NEW CONSEQUENCES OF PFA(T^*)

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Abstract. Let T^* be an almost Suslin tree, that is, an Aronszajn tree with no stationary antichains. Krueger [Kru20b] introduced a forcing axiom, PFA(T^*), for the class of proper forcings that preserve that T^* is almost Suslin. He showed that PFA(T^*) implies several well-known consequences of the Proper Forcing Axiom (PFA), including Suslin's Hypothesis and the P-ideal dichotomy. We extend this list by proving that PFA(T^*) also implies the Mapping Reflection Principle (MRP) and the Open Graph Axiom (OGA). Additionally, we show that PFA(T^*) implies that all special Aronszajn trees are club-isomorphic, but it does not imply that all almost Suslin trees are club-isomorphic.

An Aronszajn tree is called *almost Suslin* if it has no stationary antichains. For such a tree T^*, Krueger introduced in [Kru20b] the forcing axiom PFA(T^*), a relativized version of PFA which applies to all proper forcings preserving that T^* is almost Suslin.

Krueger showed that PFA(T^*) already implies many of the well-known consequences of the proper forcing axiom, including the Suslin's Hypothesis, the P-ideal dichotomy, and the failure of a very weak form of club guessing. He raised the question of whether further familiar consequences of PFA also follow from PFA(T^*), specifically: the mapping reflection principle (MRP), the open graph axiom (OGA) and the non-existence of weak Kurepa trees.

A second question posed in [Kru20b], attributed to Moore, asks whether PFA(T^*) implies that all normal almost Suslin trees are club isomorphic. This is a natural analogue of the fact that PFA implies all normal Aronszajn trees are club-isomorphic; a positive answer would suggest that models of PFA(T^*) are somewhat canonical.

In this work, we address these questions. We show that PFA(T^*) imply both MRP and OGA, by constructing suitable modifications of the standard proper forcings for these principles. In contrast, we answer Moore's question negatively: starting from a model of $\diamondsuit$, we construct two almost Suslin trees T and S that are not club-isomorphic and show they remain so after forcing to obtain a model of PFA($T + S$). As a positive counterpart, we prove that PFA(T^*) does imply that any two normal special Aronszajn trees are club-isomorphic, thus establishing a partial analogue of the situation under PFA.

We note that subsequent work by Lambie-Hanson and Stejskalová [LHS25] has established that PFA(T^*) implies the non-existence of weak Kurepa trees. Furthermore, [LHS25, Fact 6.19(4)] –a fact originally remarked in [Kru20b]– can be

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used to show that PFA(T^*) implies the non-existence of ω_2-Aronszajn trees, via the standard argument from PFA.

The structure of our paper is as follows. After establishing the necessary preliminaries in Section 1, we proceed to demonstrate the main combinatorial consequences of PFA(T^*). Specifically, Section 2 is dedicated to proving that the axiom implies the Mapping Reflection Principle (MRP), and Section 3 proves that it implies the Open Graph Axiom (OGA). We then turn to structural consequences for trees in Section 4, where we resolve the question of whether PFA(T^*) implies that all almost Suslin trees are club-isomorphic.

1. Preliminaries

A set-theoretic tree is a partial order set $(T, <_T)$ such that for any node $t \in T$, the set of predecessors $\{s \in T : s <_T t\}$ is well-ordered by $<_T$. The *height* of t, written $\mathrm{ht}_T(t)$, is the unique ordinal representing the order-type of the set of its predecessors. For an ordinal α, the *α-th* level of T, written T_α, consists of the elements of T of height α. The *height of* T is the least ordinal α for which T_α is empty.

A *branch* of T is any downwards closed chain of T, and a branch is called *cofinal* if its height is equal to the height of T. For each $t \in T$ and $\xi \leq \mathrm{ht}(t)$, we let $t \restriction \xi$ to be the unique $s \in T$ of height ξ such that $s \leq_T t$. If the tree is clear from the context, then $t \perp s$ denotes the fact that t and s are incomparable in the poset $(T, \leq_T)$. The tree T is said to be *well-pruned* if for all $t \in T$, the subtree $\{s \in T : s \not\perp t\}$ has the same height as T.

For incomparable $t, s \in T$, we let $\Delta(t, s)$ to be the least ordinal α such that $t \restriction \alpha \neq s \restriction \alpha$. We leave Δ undefined if t and s are comparable. A tree is called *Hausdorff* if $\Delta(t, s)$ is always a successor ordinal.

An ω_1-tree is a tree of height ω_1 with all its levels countable. An ω_1-tree is called an *Aronszajn tree* if it has no uncountable chains, equivalently, no cofinal branches. An ω_1-tree is called a *Suslin tree* if it has no uncountable chains and no uncountable antichains. A typical argument shows that if T is a well-pruned ω_1-tree, then it is Suslin iff it has no uncountable antichains. A tree is called *normal* if it is Hausdorff, well-pruned, and every node has $\aleph_0$ immediate successors.

Definition 1.1. Let T be a tree with height ω_1. A subset $A \subseteq T$ is called a *stationary antichain* if A is an antichain and the set $\{\mathrm{ht}_T(t) : t \in A\}$ is a stationary subset of ω_1.

Definition 1.2. An *almost Suslin tree* is any Aronszajn tree without stationary antichains.

Fix an Aronszajn tree T. The remaining definitions and lemmas of this section are all from [Kru20b]. We define a relation $<_T^-$ by letting $x <_T^- y$ if $\mathrm{ht}_T(x) < \mathrm{ht}_T(y)$ and $x \not<_T y$.

Definition 1.3. Let N be a countable set such that $\delta := N \cap \omega_1$ is an ordinal. A node $x \in T_\delta$ is said to be *(N, T)-generic* if for any set $A \subseteq T$ which is a member of N and any relation $R \in \{<_T, <_T^-\}$, if $x \in A$, then there exists yRx such that $y \in A$.

Lemma 1.4. ([Kru20b, Lemma 1.4]) *Let θ be a regular cardinal with $T \in H(\theta)$, and N a countable elementary submodel of $H(\theta)$ containing T as an element. Let*

$\delta := N \cap \omega_1$. *Assume that* $x \in T_\delta$ *is* (N,T)*-generic. Let* $B \subseteq T$ *be in* N *and* $R \in \{<_T, <_T^-\}$. *If for all* yRx, $y \in B$, *then* $x \in B$.

As mentioned in the introduction, we will be mainly interested in the case that T is almost Suslin, one of the reasons being the following.

Lemma 1.5. ([Kru20b, Lemma 1.5]) *Let* θ *be a regular cardinal such that* $T \in H(\theta)$. *Let* N *be a countable elementary submodel of* $H(\theta)$ *containing* T *as an element. If* T *is almost Suslin, then for all* $x \in T_{N\cap\omega_1}$, x *is* (N,T)*-generic.*

Definition 1.6. Let T be an ω_1-tree and $\mathbb{P}$ a forcing poset. Let θ be a regular cardinal such that T and $\mathbb{P}$ are members of $H(\theta)$ and N a countable elementary submodel of $H(\theta)$ containing T and $\mathbb{P}$ as elements. A condition $q \in \mathbb{P}$ is said to be $(N,\mathbb{P},T)$*-generic* if q is $(N,\mathbb{P})$-generic, and whenever $x \in T_{N\cap\omega_1}$ is (N,T)-generic, then q forces that x is $(N[\dot{G}_{\mathbb{P}}],T)$-generic.

