A Dynamical Néron–Ogg–Shafarevich Criterion via Orbital Arboreal Representations

J.R. Pérez-Buendía[1*]

[1*]Secihti-CIMAT, Unidad Mérida, Mérida, Yucatán, México.

Corresponding author(s). E-mail(s): rogelio.perez@cimat.mx;

Abstract

Let K be a non-archimedean local field and $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ a rational endomorphism of degree $d \geq 2$ defined over K. In the tame case ($p \nmid d$) we give a concise local criterion for strict good reduction on the natural residual étale locus: there exists a nonempty open $U_k \subset \mathbb{P}^1_k \setminus \mathbf{PC}(\tilde{\varphi})$ such that for every $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \in U_k$ the reduced level-1 fiber has degree d and is étale; equivalently, the fiber polynomial has unit leading coefficient and unit discriminant. In particular, all backward preimage extensions $K(X_n(x))/K$ are unramified for all $n \geq 1$. This work provides an orbital refinement of the pointwise criterion of Benedetto, framing it in terms of a canonical, orbit-invariant Galois object. We provide complete proofs and explicit examples over $\mathbb{Q}_p$.

Keywords: Non-archimedean dynamics, arithmetic dynamics, good reduction, arboreal representations, Galois representations, Néron-Ogg-Shafarevich criterion

MSC Classification: 37P05 , 11S15 , 37P20 , 14H25

1 Introduction

Good reduction precludes the emergence of new ramification in the Galois tower, thereby facilitating the analysis of rational dynamical systems over local fields. This phenomenon establishes connections among non-archimedean analysis, arithmetic geometry, and Galois theory. A central principle, analogous to the Néron-Ogg-Shafarevich criterion for abelian varieties, asserts that good reduction is characterized by the absence of ramification in the natural towers associated with the system (see, e.g., [3–5, 12]). For results on degree bounds for attaining potentially good reduction after a finite extension, see [5]. In addition to abelian varieties, p-adic NOS-type criteria have been established for curves via the p-adic unipotent fundamental group [2], and for $K3$ surfaces using monodromy and crystalline techniques [7, 9, 10].

A rigorous Galois criterion for rational maps, however, must avoid a common pitfall: quantifying over *all* $x \in \mathbb{P}^1(\overline{K})$ fails, as a point x may itself generate a ramified extension $K(x)/K$. This would make the entire tower $K(X_n(x))/K$ ramified, regardless of the dynamics of φ. A precise criterion must therefore restrict the test points x to a suitable unramified integral locus where the dynamics truly determine the ramification.

In this paper, we address this by introducing the *orbital colimit* $\mathcal{G}_{O^+(x)} := \varinjlim_n \mathrm{Im}(\rho_{\varphi^n(x)})$, which is the direct limit of the pointwise arboreal representations along a forward orbit $O^+(x)$. We then prove a precise version of the Galois criterion (Theorem 3.1): φ has *strict good reduction* if and only if the inertia subgroup $I_v \subset G_K$ acts trivially on the preimage towers $X_n(x)$ for all integral points $x \in \mathbb{P}^1(\mathcal{O}_K)$ lying in the residual open set $U_k = \mathbb{P}^1_k \setminus \mathrm{PC}(\tilde{\varphi})$ (where the reduced fibers have degree d and are étale).

Comparison with prior work and framing.

Benedetto's criteria [4, 5] focus on attaining potentially good reduction after a finite extension. Our main result (Theorem 3.1) provides an *operational reformulation* of the pointwise criterion for strict good reduction [3, Corollary 5.52] on the natural residual locus $U_k = \mathbb{P}^1_k \setminus \mathrm{PC}(\tilde{\varphi})$. A similar orbital approach appears in Adams [1], who uses the p-adic unipotent fundamental group to derive a stronger notion of good reduction via monodromy. Conceptually, our main novelty is the use of the orbital colimit $\mathcal{G}_{O^+(x)}$ (Section 4) as a *canonical orbital packaging* of arboreal Galois data. This removes basepoint artifacts and yields a uniform, iteration-compatible *Dynamical Néron–Ogg–Shafarevich criterion*, establishing that trivial inertia on this object is precisely equivalent to strict good reduction.

2 Preliminaries

2.1 Good reduction of rational maps

Let K be a non-archimedean local field, with $\mathcal{O} = \mathcal{O}_K$ its ring of integers, $\mathfrak{m}$ its maximal ideal, and $k = \mathcal{O}/\mathfrak{m}$ its residue field (assumed perfect). We write

$$\varphi([X:Y]) = [F(X,Y) : G(X,Y)], \quad F, G \in K[X,Y] \text{ homogeneous, coprime.}$$

We say φ has *strict good reduction* at v if there exists a model with $F, G \in \mathcal{O}[X,Y]$ coprime such that the reduction $\tilde{\varphi} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ satisfies $\deg \tilde{\varphi} = \deg \varphi = d$. This is equivalent to the existence of such a model with $\mathrm{Res}(F,G) \in \mathcal{O}^\times$; see, e.g., [12, Prop. 2.15–2.17].

We write $\mathrm{Res}(F,G)$ for the resultant of homogeneous coprime forms F, G defining φ, $\mathrm{Disc}(f)$ for the discriminant of a separable polynomial f, and $\deg(\tilde{\varphi})$ for the degree of the reduction of φ in any integral model. Reductions of maps/polynomials are denoted by a tilde (e.g. $\tilde{\varphi}$, $\tilde{P}$), and reductions of points by a bar (e.g. $\bar{x}$). For iterates, we consistently write $\tilde{\varphi}^m(\mathrm{Crit}(\tilde{\varphi}))$. We use $\mathrm{Br}(\cdot)$ for branch loci, and $\varprojlim$, $\varinjlim$ for projective/direct limits.

2.2 Preimage towers and Galois action

For $x \in \mathbb{P}^1(\overline{K})$ and $n \geq 0$, define

$$X_n(x) = \{y \in \mathbb{P}^1(\overline{K}) : \varphi^n(y) = x\}.$$

If $\varphi^n(T) = P_n(T)/Q_n(T)$ with $P_n, Q_n \in K[T]$ coprime, then $X_n(x)$ are the roots of $F_{n,x}(T) = P_n(T) - x\,Q_n(T) \in K(x)[T]$ (in particular, P_1, Q_1 represent φ at level $n = 1$). The Galois group $G_{K(x)} = \mathrm{Gal}(\overline{K}/K(x))$ acts by permuting $X_n(x)$. The transition maps

$$\pi_{n+1} : X_{n+1}(x) \to X_n(x), \quad \xi \mapsto \varphi(\xi),$$

are $G_{K(x)}$-equivariant, defining the projective limit $X_\infty(x) = \varprojlim_n X_n(x)$ and the representation

$$\rho_x : G_{K(x)} \longrightarrow \mathrm{Aut}(X_\infty(x)).$$

2.3 Functoriality and iteration

If $y = \varphi(x)$, then $K(y) \subseteq K(x)$, which induces an *inclusion* $G_{K(x)} \hookrightarrow G_{K(y)}$. The *shift map*

$$s_{x\to y} : X_\infty(x) \to X_\infty(y), \qquad (\xi_0, \xi_1, \dots) \mapsto (\xi_1, \xi_2, \dots),$$

is continuous and $G_{K(x)}$-equivariant. This induces a natural, continuous homomorphism

$$r_{x\to y} : \mathrm{Aut}(X_\infty(x)) \to \mathrm{Aut}(X_\infty(y))$$

(not necessarily surjective) making the diagram commute:

$$\begin{tikzcd}
G_{K(x)} \arrow[r, "\rho_x"] \arrow[d, hook] & \mathrm{Aut}(X_\infty(x)) \arrow[d, "r_{x\to y}"] \\
G_{K(y)} \arrow[r, "\rho_y"] & \mathrm{Aut}(X_\infty(y)).
\end{tikzcd}$$

