

Generating pivot Gray codes for spanning trees of complete graphs in constant amortized time

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Abstract. We present the first known pivot Gray code for spanning trees of complete graphs, listing all spanning trees such that consecutive trees differ by pivoting a single edge around a vertex. This pivot Gray code thus addresses an open problem posed by Knuth in *The Art of Computer Programming, Volume 4* (Exercise 101, Section 7.2.1.6, [Knuth, 2011]), rated at a difficulty level of 46 out of 50, and imposes stricter conditions than existing revolving-door or edge-exchange Gray codes for spanning trees of complete graphs. Our recursive algorithm generates each spanning tree in constant amortized time using $O(n^2)$ space. In addition, we provide a novel proof of Cayley’s formula, n^{n-2} , for the number of spanning trees in a complete graph, derived from our recursive approach. We extend the algorithm to generate edge-exchange Gray codes for general graphs with n vertices, achieving $O(n^2)$ time per tree using $O(n^2)$ space. For specific graph classes, the algorithm can be optimized to generate edge-exchange Gray codes for spanning trees in constant amortized time per tree for complete bipartite graphs, $O(n)$ -amortized time per tree for fan graphs, and $O(n)$ -amortized time per tree for wheel graphs, all using $O(n^2)$ space.

1 Introduction. A *complete graph* on n vertices, denoted by K_n , is a simple labeled graph in which every pair of distinct vertices is connected by exactly one edge. In other words, each vertex in a complete graph is adjacent to every other vertex. For example, the complete graphs on one to five vertices, that is K_1 to K_5 , are shown in Figure 1.1:

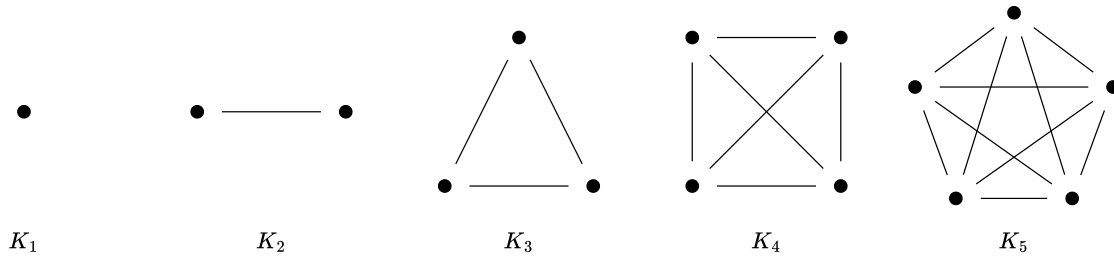


Figure 1.1: Complete graphs from K_1 to K_5 .

A *spanning tree* of a simple graph G is a subgraph that includes all vertices of G and forms a tree. In other words, it is a subgraph of G that connects all vertices of G without cycles. As an example, Figure 1.2 illustrates the 16 spanning trees of K_4 .

The number of spanning trees in a complete graph K_n with n vertices is given by Cayley’s formula [5]. This formula, named after the mathematician Arthur Cayley, gives the number of labeled spanning trees in K_n as:

$$n^{n-2}.$$

This enumeration sequence corresponds to sequence A000272 in the Online Encyclopedia of Integer Sequences [29]. For example, the number of spanning trees for complete graphs K_1 to K_8 is 1, 1, 3, 16, 125, 1296, 16807, and 262144, respectively. Cayley provided an informal proof of the formula for the number of spanning trees in a

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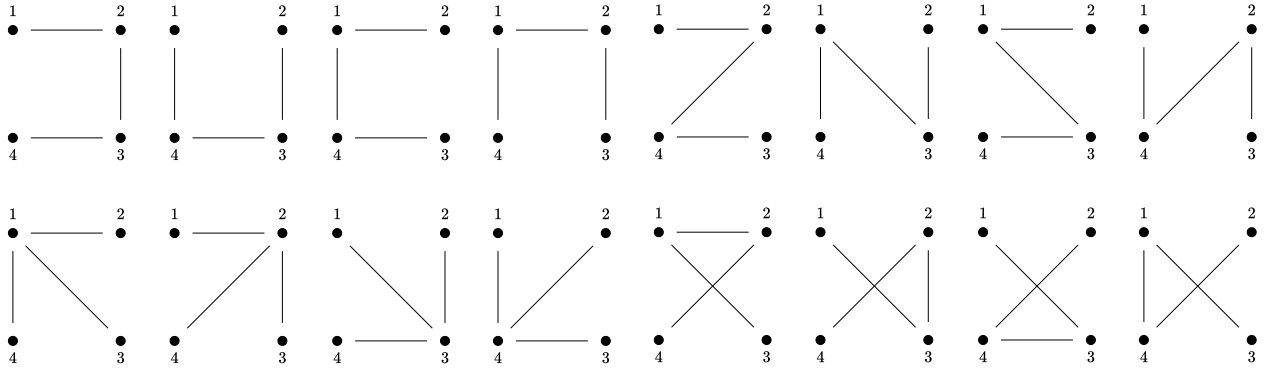


Figure 1.2: The 16 spanning trees of K_4 .

complete graph K_n [5]. Later, Prüfer [30] and Kirchhoff [18] offered simpler proofs by establishing a one-to-one correspondence between spanning trees and Prüfer sequences, and by applying the matrix-tree theorem, respectively. Mütze [27], however, commented that in *The Art of Computer Programming, Volume 4* [20] (Ex. 101 in Sec. 7.2.1.6), Knuth possibly sought a more fine-grained explanation of Cayley's formula for the number of spanning trees in complete graphs.

The study of spanning trees of complete graphs and other graphs has a long history and has traditionally attracted considerable interest from mathematicians and graph theorists [3, 5, 6, 10, 18, 20, 23, 26, 27, 28, 30, 31, 32, 36, 38, 42]. The task of efficiently enumerating all spanning trees of graphs is fundamental in computer science and finds applications across diverse scientific fields including epidemiology, networking, chemistry, biology, astronomy, archaeology, image processing, and social media. Notably, one of the earliest known works on enumerating all spanning trees of a graph was conducted by the German physicist Wilhelm Feussner in 1902, motivated by applications in electrical network analysis [9]. For more applications on spanning trees of graphs, see [6, 20].

One of the most important aspects of combinatorial generation is to list the instances of a combinatorial object so that consecutive instances differ by a specified *closeness condition* involving a constant amount of change. Lists of this type are called *Gray codes*. This terminology is due to the eponymous *binary reflected Gray code* (BRGC) by Frank Gray, which orders the 2^n binary strings of length n so that consecutive strings differ by one bit. For example, when $n = 4$ the order is

0000, 1000, 1100, 0100, 0110, 1110, 1010, 0010, 0011, 1011, 1111, 0111, 0101, 1101, 1001, 0001.

Variations that reverse the entire order or the individual strings are also commonly used in practice and in the literature. For more information about binary reflected Gray code and its variations, see [33, 34, 40]. We note that the order above is *cyclic* because the last and first strings also differ by the closeness condition, and this property holds for all n . The BRGC listing is a *1-Gray code* in which consecutive strings differ by one symbol change. In this paper, we utilize a generalized version of the binary reflected Gray code to construct Gray codes for spanning trees of complete graphs and general graphs.

An interesting problem related to graphs is thus to discover a Gray code for spanning trees of graphs. But before addressing this problem, we first define the nature of changes allowed in Gray codes for spanning trees of graphs. An *edge-exchange Gray code* for the spanning trees of a graph is a list of all spanning trees of the graph such that each consecutive spanning trees differ by the removal of one edge and the addition of another. For complete graphs, this problem was posed as an open problem by Knuth in *The Art of Computer Programming, Volume 4* [20] (Ex. 101 in Sec. 7.2.1.6), with a difficulty rating of 46 out of 50:

- Is there a simple revolving-door way to list all n^{n-2} spanning trees of the complete graph K_n ? (The order produced by Algorithm S is quite complicated.)

The *revolving-door* property, as described by Knuth, refers to an edge-exchange Gray code in which consecutive spanning trees differ by the substitution of one edge for another [20]. Cummins first demonstrated that the

spanning trees of any connected graph can be listed cyclically via edge exchanges, thereby proving the existence of edge-exchange Gray codes for spanning trees of general graphs [8]. Later, Kamae [13], Kishi and Kajitani [19], and Holzmänn and Harary [12] provided more detailed proofs of the existence of edge-exchange Gray codes for spanning trees of graphs, with extensions to matroids. However, these results are existential and do not provide algorithms for generating such Gray codes. There are also many algorithms for generating spanning trees of graphs [1, 3, 6, 7, 10, 11, 14, 15, 16, 17, 22, 23, 26, 37, 42], however, the spanning trees generated by these algorithms do not result in an edge-exchange Gray code. The problem of generating an edge-exchange Gray code for spanning trees of an arbitrary graph was resolved by Smith [39], and more recently by Merino, Mütze, and Williams [25], and Merino and Mütze [24]. Notably, Smith’s algorithm corresponds to the Algorithm S referenced in Knuth’s open problem. Their algorithm produces the edge-exchange Gray code for K_5 as shown in Figure 1.3:

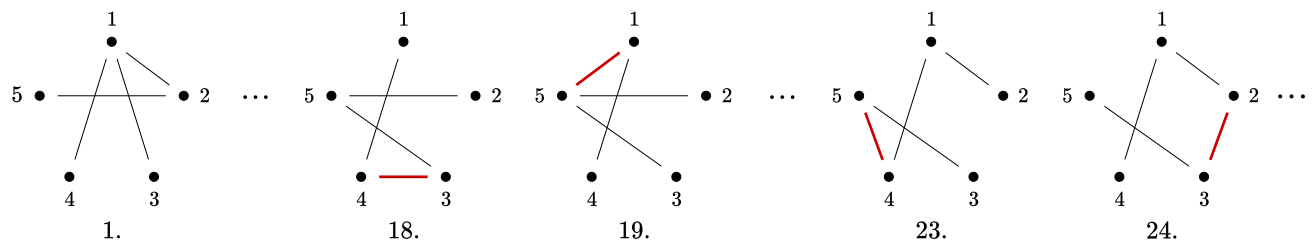


Figure 1.3: Edge-exchange Gray code for spanning trees of K_5 by Smith’s algorithm (Algorithm S in [20]). Only part of the listing is shown.

Interestingly, edge-exchange Gray codes of a graph G correspond to Hamilton paths on the 0/1-polytope that is obtained as the convex hull of the characteristic vectors of all spanning trees of G [35]. Among these algorithms, Smith’s algorithm is the only one that generates the Gray code listing in constant amortized time per tree. The algorithms of Merino, Mütze, and Williams [25] and Merino and Mütze [24] generate edge-exchange Gray codes in $O(m \log n (\log \log n)^3)$ and $O(m \log n)$ time per tree, respectively.

Cameron, Grubb and Sawada [4] proposed the concept of pivot Gray code (also known as a *strong revolving-door Gray code* [20]) for spanning trees of graphs, which imposes stricter constraints on the transitions allowed in Gray codes. A *pivot Gray code* for the spanning trees of a graph is an edge-exchange Gray code with the additional requirement that the removed and added edges share a common vertex. For example, the edge-exchange Gray code generated by Smith’s algorithm is not a pivot Gray code, as the transitions from tree 18 to tree 19, and the transitions from tree 23 to tree 24 in Figure 1.3 for K_5 generated by their algorithm lack the pivot property.

Cameron et al. [4] developed a greedy strategy to list the spanning trees of a fan graph in a pivot Gray code order. For example, the fan graph F_4 with four vertices and its pivot Gray code, generated by Cameron et al.’s algorithm, are presented in Figure 1.4. The edge-exchange Gray codes provided by Merino, Mütze, and Williams [25] and Merino and Mütze [24] are not pivot Gray codes for complete graphs. The algorithm by Cameron et al. can be optimized to generate the pivot Gray codes for fan graphs in constant amortized time per tree. They also generalized the algorithm to produce pivot Gray codes for wheel graphs. More recently, Behrooznia and Mütze [2] provided an algorithm that generates pivot Gray codes for outerplanar graphs (which include fan graphs) in $O(n \log n)$ time per tree. Cameron et al., and Behrooznia and Mütze posed the following open problems regarding complete graphs:

- It remains an open question to find similar results (pivot Gray codes) for other classes of graphs, including the n -cube and the complete graph [4].
- Knuth [20] asked in Exercise 101 in Section 7.2.1.6 whether the complete graph K_n admits a “nice” Gray code listing of its spanning trees. One interpretation of this question, in the spirit of Problem 1, could be: Is there a pivot-exchange Gray code for the spanning trees of K_n ? Alternatively, is there a Gray code listing that provides a more fine-grained explanation of Cayley’s formula n^{n-2} for the number of spanning trees [2]?

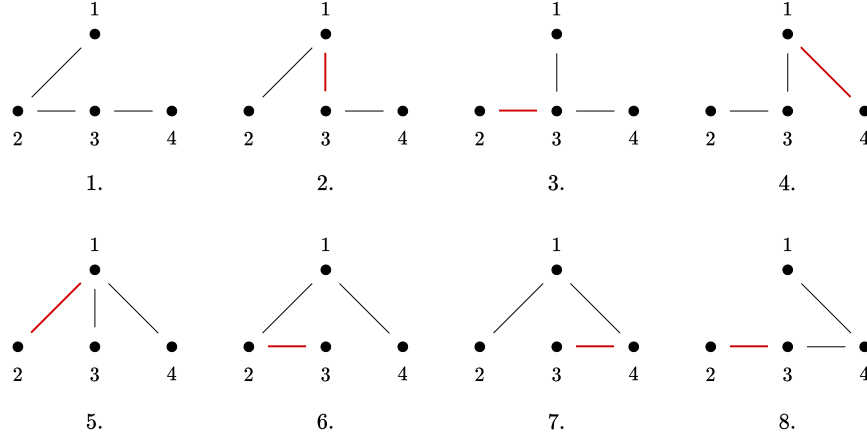


Figure 1.4: Pivot Gray code for spanning trees of F_4 .