Assume that q is $(N,\mathbb{P})$-generic. Then the following property is easily seen to be equivalent to q being $(N,\mathbb{P},T)$-generic: for any node $x \in T_{N\cap\omega_1}$ which is (N,T)-generic, a $\mathbb{P}$-name $\dot{A} \in N$ for a subset of T, and relation $R \in \{<_T, <_T^-\}$, if $r \leq q$ forces that $x \in \dot{A}$, then there exists yRx and $s \leq r$ such that s forces that $y \in \dot{A}$.

Definition 1.7. Let T be an ω_1-tree and $\mathbb{P}$ a forcing poset. We say that $\mathbb{P}$ is T*-proper* if for all large enough regular cardinals θ with $\mathbb{P}$ and T in $H(\theta)$, there are club many N in $[H(\theta)]^\omega$ such that N is an elementary submodel of $H(\theta)$ containing T and $\mathbb{P}$ as elements satisfying that for all $p \in N \cap \mathbb{P}$, there is $q \leq p$ which is $(N,\mathbb{P},T)$-generic.

Definition 1.8. Define $\mathrm{PFA}(T)$ to be the statement that for any T-proper forcing poset $\mathbb{P}$, whenever $\langle D_\alpha : \alpha < \omega_1\rangle$ is a sequence of dense subsets of $\mathbb{P}$, then there exists a filter G on $\mathbb{P}$ such that for all $\alpha < \omega_1, G \cap D_\alpha \neq \emptyset$.

Theorem 1.9. *If* T^* *is an almost Suslin tree, and there is a supercompact cardinal, then there is a forcing poset which forces* $\mathrm{PFA}(T^*)$ *and* $\kappa = \omega_2$.

For the rest of this paper T^* will always denote an almost Suslin tree.

2. Mapping reflection principle

In this section we shall prove that $\mathrm{PFA}(T^*)$ implies Moore's Mapping Reflection Principle. We recall the relevant definitions.

Let X be an uncountable set. We equip $[X]^\omega$ with the Ellentuck topology obtained by declaring the sets $[a,A] := \{Y \in [X]^\omega : a \subseteq Y \subseteq A\}$ to be open for all $a \in [X]^{<\omega}$ and $A \in [X]^\omega$. Let θ be a regular cardinal such that $[X]^\omega \in H(\theta)$ and let M be a countable elementary submodel of $H(\theta)$ containing $[X]^\omega$ as an element. We say that a subset $\Sigma \subseteq [X]^\omega$ is M*-stationary* if $\Sigma \cap E \cap M \neq \emptyset$ for any club $E \subseteq [X]^\omega$ in M.

Definition 2.1. An *open stationary* set mapping Σ is a map whose domain is a club subset of countable elementary submodels of $H(\theta)$ containing $[X]^\omega$ as an element and such that $\Sigma(M) \subseteq [X]^\omega$ is open (in the Ellentuck topology) and M-stationary for all M in its domain.

Definition 2.2. An open stationary set mapping Σ *reflects* if there exists a continuous $\in$-increasing sequence $\langle N_\xi : \xi < \omega_1\rangle$ such that, for all $0 < \nu < \omega_1$, $N_\nu \in \mathrm{dom}(\Sigma)$

there exists $\nu_0 < \nu$ such that $N_\xi \cap X \in \Sigma(N_\nu)$ for all $\nu_0 < \xi < \nu$ (observe that the last condition is only meaningful for limit ordinals). We say that $\langle N_\xi : \xi < \omega_1 \rangle$ is a reflecting sequence for Σ.

Definition 2.3. The Set Mapping Reflection Principle MRP is the assertion that all open stationary set mappings reflect.

We shall prove that there is T^*-proper forcing which adds a reflecting sequence. There are several proper posets in the literature for adding such a sequence (see [Moo05], [Moh21]). However, we were unable to prove that any of them are T^*-proper. Thus, we will prove that a variant of Moore's forcing is as required.

Definition 2.4. Let Σ be an open stationary set mapping (with X, θ as before). Define $\mathbb{P}_\Sigma$ to be the forcing poset whose conditions are all pairs $p = (f_p, F_p)$ where $f_p : \alpha_p + 1 \to \mathrm{dom}(\Sigma)$ is a continuous $\in$-increasing map, $\alpha_p \in \omega_1$ and F_p is a countable subset of X such that for all $0 < \nu \leq \alpha_p$, there exists $\nu_0 < \nu$ such that $f_p(\xi) \cap X \in \Sigma(f_p(\nu))$ for all $\nu_0 < \xi < \nu$, ordered by $(f_q, F_q) \leq (f_p, F_p)$ if f_q is an extension of f_p, $F_p \subseteq F_q$ and $F_p \in f_q(\xi)$ for all $\alpha_p < \xi \leq \alpha_q$.

The next two lemmas will help us prove that $\mathbb{P}_\Sigma$ is T^*-proper.

Lemma 2.5. *Let θ^* be a regular cardinal such that $\mathbb{P}_\Sigma, H(|\mathbb{P}_\Sigma|^+), \Sigma$ and T^* are members of $H(\theta^*)$, and let M be a countable elementary submodel of $H(\theta^*)$ containing $\mathbb{P}_\Sigma, H(|\mathbb{P}_\Sigma|^+), \Sigma$ and T^* as elements. Let $p \in M \cap \mathbb{P}_\Sigma, \dot{A} \in M$ be a $\mathbb{P}_\Sigma$-name for a subset of T^* and $y \in M \cap T^*$. Then either:*

(i) *there exists $q \leq p$ in M such that $f_q(\xi) \cap X \in \Sigma(M \cap H(\theta))$ for $\alpha_p < \xi \leq \alpha_q$ and $q \Vdash y \in \dot{A}$, or*
(ii) *there exists $F \in [X]^\omega$ such that $(f_p, F_p \cup F) \Vdash y \notin \dot{A}$.*

Proof. Suppose (i) fails. Let E denote the collection of all sets of the form $N = N^* \cap X$ where N^* is a countable elementary submodel of $H(|\mathbb{P}_\Sigma|^+)$ containing $H(\theta), p, \dot{A}, y, T^*$ and $\mathbb{P}_\Sigma$ as elements. It follows that $E \subseteq [X]^\omega$ is a club in $M \cap H(\theta)$. Since $\Sigma(M \cap H(\theta))$ is open and $M \cap H(\theta)$-stationary there is an $N \in E \cap \Sigma(M \cap H(\theta)) \cap M$ and there is an $F \in [N]^{<\omega}$ such that $[F, N] \subseteq \Sigma(M \cap H(\theta))$. We claim that $(f_p, F_p \cup F) \Vdash y \notin \dot{A}$. To prove the claim. Suppose for a contradiction that this is not the case. Then there is a $r \leq (f_p, F_p \cup F)$ such that $r \Vdash y \in \dot{A}$. Hence, by elementarity, there is a $q \in N^* \cap \mathbb{P}_\Sigma$ such that $q \Vdash y \in \dot{A}$. By the definition of the order, it follows that $f_q(\xi) \in [F, N] \subseteq \Sigma(M \cap H(\theta))$ for all $\alpha_p < \xi \leq \alpha_q$. Thus, q is a witness that (i) holds which is a contradiction.