By iteration, for $x_n = \varphi^n(x)$, we obtain a direct system of images $\mathrm{Im}(\rho_{x_n})$.

2.4 Orbital colimit

Define the *orbital colimit*

$$\mathcal{G}_{O^+(x)} := \varinjlim_n \mathrm{Im}(\rho_{\varphi^n(x)}), \qquad \rho_{O^+(x)} : G_{K(O^+(x))} \to \mathcal{G}_{O^+(x)},$$

where $K(O^+(x)) = \bigcup_n K(\varphi^n(x))$. This is a canonical algebraic object associated with the orbit. We endow $\mathcal{G}_{O^+(x)}$ with the direct limit topology of the profinite groups $\mathrm{Im}(\rho_{\varphi^n(x)})$. Thus, it is a profinite group.

2.5 Two technical lemmata

Lemma 2.1 (Unit resultant persists under iteration) *If* $\mathrm{Res}(F,G) \in \mathcal{O}^\times$ *for an integral presentation of* φ*, then for every* $n \geq 1$ *there exists an integral presentation* (P_n, Q_n) *of* φ^n *with* $\mathrm{Res}(P_n, Q_n) \in \mathcal{O}^\times$*.*

Proof Good reduction is stable under iteration: if φ admits an integral presentation with $\mathrm{Res}(F,G) \in \mathcal{O}^\times$ and $\deg \tilde{\varphi} = d$, then so does each iterate φ^n; see Silverman [12, Prop. 2.17]. In particular, for every $n \geq 1$ there exist coprime $P_n, Q_n \in \mathcal{O}[T]$ representing φ^n with $\mathrm{Res}(P_n, Q_n) \in \mathcal{O}^\times$. □

Lemma 2.2 (Unit discriminant ⇒ unramified) *Let* $f(T) \in \mathcal{O}[T]$ *be a separable polynomial with unit leading coefficient. If* $\mathrm{Disc}(f) \in \mathcal{O}^\times$*, then the splitting field of* f *over* K *is unramified. In particular, for any set* Y *of roots of* f*,* $K(Y)/K$ *is unramified.*

Proof Since K is a local field, its residue field k is perfect. $\mathrm{Disc}(f) \in \mathcal{O}^\times$ implies the reduction $\tilde{f} \in k[T]$ is separable. Moreover, the unit leading coefficient ensures that $\deg \tilde{f} = \deg f$, so the reduction has the correct degree and Dedekind's criterion applies directly. By Dedekind's criterion, this implies the different is trivial, and thus the prime $\mathfrak{m}$ does not ramify in the splitting field; see, e.g., [11, Ch. III]. Moreover, a compositum of unramified extensions remains unramified, so the full splitting field of f over K is unramified as well. □

Definition 2.3 (Postcritical set) For the reduction $\tilde{\varphi} : \mathbb{P}^1_k \to \mathbb{P}^1_k$, define the reduced postcritical set

$$\mathrm{PC}(\tilde{\varphi}) \;:=\; \bigcup_{m \geq 1} \tilde{\varphi}^{\,m}\big(\mathrm{Crit}(\tilde{\varphi})\big) \;\subset \mathbb{P}^1_k.$$

Remark 2.4 Equivalently, $\mathrm{Crit}(\tilde{\varphi})$ is the ramification divisor; in homogeneous coordinates it is cut out by $F_X G_Y - F_Y G_X$ (subscripts denote partial derivatives), for a reduced integral model $\tilde{\varphi}([X:Y]) = [\tilde{F} : \tilde{G}]$.

Proposition 2.5 (Étale fibers for all iterates ⇔ non-postcritical) *Let* $\tilde{\varphi} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ *be the reduction of* φ*, with* k *perfect. For* $\bar{x} \in \mathbb{P}^1_k$*, the following are equivalent:*

(a) For every $n \geq 1$*, the fiber of* $\tilde{\varphi}^{\,n}$ *over* $\bar{x}$ *is étale.*

(b) $\bar{x} \notin \mathrm{PC}(\tilde{\varphi}) = \bigcup_{m \geq 1} \tilde{\varphi}^{\,m}\big(\mathrm{Crit}(\tilde{\varphi})\big)$*.*

Proof Recall the standard chain rule for branch loci on $\mathbb{P}^1$:

$$\mathrm{Crit}\big(\tilde{\varphi}^{\,n}\big) = \bigcup_{j=0}^{n-1} (\tilde{\varphi}^{\,j})^{-1}\big(\mathrm{Crit}(\tilde{\varphi})\big),$$

$$\mathrm{Br}\big(\tilde{\varphi}^{\,n}\big) = \tilde{\varphi}^{\,n}\big(\mathrm{Crit}(\tilde{\varphi}^{\,n})\big) = \bigcup_{m=1}^{n} \tilde{\varphi}^{\,m}\big(\mathrm{Crit}(\tilde{\varphi})\big) \;\subset\; \mathrm{PC}(\tilde{\varphi}).$$

Since k is perfect, étaleness of the fiber over $\bar{x}$ for $\tilde{\varphi}^n$ is equivalent to $\bar{x} \notin \mathrm{Br}(\tilde{\varphi}^n)$. ($\neg$(b)$\Rightarrow$ $\neg$(a)) If $\bar{x} \in \mathrm{PC}(\tilde{\varphi})$, then $\bar{x} \in \mathrm{Br}(\tilde{\varphi}^m)$ for some $m \geq 1$, so the fiber of $\tilde{\varphi}^m$ over $\bar{x}$ is not étale. (a)$\Rightarrow$(b) contrapositively: if for some $n \geq 1$ the fiber of $\tilde{\varphi}^n$ over $\bar{x}$ is not étale, then $\bar{x} \in \mathrm{Br}(\tilde{\varphi}^n) \subset \mathrm{PC}(\tilde{\varphi})$, by finitude of $\mathrm{PC}(\tilde{\varphi})$ in $\mathbb{P}^1$ (Silverman [12, Thm. 5.4]). □

Remark 2.6 In particular, $U_k := \mathbb{P}^1_k \setminus \mathrm{PC}(\tilde{\varphi})$ is a nonempty Zariski open subset on which every fiber of $\tilde{\varphi}^n$ is étale for all $n \geq 1$.