In this paper, we address these open problems and Knuth's open problem by providing the first known pivot Gray code for spanning trees of complete graphs. We also provide a simpler proof of Cayley's formula for the number of spanning trees of complete graphs. Our algorithm generates each spanning tree in constant amortized time using $O(n^2)$ space. We further extend our algorithm to generate edge-exchange Gray codes for general graphs with n vertices, achieving $O(n^2)$ time per tree using $O(n^2)$ space. The runtime can be optimized for specific classes of graphs, including complete bipartite graphs, fan graphs, and wheel graphs, which can be generated in constant amortized time, $O(n)$ -amortized time, and $O(n)$ -amortized time per tree, respectively, using $O(n^2)$ space.

The rest of the paper is outlined as follows. In Section 2, we describe a simple recursive algorithm for generating pivot Gray codes for spanning trees of complete graphs. Then in Section 3, we generalize our algorithm to generate edge-exchange Gray codes for general graphs. We provide some final remarks, including lemmas on related problems, in Section 4.

2 Pivot Gray codes for spanning trees of complete graphs. In this section, we present an algorithm to generate the first known pivot Gray code for spanning trees of K_n . We first introduce a novel encoding for spanning trees of a graph. We then describe a recursive algorithm to generate pivot Gray codes for spanning trees of K_n , which relies on generating Gray codes for k -ary strings. Finally, we present a simple algorithm to efficiently generate a Gray code for k -ary strings by leveraging an algorithm for generating Gray codes for reflectable languages.

2.1 Encoding spanning trees in a graph. In our proposed representation for spanning trees of a graph, each spanning tree is represented by an h -tuple $T = (t_1, t_2, \dots, t_h)$, where h is the height of the rooted tree corresponding to the spanning tree rooted at vertex 1. The ℓ -th element t_ℓ of T encodes the connections between vertices at level ℓ and those at level $\ell - 1$ of the rooted tree.

Suppose the rooted tree from level 0 to level $\ell - 1$ contains $q_{\ell-1}$ vertices, with p vertices at level $\ell - 1$. The ℓ -th element t_ℓ of T is a string of length $n - q_{\ell-1}$, that is $t_\ell = a_1 a_2 \dots a_{n-q_{\ell-1}}$, where each character a_i ranges from 0 to p . Each character a_i indicates whether the i -th smallest remaining vertex (from the $n - q_{\ell-1}$ vertices not yet assigned to levels 0 through $\ell - 1$) connects to a vertex at level $\ell - 1$. If $a_i = 0$, then this vertex is not connected to any vertex at level $\ell - 1$. Otherwise if $a_i \neq 0$, then it connects to the (a_i) -th smallest vertex at level $\ell - 1$. For example, the tuple $(00100, 1100, 22)$ corresponds to the spanning tree of K_6 with its associated rooted tree shown in Figure 2.1. Note that in order to have a valid spanning tree, each vertex at level $\ell > 0$ must have a parent and thus $t_\ell \neq 0^{n-q_{\ell-1}}$.

2.2 Pivot Gray code construction. Our recursive algorithm leverages Gray codes for k -ary strings to enumerate all spanning trees of K_n . We first provide an overview of the algorithm, with details and optimizations described in the next subsection. The general approach is to generate all possible tuples T corresponding to the spanning trees of K_n , while minimizing differences between consecutive tuples to ensure a pivot Gray code for the spanning trees.

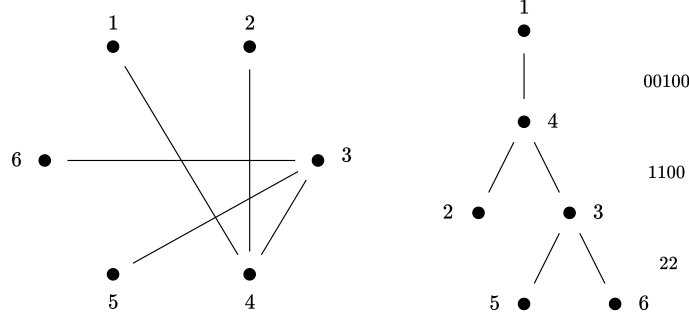


Figure 2.1: The spanning tree and rooted tree of K_6 that corresponds to the tuple $T = (00100, 1100, 22)$.

To achieve this, our algorithm recursively generates all possible rooted trees rooted at vertex 1 at level zero, with a fixed subtree $S_{\ell-1}$ spanning levels 0 to $\ell-1$. Each generated rooted tree includes $S_{\ell-1}$ as its subtree and extends it by connecting the remaining vertices (those not in $S_{\ell-1}$) to vertices at level $\ell-1$ in all valid configurations, ensuring that consecutive trees differ by a pivot change. Initially, S_0 only contains vertex 1, and the algorithm generates all rooted trees containing vertices $\{2, 3, \dots, n\}$ with S_0 as the subtree at level 0, maintaining the pivot Gray code property. The algorithm recursively applies this process until all rooted trees with a given subtree $S_{\ell-1}$ are exhaustively generated at level ℓ . It then backtracks to level $\ell-1$, generates the next configuration of $S_{\ell-1}$ with the same subtree $S_{\ell-2}$, and recursively generates all rooted trees with the updated $S_{\ell-1}$ at level ℓ . This continues until all configurations at each level are exhausted.

Our algorithm begins with the spanning tree $1-2-\dots-n$. As an example, Figure 2.2 illustrates a pivot Gray code listing for K_4 , starting with the spanning tree $1-2-3-4$. At level three of a rooted tree rooted at vertex 1, only one configuration exists for the given subtree S_2 (e.g., vertex 4 connected to vertex 3). The algorithm backtracks to level two, where two configurations with the same subtree S_1 have not yet appeared (vertex 2 connected to both vertices 3 and 4, and vertex 2 connected to vertex 4). A pivot change is applied (for example, reassigning the parent of vertex 4 from vertex 3 to vertex 2), and the algorithm attempts to generate all configurations at level three. For the configuration with vertex 2 as the parent of vertices 3 and 4, no vertices remain at level three, so no further configurations are generated. Then, for the configuration with vertex 2 as the parent of vertex 4, level three has one configuration (vertex 3 connected to vertex 4). After exhausting all configurations with the given S_1 and S_2 at levels two and three, the algorithm backtracks to level one, generates the next configuration of S_1 , and recursively generates all rooted trees with the updated S_1 at level two and subsequently level three. This process continues until all spanning trees of K_4 are generated, as shown in Figure 2.2.

The question then becomes: How can we ensure that repeatedly applying pivot changes generates all possible configurations. We first prove the following lemmas.

LEMMA 2.1. *Let $T = (t_1, t_2, \dots, t_h)$ be the tuple corresponding to a spanning tree of K_n rooted at vertex 1, with p vertices at level $\ell-1$ and $q_{\ell-1}$ vertices across levels 0 to $\ell-1$. To exhaustively generate all spanning trees of K_n rooted at vertex 1 that share the same subtree $S_{\ell-1}$ spanning levels 0 to $\ell-1$ (i.e., those whose corresponding tuples are of the form $T' = (t_1, t_2, \dots, t_{\ell-1}, t'_\ell, t'_{\ell+1}, \dots, t'_h)$) exactly once, it is necessary to list every $(p+1)$ -ary string of length $n - q_{\ell-1}$ exactly once for t'_ℓ , excluding the all-zero string $0^{n-q_{\ell-1}}$.*

Proof. If $q_{\ell-1} > 0$, then since the spanning tree is connected, at least one vertex at level ℓ must connect to a vertex at level $\ell-1$ and thus $t'_\ell \neq 0^{n-q_{\ell-1}}$. Moreover, since K_n is a complete graph, all possible connections from the $n - q_{\ell-1}$ vertices not in levels 0 to $\ell-1$ to the p vertices at level $\ell-1$ are valid. These possible connections correspond to all possible $(p+1)$ -ary strings of length $n - q_{\ell-1}$ for t'_ℓ except $0^{n-q_{\ell-1}}$. $\square$

LEMMA 2.2. *Let $T = (t_1, t_2, \dots, t_h)$ be the tuple corresponding to a spanning tree of K_n rooted at vertex 1 with $q_{\ell-1}$ vertices across levels 0 to $\ell-1$. Suppose t'_ℓ is obtained by either changing a symbol of t_ℓ , or by swapping a 0 and an $r > 0$ of t_ℓ when $t_\ell = 0^{n-q_{\ell-1}-b-1}r0^b$ for some $b \geq 0$. Then there exists a tuple $T' = (t_1, t_2, \dots, t_{\ell-1}, t'_\ell, t'_{\ell+1}, \dots, t'_j)$ corresponding to a spanning tree of K_n rooted at vertex 1 that is obtainable from T by a single pivot change.*

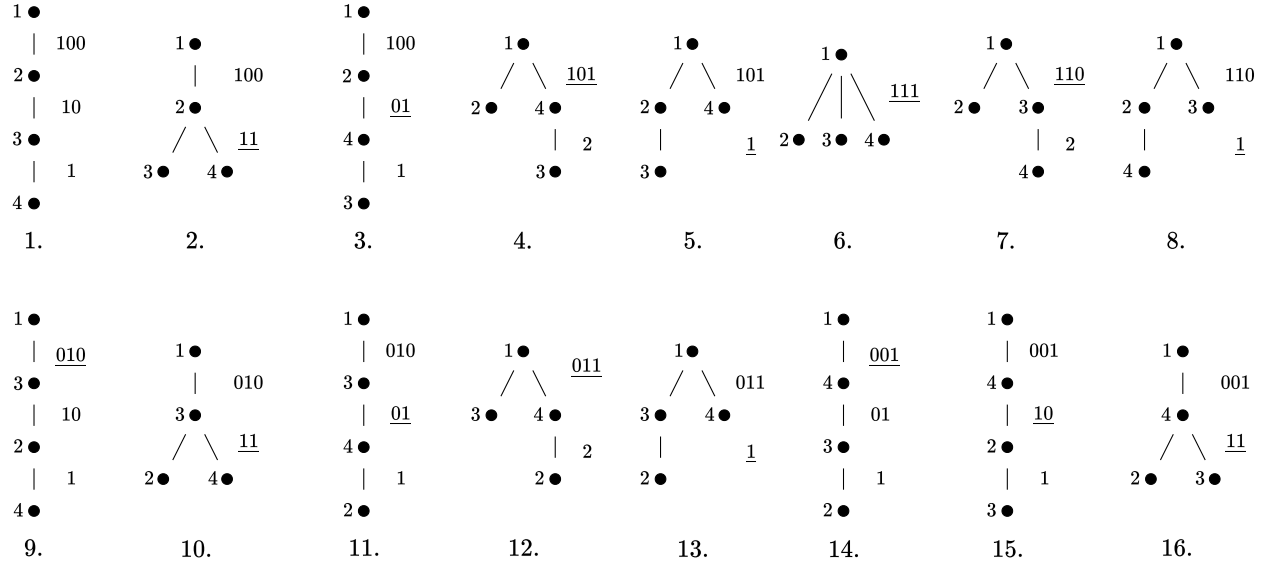


Figure 2.2: Pivot Gray code for spanning trees of K_4 .

Proof. Let $t_\ell = a_1 a_2 \cdots a_{n-q_{\ell-1}}$, where a_v in t_ℓ corresponds to the connection of a vertex v to a vertex at level $\ell - 1$, where v is not in $S_{\ell-1}$. We consider the following four cases for a pivot change on an edge involving a vertex at level $\ell - 1$, while preserving the subtree $S_{\ell-1}$ spanning levels 0 to $\ell - 1$:

- v at level $q > \ell$ disconnects from its parent w and connects to a vertex u at level $\ell - 1$: Initially, $a_v = 0$, and it updates to $a_v = r$ for some $r > 0$, requiring a single symbol change in t_ℓ ;
- v at level ℓ disconnects from its parent u at level $\ell - 1$ and connects to a vertex w at level $q \geq \ell$: Initially, $a_v = r$ for some $r > 0$, and it updates to $a_v = 0$, requiring a single symbol change in t_ℓ ;
- v at level ℓ disconnects from its parent u at level $\ell - 1$ and connects to another vertex w at level $\ell - 1$: Initially, $a_v = r$ for some $r > 0$, and it updates to $a_v = j$ for some $j > 0$ where $j \neq r$, requiring a single symbol change in t_ℓ ;
- A vertex u at level $\ell - 1$ disconnects from its child v at level ℓ and connects to a vertex $w \neq v$ at level $q > \ell$, with v being the only vertex at level ℓ : Let a_w be the character in t_ℓ that corresponds to the connection of vertex w to a vertex at level $\ell - 1$. Initially, $a_v = r$ for some $r > 0$, and $a_w = 0$ since w is at level $q > \ell$; they update to $a_v = 0$ and $a_w = r$, requiring a swap between a 0 and a non-zero character in t_ℓ . To maintain the connectivity, w must be a descendant of v .