□

Lemma 2.6. *Let θ^* be a regular cardinal such that $\mathbb{P}_\Sigma, H(|\mathbb{P}_\Sigma|^+), \Sigma$ and T^* are members of $H(\theta^*)$, and let M be a countable elementary submodel of $H(\theta^*)$ containing $\mathbb{P}_\Sigma, H(|\mathbb{P}_\Sigma|^+), \Sigma$ and T^* as elements. Let $\delta = M \cap \omega_1$. Suppose that $x \in T^*_\delta$, $p \in M \cap \mathbb{P}_\Sigma, \dot{A} \in M$ is a $\mathbb{P}_\Sigma$-name for a subset of T^* and $R \in \{<_{T^*}, <^-_{T^*}\}$. Then either:*

(i) *there exists yRx and $q \leq p$ in M such that $f_q(\xi) \cap X \in \Sigma(M \cap H(\theta))$ for $\alpha_p < \xi \leq \alpha_q$ and $q \Vdash y \in \dot{A}$, or*
(ii) *there exists $F \in [X]^\omega$ such that $(f_p, F_p \cup F) \Vdash x \notin \dot{A}$.*

Proof. If (i) fails, then it follows, from Lemma 2.5, that for all yRx there exists $F_y \in [X]^\omega$ such that $(f_p, F_p \cup F_y) \Vdash y \notin \dot{A}$. Let B be the set of $z \in T^*$ for which there exists a countable set $F \in [X]^\omega$ such that $(f_p, F_p \cup F) \Vdash z \notin \dot{A}$. Notice that $B \in M$ since it is definable from parameters in M. By assumption for all $yRx, y \in B$. Applying 1.4, it follows that $x \in B$. So there exists a countable set $F \in [X]^\omega$ such that $(f_p, F_p \cup F) \Vdash x \notin \dot{A}$. □

We are ready to prove the main Theorem of the section.

Theorem 2.7. PFA(T^*) *implies* MRP.

Proof. We shall prove that $\mathbb{P}_\Sigma$ is T^*-proper and adds a reflecting sequence. To do this, fix a large enough regular cardinal θ^* and a countable elementary substructure M of $H(\theta^*)$ such that $\mathbb{P}_\Sigma, \Sigma, T^*$ and $H(|\mathbb{P}_\Sigma|^+)$ are all in M. Consider $p \in \mathbb{P}_\Sigma \cap M$. Let $\delta = M \cap \omega_1$. Let $\langle (D_n, x_n, \dot{A}_n, i_n) : n < \omega \rangle$ be an enumeration of all 4-tuples $(D, x, \dot{A}, i)$ where D is a dense subset of $\mathbb{P}_\Sigma$ in M, $x \in T^*_\delta$, $\dot{A}$ is a $\mathbb{P}_\Sigma$-name for a subset of T^* and $i \in 3$. We will construct recursively a partial function $G : \omega \to ([X]^\omega \cup T^* \restriction \delta)$ and a sequence of conditions $\langle p_n : n < \omega \rangle$ such that:

(i) $p_{n+1} \leq p_n$;
(ii) $p_{n+1} \in D_n$ whenever $i_n = 0$;
(iii) $p_n \in \mathbb{P}_\Sigma \cap M$;
(iv) $\forall \alpha_{p_n} < \xi \leq \alpha_{p_{n+1}}$, $f_{p_{n+1}}(\xi) \cap X \in \Sigma(H(\theta) \cap M)$;
(v) G satisfies the following:
 (a) If $G(n) \in T^* \restriction \delta$, then $p_{n+1} \Vdash G(n) \in \dot{A}_n$, otherwise
 (b)$(f_{p_n}, F_{p_n} \cup G(n)) \Vdash x_n \notin \dot{A}_n$.

Let $p_0 := p$. Suppose that p_n and G (up to n) have been defined. We proceed by cases. First suppose $i_n = 0$. In this case $n \notin \mathrm{dom}(G)$. Let E_n be the set of all elements of the form $N = N^* \cap X$ where N^* is a countable elementary submodel of $H(|\mathbb{P}_\Sigma|^+)$ which contains $H(\theta), \mathbb{P}_\Sigma, D_n, \dot{A}_n$, T^* and p_n as elements. Notice that $E_n \subseteq [X]^\omega$ is a club in $M \cap H(\theta)$. Using that $\Sigma(M \cap H(\theta))$ is open and $(M \cap H(\theta))$-stationary, we can find $N_n \in \Sigma(M \cap H(\theta)) \cap E_n \cap M$, and a finite set $F_n \subseteq N_n$ such that $[F_n, N_n] \subseteq \Sigma(M \cap H(\theta))$. Let N^*_n be a countable elementary submodel of $H(|\mathbb{P}_\Sigma|^+)$ witnessing that $N_n \in E_n$. Let p_{n+1} be any extension of $(f_{p_n}, F_{p_n} \cup F_n)$ in $D_n \cap M$ (such an extension exists by elementarity). Next suppose, $i_n \neq 0$. If $i_n = 1$, then set $R = <_{T^*}$ and otherwise set $R = <^-_{T^*}$. By Lemma 2.6, there are two possibilities, either there is an $r \leq p_n$ in M and yRx such that $f_r(\xi) \cap X \in \Sigma(H(\theta) \cap M)$ for $\alpha_{p_n} < \xi \leq \alpha_r$ and $r \Vdash y \in \dot{A}$. In this case set $G(n) := y$ and $p_{n+1} := r$. Or there is a $G_n \in [X]^\omega$ such that $(f_{p_n}, F_{p_n} \cup G_n) \Vdash x_n \notin \dot{A}_n$. In this case, set $p_{n+1} := p_n$ and $G(n) := G_n$.

Let $f_q := \bigcup_{n \in \omega} f_{p_n} \cup \{(\alpha, M \cap H(\theta))\}$ where $\alpha := \sup_{n \in \omega} \alpha_{p_n}$ and

$$F_q := \bigcup_{n \in \omega} F_{p_n} \cup \bigcup \{G(n) : G(n) \in [X]^\omega\}.$$

Let $q := (f_q, F_q)$. We claim that q is $(N, \mathbb{P}_\Sigma, T^*)$-generic. First we check that q is a condition. Notice that for each, $x \in M \cap X$ the set $D_x := \{r \in \mathbb{P}_\Sigma : x \in \bigcup \mathrm{ran}(f_r)\}$ is an open dense set in M. Then it follows from the construction that $\bigcup_{\xi < \alpha} f_q(\xi) = M \cap H(\theta)$. We infer from clause (iii) that $f_q(\xi) \cap X \in \Sigma(M \cap H(\theta))$ for $\alpha_p < \xi < \alpha$. Thus, q is a condition and by definition $q \leq p_n$ for $n \in \omega$. By construction $q \in D$ for every dense open subset in M. Hence, q is $(M, \mathbb{P}_\Sigma)$-generic. It remains to show that q is $(M, \mathbb{P}_\Sigma, T^*)$-generic. To do this, let D be a dense open set in M (arbitrary), $x \in T^*_\delta$, $\dot{A} \in M$ a $\mathbb{P}_\Sigma$-name for a subset of T^* and $R \in \{<_{T^*}, <^-_{T^*}\}$. Set $i = 1$ if $R = <_{T^*}$ and $i = 2$ otherwise. Fix n such that $(D_n, x_n, \dot{A}_n, i_n) = (D, x, \dot{A}, i)$. At stage n of the construction we consider two possibilities. The first was the existence of a yRx and $r \leq p_n$ such that $r \Vdash y \in \dot{A}$. In this case $G(n)Rx$ and $p_{n+1} \Vdash G(n) \in \dot{A}$. Thus, $G(n)Rx$ and $q \Vdash G(n) \in \dot{A}$ as required. The second possibility is that there exists an element in $[X]^{\omega}$, called $G(n)$, such that $(f_{p_n}, F_{p_n} \cup G(n)) \Vdash x \notin \dot{A}$. Since $q \leq (f_{p_n}, F_{p_n} \cup G(n))$ then it follows that $q \Vdash x \notin \dot{A}$.