Remark 2.7 If $\tilde{\varphi}$ is purely inseparable, then every iterate is ramified everywhere and $\mathrm{PC}(\tilde{\varphi}) = \mathbb{P}^1_k$, so the statement is vacuous and consistent with the proposition.

3 Main Result

Throughout this section and the remainder of the paper, we work in the tame case, assuming that the residue characteristic p does not divide the degree of the map d (i.e., $p \nmid d$). This ensures that all residual ramification is tame, which is essential for our criterion.

Good reduction is characterized not by all points, but specifically by those integral points whose fibers avoid the reduced postcritical locus.

Theorem 3.1 (Galois criterion — strict local version) *Let K be a non-archimedean local field with ring of integers $\mathcal{O}_K$ and residue field k. For $\varphi \in K(z)$ of degree $d \geq 2$, the following are equivalent:*

(1) φ has strict good reduction (i.e. there exists an integral model with $\mathrm{Res} \in \mathcal{O}_K^\times$ *and* $\deg \tilde{\varphi} = d$*).*

(2) There exists a nonempty open $U_k \subset \mathbb{P}^1_k$ with $U_k \cap \mathrm{PC}(\tilde{\varphi}) = \varnothing$ such that, for every $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \in U_k$, the reduced level-1 fiber

$$\tilde{F}_{1,\bar{x}}(T) = \tilde{P}_1(T) - \bar{x}\,\tilde{Q}_1(T)$$

has degree d and is separable. Equivalently, there exists an integral model *such that the leading coefficient of $F_{1,x}$ is a unit and* $\mathrm{Disc}(F_{1,x}) \in \mathcal{O}_K^\times$.

In this case, for all $n \geq 1$ and all such x, the polynomials $F_{n,x}(T)$ are separable with $\mathrm{Disc}(F_{n,x}) \in \mathcal{O}_K^\times$*, hence $K(X_n(x))/K$ is unramified and I_v acts trivially on $X_\infty(x)$.*

Corollary 3.2 (Moduli version, $\mathrm{PGL}_2(K)$-invariant) *Let* $R_v(\varphi) := \min_{M \in \mathrm{PGL}_2(K)} v(\mathrm{Res}(\varphi^M))$ *be the valuation of the minimal resultant. Then $R_v(\varphi) = 0$ (i.e., φ has good reduction in the moduli sense) if and only if there exists a conjugate $M \in \mathrm{PGL}_2(K)$ such that condition (2) of Theorem 3.1 holds for φ^M.*

Proof By definition, $R_v(\varphi) = 0$ if and only if there exists $M \in \mathrm{PGL}_2(K)$ with $\mathrm{Res}(\varphi^M) \in \mathcal{O}_K^\times$ and $\deg \widetilde{\varphi^M} = d$, i.e., φ^M has strict good reduction. Applying Theorem 3.1 to φ^M shows that condition *(2)* holds precisely in this case. This is equivalent to $R_v(\varphi) = 0$, as claimed. □

Remark 3.3 (Relative inertia) Suppose one insists on quantifying over *all* $x \in \mathbb{P}^1(\overline{K})$. In that case, the correct statement involves the relative inertia group $I_{v(x)} \subset G_{K(x)}$ for the appropriate valuation $v(x)$ on $K(x)$. Theorem 3.1 provides a cleaner criterion by fixing the base field K and restricting the test points x to the unramified integral locus.

Remark 3.4 (Wild case $p \mid d$) If $p \mid d$, the situation is substantially more complicated. Inseparability in the reduction $\tilde{\varphi}$ may force the postcritical set to be the entire line ($\mathrm{PC}(\tilde{\varphi}) = \mathbb{P}^1_k$), making our criterion vacuous. The wild case requires a separate, deeper analysis of ramification breaks and is beyond the scope of this paper.

4 Orbital open image: a canonical formulation

In the classical (pointwise) framework, one fixes a basepoint $x \in \mathbb{P}^1(\overline{K})$ and studies the arboreal representation

$$\rho_x : \; G_{K(x)} \longrightarrow \mathrm{Aut}\big(X_\infty(x)\big),$$

asking whether the image is "large" (e.g., open of finite index) in a natural automorphism group (typically the automorphism group of a regular d-ary rooted tree when no local branching obstructions arise).

Remark 4.1 (Intuition) Heuristically, the orbital colimit assembles the "local monodromies" of preimage trees along the forward orbit into a single canonical Galois object. The connecting maps are induced by the shift $s_{x_n \to x_{n+1}}$, so the colimit may be viewed as the direct limit of arboreal images measured on successive basepoints along the orbit. This eliminates basepoint dependence while retaining all dynamical ramification data.

Proposition 4.2 (Canonicity and independence of basepoint) *Let $x \in \mathbb{P}^1(\overline{K})$ and set $x_n = \varphi^n(x)$. Then:*

- *(a) $K(O^+(x)) = K(O^+(x_n))$ for all $n \geq 0$; hence $G_{K(O^+(x))}$ is canonically attached to the orbit.*
- *(b) The directed system $\{\mathrm{Im}(\rho_{x_m}),\, r_{m\to m+1}\}_{m\geq 0}$ has colimit $\mathcal{G}_{O^+(x)}$ which is independent of n.*
- *(c) If $M \in \mathrm{PGL}_2(K)$, then $\rho_{O^+(Mx)}$ is naturally isomorphic to the transport of $\rho_{O^+(x)}$ via M; in particular, the* conjugacy class over K *of $\rho_{O^+(x)}$ is a conjugacy-invariant of the orbit in moduli.*

Proof (a) For every $m \geq 0$ one has $\varphi^{n+m}(x) = \varphi^m(x_n)$, hence

$$K\big(O^+(x)\big) = \bigcup_{m\geq 0} K\big(\varphi^m(x)\big) = \bigcup_{m\geq 0} K\big(\varphi^m(x_n)\big) = K\big(O^+(x_n)\big).$$

Hence the absolute Galois groups $G_{K(O^+(x))}$ and $G_{K(O^+(x_n))}$ are canonically identified. (b) The directed system $\{\mathrm{Im}(\rho_{x_m}),\, r_{m\to m+1}\}_{m\geq 0}$ is cofinal with each of its tails $\{m \geq n\}$. Cofinality identifies canonically the colimit over $m \geq 0$ with the colimit over $m \geq n$, showing that $\mathcal{G}_{O^+(x)}$ is independent of n. (c) For $M \in \mathrm{PGL}_2(K)$, write $\varphi^M = M \circ \varphi \circ M^{-1}$. Then $O^+(Mx) = M\big(O^+(x)\big)$. At each level m, M induces $X_\infty(x_m) \xrightarrow{\sim} X_\infty((Mx)_m)$ intertwining the G_K-actions and respecting the connecting maps. Passing to the colimit yields a natural isomorphism between $\rho_{O^+(x)}$ and $\rho_{O^+(Mx)}$. In particular, the conjugacy class over K of $\rho_{O^+(x)}$ is an invariant of the orbit in moduli. □

4.1 The target orbital symmetry group

When no critical collisions occur in the backward trees along the orbit (the "regular" situation), each $X_\infty(x_n)$ is a regular d-ary rooted tree and their automorphism groups are (non-canonically) isomorphic to the standard profinite wreath limit $\mathrm{Aut}(T_d)$. In general, branching at critical preimages imposes relations; we isolate the canonical target as follows.