These four cases are illustrated in Figure 2.3. Each case corresponds to applying a pivot change to an edge involving a vertex at level $\ell - 1$. The first three cases (cases (a), (b), and (c)) correspond to a single symbol change in t_ℓ , and the last case (case (d)) corresponds to the swapping of a 0 and an $r > 0$ in t_ℓ when $t_\ell = 0^{n-q_{\ell-1}-b-1}r0^b$ for some $b \geq 0$. $\square$

As previously described, our recursive algorithm generates pivot Gray codes for K_n by recursively enumerating all possible rooted trees rooted at vertex 1, with a given subtree $S_{\ell-1}$ fixed for levels 0 to $\ell - 1$. By Lemma 2.1, this corresponds to generating all possible $(p + 1)$ -ary strings for t_ℓ in the tuple $T = (t_1, t_2, \dots, t_h)$, excluding $0^{n-q_{\ell-1}}$, while keeping t_1 to $t_{\ell-1}$ unchanged. Thus, the algorithm can be interpreted as traversing listings of $(p + 1)$ -ary strings of length $n - q_{\ell-1}$ that satisfy the following conditions:

- Consecutive strings differ by a single symbol change, or by swapping a 0 and a non-zero character, which both strings have exactly one non-zero character (Lemma 2.2);

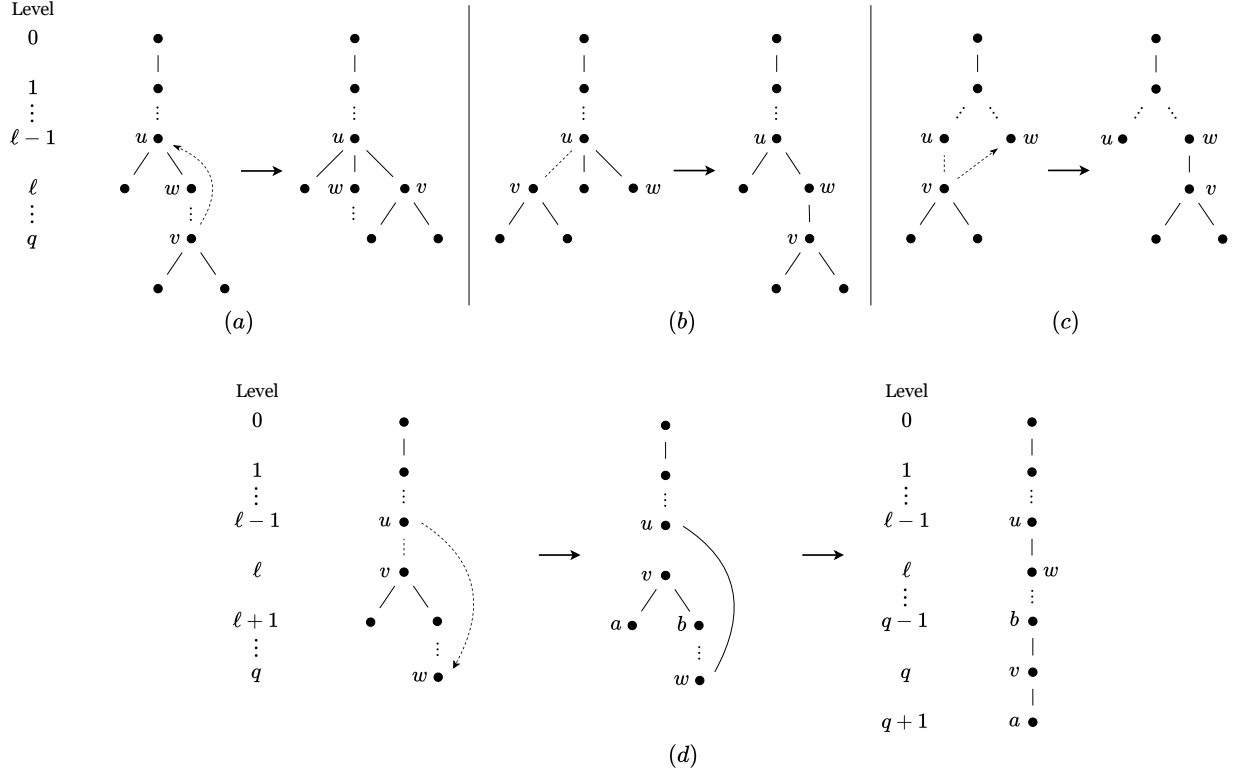


Figure 2.3: The four cases for a pivot change on an edge involving a vertex at level $\ell - 1$, while preserving the subtree $S_{\ell-1}$ spanning levels 0 to $\ell - 1$.

2. The listing exhaustively includes all possible $(p + 1)$ -ary strings of length $n - q_{\ell-1}$. The listing must be able to start from any valid $(p + 1)$ -ary string. This is necessary because a pivot change at level ℓ may require strings at levels greater than ℓ to begin with an arbitrary $(p + 1)$ -ary string (see tree 14 in Figure 2.2);
3. The string $0^{n-q_{\ell-1}}$ is excluded.

Such listings can be produced using Gray codes for k -ary strings. There are many algorithms for generating 1-Gray codes for the set of k -ary strings [27, 31, 32, 33, 34, 40, 41]. With minor modifications, many of these algorithms can be adapted to meet the above conditions. For example, consider the following cyclic 1-Gray code listings L_2^3 , L_2^2 , L_3^1 , and L_2^1 for the sets of binary strings of length three, binary strings of length two, ternary strings of length one, and binary strings of length one, respectively:

- L_2^3 : 100, 101, 111, 110, 010, 011, 001, 000;
- L_2^2 : 10, 11, 01, 00;
- L_3^1 : 2, 1, 0;
- L_2^1 : 1, 0.

Let L' denote the sequence of strings of length j obtained by removing the string 0^j from a listing L . By removing the string 0^j from each of the above listings, we obtain the following sequences that satisfy the conditions:

- $L_2^{3'}$: 100, 101, 111, 110, 010, 011, 001;
- $L_2^{2'}$: 10, 11, 01;
- $L_3^{1'}$: 2, 1;

Algorithm 2.1 Generating pivot Gray codes for spanning trees of complete graphs

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1: procedure GENS( $V, t_{\ell-1} = a_1 a_2 \cdots a_{|V|}$ )
2:    $P \leftarrow v_i \in V$  where  $a_i > 0$ 
3:    $C \leftarrow v_i \in V$  where  $a_i = 0$ 
4:   if  $|C| = 0$  then
5:     Print spanning tree
6:   return
7:    $t_\ell = b_1 b_2 \cdots b_{|C|} \leftarrow 0^{|C|}$ 
8:   for  $c_i \in C$  do
9:     if  $\pi(c_i) \in P$  then  $b_i \leftarrow j$  where  $\pi(c_i) = p_j \in P$ 
10:  GENS( $C, t_\ell$ )
11:  while NEXTK( $t_\ell, |P| + 1$ )  $\neq \emptyset$  do
12:     $t_\ell, \delta \leftarrow \text{NEXTK}(t_\ell, |P| + 1)$ 
13:    if  $|\delta| > 1$  then
14:       $\pi(c_{\delta_1}) \leftarrow \pi(c_{\delta_0})$  ▷ case (d)
15:      LIFT( $c_{\delta_1}$ )
16:    else if  $b_{\delta_0} > 0$  then
17:       $\pi(c_{\delta_0}) \leftarrow p_{b_{\delta_0}} \in P$  ▷ case (a) and (c)
18:    else
19:       $\pi(c_{\delta_0}) \leftarrow c_j \in C$  where  $b_j > 0$  ▷ case (b)
20:  GENS( $C, t_\ell$ )

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- $L_2^{1'}$: 1.

Note that L_2^3 , L_2^2 , L_3^1 , and L_2^1 are cyclic, meaning the first and last strings also differ by a single symbol change or a swap between a 0 and a non-zero character.

We now demonstrate how our algorithm uses the listings $L_2^{3'}$, $L_2^{2'}$, $L_3^{1'}$, and $L_2^{1'}$ to construct a pivot Gray code for the spanning trees of K_4 . Initially, vertex 1 is at level zero of the rooted tree, and the algorithm recursively enumerates all possible strings for t_ℓ in the tuple $T = (t_1, t_2, \dots, t_h)$. For the connection between levels zero and one, since level zero only contains vertex 1, then $n - q_0 = n - 1 = 3$, and thus t_1 is a binary string of length three. Thus, our algorithm enumerates each string in $L_2^{3'}$. The first string in $L_2^{3'}$ is 100, so $t_1 = 100$, which corresponds to vertex 2 at level one connecting to vertex 1 at level zero. Next, the algorithm recursively considers level two, where $n - q_1 = n - 2 = 2$ since the remaining vertices are vertices 3 and 4, and level one contains only vertex 2. Thus, t_2 is a binary string of length two, and the algorithm enumerates each string in $L_2^{2'}$. The first string in $L_2^{2'}$ is 10, so $t_2 = 10$, which corresponds to vertex 3 at level two connecting to vertex 2 at level one. The algorithm then recursively considers level three, with the only remaining vertex 4, and level two containing only vertex 3. Thus, t_3 is a binary string of length one, and the algorithm enumerates each string in $L_2^{1'}$. The only string in $L_2^{1'}$ is 1, so $t_3 = 1$ which corresponds to vertex 4 at level three connecting to vertex 3 at level two. Since $L_2^{1'}$ contains only one string, all strings in $L_2^{1'}$ are exhausted. The algorithm then backtracks to level two and generates the next string in $L_2^{2'}$ after 10, which is 11. This implies that vertex 2 at level one is connected to both vertices 3 and 4 at level two. This process continues and the resulting listing of spanning trees for K_4 is shown in Figure 2.2. Note that in some cases, the algorithm may not start with the first string in these sequences. For example, in the transition from tree 13 to tree 14 in Figure 2.2, the pivot change disconnects the parent of vertex 3 and connects vertex 3 to vertex 4. At level two of tree 14, the connection between vertices 4 and 3 corresponds to $t_2 = 01$, which is not the first string in $L_2^{2'}$. In such cases, the algorithm starts with 01 in $L_2^{2'}$ and continues generating strings until it reaches the end of the sequence, then cycles to the front of the sequence to exhaust all strings in $L_2^{2'}$. Since $L_2^{2'}$ is cyclic, consecutive strings differ by a single symbol change or by swapping a 0 and a non-zero character, ensuring the resulting listing remains a Gray code.

Our algorithm GENS generates a pivot Gray code for spanning trees of K_n takes two parameters, where V is the set of vertices not yet included in the rooted tree up to level $\ell - 2$, and $t_{\ell-1} = a_1 a_2 \cdots a_{|V|}$ is a k -ary string encoding parent assignments for vertices at level $\ell - 1$. A parent pointer is maintained for each vertex v , where $\pi(v)$ denotes its parent in the rooted tree. The algorithm initializes with the spanning tree $1 - 2 - \cdots - n$,

setting $\pi(v) = v - 1$ for each vertex $v \geq 2$ and $\pi(1) = 0$ (indicating the root). The algorithm maintains two sets: $P = \{v_i \in V \mid a_i > 0\}$, containing vertices at level $\ell - 1$, and $C = \{v_i \in V \mid a_i = 0\}$, containing vertices not in levels 0 to $\ell - 1$. If $|C| = 0$, then the rooted tree includes all n vertices of K_n , so the algorithm prints the spanning tree and returns. Otherwise, our algorithm updates $t_\ell = b_1 b_2 \cdots b_{|C|}$ by checking the current connections between vertices in P and vertices in C . It then calls $\text{GENS}(C, t_\ell)$ to recursively construct rooted trees with the same subtree $S_{\ell-1}$. Next, the algorithm iteratively updates t_ℓ using $\text{NEXTK}(t_\ell, |P| + 1)$, which generates the next k -ary string in Gray code order with base $k = |P| + 1$. The function NEXTK returns the updated t_ℓ and the indices δ which stores the positions changed from the previous string. Efficient generation of k -ary Gray codes is discussed in Section 2.3. If $|\delta| > 1$, then this corresponds to case (d) of Lemma 2.2, the algorithm sets the parent of $c_{\delta_1} \in C$ to the parent of $c_{\delta_0} \in C$. We then “lift” the vertex $c_{\delta_1} \in C$ to level ℓ and updating the parent pointers of ancestors of c_{δ_1} (line 15, Algorithm 2.1 by calling $\text{LIFT}(c_{\delta_1})$). If $|\delta| = 1$, there are two cases:

- If b_{δ_0} is updated to a value $r > 0$, then this case corresponds to cases (a) or (c) of Lemma 2.2, the algorithm sets the parent of c_{δ_0} to $p_r \in P$;
- If b_{δ_0} is updated to 0, then this case corresponds to case (b) of Lemma 2.2, the algorithm sets the parent of c_{δ_0} to a vertex $c_j \in C$ where $b_j > 0$ (a sibling at level ℓ with a parent).

The algorithm then calls $\text{GENS}(C, t_\ell)$ to recursively construct rooted trees with the updated subtree S_ℓ . Pseudocode of the algorithm is shown in Algorithm 2.1. To run the algorithm, we make the initial call $\text{GENS}(\{1, 2, 3, \dots, n\}, (10^{n-1}))$ which sets $V = \{1, 2, 3, \dots, n\}$, and $t_0 = 10^{n-1}$. A complete Python implementation of the algorithm, incorporating optimizations discussed in later subsections, is provided in Appendix A.1.

THEOREM 2.3. *The algorithm GENS generates a pivot Gray code for the spanning trees of K_n .*

Proof. By Lemma 2.2, the algorithm GENS ensures that consecutive spanning trees differ by a single pivot change. To establish the pivot Gray code property, it suffices to show that each spanning tree of K_n appears exactly once in the listing. We prove this by contradiction.

Suppose there exists a spanning tree of K_n that does not appear in the listing generated by GENS. Such a spanning tree can be encoded as a tuple $T = (t_1, t_2, \dots, t_h)$. By Lemma 2.1, the absence of T in the listing implies that no spanning tree in the listing has a subtree S_{h-1} , spanning levels 0 to $h - 1$, that matches the first $h - 1$ components $(t_1, t_2, \dots, t_{h-1})$ of T . Otherwise, GENS would exhaustively generate a Gray code for t_h (via GENK) to include T . Applying the same argument recursively, no spanning tree in the listing has a subtree S_{h-2} , spanning levels 0 to $h - 2$, matching $(t_1, t_2, \dots, t_{h-2})$. Continuing applying the same reasoning implies that no spanning tree has a subtree S_0 at level zero matching t_1 . However, at level zero, GENS generates a Gray code of length $n - 1$ (via GENK) that includes all binary strings except 0^{n-1} , covering all possible t_1 for spanning trees of K_n , a contradiction. $\square$

Storing the entire Gray code listing for k -ary strings, however, requires exponential space. In the next subsection, we present a simple algorithm that generates the next k -ary string in the Gray code listing using only a linear amount of space.

2.3 Efficient generation of the underlying Gray codes for k -ary strings and the pivot Gray codes for spanning trees of K_n . To generate a Gray code for k -ary strings, excluding the string $0^{n-q_{\ell-1}}$, that satisfies the three conditions discussed in Section 2.2, we adopt the reflectable language framework described in [21]. In [21], Li and Sawada developed a framework for generating Gray codes for reflectable languages. A language L over an alphabet Σ is reflectable if, for every $i > 1$, there exist two symbols $x_i, y_i \in \Sigma$ such that if $w_1 w_2 \cdots w_{i-1}$ is a prefix of a word in L , then both $w_1 w_2 \cdots w_{i-1} x_i$ and $w_1 w_2 \cdots w_{i-1} y_i$ are also prefixes of words in L . By reflecting the order of the children and using the special symbols x_i and y_i as the first and last children of each node at level $i - 1$, Li and Sawada devised a generic recursive algorithm to generate Gray codes for reflectable languages, including combinatorial objects such as k -ary strings, restricted growth functions, and k -ary trees. For example, their algorithm generates the binary reflected Gray code discussed in Section 1 when setting $x_i = 0$ and $y_i = 1$ for every level $i > 1$.