We now show that $\mathbb{P}_\Sigma$ adds a reflecting sequence. We claim that, for each $\alpha < \omega_1$, the set $D_\alpha := \{p \in \mathbb{P}_\Sigma : \alpha \in \mathrm{dom}(p)\}$ is dense open. To see this, notice that the sets $D_x := \{p \in \mathbb{P}_\Sigma : x \in \bigcup \mathrm{ran}(p)\}$ are dense open for every $x \in H(\theta)$. It follows that $\mathbb{1} \Vdash \check{H}(\theta) = \bigcup_{p \in \dot{G}} \mathrm{ran}(p)$. Since $\mathbb{P}_\Sigma$ is proper, then ω_1 is preserved and hence, $\mathbb{1} \Vdash \check{\omega_1} \subseteq \mathrm{dom}(\bigcup \dot{G})$. This concludes the proof of the Theorem. □

3. Open Graph Axiom

The purpose of this section is to prove that PFA(T^*) implies OGA. We recall the relevant definitions.

Definition 3.1. Let (X, d) be a separable metric space. An open graph on X is a structure $\mathcal{G} = (X, E)$ where the edge set E is viewed as an open symmetric subset of $X^2 \setminus \{(x, x) : x \in X\}$.

Definition 3.2. The open graph axiom OGA is the statement that whenever $\mathcal{G} = (X, E)$ is an open graph on a separable metric space (X, d), then either there is an uncountable subset $Y \in [X]^{\omega_1}$ such that $[Y]^2 \subseteq E$ or there is decomposition $X = \bigcup_{n \in \omega} X_n$ with $[X_n]^2 \cap E = \emptyset$ for all $n \in \omega$.

Fix an open graph $\mathcal{G} = (X, E)$ on a separable metric space (X, d). We shall prove that, assuming PFA(T^*), one of the two alternatives of OGA holds for $\mathcal{G}$. If the second alternative holds, then there is nothing to do. Hence, suppose the second alternative fails. We will prove that there is a T^*-proper forcing which adds an uncountable subset $Y \subseteq X$ such that $[Y]^2 \subseteq E$. There are several proper forcing in the literature that add such a set. We shall prove that a small variant of Todorcevic's forcing in [Tod11] is as required.

Let $\mathcal{I}$ be the (proper) σ-ideal consisting of all subsets Y of X that are countably chromatic i.e. there is a decomposition $Y = \bigcup_{n \in \omega} Y_n$ with $[Y_n]^2 \cap E = \emptyset$ for all $n \in \omega$. Furthermore, by removing a relatively open subset of X, we may assume that no nonempty open subset of X belongs to $\mathcal{I}$.

Let $\mathcal{H} := \mathcal{I}^+$ denote the corresponding co-ideal i.e. the set of elements not in $\mathcal{I}$.

Definition 3.3. For each $n > 0$, the Fubini power $\mathcal{H}^n$ is the co-ideal on X^n defined recursively by $\mathcal{H}^1 := \mathcal{H}$ and for $n > 1$,

$$\mathcal{H}^n = \{W \subseteq X^n : \{\overline{x} \in X^{n-1} : W_{\overline{x}} \in \mathcal{H}\} \in \mathcal{H}^{n-1}\},$$

where for $W \subseteq X^n$ and $\overline{x} \in X^{n-1}$,

$$W_{\overline{x}} = \{y \in X : \overline{x}^\frown y \in W\}.$$

The following Lemma appears as [Tod11, Lemma 5.1].

Lemma 3.4. *Suppose $\mathcal{G} = (X, E)$ is an open graph on a separable metric space (X, d) which is not countably chromatic and that n is a positive integer. Let $W \in \mathcal{H}^n(\mathcal{G})$ and let*

$$\partial W := \{\overline{x} \in W : (\forall \epsilon > 0)(\exists \overline{y} \in W)(\forall i < n)[(x_i, y_i) \in E \wedge d(x_i, y_i) < \epsilon]\}$$

Then $W \setminus \partial W \notin \mathcal{H}^n(\mathcal{G})$.

Definition 3.5. Let θ be a regular cardinal such that $(X,d), \mathcal{G}, \mathcal{I}$ and T^* belong to $H(\theta)$ and fix $<_w$ a well-order of $H(\theta)$.

(i) Let $X \in H(\theta)$, by $\mathcal{SK}(X)$ we denote the Skolem closure of X (where the set of Skolem functions is defined using the well-order $<_w$ of $H(\theta)$).
(ii) If $M \prec H(\theta)$ is countable, by M^+ we denote $\mathcal{SK}(M \cup \{M\})$.

We are now ready to introduce our forcing notion. It is worth mentioning that while the idea of using successors of models in side conditions is a quite natural way to prove $(N, \mathbb{P}, T^*)$-genericity, it has been brought to our attention that this notion has already been considered by [KT20] in a similar context.

Definition 3.6. Let $\mathbb{P}_{\mathcal{G}}$ be the collection of all pairs $p = (\mathcal{N}_p, f_p)$ satisfying the following conditions:

(i) $\mathcal{N}_p = \{N_0, \dots, N_m\}$ has the following properties:
   (a) For each $i \leq m$, $N_i \prec (H(\theta), \in, <_w)$ (countable) containing $(X,d), \mathcal{G}, \mathcal{I}$ and T^* as elements.
   (b) $N_i \in N_i^+ \in N_{i+1}$ for $i < m$.
(ii) $f_p : \mathcal{N}_p \to X$ such that
   (a) For each $i \leq m$, $f_p(N_i) \in N_{i+1} \setminus N_i^+$ (where $N_{m+1} = X$).
   (b) For each $i \leq m$, $f_p(N_i) \notin \bigcup(\mathcal{I} \cap N_i^+)$.
   (c) For each $i < j \leq m$, $(f_p(N_i), f_p(N_j)) \in E$.

Let $q \leq p$ if $f_p \subseteq f_q$.

The next Lemma is straightforward and appears implicit in [Tod11]. We just write it to have it for future reference.

Lemma 3.7. *Let $\{N_0, \dots, N_m\}$ be an increasing $\in$-chain of countable elementary submodels of $H(\theta)$ containing all relevant parameters and let $(x_0, \dots, x_m) \in X^{m+1}$ be such that $x_i \in N_{i+1} \setminus N_i$ (where $N_{m+1} = X$) and $x_i \notin \bigcup(N_i \cap \mathcal{I})$ for $i \leq m$. If $W \subseteq X^{m+1}$ is such that $W \in N_0$ and $(x_0, \dots, x_m) \in W$, then $W \in \mathcal{H}^{m+1}$.*

Theorem 3.8. *The axiom* PFA(T^*) *implies* OGA.