Definition 4.3 (Admissible orbital symmetry) Let $\mathbf{A}_n := \mathrm{Aut}\big(X_\infty(x_n)\big)$ and let $r_n : \mathbf{A}_n \to \mathbf{A}_{n+1}$ be the continuous homomorphisms induced by the shift $s_{x_n \to x_{n+1}}$. Define the *orbital symmetry group* as the direct limit

$$\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big) := \varinjlim_n \mathbf{A}_n \quad \text{(colimit in topological groups, or its profinite completion if desired).}$$

In the regular case (no critical preimages at any level), each r_n is an isomorphism and $\mathrm{Aut}^{\mathrm{orb}}(O^+(x)) \simeq \mathrm{Aut}(T_d)$, the profinite automorphism group of the infinite rooted d-ary tree.

Thus $\rho_{O^+(x)}$ maps canonically to $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$, and in the regular case the target is the whole profinite automorphism group $\mathrm{Aut}(T_d)$.

4.2 Classical vs. orbital open image

Classically, one fixes x and asks for open image of ρ_x in $\mathrm{Aut}(X_\infty(x))$ (cf. Odoni–Jones; see [8]). The drawback is the basepoint dependence. The orbital setting removes it:

Conjecture 4.4 (Orbital Open Image, regular form) *Let K be a global field, and let $\varphi \in K(z)$ be a rational map of degree $d \geq 2$ which is* non-special *(i.e. not conjugate over $\overline{K}$ to a power map, Chebyshev, or a Lattès map). Let $O^+(x)$ be a forward orbit which is infinite and such that no x_n lies on the postcritical set. If the backward trees along $O^+(x)$ are regular (no critical collisions at any level), then the canonical orbital representation*

$$\rho_{O^+(x)} : \; G_{K(O^+(x))} \longrightarrow \mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big) \;\simeq\; \mathrm{Aut}(T_d)$$

has open image *(equivalently: finite index).*

Remark 4.5 (General form with critical branching) In the presence of critical preimages, the natural target is the admissible orbital symmetry group $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$ of Definition 4.3, which encodes the wreath recursion with the local multiplicity data. The expected statement is: *Conjecture (general).* Under the same non-special hypotheses, for any infinite forward orbit $O^+(x)$ with x_n outside the postcritical set, the image of $\rho_{O^+(x)}$ is *open* in $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$.

Proposition 4.6 (Transfer between pointwise and orbital pictures) *Fix x and write $x_n = \varphi^n(x)$. Let $\phi_n : \mathbf{A}_n \to \mathrm{Aut}^{\mathrm{orb}}(O^+(x))$ be the canonical map to the colimit, where $\mathbf{A}_n = \mathrm{Aut}(X_\infty(x_n))$. Then:*

(a) If for some n the pointwise image $\mathrm{Im}(\rho_{x_n})$ is open in $\mathbf{A}_n$ and the image $\phi_n(\mathbf{A}_n)$ is open in $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$ (e.g., in the regular case), then the orbital image $\mathrm{Im}(\rho_{O^+(x)})$ is open in $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$.

(b) Conversely, if $\mathrm{Im}(\rho_{O^+(x)})$ is open in $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$, each connecting map $r_m : \mathbf{A}_m \to \mathbf{A}_{m+1}$ has open image, and there exists n_0 such that the kernel $\ker(\phi_{n_0})$ is finite, then $\mathrm{Im}(\rho_{x_n})$ is open in $\mathbf{A}_n$ for all sufficiently large n.

In the regular case (no critical collisions), each r_n is an isomorphism and every ϕ_n is injective, so the hypothesis $\ker(\phi_{n_0})$ finite holds automatically.

Proof Write $\mathbf{A}_n := \mathrm{Aut}\big(X_\infty(x_n)\big)$ and $\Gamma_n := \mathrm{Im}(\rho_{x_n}) \le \mathbf{A}_n$. Let $\phi_n : \mathbf{A}_n \to \mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$ be the canonical continuous homomorphism into the orbital colimit (topological direct limit, or its profinite completion when taken). By construction,

$$\rho_{O^+(x)} = \phi_n \circ \rho_{x_n} \quad \text{for every } n,$$

and therefore

$$\mathrm{Im}(\rho_{O^+(x)}) = \phi_n(\Gamma_n) \subseteq \phi_n(\mathbf{A}_n) \subseteq \mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big). \tag{1}$$

(a) Assume that for some n the subgroup Γ_n is open in $\mathbf{A}_n$, and that $\phi_n(\mathbf{A}_n)$ is open in $\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$. Consider the quotient map

$$\pi : \ \mathbf{A}_n \longrightarrow \mathbf{A}_n / \ker(\phi_n),$$

which induces a topological isomorphism $\mathbf{A}_n/\ker(\phi_n) \xrightarrow{\sim} \phi_n(\mathbf{A}_n)$. The image $\pi(\Gamma_n) \cong \phi_n(\Gamma_n)$ is open in $\mathbf{A}_n/\ker(\phi_n)$, hence open in $\phi_n(\mathbf{A}_n)$. Since $\phi_n(\mathbf{A}_n)$ is open in $\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$, any open subset of $\phi_n(\mathbf{A}_n)$ is open in $\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$ as well. By (1), $\phi_n(\Gamma_n) = \mathrm{Im}(\rho_{O^+(x)})$ is therefore open in $\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$. *(b)* Conversely, assume that $\mathrm{Im}(\rho_{O^+(x)})$ is open in $\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$ and that each connecting map

$$r_m : \ \mathbf{A}_m \longrightarrow \mathbf{A}_{m+1}$$

has open image. Let $B_n := \phi_n(\mathbf{A}_n) \le \mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$. From (1) we have

$$\phi_n(\Gamma_n) = \mathrm{Im}(\rho_{O^+(x)}) \cap B_n,$$

which is open in B_n (with the subspace topology), since $\mathrm{Im}(\rho_{O^+(x)})$ is open in the ambient group $\mathrm{Aut}^{\mathrm{orb}}\big(O^+(x)\big)$. Via the quotient isomorphism $\mathbf{A}_n/\ker(\phi_n) \xrightarrow{\sim} B_n$ induced by ϕ_n, the preimage