LEMMA 2.4. *The set of k -ary strings of length j , excluding the string 0^j , is a reflectable language when $k > 2$.*

Proof. By setting $x_i = k - 1$ and $y_i = k - 2$ for each level i , k -ary strings of length j are reflectable. $\square$

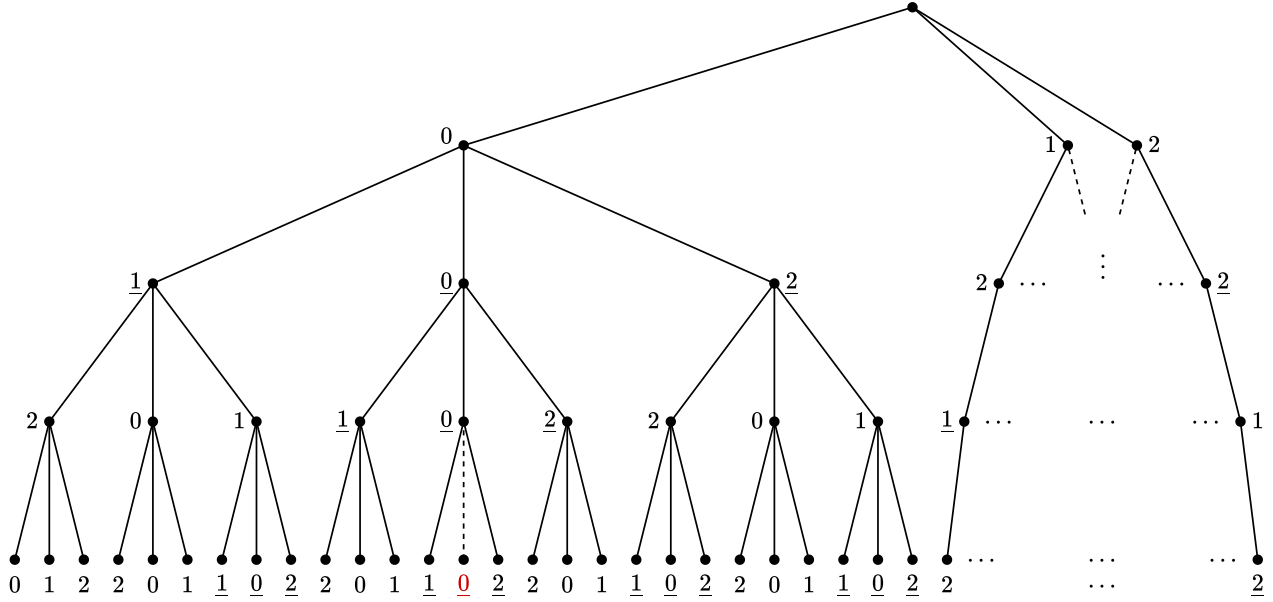


Figure 2.4: Reflectable Gray code of ternary strings of length four generated by GENK, starting with 0120.

Since the set of k -ary strings of length j , excluding the string 0^j , is a reflectable language, we can leverage the reflectable language framework described in [21] to generate a 1-Gray code for this set in constant amortized time per string using only a linear amount of space. The framework is also quite versatile, allowing the Gray code to start with any arbitrary string. This is particularly useful in the construction of our pivot Gray code for spanning trees of K_n , where by the second condition in Section 2.2, a pivot change at level ℓ may require strings at levels greater than ℓ to begin with an arbitrary k -ary string of length j .

Our algorithm for generating a 1-Gray code for k -ary strings of length j , excluding 0^j , maintains a global direction array d , where d_i indicates the ordering direction for the i -th character. If $d_i = 1$, the children of the i -th character start with a_i and increment by 1 (modulo k) until reaching $a_i - 1$. If $d_i = -1$, the children start with $a_i - 1$ and decrement by 1 (modulo k) until reaching a_i . The algorithm avoids generating the string 0^j . Two special cases ensure the listing remains a 1-Gray code when $k > 2$ by preventing $x_i = 0$ or $y_i = 0$, which would disrupt the reflectable property if 0^j were removed:

- If the i -th character of the initial string $\alpha = a_1 a_2 \cdots a_j$ has $a_i = 1$, we set $d_i = -1$, so its children start with $y_i = 1$ and end with $x_i = 2$;
- If the i -th character of the initial string $\alpha = a_1 a_2 \cdots a_j$ has $a_i = 0$, we set $d_i = 1$ for the first $2k$ characters, updating a_i from 0 to $k - 1$, and then from $k - 1$ to $k - 2$ for subsequent strings. We then set $x_i = k - 1$ and $y_i = k - 2$, and apply the usual reflectable strategy for the remaining updates of a_i .

As an example, our algorithm generates a 1-Gray code for ternary strings of length four excluding the string 0000, starting with the initial string 0120 as shown below:

0120, 0121, 0122, 0102, 0100, 0101, 0111, 0110, 0112, 0012, 0010, 0011, 0001, 0002, 0022, 0020,
0021, 0022, 0220, 0222, 0202, 0200, 0201, 0211, 0210, 0212, 1212, 1210, 1211, 1201, 1200, 1202,
1222, 1220, 1221, 1021, 1020, 1022, 1002, 1000, 1001, 1011, 1010, 1012, 1112, 1110, 1111, 1101,
1100, 1102, 1122, 1120, 1121, 2121, 2120, 2122, 2102, 2100, 2101, 2111, 2110, 2112, 2012, 2010,
2011, 2001, 2000, 2002, 2022, 2020, 2021, 2221, 2220, 2222, 2202, 2200, 2201, 2211, 2210, 2212.

The recursive construction tree is shown in Figure 2.4. Pseudocode of the algorithm is shown in Algorithm 2.2. To run the algorithm, we make the initial call $\text{GENK}(i, k, \alpha)$, where i is the current level of the recursive computation

Algorithm 2.2 Generating k -ary strings starting with α in reflectable Gray code order.

```

1: procedure GENK( $i, k, \alpha = a_1 a_2 \cdots a_j$ )
2:   for  $r = 1$  to  $k$  do
3:     if  $i = |\alpha|$  then
4:       if  $s \neq 0$  then
5:         Output  $\alpha, \delta$ 
6:          $\delta \leftarrow \emptyset$ 
7:       else
8:         GENK( $i + 1, k, \alpha$ )
9:       if  $a_{i+1} = k - 1$  then  $d_{i+1} \leftarrow 1$ 
10:      else  $d_{i+1} \leftarrow -d_{i+1}$ 
11:    if  $r < k$  then
12:       $s \leftarrow s + ((a_i + d_i + k) \bmod k) - a_i$ 
13:       $a_i \leftarrow (a_i + d_i + k) \bmod k$ 
14:       $\delta \leftarrow \delta, i$ 

```

tree (initialized to 1), k is the base of the k -ary strings, and α is the initial string of the listing. A complete Python implementation of an advanced of GENK, namely GENMR, which generates Gray codes for mixed-radix strings, is provided in Appendix B. This implementation can generate Gray codes for k -ary strings by setting $k_i = k$ for all positions.

LEMMA 2.5. *Consecutive strings generated by GENK differ by a single symbol change, or by swapping a 0 and a 1 which the strings have only one single non-zero character.*

Proof. If $k > 2$, the set of k -ary strings of length j , excluding the string 0^j , forms a reflectable language. The algorithm GENK selects x_i and y_i as non-zero characters, ensuring the listing is a 1-Gray code by the results in [21]. Thus, consecutive strings differ by a single symbol change. Otherwise, if $k = 2$, observe that the set of binary strings of length j , including the string 0^j , is a reflectable language, and GENK generates a 1-Gray code for this set. When the string 0^j is removed, there are two cases: if 0^j is at the end of the listing, its removal does not affect the Gray code property; otherwise, if 0^j appears between two strings, say $0^x 10^{j-x-1}$ and $0^y 10^{j-y-1}$ where $x \neq y$, these strings differ by a swap between a 0 and a 1. $\square$

The algorithm GENK efficiently generates pivot Gray codes for spanning trees of K_n . For example, the algorithm GENK produces the following 1-Gray code listings for k -ary strings of length j , starting with an initial string α :

- $j = 3, k = 2, \alpha = 100$: 100, 101, 111, 110, 010, 011, 001;
- $j = 2, k = 2, \alpha = 10$: 10, 11, 01;
- $j = 2, k = 2, \alpha = 11$: 11, 01, 10;
- $j = 1, k = 3, \alpha = 2$: 2, 1;
- $j = 1, k = 3, \alpha = 1$: 1, 2.

The resulting listing of spanning trees for K_4 , generated using these underlying Gray codes for k -ary strings of length j by GENS (Algorithm 2.1), is shown in Figure 2.2.

We now prove the complexity of our algorithm GENS (Algorithm 2.1) for generating pivot Gray codes for spanning trees of K_n . As standard for generation algorithms, the time required to output a spanning tree is not part of the analysis.

THEOREM 2.6. *Spanning trees of complete graphs K_n can be generated in pivot Gray code order in constant amortized time per tree, using $O(n^2)$ space.*

Proof. The algorithm GENS recursively constructs spanning trees of K_n by enumerating k -ary strings for t_ℓ in the tuple $T = (t_1, t_2, \dots, t_h)$, where t_ℓ encodes connections from level ℓ to level $\ell - 1$. By Lemma 2.5, the function

GENK generates 1-Gray code listings for k -ary strings of length $|t_\ell|$, excluding $0^{|t_\ell|}$, in constant amortized time per string, as established in [21]. Using a function iterator, NEXTK updates t_ℓ in constant amortized time per string by tracking the position that differs from the previous string, which is achieved by storing the last level of the computation tree where the character is not x_i or y_i .

The algorithm GENS has at most $n - 1$ levels in the rooted tree. The space required by GENK includes the direction array d of length at most n , which uses $O(n)$ space for each level ℓ , thus utilizing $nO(n) = O(n^2)$ space. For GENS, let $t = |t_{\ell-1}|$. Updating sets P and C (lines 2 and 3 of Algorithm 2.1) takes $O(t)$ time when P and C are maintained as linked lists. Computing $t_\ell = b_1 b_2 \cdots b_{|C|}$ which describes connections between P and C , can be optimized to $O(t)$ by maintaining a hash table m that maps $\pi(c_i) = p_j \in P$ to j . The details of the hash table m are provided in Appendix A. Thus, operations before the while loop (lines 2–10 of Algorithm 2.1) require $O(t)$ time. However, the while loop (lines 11–20 of Algorithm 2.1) generates $(|P| + 1) \cdot |C| - 1$ configurations. In the worst case, it produces at least $t - 1$ configurations when $|C| = 1$, which differs by a constant factor; otherwise, the number of configurations grows exponentially and is far larger than t . This amortizes the $O(t)$ time to constant time per tree.

Within the while loop, the lift operation (line 15 of Algorithm 2.1) requires $O(t)$ time in the worst case. However, this operation is only triggered by a swap between a 0 and a non-zero character, which occurs at most once per 1-Gray code listing of 2^t binary strings ($k = 2$). Thus, its cost is amortized to constant time per tree. For the operation in line 19 of Algorithm 2.1, which connects a disconnected $c_i \in C$ to its sibling, a hash table C' with chaining via a doubly linked list can be maintained to efficiently store and access the siblings of c_i at level ℓ in constant time. The implementation details are provided in Appendix A. Other operations, such as updating π and recursive calls, require constant time and use $O(n)$ space for the hash table π , tuple T , and sets P and C . Therefore, GENS generates each spanning tree of K_n in constant amortized time per tree, using $O(n^2)$ space. $\square$

Details of the optimizations for the algorithms GENS and GENK are provided in Appendix A. The spanning trees of K_5 generated by GENS and GENK are listed in Appendix D.

2.4 Enumeration. The number of spanning trees in a complete graph K_n is given by Cayley's formula, which states that there are n^{n-2} spanning trees for K_n . As previously mentioned, Cayley provided an informal proof of this formula for K_n [5]. Later, Prüfer [30] and Kirchhoff [18] offered simpler proofs by establishing a one-to-one correspondence between spanning trees and Prüfer sequences and by applying the matrix-tree theorem, respectively. Mütze [27], however, commented that in *The Art of Computer Programming, Volume 4* [20] Knuth sought a more detailed combinatorial explanation of Cayley's formula. This subsection thus provides a simpler proof of Cayley's formula for the number of spanning trees in complete graphs, based on observations from our recursive algorithm.

THEOREM 2.7. *The number of spanning trees in a complete graph K_n is n^{n-2} .*

Proof. In Algorithm 2.1, GENS takes V as an input, which is a set that contains vertices not in levels 0 to $\ell - 2$ of the rooted tree. It computes sets P and C , which contain the vertices at level $\ell - 1$ and the vertices not at levels 0 to $\ell - 1$, respectively. We now derive the total number of spanning trees that can be formed given a subtree spanning levels 0 to $\ell - 2$ and the set of remaining vertices V .