Proof. We shall prove that $\mathbb{P}_{\mathcal{G}}$ is T^*-proper and adds an uncountable subset Y of X with $[Y]^2 \subseteq E$.

Let θ^* be a large enough regular cardinal such that $\mathbb{P}_{\mathcal{G}}, T^*, H(\theta), <_w, \mathcal{I}, \mathcal{H}, (X,d)$ and $\mathcal{G}$ belong to $H(\theta^*)$. Let $\overline{M} \prec H(\theta)$ countable which contains all relevant parameters. Consider $p \in \overline{M} \cap \mathbb{P}_{\mathcal{G}}$. We will find $q \leq p$ which is $(M, \mathbb{P}_{\mathcal{G}}, T^*)$-generic. Set $M := \overline{M} \cap H(\theta)$ and $\delta = M \cap \omega_1$. Define $q = (\mathcal{N}_p \cup \{M\}, f_p \cup \{(M, q(M))\})$ where $q(M)$ is any element of $X \setminus \bigcup(\mathcal{I} \cap M^+)$ and such that $(q(M), p(N)) \in E$ for any $N \in \mathcal{N}_p$. Let us see that such a $q(M)$ exists. Let $\overline{N}$ be the maximum element of $\mathcal{N}_p$. Fix a countable basis $\mathcal{B}$ of (X,d) in M and pick a basic open set $U \in \mathcal{B}$ such that $p(\overline{N}) \in U$ and $(p(N), x) \in E$ for all $N \in \mathcal{N}_p \cap \overline{N}$ and $x \in U$. Let Z be the set of all elements of $x \in U$ such that for any $y \in U$ $(x,y) \notin E$. Notice that $Z \in \overline{N} \cap \mathcal{I}$. It follows, from clause (ii)(b) of the definition of the forcing, that $p(\overline{N}) \notin Z$. Pick $y \in U$ such that $(y, p(\overline{N})) \in E$. Since E is open there is a $V \in \mathcal{B}$ such that $y \in V \subseteq U$ and $V \times \{x\} \subseteq E$. Let $q(M)$ be any element in $V \setminus (\bigcup(M^+ \cap \mathcal{I}) \cup M^+)$ (this is possible since $\mathcal{B} \subseteq \mathcal{H}$).

We claim that q is a $(M, \mathbb{P}_{\mathcal{G}}, T^*)$-generic. Fix a dense open set $D \in M$, $a \in T^*_\delta$ and $\dot{A} \in M$ a $\mathbb{P}_{\mathcal{G}}$-name for a subset of T^* and a relation $R \in \{<_{T^*}, <^-_{T^*}\}$. Let

$r \leq q$. By extending r further, if necessary, we may assume that $r \in D$, r decides whether or not $a \in \dot{A}$. If r forces that $a \notin \dot{A}$, then replace $\dot{A}$ in what follows by the canonical $\mathbb{P}_{\mathcal{G}}$-name for T^*. Thus, we may assume without loss of generality that r forces that $a \in \dot{A}$.

We need to find $s \in D \cap M$ and bRa such that s is compatible with r and $s \Vdash b \in \dot{A}$. Let $r_M := (\mathcal{N}_r \cap M, f_r \upharpoonright \mathcal{N}_r \cap M)$. It follows that $r_M \in \mathbb{P}_{\mathcal{G}} \cap M$ and r is an end-extension of r_M. Let $m = |\mathcal{N}_r \setminus M|$ and $M = N_0, \dots, N_{m-1}$ denote the increasing enumeration of $\mathcal{N}_r \setminus M$. Let W be the set of all m-tuples $(x_0, \dots, x_{m-1})$ such that there is an end-extension t of r_M in D such that

(a) $|\mathcal{N}_t \setminus \mathcal{N}_{r_M}| = m$
(b) if $K_0, \dots, K_{m-1}$ is the increasing enumeration of $\mathcal{N}_t \setminus \mathcal{N}_{r_M}$ then $t(K_i) = x_i$ for $i < m$.
(c) $t \Vdash a \in \dot{A}$.

Using that E is open, fix basic open sets $U_i (i < m)$ in $\mathcal{B}$ such that $r(N_i) \in U_i$ and $U_i \times U_j \subseteq E$ for $i \neq j \in m$. Here is the key point of the definition of the forcing, since $a \in N_0^+$, then $W \in N_0^+$ and $(r(N_0), \dots, r_{m-1}(N_{m-1})) \in W$. Thus it follows, from Lemma 4.7 and Lemma 4.4, that $\partial W \in \mathcal{H}^m$ and $(r(N_0), \dots, r_{m-1}(N_{m-1})) \in \partial W$. Pick $\varepsilon > 0$ such that $B_\varepsilon(r(N_i)) \subseteq U_i$ for all $i < m$. From the definition of ∂W we infer that there is a tuple $(x_0, \dots, x_{m-1}) \in W$ such that $(r(N_i), x_i) \in E$ and $d(p(N_i), x_i) < \varepsilon$ for all $i < m$. Using, once more, that E is open find basic open sets $V_i (i < m)$ such that $x_i \in V_i \subseteq U_i$ and $V_i \times \{r(N_i)\} \subseteq E$ for $i < m$. Let t be a witness that $(x_0, \dots, x_{m-1}) \in W$ (we do not claim that t is compatible with r).

Let B denote the set of nodes $c \in T^*$ such that there is an $s \in D$ such that $s \Vdash c \in \dot{A}$ and $(s(K_0), \dots, s(K_{m-1})) \in V_1 \times \cdots \times V_{m-1}$. Notice that t witness that $a \in B$. Since a is (M, T^*)-generic, there exists bRa such that $b \in B$. By elementarity, we can find such a b in M. Let s in M be a witness that $b \in B$. We claim that s and r are compatible. Let $t := s \cup r$. We will show that $t \in \mathbb{P}_{\mathcal{G}}$. Then clearly $t \leq r, s$. It is easy to check that t satisfies all properties for being in $\mathbb{P}_{\mathcal{G}}$ except perhaps clause (ii)(c). To see this, it is sufficient to show that for any $i, j < m$, $(r(N_i), s(K_j)) \in E$. We proceed by cases. On one hand, if $i \neq j$, then $(r(N_i), s(K_j)) \in U_i \times U_j \subseteq E$. On the other hand, if $i = j$, then $(r(N_i), s(K_j)) \subseteq \{r(N_i)\} \times V_i \subseteq E$. This completes the proof that q is $(M, \mathbb{P}_{\mathcal{G}}, T^*)$-generic.

Finally, observe that for each $\alpha < \omega_1$, the set $D_\alpha := \{p : \alpha \in \bigcup \mathrm{dom}(p)\}$ is dense-open. Applying PFA(T^*), to $\mathbb{P}_{\mathcal{G}}$ and to the above sequence of dense open sets, we get a filter G such that $Y := \mathrm{ran}(\bigcup G)$ is an uncountable subset of X which generates a complete subgraph of $\mathcal{G}$. This concludes the proof of the Theorem. $\square$

4. Club isomorphisms

A further question posed in [Kru20b], which is attributed to Moore, is whether PFA(T^*) implies that any two normal almost Suslin trees are club-isomorphic. This is a natural question, since a positive answer would imply that the choice of T^* is somewhat canonical. We answer this negatively by showing that it is consistent with PFA(T^*) that not all normal almost Suslin trees are club-isomorphic. As a counterpart to this "negative" result, we show that PFA(T^*) does imply that any two normal special Aronszajn trees are club-isomorphic.