$$\pi_n^{-1}\big(\phi_n(\Gamma_n)\big) = \Gamma_n \cdot \ker(\phi_n)$$

is an open subgroup of $\mathbf{A}_n$. Thus $\Gamma_n \cdot \ker(\phi_n)$ is open in $\mathbf{A}_n$ for every n. By hypothesis, there exists n_0 such that $\ker(\phi_{n_0})$ is finite. Applying the previous argument with $n = n_0$, we get that $\Gamma_{n_0} \cdot \ker(\phi_{n_0})$ is open in $\mathbf{A}_{n_0}$. Since $\ker(\phi_{n_0})$ is finite, the index

$$\big[\Gamma_{n_0} \cdot \ker(\phi_{n_0}) : \Gamma_{n_0}\big]$$

is finite, hence Γ_{n_0} itself is an open subgroup of $\mathbf{A}_{n_0}$. Let $r_{n_0,m} = r_{m-1} \circ \cdots \circ r_{n_0} : \mathbf{A}_{n_0} \to \mathbf{A}_m$. For $m > n_0$, the image $r_{n_0,m}(\Gamma_{n_0})$ is open in $r_{n_0,m}(\mathbf{A}_{n_0})$; and since each r_k has open image, the composite map $r_{n_0,m}$ also has open image, so $r_{n_0,m}(\mathbf{A}_{n_0})$ is an open subgroup of

$\mathbf{A}_m$. Consequently, $r_{n_0,m}(\Gamma_{n_0})$ is open in $\mathbf{A}_m$. By functoriality (the diagram defining $r_{x\to y}$ commutes with the inclusions of Galois groups), we have

$$r_{n_0,m}(\Gamma_{n_0}) \subseteq \Gamma_m.$$

Thus Γ_m contains an open subgroup of $\mathbf{A}_m$, hence Γ_m is open in $\mathbf{A}_m$. This holds for all sufficiently large m, and the proof is complete. □

Remark 4.7 (Special maps and small images) For the special families (power, Chebyshev, Lattés), the admissible orbital symmetry group is a proper closed subgroup of $\mathrm{Aut}(T_d)$ (e.g. virtually abelian or coming from endomorphisms of elliptic curves), and one cannot expect open image in $\mathrm{Aut}(T_d)$. The formulation above isolates the *right* target in all cases via $\mathrm{Aut}^{\mathrm{orb}}(O^+(x))$.

4.3 Local criteria and openness outside a finite set of places

By Theorem 3.1, for all but finitely many non-archimedean places v where φ has strict good reduction and the reduced orbit avoids the postcritical set, the inertia I_v acts trivially on each fiber along the orbit. Hence, outside a finite set S, the local images are unramified; openness thus reduces to global (Frobenius) permutation statistics in the admissible orbital symmetry group (cf. [6, 12, 13]).

Adelic outlook.

The orbital formalism suggests an adelic packaging: outside a finite set S of places of bad reduction or postcritical collision, inertia acts trivially on the fibers, so the local images are unramified and controlled by Frobenius statistics in the admissible orbital symmetry group. It is natural to expect that (under the usual non-special hypotheses) the global orbital image is open in the restricted product of local targets, with equidistribution governed by Chebotarev-type principles.

4.4 The sheaf of orbital colimits and unramifiedness

Unless otherwise stated, orbits are considered in $\mathbb{P}^1(\overline{K})$. For the local criterion, we restrict to integral representatives $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \notin \mathrm{PC}(\tilde{\varphi})$. Let $\mathcal{O}_\varphi$ be the *orbit groupoid* of φ, whose objects are forward orbits $O^+(x)$ in $\mathbb{P}^1(\overline{K})$ and whose arrows are orbit isomorphisms induced by φ (e.g. base changes $x \mapsto \varphi^n(x)$) and by transport $x \mapsto \sigma(x)$ with $\sigma \in G_K$. Define the functor

$$\mathscr{G}_\varphi : \ \mathcal{O}_\varphi^{\mathrm{op}} \longrightarrow \mathbf{Grp}, \qquad O^+(x) \longmapsto \mathcal{G}_{O^+(x)} := \varinjlim_n \mathrm{Im}(\rho_{\varphi^n(x)}),$$

where arrows act via the connecting homomorphisms $r_{x_n\to x_{n+1}}$ and via transport of structure along $\sigma \in G_K$. In other words, $\mathscr{G}_\varphi$ is the *sheaf of orbital colimits* on the orbit groupoid; its fiber over $O^+(x)$ is exactly the orbital colimit from Section 4. To obtain a continuous topological action, one may take the profinite completion fiberwise.

Definition 4.8 (Unramified at v over the open locus) Let v be a non-archimedean place of K with inertia $I_v \subset G_K$. We say that $\mathscr{G}_\varphi$ is *unramified at v over the integral open locus* if for every orbit $O^+(x)$ with a representative $x \in \mathbb{P}^1(\mathcal{O}_K)$ whose reduction $\bar{x}$ avoids the reduced postcritical set $\mathrm{PC}(\tilde{\varphi})$, the action of I_v on the fiber $\mathcal{G}_{O^+(x)}$ is trivial.

This definition does not depend on the choice of a representative of the orbit (Proposition 4.2).

Lemma 4.9 (Uniform inertia triviality) *If φ has strict good reduction, then I_v acts trivially on $\mathcal{G}_{O^+(x)}$ for* every *admissible orbit $O^+(x)$.*

Proof By Thm 3.1, I_v acts trivially on each $X_n(x)$. The shifts induce identity on images, so the colimit is trivial uniformly. □

Lemma 4.10 (Inertia on transition maps) *Let $x \in \mathbb{P}^1(\mathcal{O}_K)$ with reduction $\overline{x} \in U_k = \mathbb{P}^1_k \setminus PC(\tilde{\varphi})$, and let $x_n = \varphi^n(x)$. Let $r_{n\to n+1} : \mathrm{Im}(\rho_{x_n}) \to \mathrm{Im}(\rho_{x_{n+1}})$ be the transition homomorphism in the orbital colimit. Then $r_{n\to n+1}$ is injective when restricted to the image of the inertia group I_v.*

Proof By definition of U_k and the proof of Theorem 3.1, if $\overline{x} \in U_k$, then $\overline{x_n} \in U_k$ for all $n \geq 0$. Consequently, for any n, the entire arboreal extension $K(X_\infty(x_n))/K$ is unramified. This implies that the inertia group I_v acts trivially on the tower $X_\infty(x_n)$. Therefore, the image of I_v under ρ_{x_n} is the trivial subgroup $\{id\} \subset \mathrm{Im}(\rho_{x_n})$. This holds for both x_n and x_{n+1}. The map $r_{n\to n+1}$ thus restricts to a map from $\{id\}$ to $\{id\}$, which is trivially injective (its kernel is $\{id\}$). □

Proposition 4.11 (Good reduction as unramifiedness of the sheaf) *With the notation above, the following are equivalent:*

(a) *φ has* strict good reduction *at v.*

(b) *The sheaf $\mathscr{G}_\varphi$ is* unramified at v over the integral open locus.