Let $k = |P|$ and $n = |C|$. Define $f(k, n)$ as the number of spanning trees that can be formed given a subtree spanning levels 0 to $\ell - 2$, with P as the set of vertices at level $\ell - 1$ and C as the set of vertices not in levels 0 to $\ell - 1$. If $n = 0$, there is only one possible spanning tree, where all vertices in V are at level ℓ . If $n > 0$, we select i vertices from C to connect to vertices in P , where each selected vertex can connect to any of the k vertices in P . By the product rule, there are $\binom{n}{i} k^i$ possible connections between vertices at level $\ell - 1$ and vertices at level ℓ . For the $n - i$ vertices in C not selected to connect to P , there are $f(i, n - i)$ ways to form a spanning tree with the i vertices at level ℓ . Summing over all possible values of i from 0 to n , the total number of spanning trees is given by the recursive formula:

$$f(k, n) = \begin{cases} 1, & \text{if } n = 0; \\ \sum_{i=1}^n \binom{n}{i} k^i f(i, n - i), & \text{otherwise.} \end{cases}$$

We now prove by strong induction on n that

$$f(k, n) = k(k + n)^{n-1}.$$

For the base case, when $n = 0$:

$$\begin{aligned} f(k, 0) &= 1 \\ &= k(k + 0)^{-1} \end{aligned}$$

For $n = 1$:

$$\begin{aligned} f(k, 1) &= \sum_{i=1}^1 \binom{1}{i} k^i f(i, 1-i) \\ &= \binom{1}{1} k^1 f(1, 0) \\ &= k \cdot 1 \\ &= k(k + 1)^0. \end{aligned}$$

Assume by strong induction that $f(k, t) = k(k + t)^{t-1}$ for all $t < n$. For $t = n$:

$$\begin{aligned} f(k, n) &= \sum_{i=1}^n \binom{n}{i} k^i f(i, n-i) \\ &= \sum_{i=1}^n \binom{n}{i} k^i \cdot i n^{n-i-1} \end{aligned}$$

Extract $\frac{n}{i}$ from $\binom{n}{i} = \frac{n}{i} \binom{n-1}{i-1}$ for $i \geq 1$:

$$\begin{aligned} f(k, n) &= \sum_{i=1}^n \binom{n-1}{i-1} \cdot \frac{n}{i} \cdot k^i \cdot i n^{n-i-1} \\ &= \sum_{i=1}^n \binom{n-1}{i-1} k^i \cdot n^{n-i} \\ &= k \sum_{i=1}^n \binom{n-1}{i-1} k^{i-1} \cdot n^{n-i} \end{aligned}$$

By setting $j = i - 1$:

$$f(k, n) = k \sum_{j=0}^{n-1} \binom{n-1}{j} k^j \cdot n^{n-j-1},$$

The sum $\sum_{j=0}^{n-1} \binom{n-1}{j} k^j n^{n-j-1}$ is the binomial expansion of $(k + n)^{n-1}$, so:

$$f(k, n) = k(k + n)^{n-1}.$$

For a complete graph K_{n+1} with $n + 1$ vertices, set $k = 1$ (root at level 0) and $|C| = n$ (remaining vertices), so:

$$f(1, n) = 1 \cdot (1 + n)^{n-1} = (n + 1)^{n-1}.$$

Adjusting for the complete graph K_n , we correct the index: $f(1, n - 1) = 1 \cdot (1 + (n - 1))^{n-2} = n^{n-2}$, which matches Cayley's formula for K_n . $\square$

3 Edge-exchange Gray codes for spanning tree of general graphs. With modifications, our algorithm GENS can be extended to generate edge-exchange Gray codes for connected general graphs. The approach for generating edge-exchange Gray codes for general graphs is similar to that for generating pivot Gray codes for complete graphs. However, a key difference arises between complete and general graphs. In GENS,

given sets P and C in the rooted tree, where P contains vertices at level $\ell - 1$ and C contains vertices not in levels 0 to $\ell - 1$, every vertex in C can connect to any vertex in P in a complete graph, yielding $|P||C|$ possible configurations for connections between levels $\ell - 1$ and ℓ . In general graphs, however, not all such connections may exist, as an edge between $p \in P$ and $c \in C$ may be absent in the original graph. Thus, additional handling is required to account for valid connections between vertices in P and C .

Below, we describe three mechanisms to modify our algorithm GENS to generate edge-exchange Gray codes for connected general graphs. We illustrate the handling required for each case using the example graph G shown in Figure 3.1.

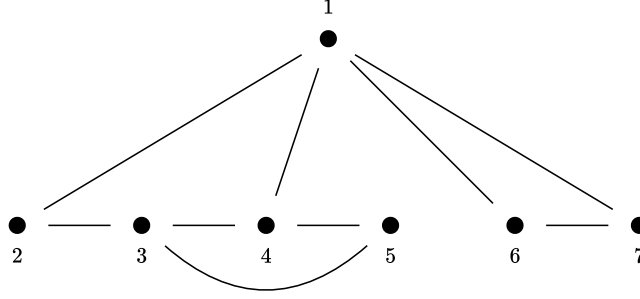


Figure 3.1: A connected general graph G with seven vertices.

The three modifications to GENS are discussed in the following three subsections.

3.1 Mixed-radix string representation. As previously noted, given sets P and C in the rooted tree, where P contains vertices at level $\ell - 1$ and C contains vertices not in levels 0 to $\ell - 1$, not all connections between vertices in P and C may exist, as an edge between $p \in P$ and $c \in C$ may be absent in the original graph. The challenge is to represent all valid configurations when some connections are not possible.

Consider the spanning tree of the graph G in Figure 3.1 and its corresponding rooted tree shown in Figure 3.2.

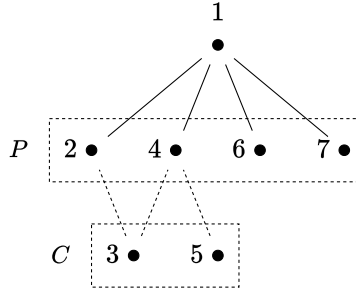


Figure 3.2: The rooted tree that corresponds to a spanning tree of G .

At levels 1 and 2 of the rooted tree, we have $P = \{2, 4, 6, 7\}$ and $C = \{3, 5\}$. In this case, vertex 3 has two possible connections to vertices in P , while vertex 5 has only one possible connection. Unlike in complete graphs, where t_ℓ is a $|C|$ -length string with each character a_i ranging from 0 to $|P|$, here each a_i ranges from 0 to the number of valid connections for the corresponding vertex in C . We thus use mixed-radix strings to encode parent assignments between vertices in P and C for general graphs, unlike using k -ary strings for complete graphs.

A *mixed-radix string* is a sequence where each position has its own base (or radix), determining the range of valid values for that position. Each element $t_\ell = a_1 a_2 \cdots a_{n-q_{\ell-1}}$ in the tuple T is now a mixed-radix string of length $n - q_{\ell-1}$, where each a_i ranges from 0 to the number of vertices in P adjacent to the i -th smallest vertex in C . If $a_i = 0$, then the i -th smallest vertex at level ℓ is not connected to any vertex at level $\ell - 1$. Otherwise if $a_i > 0$, then it is connected to the (a_i) -th smallest vertex in P that is adjacent to the i -th smallest vertex in C in the original graph G . For example, in Figure 3.1, if $P = \{2, 4, 6, 7\}$ and $C = \{3, 5\}$ where vertex 3 is adjacent to vertices 2 and 4 in G , and vertex 5 is adjacent to vertex 4 in G , then t_ℓ is a string of length two with

Algorithm 3.1 Generating mixed-radix strings starting with α in reflectable Gray code order.

```

1: procedure GENMR( $i, k = k_1 k_2 \dots k_j, \alpha = a_1 a_2 \dots a_j$ )
2:    $i' \leftarrow m[i]$ 
3:   for  $r = 1$  to  $k_{i'}$  do
4:     if  $i = |\alpha|$  then
5:       if  $s \neq 0$  then
6:         Output  $\alpha, \delta$ 
7:          $\delta \leftarrow \emptyset$ 
8:       else
9:         GENMR( $i + 1, k, \alpha$ )
10:       $j' \leftarrow m[i + 1]$ 
11:      if  $\alpha_{j'} = k_{j'} - 1$  then  $d_{j'} \leftarrow 1$ 
12:      else  $d_{j'} \leftarrow -d_{j'}$ 
13:    if  $r < k_{i'}$  then
14:       $s \leftarrow s + ((\alpha_{i'} + d_{i'} + k_{i'}) \bmod k_{i'}) - \alpha_{i'}$ 
15:       $\alpha_{i'} \leftarrow (\alpha_{i'} + d_{i'} + k_{i'}) \bmod k_{i'}$ 
16:       $\delta \leftarrow \delta, i$ 

```

$a_1 \in \{0, 1, 2\}$ (representing no connection or connections to 2 or 4) and $a_2 \in \{0, 1\}$ (representing no connection or a connection to 4). The possible configurations for vertices 3 and 5 can be represented by a mixed-radix string of length two, with the first digit in base three (for vertex 3's connections) and the second digit in base two (for vertex 5's connection). The valid mixed-radix strings representing these configurations are 01, 10, 11, 20, 21. Note that the string 00 is omitted from the listing, as a spanning tree requires at least one connection between levels $\ell - 1$ and ℓ .

To generate a Gray code for spanning trees of general graphs, we generate a Gray code for mixed-radix strings representing connections between vertices in P and C , satisfying the three conditions in Section 2.2.

LEMMA 3.1. *The set of mixed-radix strings of length j is a reflectable language.*

Proof. Let k_i denote the radix of the character a_i in a mixed-radix string $a_1 a_2 \dots a_j$ of length j , where $0 \leq a_i < k_i$ for each i . By setting $x_i = k_i - 1$ and $y_i = k_i - 2$ for each i , observe that mixed-radix strings of length j are reflectable. $\square$

The algorithm GENK is modified to create GENMR, which generates a Gray code for mixed-radix strings by replacing the parameter k with an array $(k_1, k_2, \dots, k_j)$, where $k_i \geq 1$ specifies the range of each digit a_i in a string $\alpha = a_1 a_2 \dots a_j$, with $a_i \in \{0, 1, \dots, k_i - 1\}$. To ensure spanning tree connectivity, the all-zero string 0^j is excluded, requiring at least one non-zero digit. To omit 0^j , we apply a simple trick when there exists an index $i < j$ with $k_i \geq 2$ and the last position has $k_j = 2$. In this case, we swap the levels of i and j when we generate the reflectable Gray code for mixed-radix strings. This swap ensures that the string 0^j appears between the strings $0^{j-1}x$ and $0^{j-1}y$, where $x \neq y$, allowing us to remove 0^t while maintaining a 1-Gray code. A graphical interpretation of this simple trick is illustrated in Figure 3.3. It is also worth noting that if there exists $k_i = 1$ for some position i , then $a_i = 0$ is the only possibility, which does not affect the 1-Gray code property of the resulting listing. Pseudocode of the algorithm is shown in Algorithm 3.1. To run the algorithm, we make the initial call GENMR(i, k, α), where i is the current level of the recursive computation tree (initialized to 1), $k = (k_1, k_2, \dots, k_j)$ is the array of radices for each digit in α , and α is the initial string of the listing. A complete Python implementation of GENMR is provided in Appendix B.

LEMMA 3.2. *Consecutive strings generated by GENMR differ by a single symbol change, or by swapping a 0 and a 1 which the strings have only one single non-zero character.*

Proof. The proof follows a similar approach to that of Lemma 2.5. If $k_i = 2$ for all i , the listing is binary, and GENMR generates the same listing as GENK in Algorithm 2.2. Thus, consecutive strings differ by a single bit change or a swap between a 0 and a 1. If there exists some $k_i = 1$, then $a_i = 0$, fixing the digit 0 at that level i , and the fixed digit 0 does not affect the 1-Gray code property. Otherwise, if there exists some $k_i > 2$,

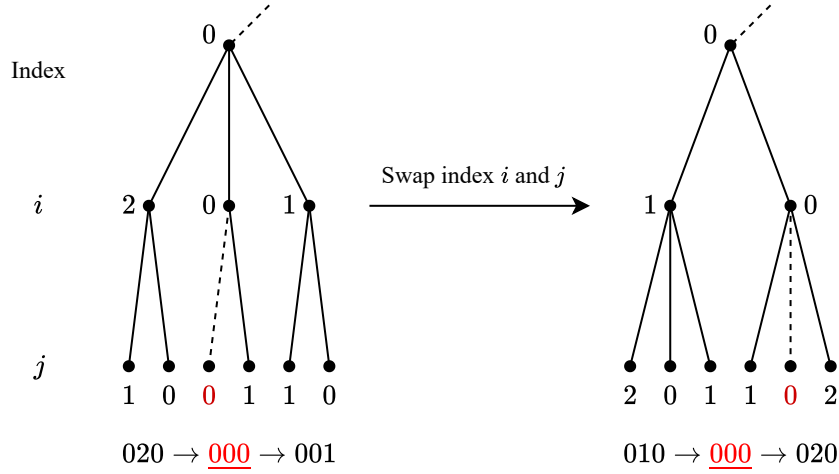


Figure 3.3: Swapping the ordering of c_i and c_j in C to generate a reflectable Gray code.

our simple trick on swapping c_i with c_j in C with $k_j = 2$ ensures that the last character has base $k_j > 2$. By Lemma 3.1, matrix-radix strings form a reflectable language, and GENMR produces a 1-Gray code when the string 0^j is included in the listing. Observe that 0^j appears between strings $0^{j-1}x$ and $0^{j-1}y$ when $k_j > 2$, where $x \neq y$ and $x, y \neq 0$. Removing the string 0^j maintains the 1-Gray code property, as the transition from $0^{j-1}x$ to $0^{j-1}y$ involves a single symbol change in the last position. $\square$

3.2 Connected components. The second modification addresses the connectivity of subtrees. Consider the initial spanning tree of the graph G in Figure 3.1. In our algorithm GENS for complete graphs, the initial spanning tree is the path $1 - 2 - \dots - n$, corresponding to the tuple $T = (10^{n-1}, 10^{n-2}, \dots, 1)$. However, such an initial tree is not feasible for G . For instance, if vertex 1 is only connected to vertex 2, vertices 6 and 7 become disconnected from the rest of G . Thus, we design a mechanism to ensure that the spanning tree accounts for the connectivity of each component in C with respect to the vertices in V .

Given a vertex set C at level ℓ , we apply depth-first search or breadth-first search to identify the connected components $\{C_1, C_2, \dots, C_m\}$ within C . Each connected component C_i must have at least one vertex connected to a parent in P ; otherwise, it would be disconnected from the rest of the graph, preventing the formation of a spanning tree. We denote $t_{\ell,i}$ as a mixed-radix string of length $|C_i|$, where each character represents the connection of the corresponding vertex in C_i to a vertex at level $\ell - 1$. To ensure connectivity, the string $0^{|C_i|}$, which indicates no connections to P , is excluded for each $t_{\ell,i}$. We then apply the algorithm GENMR to generate all possible mixed-radix strings $t_{\ell,i}$ for each connected component C_i , excluding $0^{|C_i|}$. Thus, GENMR enables the generation of all possible connection configurations for the rooted tree, ensuring that each component C_i maintains at least one connection to P . As an example, consider the initial spanning tree for the graph G . At level zero, $P = \{1\}$, and we consider level one. The algorithm performs a depth-first search or a breadth-first search to identify the connected components in C , yielding $C_1 = \{2, 3, 4, 5\}$ and $C_2 = \{6, 7\}$. For C_1 , only vertices 2 and 4 are adjacent to vertex 1 in P , so $t_{1,1}$ is a mixed-radix string of length four, with characters for vertices 2 and 4 in $\{0, 1\}$ (no connection or connection to vertex 1) and characters for vertices 3 and 5 in $\{0\}$ (no connection possible). For C_2 , both vertices 6 and 7 are adjacent to vertex 1, so $t_{1,2}$ is a mixed-radix string of length two, with each character in $\{0, 1\}$ (no connection or connection to vertex 1). The valid strings for each component exclude $0^{|C_i|}$, ensuring at least one connection to P .