Let us begin with the "positive" result. Recall that an Aronszajn tree is called *normal* if it has a unique minimal element, it is Hausdorff, and every node has $\aleph_0$ extensions in any level above it. Two ω_1-trees T and S are called *club-isomorphic* if for some club $C \subseteq \omega_1$, $T \restriction C \cong S \restriction C$. Abraham and Shelah [AS85] showed that for any two normal Aronszajn trees there is a proper forcing that introduces a club isomorphism between them, therefore PFA implies that all normal Aronszajn trees are club-isomorphic. Abraham-Shelah's forcing is defined as follows: Fix T and S normal Aronszajn trees, and let $\mathbb{P}(T,S)$ be the forcing notion consisting of all the pairs $p = (A_p, f_p)$ where $A_p \in [\omega_1]^{<\omega}$, and f_p is a finite function such that $\operatorname{dom}(f_p)$ is a downwards closed subtree of $T \restriction A_p$ in which every maximal branch has the same length, $\operatorname{ran}(f_p)$ is a downwards closed subtree of $S \restriction A_p$, f_p is increasing, and for every $x \in T$, $\operatorname{ht}_T(x) = \operatorname{ht}_S(f(x))$. Let $(A_q, f_q) \le (A_p, f_p)$ if $A_q \supseteq A_p$ and $f_q \supseteq f_p$. The following proposition can be found in [Kru25], and it is also implicit in [AS85].

Proposition 4.1. *Let $p, q \in \mathbb{P}(T,S)$ and $\alpha < \beta < \omega_1$ be such that:*

- *$\alpha \in A_p$ and $\beta \in A_q$.*
- *$A_p \subseteq \beta$ and $A_p \cap \alpha = A_q \cap \beta$.*
- *$p \restriction \alpha = q \restriction \beta$.*
- *Every two elements $\operatorname{dom}(f_q) \cap T_\beta$ split before α and every two elements of $\operatorname{ran}(f_q) \cap S_\beta$ split before α.*
- *Every element of $\operatorname{dom}(f_p) \cap T_\alpha$ is incomparable with every element of $\operatorname{dom}(f_q) \cap T_\beta$, every element of $\operatorname{ran}(f_p) \cap S_\alpha$ is incomparable with every element of $\operatorname{ran}(f_q) \cap S_\beta$.*

Then p and q are compatible.

Lemma 4.2. *If T and S are special Aronszajn trees, then $\mathbb{P}(T,S)$ is T^*-proper.*

Proof. Let $\mathbb{P} := \mathbb{P}(T,S)$. Let $g : T \cup S \to \omega$ witness that both T and S are special. Let θ be a large enough regular cardinal and let $N \prec H(\theta)$ be countable such that $T, S, \mathbb{P}, g \in N$. Fix $p \in \mathbb{P} \cap N$ and define $\delta := N \cap \omega_1$. Let $q \le p$ be any extension such that $A_q := A_p \cup \{\delta\}$. We claim that q is $(N, \mathbb{P}, T^*)$-generic. To see this, fix a dense set $D \in M$, $a \in T^*_\delta$, $\dot{A} \in M$ a $\mathbb{P}$-name for a subset of T^*, and a relation $R \in \{<_{T^*}, <^-_{T^*}\}$. Let $r \le q$. By extending r further, if necessary, we may assume that $r \in D$, r decides whether or not $a \in \dot{A}$. If r forces that $a \notin \dot{A}$, then replace $\dot{A}$ in what follows by the canonical $\mathbb{P}$-name for T^*. Thus, we may assume without loss of generality that r forces that $a \in \dot{A}$.

We need to find $s \in D \cap M$ and $b \mathrel{R} a$ such that s is compatible with r and $s \Vdash b \in \dot{A}$.

Let $n = |\operatorname{dom}(f_r) \cap T_\delta|$, and $(\hat{t}_0, \dots, \hat{t}_{n-1})$ be some enumeration of $\operatorname{dom}(f_r) \cap T_\delta$. For each $i < n$, let $p_i := g(\hat{t}_i)$ and $q_i := g(f_r(\hat{t}_i))$. Let $r_0 := r \restriction \delta$, and let α be such that $\max(A_{r_0}) < \alpha < \delta$, and such that for all $i \neq j < n$, both $\hat{t}_i$ with $\hat{t}_j$ and $\hat{s}_i$ with $\hat{s}_j$ split before α. For every $i < n$, let $\check{t}_i \in T_\alpha$ and $\check{s}_i \in S_\alpha$ be such that $\check{t}_i <_T \hat{t}_i$ and $\check{s}_i <_S f_r(\hat{t}_i)$. Now consider the set C of all tuples $(z, s, \beta, t_0, \dots, t_{n-1}) \in T^* \times \mathbb{P} \times \omega_1 \times T^n$ such that:

- $s \in D$ and $s \Vdash z \in \dot{A}$.
- $A_s \setminus \alpha \neq \emptyset$ and $\min(A_s \setminus \alpha) = \beta$.
- $|\operatorname{dom}(f_s) \cap T_\beta| = n$.
- $\operatorname{dom}(f_s) \cap T_\beta = \{t_i : i < n\}$.
- For all $i \neq j < n$, $\check{t}_i$ and t_j are T-incomparable, and $\check{s}_i$ and $f_s(t_j)$ are S-incomparable.
- For all $i < n$, $g(t_i) = p_i$ and $g(f_s(t_i)) = q_i$.

Note that $(x, r, \delta, \hat{t}_0, \dots, \hat{t}_{n-1}) \in C$. Thus $B := \{z : (\exists s, \beta, \vec{t})((z, s, \beta, \vec{t}) \in C)\}$ is nonempty, it is in N, and $x \in B$. Since x is (N, T^*)-generic, there exists $y \in B$ such that $y \, R \, x$, so $(y, s, \beta, t_0, \dots, t_{n-1}) \in C$ for some $s, \beta, t_0, \dots, t_{n-1} \in N$. Now note that for every $i, j < n$, t_i is T-incomparable with $\hat{t}_j$ and $f_s(t_i)$ is S-incomparable with $f_r(\hat{t}_j)$; if $i \neq j$ this is immediate from the definition of C, and if $i = j$, then it follows from $\beta < \delta$, $g(t_i) = g(\hat{t}_i)$ and $g(f_s(t_i)) = g(f_r(\hat{t}_i))$, again by the definition of C. Thus Lemma 4.1 implies that s is compatible with r. By definition of C, $s \Vdash y \in \dot{A}$ and $s \in D$. This finishes the proof. $\square$

Corollary 4.3. PFA(T^*) *implies that all special normal Aronszajn trees are club-isomorphic.*

4.1. **Not club-isomorphic almost Suslin trees.** We dedicate this section to show that, for some carefully chosen T^*, there is a model of PFA(T^*) in which not all normal almost Suslin trees are club-isomorphic. Let us fix some notation first. If T and S are trees, $S + T$ denotes the tree given by putting S and T side by side, and $S \otimes T$ the set $\{(s, t) \in S \times T : \mathrm{ht}_S(s) = \mathrm{ht}_T(t)\}$ under the product order, which is again a tree, and it is Aronszajn whenever both T and S are ω_1-trees and at least one of them is Aronszajn.