In particular, under (a)/(b) and for every admissible fiber, the extensions $K(X_n(x))/K$ are unramified for all $n \geq 1$.

Proof (a) ⇒ (b). Assume φ has strict good reduction. By Lemma 4.9, I_v acts trivially on every fiber $\mathcal{G}_{O^+(x)}$ over the open locus. This is the definition of unramifiedness. (b) ⇒ (a). Assume $\mathscr{G}_\varphi$ is unramified at v over the integral open locus. By definition, for every $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \notin \mathrm{PC}(\tilde{\varphi})$, the inertia I_v acts trivially on $\mathcal{G}_{O^+(x)}$, hence in particular on the first level $X_1(x)$, which is a canonical quotient of the orbital structure. Therefore the level–1 fiber polynomial

$$F_{1,x}(T) = P_1(T) - x\,Q_1(T)$$

has unit leading coefficient and unit discriminant; in particular, $\widetilde{F_{1,\bar{x}}}$ has degree d and is separable for all such $\bar{x}$. Varying $\bar{x}$ outside the finite set $\mathrm{PC}(\tilde{\varphi})$ shows there is a nonempty Zariski open $U_k \subset \mathbb{P}^1_k$ on which every reduced level–1 fiber has degree d and is separable. We claim this forces $\deg\tilde{\varphi} = d$ for some integral model, and hence strict good reduction.

Indeed, if no integral model had $\deg\tilde{\varphi} = d$, then for every integral presentation (F, G) the reductions $(\tilde{F}, \tilde{G})$ would share a nontrivial common factor, so the induced map on $\mathbb{P}^1_k$ would fail to have degree d. Equivalently, away from a finite subset of $\mathbb{P}^1_k$, the reduced fibers would have degree $< d$ or fail to be étale. This contradicts the existence of a nonempty open U_k where all reduced fibers have degree d and are separable. Therefore, there exists an integral model with $\deg\tilde{\varphi} = d$. For such a model, $\mathrm{Res}(F, G) \in \mathcal{O}_K^\times$ (see e.g. [12, Prop. 2.15–2.17]), which is the definition of strict good reduction. □

Remark 4.12 (Étalé space and the G_K-action) The disjoint union $\coprod_O \mathcal{G}_O$ is canonically viewed as the *étalé space* of the functor $\mathscr{G}_\varphi$ over the groupoid $\mathcal{O}_\varphi$, and G_K acts by permuting the fibers and transporting structure among them. The "unramifiedness" in Proposition 4.11 is equivalent to the triviality of I_v on this total space over the integral open locus.

5 Proofs of Theorem 3.1

Proof of (1) ⇒ (2) Assume φ has strict good reduction (1). Choose an integral model (F, G) such that $\mathrm{Res}(F, G) \in \mathcal{O}^\times$. The reduced postcritical set $\mathrm{PC}(\tilde{\varphi})$ is finite. Let $U_k = \mathbb{P}^1_k \setminus \mathrm{PC}(\tilde{\varphi})$, a nonempty Zariski open set. Let $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \in U_k$. By definition of $\mathrm{PC}(\tilde{\varphi})$, $\bar{x}$ is not postcritical. Hence (by Prop. 2.5) the reduced fiber $\tilde{F}_{1,\bar{x}}(T) = \tilde{P}_1(T) - \bar{x}\,\tilde{Q}_1(T)$ has degree d and is étale. Therefore the leading coefficient of $F_{1,x}$ is a unit and $\mathrm{Disc}(F_{1,x}) \in \mathcal{O}^\times$, establishing (2). For the final claim, note that by Lemma 2.1, $\mathrm{Res}(P_n, Q_n) \in \mathcal{O}^\times$ for all n. Since $\bar{x} \notin \mathrm{PC}(\tilde{\varphi})$, by Proposition 2.5, each reduced fiber $\tilde{F}_{n,\bar{x}}$ is also étale for all $n \geq 1$. Thus $\mathrm{Disc}(F_{n,x}) \in \mathcal{O}^\times$ for all n, and by Lemma 2.2 we get that $K(X_n(x))/K$ is unramified. □

Proof of (2) ⇒ (1) Assume condition (2) holds: there exists a nonempty open $U_k \subseteq \mathbb{P}^1_k$ such that for every $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \in U_k$, the reduced level-1 fiber $\tilde{F}_{1,\bar{x}}(T) = \tilde{P}_1(T) - \bar{x}\tilde{Q}_1(T)$ has degree d and is separable. Equivalently, the leading coefficient of $F_{1,x}$ is a unit and $\mathrm{Disc}(F_{1,x}) \in \mathcal{O}_K^\times$.

By Proposition 2.5, the condition that the reduced level-1 fiber is separable of degree d on U_k implies $U_k \cap \mathrm{PC}(\tilde{\varphi}) = \varnothing$. Since $\mathrm{PC}(\tilde{\varphi})$ is finite (Silverman [12, Thm. 5.4]), U_k is a nonempty Zariski open subset of $\mathbb{P}^1_k$ on which the reduction $\tilde{\varphi}$ has degree d and is étale at level 1.

We claim this forces strict good reduction. Indeed, if no integral model had $\deg\tilde{\varphi} = d$, then for every integral presentation (F, G) the reductions $(\tilde{F}, \tilde{G})$ would share a nontrivial common factor, so the induced map on $\mathbb{P}^1_k$ would fail to have degree d. Equivalently, away from a finite subset of $\mathbb{P}^1_k$, the reduced fibers would have degree $< d$ or fail to be separable. This contradicts the existence of a nonempty open U_k where all reduced level-1 fibers have degree d and are separable. Therefore, there exists an integral model with $\deg\tilde{\varphi} = d$. For such a model, $\mathrm{Res}(F, G) \in \mathcal{O}_K^\times$ (see, e.g., [12, Prop. 2.15–2.17]), which is the definition of strict good reduction.

