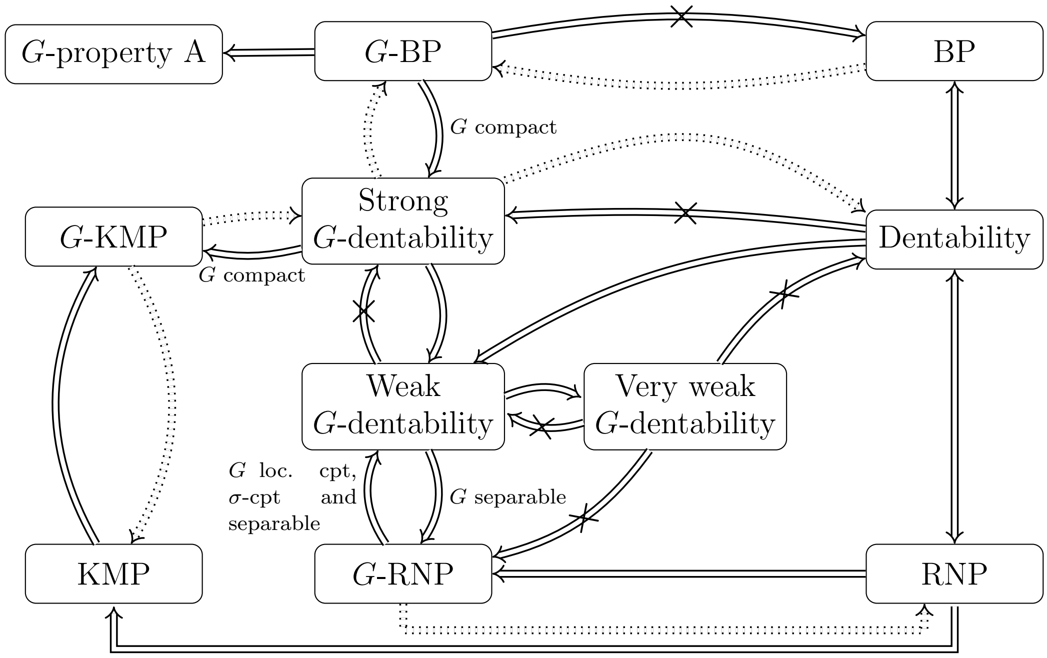

Diagram 2: This diagram summarizes all the implications among the properties we have considered in the paper. Dotted arrows denote cases where it remains unclear whether the implication holds, whereas crossed arrows mark implications that generally fail to hold; counterexamples are provided throughout the text.

The reader can see from the diagram that the strongest results are obtained for G-Banach spaces where G is compact and separable. In this case, G-BP implies all the other G-properties, and weak G-dentability is moreover equivalent with the G-RNP. Since the diagram does not make this clear, let us note that the implications from G-BP to weak G-dentability and the G-RNP do not hold in general. We state it explicitly as follows.

Fact 8.1. L_1 has, for the isometry group $G = \mathrm{Iso}(L_1)$, the G-BP by Theorem 7.9, yet fails to have the G-RNP by Theorem 3.21. Moreover, L_1 is not weakly G-dentable since L_1 is almost-transitive, by the proof of Theorem 7.9, every closed bounded convex G-invariant subset of L_1 is a closed ball around 0, which is not dentable. In particular, L_1 with $G = \mathrm{Iso}(L_1)$ serves as an example of a G-Banach space that is very weakly G-dentable (see Theorem 3.7) but not weakly G-dentable.

Problem 8.2. Decide whether the implications of the dotted arrows hold true.

One of the most mysterious implications is the one between the classical Bishop-Phelps property and the G-Bishop-Phelps property. Notice that it is the only instance of the phenomenon, among the properties we have considered here, that the classical property does not obviously imply its G-analogue. It is possibly also related to the apparent difficulty of showing that strong G-dentability implies the G-BP (for a compact group G). It would be certainly very interesting to find an example of G-Banach space with the BP which is strongly G-dentable but fails the G-BP, thereby disproving both implications at once.

It is not surprising that the question whether the G-KMP implies strong G-dentability remains open, given that even in the classical setting the long-standing problem of whether the KMP implies the RNP is still unresolved (see, for instance, [38, 39]). This indicates that the G-analogue will likewise require further investigation, a direction we hope to explore in future work.

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