Our strategy to obtain the consistency of PFA(T^*) + "not all normal almost Suslin trees are club-isomorphic" relies on the following lemma.

Lemma 4.4. *Assume there is a supercompact cardinal. Assume there are two normal almost Suslin trees, T and S, such that $T \otimes T$ is again almost Suslin but $S \otimes S$ is not. Let $T^* := (T \otimes T) + S$. Then there is a model of* PFA(T^*) *in which S and T are not club isomorphic.*

Proof. Suppose that we are in a model in which there are T and S as described. Let $T^* := (T \otimes T) + S$, clearly T^* is also almost Suslin. Now construct a model of PFA(T^*) by iterating T^*-proper forcings (using a Laver function as a bookkeping device) as in [Kru20b]. Since T^* remains almost Suslin, so do T, S and $T \otimes T$, and since stationary sets are preserved, $S \otimes S$ is still not almost Suslin. Thus S and T cannot be club-isomorphic. $\square$

Remark 4.5. *The tree we will use for T will be a coherent tree. By Corollary 4.7 and the fact that coherence is preserved by generic extensions preserving ω_1, we see that for such a T it is enough to consider $T^* := T + S$, rather then $T^* := (T \otimes T) + S$, in the previous proof.*

We now show how to get a model in which there are T and S as in Lemma 4.4.

As mentioned, T will be a *coherent tree*: an uncountable $T \subseteq \omega^{<\omega_1}$ that is closed under taking initial segments and such that for all $t, s \in T$, if $\mathrm{dom}(t) \leq \mathrm{dom}(s)$ then $\{\xi < \mathrm{dom}(t) : t(\xi) \neq s(\xi)\}$ is finite.

Lemma 4.6. *Let T be a coherent Aronszajn tree, $\Gamma \subseteq \omega_1$ a stationary set, and $\{(x_\alpha, y_\alpha) : \alpha \in \Gamma\}$ be such that $\mathrm{dom}(x_\alpha) = \mathrm{dom}(y_\alpha) = \alpha$ for all $\alpha \in \Gamma$. Then there is stationary $\Gamma' \subseteq \Gamma$ such that for all $\alpha < \beta$ in Γ', $x_\alpha \sqsubseteq x_\beta$ iff $y_\alpha \sqsubseteq y_\beta$.*

Proof. We may assume that Γ consists of limit ordinals. By coherence, for each $\alpha \in \Gamma$ there is $\gamma(\alpha) < \alpha$ such that $x_\alpha \restriction [\gamma(\alpha), \alpha) = y_\alpha \restriction [\gamma(\alpha), \alpha)$. Using Fodor's lemma, there is a stationary set $\Gamma' \subseteq \Gamma$ and $\gamma < \min(\Gamma')$ such that $\gamma(\alpha) = \gamma$ for all $\alpha \in \Gamma'$. Since T_γ is countable, there are $x, y \in T_\gamma$ such that $\Gamma'' := \{\alpha \in \Gamma' : x \sqsubseteq$

$x_\alpha, y \sqsubseteq y_\alpha\}$ is stationary. Then $x^\frown t \mapsto y^\frown t$ is a $\sqsubseteq$-isomorphism from $\{x_\alpha : \alpha \in \Gamma''\}$ onto $\{y_\alpha : \alpha \in \Gamma''\}$ $\square$

Note that the previous lemma easily generalize to finite tuples, and thus the following Corollary also holds for any finite product $T \otimes \cdots \otimes T$.

Corollary 4.7. *If T is a coherent Aronszaj tree, then T is almost Suslin iff $T \otimes T$ is almost Suslin.*

Definition 4.8. For a tree S, its *derived trees of dimension n* are the trees of the form $S(x_1) \otimes \dots S(x_n)$ where $(x_1, \dots, x_n)$ are distinct elements of the same height, and $S(x_i) := \{s \in S : x_i \leq_S s\}$.

It is a standard fact that $\diamondsuit$ implies the existence of a coherent Suslin tree. See for example [Tod84]. By Corollary 4.7, to obtain our desired model it is enough find a model of $\diamondsuit$ plus the existence of a Suslin tree S such that all its derived trees of dimension 2 are special; this implies that $S \otimes S$ is not almost Suslin. One way to obtain such a model is to start with a model of $\diamondsuit$ and use Krueger's forcing [Kru20a, Theorem 4.4] to introduce such a tree S. This works because Krueger's forcing has size ω_1 (in a model of CH), and thus it preserves $\diamondsuit$. Other option, which seems more natural to us, is to show that $\diamondsuit$ already implies the existence of a tree like S. We take this approach.

Jensen and Johnsbråten [JJ74] used $\diamondsuit$ to construct a Suslin tree S with the property that $\{(s,t) \in S \otimes S : s \neq t\}$ is special, as mentioned in [Tod84, 9.15 (vi)]. In particular, it follows that every derived tree of dimension at least 2 is special. However, their original proof is quite involved, and does not explicitly stablish this fact. For clarity and completeness, we present a variant of their construction. Our approach yields a proof of a stronger statement, confirming a conjecture described as "quite plausible" by Scharfenberger-Fabian in [SF10, Sec3.6].

Theorem 4.9. *Assume $\diamondsuit$. For every $1 \leq d < \omega$, there is a normal Suslin tree S such that:*

- *All its derived trees of dimension at most d are Suslin.*
- *All its derived trees of dimension greater than d are special.*

Proof. Fix $d \geq 1$. First observe that it is enough to make sure that all derived trees of dimension exactly d are Suslin, and all derived trees of dimension exactly $d+1$ are special.

Let us fix some notation. For $1 \leq n < \omega$, let $S^{(n)}$ denote the $\bar{x} \in \bigotimes_{i<n} S$ that are injective, i.e., without repetitions. Similarly define $S^{(n)}_\alpha$. We also understand that if $\bar{x} \in S^{(n)}$, then $\bar{x} = (x_0, \dots, x_{n-1})$. We order $S^{(n)}$ with the partial order induced from $\bigotimes_{i<n} S$, which we will denote simply by $<_S$.

Fix $\langle A_\delta : \delta < \omega_1 \rangle$ a $\diamondsuit$-sequence for subsets of ω_1^d. We will construct S by recursion on the levels, making sure that for each $\alpha < \omega_1$, $S \restriction \alpha$ is a countable ordinal as a set. Together with S we will define $f : \bigotimes_{i<d+1} S \to \mathbb{Q}_{\geq 0}$ satisfying the following:

(i) $f(\bar{x}) = 0$ iff $\bar{x}$ is not injective, i.e., $\bar{x} \notin S^{(d+1)}$.

(ii) For all $\bar{x}, \bar{y} \in S^{(d+1)}$, if $\bar{x} <_S \bar{y}$ then $f(\bar{x}) < f(\bar{y})$.

(iii) Suppose $\alpha < \beta$, and $\bar{x} \in S^{(n)}_\alpha$ with $n \geq d$. Then for all $I \in [n]^d$ and $\sigma : I \to S_\beta$ such that for all $i \in I$, $x_i <_S \sigma(i)$, and for all $\varepsilon \in \mathbb{Q}^+$, there is $\bar{y} \in S^{(n)}_\beta$ such that $\bar{x} <_S \bar{y}$, $y_i = \sigma(i)$ for $i \in I$, and for all $i_0, \ldots, i_d < n$ without repetitions, $f(x_{i_0}, \ldots, x_{i_d}) < f(y_{i_0}, \ldots, y_{i_d}) < f(x_{i_0}, \ldots, x_{i_d}) + \varepsilon$.