Finally, having established strict good reduction, Lemma 2.1 implies that the resultant of φ^n is a unit for all n. Consequently, $\mathrm{Disc}(F_{n,x}) \in \mathcal{O}_K^\times$ for all x in the integral open locus, and by Lemma 2.2, the extensions $K(X_n(x))/K$ are unramified for all $n \geq 1$. □

6 Examples over $\mathbb{Q}_p$

Example 6.1 (Good reduction: $\varphi(z) = z^2 - 1$ for $p \neq 2$) Consider $F(X,Y) = X^2 - Y^2$ and $G(X,Y) = Y^2$, so that φ is given by $z \mapsto z^2 - 1$. The resultant $\mathrm{Res}(F,G)$ equals $1 \in \mathbb{Z}_p^\times$, confirming that φ has strict good reduction. The reduction is $\tilde{\varphi}(z) = z^2 - 1$ of degree 2 over $\mathbb{F}_p$. The critical points of $\tilde{\varphi}$ are $\mathrm{Crit}(\tilde{\varphi}) = \{0, \infty\}$, and the reduced postcritical set is $\mathrm{PC}(\tilde{\varphi}) = \{-1, 0, \infty\}$. For any $x \in \mathbb{Z}_p$ such that $\bar{x} \notin \{-1, 0, \infty\}$, the reduced level-1 fiber is $\tilde{F}_{1,\bar{x}}(T) = T^2 - (\bar{x} + 1)$, which has degree 2 and is separable over $\mathbb{F}_p$ whenever $p \neq 2$. Explicitly, the fiber discriminant is $\mathrm{Disc}(F_{1,x}) = -4(x+1)$, which is a p-adic unit provided $v_p(x+1) = 0$. Therefore, $K(X_1(x))/\mathbb{Q}_p$ is unramified. By Lemma 2.1, this property persists at every level: each iterate satisfies $\mathrm{Disc}(F_{n,x}) \in \mathbb{Z}_p^\times$, ensuring that $K(X_n(x))/\mathbb{Q}_p$ remains unramified for all $n \geq 1$. This verifies, through explicit computation, that the inertia group acts trivially on each preimage tower, consistent with the local criterion for good reduction.

Example 6.2 (Good reduction: $\varphi(z) = z^2 + p$) Here $F(X,Y) = X^2 + pY^2$, $G(X,Y) = Y^2$. Then $\mathrm{Res}(F,G) = F(1,0)^2 = 1 \in \mathbb{Z}_p^\times$; thus φ has strict good reduction and $\tilde{\varphi}(z) = z^2$. We have $\mathrm{PC}(\tilde{\varphi}) = \{0, \infty\}$. If $\bar{x} \notin \{0, \infty\}$, the reduced fiber $\tilde{F}_{1,\bar{x}}(T) = T^2 - \bar{x}$ has degree 2 and is separable, so $\mathrm{Disc}(F_{1,x}) = -4(p - x) \in \mathbb{Z}_p^\times$. This aligns with Theorem 3.1.

Remark 6.3 (Why the naïve NOS fails) Even when φ has strict good reduction, unramified towers need not occur for *all* base points $x \in \mathbb{P}^1(\overline{K})$. The map $\varphi(z) = z^2 + p$ over $\mathbb{Q}_p$ already shows this. In an integral model one has $\tilde{\varphi}(z) = z^2$ and hence $\mathrm{PC}(\tilde{\varphi}) = \{0, \infty\}$. If one quantifies over all x, ramification can appear for reasons unrelated to the dynamics:

- If $v_p(x) < 0$ (so $x \notin \mathcal{O}_K$), then $\bar{x} = \infty \in \mathrm{PC}(\tilde{\varphi})$ and
$$\mathrm{Disc}\left(F_{1,x}\right) = -4\big(p - x\big)$$
has negative p-adic valuation, hence the level–1 extension is ramified.
- If $x \in \mathcal{O}_K$ but $\bar{x} = 0 \in \mathrm{PC}(\tilde{\varphi})$, then $\widetilde{F_{1,x}}(T) = T^2$ is inseparable and $\mathrm{Disc}(F_{1,x}) \notin \mathcal{O}_K^\times$, so ramification occurs again.

Thus, it is essential in Theorem 3.1 to restrict to the integral locus with $\bar{x} \notin \mathrm{PC}(\tilde{\varphi})$, where the reduced fibers are degree-d and étale—equivalently, the fiber discriminants are units and the towers are unramified.

Example 6.4 (Non-polynomial with strict good reduction via conjugation) Fix a prime p and consider $\varphi(z) = z^2 + p$ over $\mathbb{Q}_p$, which has strict good reduction (Example 6.2). Conjugate by the inversion $M(z) = 1/z \in \mathrm{PGL}_2(\mathcal{O}_{\mathbb{Q}_p})$ to obtain

$$\psi = \varphi^M = M \circ \varphi \circ M^{-1}(z) = \frac{z^2}{1 + pz^2}.$$

In homogeneous coordinates, $\psi([X:Y]) = [X^2 : Y^2 + pX^2]$ has degree 2 and $\mathrm{Res}(X^2, Y^2 + pX^2) = 1 \in \mathbb{Z}_p^\times$. Hence ψ has strict good reduction and $\tilde{\psi}(z) = z^2$. Therefore $\mathrm{PC}(\tilde{\psi}) = \{0, \infty\}$, and for every $x \in \mathbb{P}^1(\mathcal{O}_{\mathbb{Q}_p})$ with $\bar{x} \notin \{0, \infty\}$, the reduced fibers of all iterates are étale of degree 2; equivalently, the fiber polynomials have unit leading coefficient and unit discriminant, so $K(X_n(x))/\mathbb{Q}_p$ is unramified for all $n \geq 1$. This illustrates that the criterion applies uniformly to non-polynomial rational maps.

Example 6.5 (Bad reduction: $\varphi(z) = pz^2 + z$) We have $F(X,Y) = pX^2 + XY$, $G(X,Y) = Y^2$ and $\tilde{\varphi}(z) = z$, so $\deg \tilde{\varphi} = 1 < 2$ and φ has bad reduction. Moreover $\mathrm{Crit}(\tilde{\varphi}) = \varnothing$, hence $\mathrm{PC}(\tilde{\varphi}) = \varnothing$. For any $\bar{x} \in \mathbb{P}^1_k$ we get $\tilde{F}_{1,\bar{x}}(T) = T - \bar{x}$, which has degree 1 (not 2), so condition (2) *fails* (there is no open U_k where the reduced fibers all have degree 2). This matches Theorem 3.1.

7 The degree-one case (for completeness)

Our analysis focuses on $d \geq 2$, where arboreal representations exhibit nontrivial branching and Galois phenomena. For completeness, we briefly address the $d = 1$ case. For $\deg \varphi = 1$, every map is a Möbius transformation

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det M \neq 0.$$

There are no critical points, hence $\mathrm{PC}(\tilde{\varphi}) = \varnothing$. The preimage towers are linear: for any $x \in \mathbb{P}^1(\overline{K})$ and any $n \geq 1$, the level-n fiber is a single point defined over K, and if $x \in \mathbb{P}^1(\mathcal{O}_K)$ then $K(X_n(x)) = K$ for all n.