The case for complete graph can actually be considered as a special case of the algorithm to generate spanning trees for general graphs, where there is only one single connected component in a complete graph.

3.3 Two-dimensional recursion for connected components. For complete graphs, the algorithm GENS traverses the rooted tree level by level, generating pivot Gray codes for spanning trees. For general graphs, we account for connected components, introducing a second dimension to the recursion.

We design a two-dimensional recursion to iterate through all possible mixed-radix strings $t_{\ell,i}$ for each

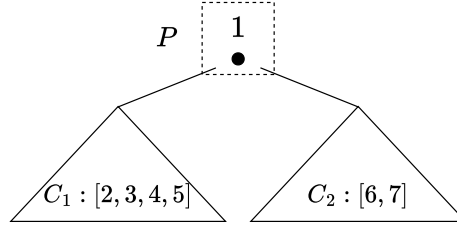


Figure 3.4: A rooted tree of G rooted at vertex 1 with two connected components $C_1 = \{2, 3, 4, 5\}$ and $C_2 = \{6, 7\}$ at level 1.

connected component C_i at level ℓ and all configurations at subsequent levels. In the first dimension, the algorithm iterates over all possible mixed-radix strings $t_{\ell,i}$ for each connected component C_i at level ℓ by invoking GENMR to generate each $t_{\ell,i}$. For each update to $t_{\ell,i}$, we recursively call GENMR to generate all possible mixed-radix strings $t_{\ell,i+1}$ for the next connected component C_{i+1} . In the second dimension, after each update to any $t_{\ell,i}$ at level ℓ , the algorithm descends to level $\ell + 1$, identifies all connected components at level $\ell + 1$ based on the current configuration at level ℓ , and recursively generates all possible mixed-radix strings $t_{\ell+1,i}$ for each connected component at level $\ell + 1$ and possible configurations at subsequent levels.

We now demonstrate how the algorithm generates spanning trees for the graph G . Initially, vertex 1 is at level 0 of the rooted tree, and the algorithm enumerates all possible strings $t_{\ell,i}$ for each connected component at level ℓ . For the connections between levels 0 and 1, we have $P = \{1\}$ and $C = \{2, 3, 4, 5, 6, 7\}$, with two connected components: $C_1 = \{2, 3, 4, 5\}$ and $C_2 = \{6, 7\}$. Since only vertices 3 and 5 in C_1 are adjacent to vertex 1 in G , the string $t_{1,1}$ for C_1 is a mixed-radix string of length four with radices $k_1 = 1$ (for vertex 2), $k_2 = 2$ (vertex 3), $k_3 = 1$ (vertex 4), and $k_4 = 2$ (vertex 5). Since both vertices 6 and 7 in C_2 are adjacent to vertex 1, the string $t_{1,2}$ for C_2 is a binary string of length two with radix $k_i = 2$ for each position. In the first dimension of the recursion, the algorithm generates all combinations of $t_{\ell,i}$, starting with $t_{1,1}$ for C_1 . For each update to $t_{1,1}$, it generates all strings in $t_{1,2}$ for C_2 and all configurations at subsequent levels. Initially, $t_{1,1} = 0100$ (vertex 3 in C_1 connects to vertex 1) and $t_{1,2} = 10$ (vertex 6 in C_2 connects to vertex 1), yielding trees 1, 2, and 3 in Figure 3.7. After exhausting all configurations for subsequent levels, the algorithm updates $t_{1,2}$ to 11 (both vertices 6 and 7 connect to vertex 1), producing trees 4, 5, and 6 in Figure 3.7. Next, it updates $t_{1,2}$ to 01 (vertex 7 connects to vertex 1), yielding trees 7, 8, and 9 in Figure 3.7. After exhausting all strings in $t_{1,2}$, the algorithm updates $t_{1,1}$ to 0101 (vertices 3 and 5 connect to vertex 1) and sets $t_{1,2} = 01$, generating trees 10 to 14 in Figure 3.7. It then updates $t_{1,2}$ to 10 (vertex 6 connects to vertex 1) and continues this process until all spanning trees of G are generated.

3.4 Analysis of the edge-exchange Gray code generation algorithm for general graphs. The algorithm GENG extends GENS (Algorithm 2.1) to generate Gray codes for spanning trees of general graphs by incorporating the changes described in above subsections. The resulting listing forms an edge-exchange Gray code for the spanning trees of a general graph G but does not form a pivot Gray code. For example, in case (b) described in Section 2.2, a vertex v at level ℓ may be the only vertex in its subtree with an edge to a vertex at level $\ell - 1$, while a descendant $x \neq v$ has edges to other vertices at level ℓ within the same connected component (e.g., tree 24 and tree 25 in Figure 3.7, also see Figure 3.6). Removing the edge (v, u) to its parent u at level $\ell - 1$ requires adding an edge (x, w) , where $w \neq v$ is in the same component, to maintain connectivity. This edge exchange does not involve a common vertex, violating the pivot Gray code property, which requires consecutive trees to differ by a single edge exchange involving a common vertex.

The following modifications are applied to GENS to produce GENG:

- At each level ℓ of the rooted tree, apply depth-first search or breadth-first search to identify the connected components in C after computing the set C (line 7 of GENG, Algorithm 3.2);
- For each vertex c_{ij} in connected component C_i , maintain a list F_{ij} of vertices in P to which c_{ij} is adjacent in the graph G (lines 10–12 of GENG);
- Introduce a function GENT (Algorithm 3.3) to recursively generate all mixed-radix strings $t_{\ell,i}$ for each connected component C_i , mirroring lines 11–20 of GENS. The function GENT takes two parameters, where i is the index of the current component C_i in C , and ℓ is the current level of the rooted tree;

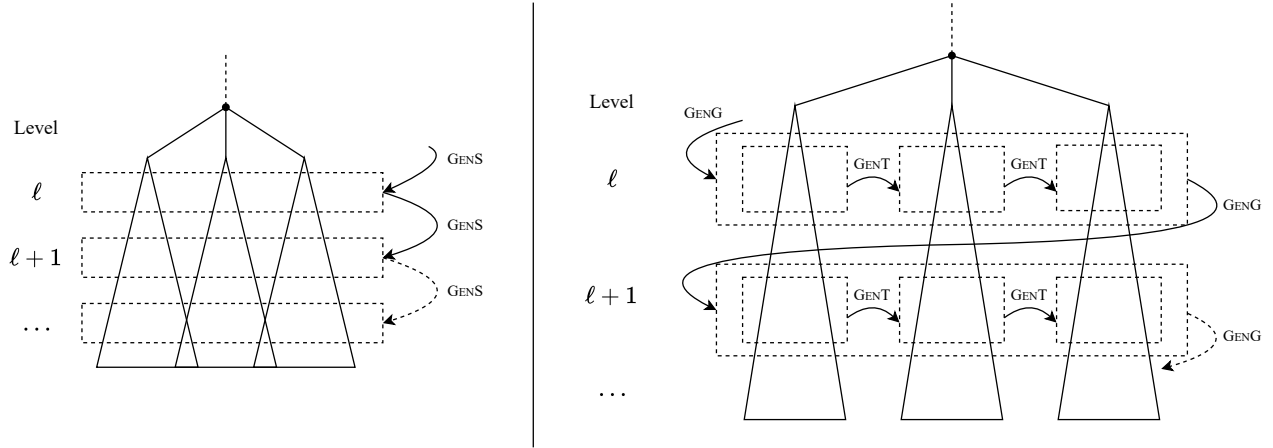


Figure 3.5: Two-dimensional recursion to iterate through all possible mixed-radix strings $t_{\ell,i}$ for each connected component C_i at level ℓ and all configurations at subsequent levels.

- Initiate the recursion by calling $\text{GENT}(1, \ell)$ to generate mixed-radix strings for all connected components at level ℓ (line 13 of GENG);
- Within GENT , recursively call $\text{GENT}(i+1, \ell)$ to generate all mixed-radix strings for the next connected component C_{i+1} (line 7 and line 21 of GENT);
- For each update to the mixed-radix string $t_{\ell,i}$ in C_i , call GENG to generate configurations for the next level $\ell+1$ when $i > |\mathcal{C}_\ell|$ where $\mathcal{C}_\ell$ is the number of connected components in level ℓ , using the union of components $\bigcup C_i$ and the concatenated string $\prod t_i$ (lines 2–5 of GENT);
- Determine the base (number of possible parents) for each vertex c_{ij} in C_i , derived from $|F_{ij}| + 1$ to account for connections within C_i (line 8 of GENT);
- Use mixed-radix strings $t_{\ell,i}$ for each connected component C_i instead of k -ary strings, with the NEXTMR function invoking GENMR to generate the next mixed-radix string (lines 10–11 of GENT);
- For case (b), where $t_{i,\delta_0} = 0$, select an edge $(c_a, c_b) \in G$ such that $c_a = c_{\delta_0}$, or c_a is a descendant of c_{δ_0} while c_b is not a descendant of c_{δ_0} , setting c_b as the parent of c_a (lines 18–20 of GENT).

Pseudocode of GENG and GENT to generate all spanning trees of general graphs is shown in Algorithm 3.2 and Algorithm 3.3. A complete Python implementation is provided in Appendix C.

THEOREM 3.3. *The algorithm GENG generates an edge-exchange Gray code for the spanning trees of a general graph G .*

Proof. The algorithm GENG , with GENT , applies the four cases in Lemma 2.2 to generate spanning trees of G . We analyze the edge exchanges in each case to confirm they produce an edge-exchange Gray code, where consecutive trees differ by exactly one edge exchange.

- Case (a): a_v updates from 0 to $r > 0$. A vertex v at level $q > \ell$ is disconnected from its parent w and connected to a vertex u at level $\ell-1$ where u is the r -th smallest available parent for v . This removes edge (v, w) and adds (v, u) , a pivot change (and thus an edge-exchange change), consistent with complete graphs in GENS ;
- Case (c): a_v updates from $r > 0$ to some $j > 0, j \neq r$. A vertex v at level ℓ is disconnected from its parent u at level $\ell-1$ and connected to another vertex w at level $\ell-1$. This removes (v, u) and adds (v, w) , a pivot change (and thus an edge-exchange change), consistent with complete graphs in GENS ;

Algorithm 3.2 Generating an edge-exchange Gray code for spanning trees of a general graph

```

1: procedure GENG( $V_{\ell-1}, t_{\ell-1} = a_1 a_2 \cdots a_{|V_{\ell-1}|}$ )
2:    $P_\ell \leftarrow v_i \in V_{\ell-1}$  where  $a_i > 0$ 
3:    $C_\ell \leftarrow v_i \in V_{\ell-1}$  where  $a_i = 0$ 
4:   if  $|C_\ell| = 0$  then
5:     Print spanning tree
6:   return
7:    $\mathcal{C}_\ell \leftarrow$  Find connected components in  $C_\ell$ 
8:   for  $C_{\ell,i} \in \mathcal{C}_\ell = \{C_{\ell,1}, \dots, C_{\ell,m}\}$  do
9:      $t_{\ell,i} = b_1 b_2 \cdots b_{|C_{\ell,i}|} \leftarrow 0^{|C_{\ell,i}|}$ 
10:    for  $c_{ij} \in C_{\ell,i}$  do
11:       $F_{\ell,ij} \leftarrow p_z \in P_\ell$  where  $(p_z, c_{ij}) \in G$ 
12:      if  $\pi(c_{ij}) \in F_{\ell,ij}$  then  $t_{\ell,ij} \leftarrow z$  where  $\pi(c_{ij}) = F_{\ell,ij}[z]$ 
13:  GENT( $1, \ell$ )

```

Algorithm 3.3 Generate all mixed-radix strings $t_{\ell,i}$ for each connected component C_i .

```

1: procedure GENT( $i, \ell$ )
2:   if  $i > |\mathcal{C}_\ell|$  then
3:      $V_\ell \leftarrow \bigcup_{i \leq |\mathcal{C}_\ell|} C_{\ell,i}$ 
4:      $t_\ell \leftarrow \prod_m t_{\ell,i}$ 
5:     GENG( $V_\ell, t_\ell$ )
6:   return
7:   GENT( $i + 1, \ell$ )
8:    $k_{\ell,i} \leftarrow \lceil |F_{\ell,ij}| + 1 \rceil_{j \in [1, \dots, |F_{\ell,i}|]}$ 
9:    $\{c_1, c_2, \dots\} \leftarrow C_{\ell,i}$ 
10:  while NEXTMR( $t_{\ell,i}, k_{\ell,i}$ )  $\neq \emptyset$  do
11:     $t_{\ell,i}, \delta \leftarrow \text{NEXTMR}(t_{\ell,i}, k_{\ell,i})$ 
12:    if  $|\delta| > 1$  then
13:       $\pi(c_{\delta_1}) \leftarrow F_{i\delta_1}[t_{\ell,i\delta_1}]$  ▷ case (d)
14:      LIFT( $c_{\delta_1}$ )
15:    else if  $k_{\delta_0} > 0$  then
16:       $\pi(c_{\delta_0}) \leftarrow F_{i\delta_0}[t_{i\delta_0}]$  ▷ case (a) and (c)
17:    else
18:      Find an edge  $(c_a, c_b) \in G$  where  $\exists \beta \geq 0, c_{\delta_0} = \pi^\beta(c_a)$  and  $\nexists \gamma, c_{\delta_0} = \pi^\gamma(c_b)$  ▷ case (b)
19:       $\pi(c_a) \leftarrow \pi(c_b)$ 
20:      LIFT( $c_a$ )
21:  GENT( $i + 1, \ell$ )

```

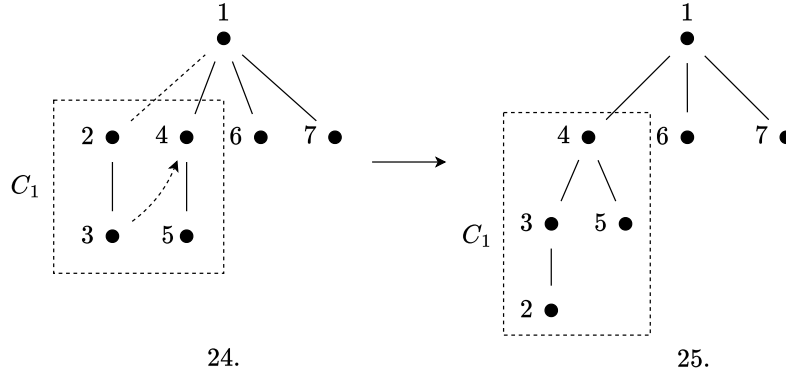


Figure 3.6: Tree 24 and tree 25 generated by GENG for G . The change between tree 24 and tree 25 is an edge-exchange but not a pivot change.