Note that property (ii) will ensure that all derived trees of dimension $d+1$ are special.

Let $S_0 := \{0\}$ and given S_α, construct $S_{\alpha+1}$ by adding $\aleph_0$ immediate successors to each node in S_α so that (i)–(iii) are satisfied. Now suppose $S \restriction \delta$ has been defined for some countable limit ordinal satisfying (i)–(iii), and all the normality requirements. We will construct a countable set B of cofinals branches through $S \restriction \delta$ satisfying the following:

(B1) For every $x \in S \restriction \alpha$ there is $b \in B$ such that $x \in B$.
(B2) If it happens that A_δ is a maximal antichain of $S(x_0) \otimes \cdots \otimes S(x_{d-1})$ for some $\bar{x} \in (S \restriction \delta)^{(d)}$, then for every $b_0, \ldots, b_{n-1} \in B$ such that for all $i < d$, $x_i \in b_i$, $\bigotimes_{i<d} b_i \cap A_\delta \neq \emptyset$.

We will define $S_\delta := \{\tilde{b} : b \in B\}$ where for every $b \in B$, $\tilde{b}$ is a node above each element of b. This will ensure that the resulting tree is normal, and has every derived tree of dimension d Suslin. Thus, it is enough to see how to construct B satisfying (B1) and (B2), and how to extend f on S_δ satisfying (i)–(iii).

Define $\mathbb{P}_\delta$ to be the poset given by the $(\bar{a}, h)$ such that for some $n \in \omega$,

- $\bar{a} \in \bigotimes_{i<n} S$. We call n its *length* and denote it by $|\bar{a}|$. Observe that we do not require $\bar{a}$ to be injective.
- $h : n^d \to \mathbb{Q}^+$ is such that for all $(i_0, \ldots, i_d) \in \mathrm{dom}(h)$, $f(a_{i_0}, \ldots, a_{i_d}) < h(i_0, \ldots, i_d)$.

The order is given by $(\bar{b}, g) \leq (\bar{a}, h)$ if $|\bar{b}| \geq |\bar{a}|$, $\mathrm{ht}(\bar{b}) \geq \mathrm{ht}(\bar{a})$, for all $i < |a|$, $b_i \geq_S a_i$, and $g \supseteq h$.

Let $N \prec H(\omega_2)$ be countable and such that $\delta, S \restriction \delta, A_\delta, \mathbb{P}_\delta \in N$. Let G be a $\mathbb{P}_\delta$-generic filter for N. For each $i < \omega$ let $b_i := \{a_i : (\bar{a}, h) \in G, |\bar{a}| > i\}$.

We claim that for each $i < \omega$, b_i is a cofinal branch through $S \restriction \delta$, and moreover, if $B := \{b_i : i < \omega\}$, then we have that (B1) and (B2) hold. To see that each b_i is a cofinal branch, note that for each $i < \omega$ and $\alpha < \delta$, the set of $(\bar{a}, h) \in \mathbb{P}_\delta$ such that $|\bar{a}| > i$ and $\mathrm{ht}(\bar{a}) \geq \alpha$, is dense and it is definable from parameters in N, thus it is in N, so G meets it. Since the verification that the clauses (B1) and (B2) hold, are similar. We will only present the proof that (B2) holds. Suppose that A_δ is a maximal antichain of $\bigotimes_{i<d} S(x_i)$, where $\bar{x} \in (S \restriction \delta)^{(n)}$. For each x_i, let $b_{k(i)}$ be such that $x_i \in b_{k(i)}$. Observe that k is injective since $\bar{x}$ is injective. Now consider the set of $(\bar{a}, h) \in \mathbb{P}_\delta$ such that $|\bar{a}| > \sup_{i<d} k(i)$, and either $(a_{k(0)}, \ldots, a_{k(d-1)}) \in A_\delta$ or it does not extend $\bar{x}$. Then this set is dense by (iii) in $S \restriction \delta$, and it is in N, thus G meets it. This easily implies what we want.

It remains to define f on S_δ. Call $\tilde{b}_i$ the node in S_δ above b_i. For every injective $(i_0, \ldots, i_d)$ we define $f(\tilde{b}_{i_0}, \ldots, \tilde{b}_{i_d}) := h(i_0, \ldots, i_d)$ where $(\bar{a}, h) \in G$ is any such that $|\bar{a}| > \sup_{i \leq d} i_d$. This is well defined because G is a filter. Of course if $(i_0, \ldots, i_d)$ is not injective we let $f(\tilde{b}_{i_0}, \ldots, \tilde{b}_{i_d}) = 0$. It follows from the definition of $\mathbb{P}_\delta$ that this will satisfy (i) and (ii). We need to show (iii).

Pick $\bar{x} \in (S \restriction \delta)^n$. We may assume $n \geq d+1$, since otherwise (iii) holds vacuously. Pick $I \in [n]^d$, $\sigma : I \to \omega$ and $\varepsilon \in \mathbb{Q}^+$. Consider $D \subseteq \mathbb{P}_\delta$ the set of $(\bar{a}, h)$ satisfying that

- $\mathrm{ran}(\sigma) \subseteq |\bar{a}|$, $\mathrm{ht}(\bar{a}) > \mathrm{ht}(\bar{x})$ and $\bar{a}$ is injective.

- If it is the case that for all $i \in I$, $x_i <_S \sigma(i)$, then there is $\sigma' : n \to \omega$ such that:
  - $\operatorname{ran}(\sigma') \subseteq |\bar{a}|$ and $\sigma' \restriction I = \sigma$.
  - For all injective $(i_0, \ldots, i_d) \in n^{d+1}$, $h(a_{\sigma'(i_0)}, \ldots, a_{\sigma'(i_d)}) < f(x_{i_0}, \ldots, x_{i_d}) + \varepsilon$.

An application if (iii) on $S \restriction \delta$ shows that D is dense. This easily implies that (iii) holds on $S \restriction (\delta + 1)$ so we are done. □

By Lemma 4.4 and Remark 4.5 we obtain the following.

Corollary 4.10. *Assuming $\diamondsuit$ there is a Suslin tree T^* and a model of* PFA(T^*) *which contains two normal almost Suslin trees that are not club-isomorphic.*

5. Concluding remarks and questions

Can Corollary 4.3 be improved to "PFA(T^*) implies that all normal Aronszajn trees without almost-Suslin subtrees are club-isomorphic" for some reasonable definition of subtree? This is motivated by Krueger's result saying that under PFA(S) all normal Aronszajn trees without Suslin subtrees are club-isomorphic (see [Kru25]). Note that a positive answer would be somewhat optimal because of the following fact: if there is an almost-Suslin tree, then there is a family of size $2^{\aleph_1}$ of pairwise not club-isomorphic normal Aronszajn trees, each of which has a stationary antichain but contains an almost Suslin subtree (see [Tod84, 5.15 (v)]).

In view of Corollary 4.10, is it consistent to have an an almost Suslin tree T^* such that after forcing to obtain PFA(T^*) all almost Suslin trees are club-isomorphic?

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