Proposition 7.1 (Degree one) *Let K be a non-archimedean local field and $\varphi \in K(z)$ have degree 1 with matrix representative M. The following are equivalent:*

(a) *φ has strict good reduction.*

(b) $\det M \in \mathcal{O}_K^\times$ *(equivalently,* $\mathrm{Res}(F,G) = \det M$ *is a unit).*

(c) *There exists a nonempty open $U_k \subset \mathbb{P}^1_k$ such that, for every $x \in \mathbb{P}^1(\mathcal{O}_K)$ with $\bar{x} \in U_k$, the reduced level–1 fiber has degree 1.*

Moreover, for every $x \in \mathbb{P}^1(\mathcal{O}_K)$ and every $n \geq 1$, one has $K(X_n(x)) = K$ and I_v acts trivially on $X_\infty(x)$.

Proof Since F, G are linear forms, $\mathrm{Res}(F,G) = ad - bc = \det M$. Thus (a)$\Leftrightarrow$(b). If $\det M \in \mathcal{O}_K^\times$, then $\tilde{M} \in \mathrm{PGL}_2(k)$ is invertible and the reduced map has degree 1, whence (c). Conversely, if (c) holds, the reduced fiber has degree 1 on a Zariski open set, forcing $\deg \tilde{\varphi} = 1$ and hence $\det M \in \mathcal{O}_K^\times$. The statement on towers follows from linearity. $\square$

Corollary 7.2 (Orbital open image is vacuous for $d = 1$) *For $\deg \varphi = 1$ and any forward orbit $O^+(x)$, each $X_\infty(x_n)$ is a 1-ary rooted tree; its root-preserving automorphism group is trivial. Hence, the admissible orbital symmetry group*

$$\mathrm{Aut}^{\mathrm{orb}}(O^+(x)) = \varinjlim_n \mathrm{Aut}\big(X_\infty(x_n)\big)$$

is trivial, and the canonical orbital representation $\rho_{O^+(x)}$ has image of index 1.

8 Remarks on Context and the ℓ-adic module

Several aspects of this framework facilitate subsequent applications and enable meaningful comparisons in both local and global contexts.

- Orbit-level packaging. The orbital colimit $\mathcal{G}_{O^+(x)}$ consolidates pointwise arboreal data along a forward orbit into a canonical object. This construction reduces dependence on the basepoint and interacts effectively with iteration and conjugation (see Section 4), which is advantageous for comparing fibers across levels and for formulating global questions.
- Local criterion on a natural open set. Formulating the local Galois criterion on the residual open locus $\mathbb{P}^1_k \setminus \mathrm{PC}(\tilde{\varphi})$ clarifies the role of the reduced postcritical set. On this locus, the reduced fibers have degree d and are étale, so the discriminants are units and the associated towers are unramified (see Theorem 3.1 and the examples). This approach isolates the genuinely dynamical contribution to ramification.
- Compatibility with global considerations. The orbit-level perspective aligns with open-image questions for arboreal representations by providing a canonical target in the regular case or an admissible symmetry group when branching occurs, which is beneficial when relating local and global statements.

8.1 The arboreal Tate module

The "crystalline" language must be refined.

- Case $\ell \neq p$: The module $T_\ell(x) = \varprojlim_n \mathbb{Z}_\ell[X_n(x)]$ is an ℓ-adic Galois representation. Theorem 3.1(2) implies $T_\ell(x)$ is unramified (as a G_K-representation) for all $x \in \mathbb{P}^1(\mathcal{O}_K)$ outside the reduced postcritical set.
- Case $\ell = p$: This is a p-adic representation. Calling it *crystalline* requires constructing a compatible Frobenius structure on the tower. If such a structure exists and the representation is crystalline, then standard p-adic Hodge theory implies potential good reduction. For degree bounds on *attaining* potentially good reduction after a finite extension (via conjugation), compare [5]. Related p-adic NOS-type criteria are known for curves via the p-adic unipotent fundamental group [2] and for $K3$ surfaces via monodromy/crystalline methods [7, 9, 10]. This lies beyond our scope; we only use ramification statements here.

Acknowledgments

The author was partially supported by the Secihti (Ciencia de Frontera) grant CF 2019/217367.

Declarations

- Funding: This work was partially supported by Secihti (Ciencia de Frontera) grant CF 2019/217367.
- Conflict of interest/Competing interests: The author declares no competing interests.
- Ethics approval and consent to participate: Not applicable.
- Consent for publication: Not applicable.
- Data availability: Not applicable.

- Materials availability: Not applicable.
- Code availability: Not applicable.
- Author contribution: J.R. Pérez-Buendía is the sole author and is responsible for all aspects of this work.

References

[1] Adams O (2022) A Dynamical Analogue of the Criterion of Néron-Ogg-Shafarevich. arXiv:2208.00359. https://arxiv.org/abs/2208.00359

[2] Andreatta F, Iovita A, Kim M (2015) A p-adic nonabelian criterion for good reduction of curves. Duke Math. J. 164(13):2597–2642. doi:10.1215/00127094-3146817

[3] R. L. Benedetto, *Dynamics in One Non-Archimedean Variable*, GSM 198, AMS, 2019. doi:10.1090/gsm/198.

[4] Benedetto RL (2014) A criterion for potentially good reduction in nonarchimedean dynamics. Acta Arith. 165(3):251–256. doi:10.4064/aa165-3-4

[5] Benedetto RL (2015) Attaining Potentially Good Reduction in Arithmetic Dynamics. Int. Math. Res. Not. IMRN 2015(22):11828–11846. doi:10.1093/imrn/rnv358

[6] Boston N, Jones R (2007) Arboreal Galois representations. Geom. Dedicata 124:27–35. doi:10.1007/s10711-006-9113-9

[7] Hernandez-Mada G (2021) A monodromy criterion for the good reduction of $K3$ surfaces. Rend. Semin. Mat. Univ. Padova 145:73–92. doi:10.4171/rsmup/50

[8] Jones R (2013) Galois representations from preimage trees: an arboreal survey. Publ. Math. Besançon Algèbre Théorie Nr. 2013:107–136. doi:10.5802/pmb.a-154

[9] Liedtke C, Matsumoto Y (2018) Good reduction of K3 surfaces. Compos. Math. 154(1):1–35. doi:10.1112/S0010437X17007400

[10] Pérez-Buendía JR (2019) A Kulikov-type classification theorem for a one parameter family of K3-surfaces over a p-adic field and a good reduction criterion. Ann. Math. Qué. 43(2):411–434. doi:10.1007/s40316-018-0110-9

[11] Serre J-P (1979) Local Fields. Graduate Texts in Mathematics, vol 67. Springer, New York, NY. ISBN: 0-387-90424-7; 978-0-387-90424-5. doi:10.1007/978-1-4757-5673-9

[12] Silverman JH (2007) The Arithmetic of Dynamical Systems. Graduate Texts in Mathematics, vol 241. Springer, New York, NY. ISBN: 978-0-387-69903-5 (hardcover); 978-1-4419-2417-9 (softcover). doi:10.1007/978-0-387-69904-2

[13] Stoll M (1992) Galois groups over $\mathbb{Q}$ of some iterated polynomials. Arch. Math. (Basel) 59(3):239–244. doi:10.1007/BF01197321