- Case (d): A bit swap updates $a_v = 1, a_w = 0$ to $a_w = 1, a_v = 0$. A vertex u_1 at level $\ell - 1$ disconnects from its child v at level ℓ , reflecting $a_v \leftarrow 0$. A vertex w at level $q > \ell$, which is a descendant of v , disconnects from $\pi(w)$ and connects to the first valid parent u_2 at level ℓ , reflecting $a_w \leftarrow 1$. This removes (v, u_1) and adds (w, u_2) where u_1 may not be u_2 , satisfying the edge-exchange Gray code property but violating the pivot Gray code property. The bit swap generated by GENMR involves only swapping a 0 and a 1, and strings have only one non-zero character (Lemma 3.2), the subtree rooted at v must remain connected after the bit swap;
- Case (b): a_v updates from $r > 0$ to 0. A vertex v at level ℓ is disconnected from its parent u at level $\ell - 1$, and the subtree rooted at v is reconnected to a vertex w in the same connected component C_i at level $q > \ell$. In general graphs, v may not be able to connect to such a vertex w which is at level $q \geq \ell$ and not in the subtree rooted at v . Removing the edge (v, u) requires adding an edge (x, w) , where x is a descendant of v , to maintain connectivity. Such a connection (x, w) must exist because there are more than one vertex in the the same connected component C_i at level ℓ , as $t_{\ell,i}$ contains more than one non-zero character (otherwise, it would result in the string $0^{[t_{\ell,i}]}$, which GENMR avoids generating). This edge exchange does not involve a common vertex when $v \neq x$, satisfying the edge-exchange Gray code property but violating the pivot Gray code property.

Thus, GENG ensures that consecutive spanning trees differ by a single edge exchange, satisfying the edge-exchange Gray code property. To establish that GENG generates all spanning trees of G exactly once, we proceed by contradiction, following a similar approach to Theorem 2.3 for complete graphs.

Suppose there exists a spanning tree T of G that does not appear in the listing generated by GENG. Such a spanning tree can be encoded as a tuple $T = ((t_{1,1}, t_{1,2}, \dots, t_{1,|C_1|}), (t_{2,1}, \dots, t_{2,|C_2|}), \dots, (t_{h,1}, t_{h,2}, \dots, t_{h,|C_h|}))$, where $|C_i|$ denotes the number of connected components at level i with respect to the subtree S_{i-1} spanning levels 0 to $i - 1$. If T is absent, no spanning tree in the listing has a subtree S_{h-1} , spanning levels 0 to $h - 1$, matching the first $h - 1$ elements $((t_{1,1}, t_{1,2}, \dots, t_{1,|C_1|}), \dots, (t_{h-1,1}, t_{h-1,2}, \dots, t_{h-1,|C_{h-1}|}))$ of T . Otherwise, GENT would exhaustively generate a Gray code for $(t_{h,1}, t_{h,2}, \dots, t_{h,|C_h|})$ to include T . Recursively applying this reasoning, no spanning tree has a subtree S_{h-2} matching the first $h - 2$ elements $((t_{1,1}, t_{1,2}, \dots, t_{1,|C_1|}), \dots, (t_{h-2,1}, t_{h-2,2}, \dots, t_{h-2,|C_{h-2}|}))$ of T . Continuing applying the same reasoning implies no spanning tree matches $((t_{1,1}, t_{1,2}, \dots, t_{1,|C_1|}))$. However, at level zero, the depth-first search or breadth-first search in GENG (line 7) identifies all connected components, and GENT generates all valid mixed-radix strings $t_{1,i}$ (via GENMR), excluding $0^{|C_i|}$, covering all possible configurations for $(t_{1,1}, t_{1,2}, \dots, t_{1,|C_1|})$. This contradicts the assumption that T is missing. Thus, all spanning trees are generated exactly once, and the listing is an edge-exchange Gray code. $\square$

THEOREM 3.4. *Spanning trees of complete graphs G with n vertices can be generated in edge-exchange Gray code order in $O(n^2)$ per tree, using $O(n^2)$ space.*

Proof. The algorithm GENG, with GENT, extends GENS (Algorithm 2.1), which generates spanning trees of a complete graph K_n in constant amortized time per tree using $O(n^2)$ space. For general graphs, GENG introduces the following additional steps that increase the time complexity while maintaining the space complexity:

- Identify connected components in C using depth-first search or breadth-first search, which takes $O(n^2)$ time in the worst case (line 7 of GENG, Algorithm 3.2);
- For each vertex $c_{i,j}$ in the connected component C_i , construct the list $F_{i,j}$ of vertices in P adjacent to $c_{i,j}$ in G , which takes $O(n^2)$ time across all vertices (lines 8–12 of GENG);
- In case (b), identify a descendant u of vertex v and an edge (u, w) to maintain connectivity, which takes $O(n^2)$ time in the worst case (lines 18–20 of GENT, Algorithm 3.3).

In the worst case, the third step may be required for each spanning tree generated. Other operations, such as updating $t_{\ell,i}$ via GENMR and recursive calls, take constant amortized time, consistent with GENS. Thus, the total time per tree is dominated by the $O(n^2)$ operations, while the space usage remains $O(n^2)$ for storing P , C , $\mathcal{C}$, t , and F . Hence, GENG generates spanning trees of a general graph G in edge-exchange Gray code order in $O(n^2)$ time per tree, using $O(n^2)$ space. $\square$

The edge-exchange Gray code for spanning trees of the Petersen graph is provided in Appendix E. Each spanning tree is represented as a rooted tree with root at vertex 1 using a *compact representation*, where a tree with n vertices is encoded as a string $a_1 a_2 \cdots a_{n-1}$ of length $n - 1$, with a_i denoting the parent of vertex $i + 1$. The compact representations of the rooted trees for graph G in Figure 3.1 are listed in Figure 3.7 as follows:

123416, 123316, 125316, 125311, 123411, 123311, 123371, 123471, 125371, 121371, 141371,
 141471, 151471, 121471, 121416, 151416, 141416, 141316, 121316, 121311, 141311, 141411,
 151411, 121411, 341411, 341311, 351411, 351416, 341316, 341416, 341471, 341371, 351471.

4 Final remarks. The algorithm GENG generates spanning trees of a general graph G with n vertices in edge-exchange Gray code order in $O(n^2)$ time per tree, using $O(n^2)$ space. Optimizations are possible when specific graph properties are known. For instance, for complete graphs K_n , GENS (Algorithm 2.1) is a specialized version of GENG that leverages the complete connectivity of K_n to achieve constant amortized time per tree. A similar optimization can be applied to complete bipartite graphs $K_{m,n}$, enabling generation in constant amortized time per tree.

THEOREM 4.1. *Spanning trees of a complete bipartite graph $K_{m,n}$ with $m + n$ vertices can be generated in edge-exchange Gray code order in constant amortized time per tree, using $O((m + n)^2)$ space.*

Proof. The proof adapts the GENS algorithm to complete bipartite graphs $K_{m,n}$. In $K_{m,n}$, with partitions A and B containing m and n vertices respectively, the vertices in set P must belong to the same partition A or B . We then partition C into two sets: one set can connect to P , and the other cannot. Most operations follow those in GENS, except for line 19 of Algorithm 2.1, where a vertex c_i at level ℓ cannot connect to another c_j at the same level due to the bipartite constraint. The strategy is to connect c_i to a child of a neighbor at level ℓ if such a child exists. To achieve constant amortized time per tree, we optimize finding valid connections using a *linked hash table*, where each vertex stores its children in a hash table with linked lists. Implementation details of the linked hash table are similar to C' discussed in Appendix A. Each vertex maintains the linked hash table with vertex indices as keys and linked lists of its children. At each level ℓ , a linked hash table C' is constructed, containing vertices $c_i \in C'$ at level ℓ that connect to a vertex at level $\ell - 1$ and have at least one child, with constant amortized time lookups and updates. When progressing to level $\ell + 1$, C' is updated in constant amortized time to maintain this property. Now in case (b) when t_ℓ sets $a_i = 0$ for a vertex c_i at level ℓ , c_i disconnects from its parent $\pi(c_i)$ by removing the entry from the parent's linked hash map. If c_i has at least one child (detected in constant time via its linked hash table), a valid connection (c_a, c_b) is identified in constant time, where $\pi(c_a) = c_i$ and c_b is the last vertex in C' . The LIFT operation takes constant time, as it modifies only c_i and c_a . Otherwise, if c_i has no children, a valid connection (c_i, c_j) is used, where $\pi(c_j)$ is the last vertex in C' and can be found in constant time. $\square$

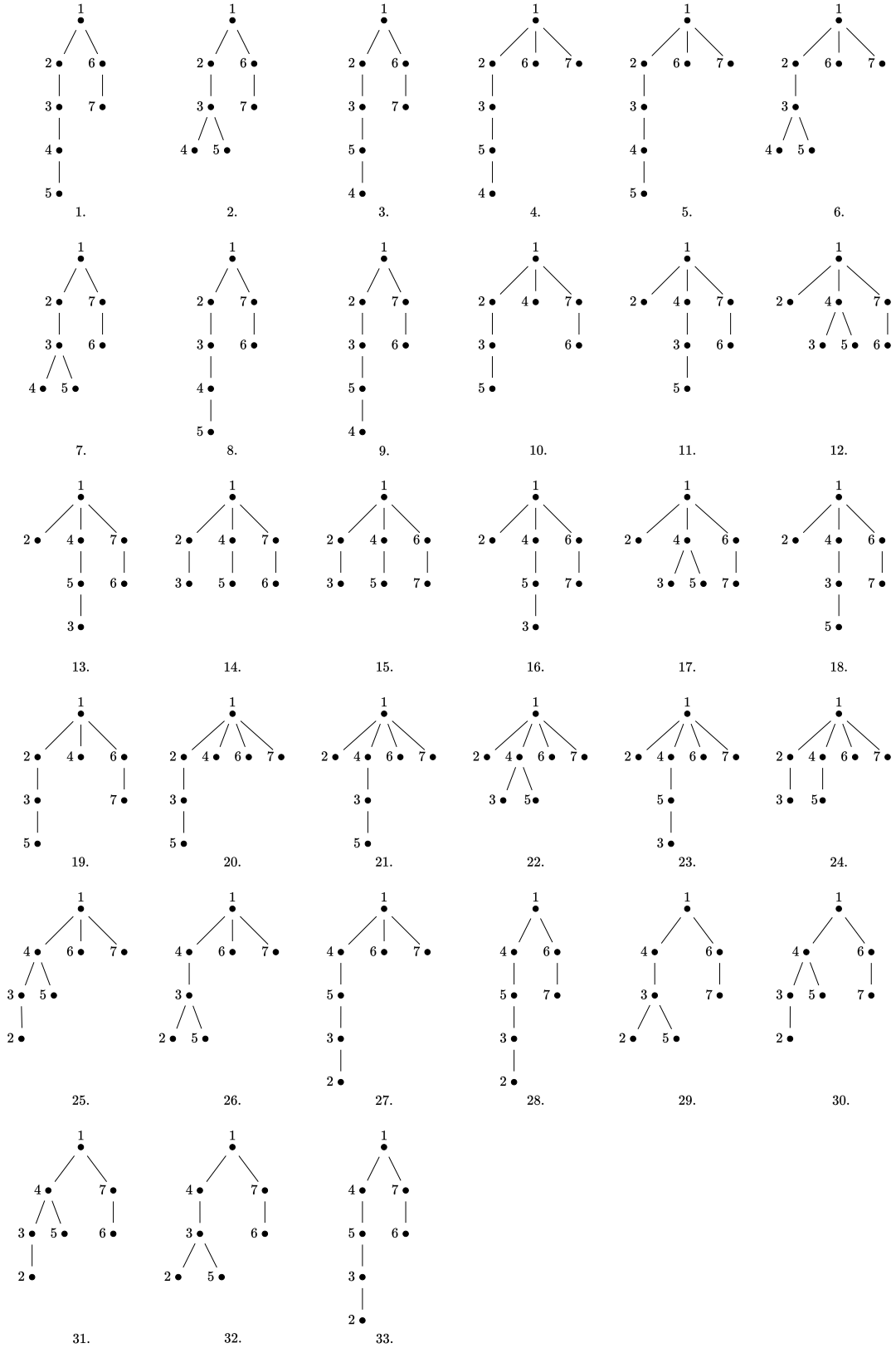


Figure 3.7: An edge-exchange Gray code for spanning trees of the graph G in Figure 3.1.

Similar strategies can be applied to generate Gray codes for spanning trees of fan graphs and wheel graphs.

THEOREM 4.2. *Spanning trees of a fan graph F_n with n vertices can be generated in edge-exchange Gray code order in $O(n)$ -amortized time per tree, using $O(n^2)$ space.*

THEOREM 4.3. *Spanning trees of a wheel graph W_n with n vertices can be generated in edge-exchange Gray code order in $O(n)$ -amortized time per tree, using $O(n^2)$ space.*

However, the Gray codes produced for $K_{m,n}$, F_n , and W_n are edge-exchange Gray codes, not pivot Gray codes, similar to those for general graphs. Future research directions include developing pivot Gray codes for other special graph classes (for example, planar graphs would be interesting). Another avenue is to develop ranking and unranking algorithms for the spanning trees generated by GENG, which may require ranking and unranking the underlying Gray codes for k -ary or mixed-radix strings.

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A Optimization for GENS. In this section, we present optimizations for GENS (Algorithm 2.1) that enable the generation of pivot Gray codes for spanning trees of K_n in constant amortized time per tree, as detailed in Theorem 2.6.

(Line 8 – 9) Computing the initial t_ℓ . The initial $t_\ell = b_1 b_2 \dots b_{|C|}$ describes the connections between level $\ell - 1$ and level ℓ . We construct a hash table m with t key-value pairs. The keys of the hash table m are the indices i of each $v_i \in V$. The values for each element in m are set to 0 by default, and the value is updated by traversing $p_i \in P$, setting $m[p_i] \leftarrow i$. Then we update each $b_i \leftarrow m[\pi(c_i)]$ where b_i indicates that $\pi(c_i) = p_{b_i}$.

(Line 19) Connect c_i to a sibling c_j at level ℓ . A hash table C' chaining with a doubly linked list can be constructed for optimizing the operation of connecting a disconnected vertex c_i to another vertex c_j in level ℓ . The structure of C' is shown in Figure A.1. C' has at most $|C|$ key-value pairs, where each key of c_i points to a tuple $(-1, -1)$ initially. A pointer C'_{-1} pointing to the last activated vertex is set to (-1) by default. For each $c_i \in C$ where $b_i > 0$, we activate c_i by setting $C'[i] \leftarrow (C'_{-1}[0], C'_{-1})$ then $C'_{-1}[0] \leftarrow i$. When a vertex connects to a $p_i \in P$, we activate this vertex by the operation described above. When a vertex c_j disconnects from $\pi(c_j) \in P$, we link $C'[C'[j][0]]$ and $C'[C'[j][1]]$ to maintain the structure of doubly linked list and set $C'[j] \leftarrow (-1, -1)$ to deactivate c_j . Because the activated vertices in C' are in level ℓ , so they must not be descendants of c_j . The valid link $(c_j, c_{C'_{-1}})$ can be found in constant time.

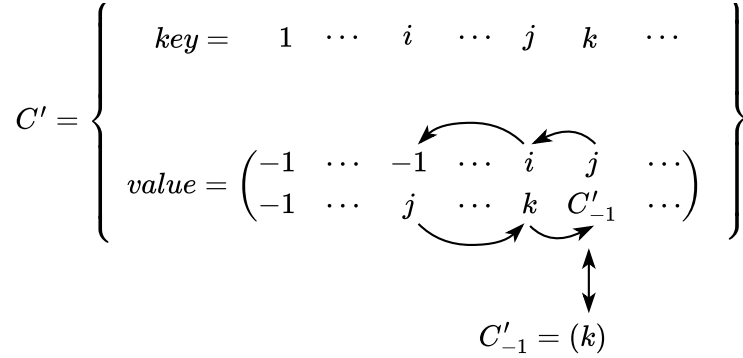


Figure A.1: Structure of hash table C' chaining with a linked list.

(Line 2 – 6) Output spanning tree. The original GENS outputs a spanning tree after traversing the input V and ensuring $|C| = 0$, which takes $O(t)$ time complexity. We optimize the output operation leveraging C' mentioned above. Note that C' contains the vertices in level ℓ . We can set a variable for $|C'|$ that records the number of activated vertices in C' can be updated after each connection and disconnection. When $|C| = |C'|$, which indicates no vertex in the next level, GENS prints the spanning tree directly, which skips the construction of P and C in the next level.

Pseudocode of the optimized algorithm is provided in Algorithm A.1. A complete Python implementation of the optimized algorithm is provided in Appendix A.1.

Algorithm A.1 Optimized algorithm for generating pivot Gray codes for spanning trees of complete graphs in constant amortized time per spanning tree.

```

1: procedure GENS( $\ell, V, t_{\ell-1}, n_c$ )
2:   if  $n_c = 0$  then
3:     Print spanning tree
4:   return
5:    $P \leftarrow v_i \in V$  where  $t_{\ell-1,i} > 0$ 
6:    $C \leftarrow v_i \in V$  where  $t_{\ell-1,i} = 0$ 
7:    $t_\ell \leftarrow 0^{|C|}$ 
8:    $C' \leftarrow \{\}$ 
9:    $m \leftarrow \{p_i \rightarrow i\}_{p_i \in P}$ 
10:  for  $c_i \in C$  do
11:    if  $\pi(c_i) \in m$  then
12:       $t_{\ell,i} \leftarrow m[\pi(c_i)]$ 
13:      Append  $i$  to  $C'$ 
14:  GENS( $\ell + 1, C, t_\ell, |C| - |C'|$ )
15:  for NEXTK( $t_\ell, |P| + 1$ )  $\neq \emptyset$  do
16:     $t_\ell, \delta \leftarrow \text{NEXTK}(t_\ell, |P| + 1)$ 
17:    if  $|\delta| > 1$  then
18:       $\pi(c_{\delta_1}) \leftarrow \pi(c_{\delta_0})$ 
19:      LIFT( $c_{\delta_1}$ )
20:      Remove  $\delta_0$  from  $C'$ 
21:      Append  $\delta_1$  to  $C'$ 
22:    else if  $t_{\ell,\delta_0} > 0$  then
23:       $\pi(c_{\delta_0}) \leftarrow p_{b_{\delta_0}} \in P$ 
24:      if  $\delta_0$  not in  $C'$  then Append  $\delta_0$  to  $C'$ 
25:    else
26:      Remove  $\delta_0$  from  $C'$ 
27:       $\pi(c_{\delta_0}) \leftarrow c_{C'_{-1}}$ 
28:  GENS( $\ell + 1, C, t_\ell, |C| - |C'|$ )

```

A.1 Python code to generate pivot Gray codes for spanning trees of complete graphs.

```

1 from k_ary import k_ary
2
3 class Node:
4     def __init__(self, val):
5         self.val = val
6         self.parent = None
7
8 class LinkedList:
9     def __init__(self):
10        self.map = {}
11        self.last = -1
12        self.len = 0
13
14    def append(self, val):
15        self.map[val] = [self.last, -1]
16        if self.last != -1:
17            self.map[self.last][1] = val
18        self.last = val
19        self.len += 1
20
21    def remove(self, val):
22        val_last = self.map[val][0]
23        val_next = self.map[val][1]
24        if val_last != -1:
25            self.map[val_last][1] = val_next
26        if val_next != -1:
27            self.map[val_next][0] = val_last
28        self.last = val_next
29        else:
30            self.last = val_last
31        del self.map[val]
32        self.len -= 1
33
34 node_list = []
35
36 def lift_node(node):
37     if node.parent is not None:
38         lift_node(node.parent)
39         node.parent.parent = node
40
41 def spanning_level(V, t_, n_c):
42     if n_c == 0:
43         yield node_list
44         return
45
46     P = [V[i] for i in range(len(V)) if t_[i] > 0]
47     C = [V[i] for i in range(len(V)) if t_[i] == 0]
48     t_l = [0 for _ in range(n_c)]
49     C_ = LinkedList()
50
51     m = {}
52     for i in range(len(P)):
53         m[P[i].val] = i+1
54
55     for i in range(len(C)):
56         if C[i].parent.val in m:
57             t_l[i] = m[C[i].parent.val]
58             C_.append(i)
59
60     yield from spanning_level(C, t_l, len(C) - C_.len)
61
62     K = k_ary([len(P) for _ in range(n_c)], t_l)

```

```

63     next(K)
64
65     for t_l, idx in K:
66         if len(idx) > 1:
67             temp = C[idx[0]].parent
68             C[idx[0]].parent = None
69             lift_node(C[idx[1]])
70             C[idx[1]].parent = temp
71             C_.append(idx[1])
72             C_.remove(idx[0])
73
74         elif t_l[idx[0]] > 0:
75             C[idx[0]].parent = P[t_l[idx[0]]-1]
76             if not idx[0] in C_.map:
77                 C_.append(idx[0])
78
79         else:
80             C_.remove(idx[0])
81             C[idx[0]].parent = C[C_.last]
82
83         yield from spanning_level(C, t_l, len(C) - C_.len)
84
85 if __name__ == '__main__':
86
87     def str_tree(nodes):
88         res = ''
89         for nd in nodes:
90             res += '{}->{}; '.format(nd.val+1, nd.parent.val+1 if nd.parent is not None else -1)
91
92         return res
93
94     print('Enter n:')
95     n = int(input())
96
97     prev = None
98     for i in range(0, n):
99         new_node = Node(i)
100         if prev is not None:
101             new_node.parent = prev
102         node_list.append(new_node)
103         prev = new_node
104     print('=====')
105
106     cnt = 0
107     for t in spanning_level(node_list, [1] + [0 for _ in range(n - 1)], n - 1):
108         print(str_tree(t))
109         cnt += 1
110
111     print('Total: ' + str(cnt))

```

B Python code to generate Gray codes for k -ary strings and mixed-radix strings.

```
1 def k_ary(k_list, alpha):
2
3     n = len(k_list)
4     d = [1 if a != 1 else -1 for a in alpha]
5     idx = []
6     s = [sum(alpha)]
7
8     order = [i for i in range(n)]
9     for i in range(n-1, -1, -1):
10         if k_list[i] > 1:
11             order[n-1], order[i] = i, n-1
12             break
13
14     def _next(i):
15         i_ = order[i]
16
17         for r in range(k_list[i_] + 1):
18
19             if i == n-1:
20                 if s[0] != 0:
21                     if len(idx) > 1 and idx[0] == idx[1]:
22                         idx.pop()
23                         yield alpha, idx
24                         idx[:] = []
25             else:
26                 i_next = order[i + 1]
27                 yield from _next(i + 1)
28                 d[i_next] = 1 if alpha[i_next] == k_list[i_next] else -d[i_next]
29
30             if r < k_list[i_]:
31                 a_b = alpha[i_]
32                 a_a = (alpha[i_] + d[i_] + k_list[i_] + 1) % (k_list[i_] + 1)
33                 s[0] += a_a - a_b
34                 alpha[i_] = a_a
35                 idx.append(i_)
36
37     yield from _next(0)
```

C Python code to generate an edge-exchange Gray code for spanning trees of a general graph.

```
1 from k_ary import k_ary
2 import numpy as np
3
4 class Node:
5     def __init__(self, val):
6         self.val = val
7         self.parent = None
8
9 node_list = []
10 adj = [[0]]
11
12 def lift_node(node):
13     if node.parent is not None:
14         lift_node(node.parent)
15         node.parent.parent = node
16
17 def get_connected_components(nodes):
18     visited = {nd.val: False for nd in nodes}
19     cc_list = []
20     for nd in nodes:
21         if not visited[nd.val]:
22             visited[nd.val] = True
23             group = [nd]
24             dfs(nd, nodes, visited, group)
25             cc_list.append(group)
26     return cc_list
27
28 def dfs(cnode, nodes, visited, group):
29     for nd in nodes:
30         if not visited[nd.val] and adj[cnode.val][nd.val] == 1:
31             visited[nd.val] = True
32             group.append(nd)
33             dfs(nd, nodes, visited, group)
34
35 def check_k(parents, children):
36     c = [0 for _ in range(len(children))]
37     p_list = [[] for _ in range(len(children))]
38     for i_c in range(len(children)):
39         n_p = 0
40         for i_p in range(len(parents)):
41             if adj[parents[i_p].val][children[i_c].val] == 1:
42                 p_list[i_c].append(parents[i_p])
43                 n_p += 1
44                 if children[i_c].parent.val == parents[i_p].val:
45                     c[i_c] = n_p
46     k_list = [len(p) for p in p_list]
47     return c, k_list, p_list
48
49 def get_connection(cnode, nodes):
50     m = {nd.val: [] for nd in nodes}
51     for nd in nodes:
52         try:
53             m[nd.parent.val].append(nd)
54         except KeyError:
55             pass
56
57     nodes_under_c = [cnode]
58     temp_nodes = [cnode]
59     while len(temp_nodes) > 0:
60         new_nodes = []
61         for tnd in temp_nodes:
62             new_nodes += m[tnd.val]
```

```

63         del m[tnd.val]
64         nodes_under_c += new_nodes
65         temp_nodes = new_nodes
66
67     nodes_other = [nd for nd in nodes
68                    if m.get(nd.val) is not None and nd.val != cnode.val]
69
70     for nd1 in nodes_under_c:
71         for nd2 in nodes_other:
72             if adj[nd1.val][nd2.val] == 1:
73                 return nd1, nd2
74
75 def spanning_level(V, t_):
76     C = [V[i] for i in range(len(V)) if t_[i] == 0]
77     if len(C) == 0:
78         yield node_list
79         return
80
81     P = [V[i] for i in range(len(V)) if t_[i] > 0]
82     C_ = get_connected_components(C)
83     F = [check_k(P, C_sub) for C_sub in C_]
84
85     def spanning_subtree(i_tree):
86         if i_tree >= len(F):
87             yield from spanning_level(sum(C_, []), sum([g[0] for g in F], []))
88             return
89
90         yield from spanning_subtree(i_tree+1)
91
92         K = k_ary(F[i_tree][1].copy(), F[i_tree][0].copy())
93         next(K)
94
95         for t_l_i, idx in K:
96             if len(idx) > 1:
97                 C_[i_tree][idx[0]].parent = None
98                 lift_node(C_[i_tree][idx[1]])
99                 C_[i_tree][idx[1]].parent = F[i_tree][2][idx[1]][t_l_i[idx[1]] - 1]
100
101             elif t_l_i[idx[0]] > 0:
102                 C_[i_tree][idx[0]].parent = F[i_tree][2][idx[0]][t_l_i[idx[0]] - 1]
103
104             else:
105                 nd1, nd2 = get_connection(C_[i_tree][idx[0]], C_[i_tree])
106                 C_[i_tree][idx[0]].parent = None
107                 lift_node(nd1)
108                 nd1.parent = nd2
109
110                 F[i_tree][0][:] = t_l_i[:]
111                 yield from spanning_subtree(i_tree+1)
112
113     yield from spanning_subtree(0)
114
115 def get_first_spanning_tree():
116     n = len(adj)
117     check_list = [True for _ in range(n)]
118     node_list[:] = [Node(i) for i in range(n)]
119     check_list[0] = False
120     for i in range(n):
121         for j in range(n):
122             if check_list[j] and adj[i][j] == 1:
123                 check_list[j] = False
124                 node_list[j].parent = node_list[i]
125
126     return node_list

```

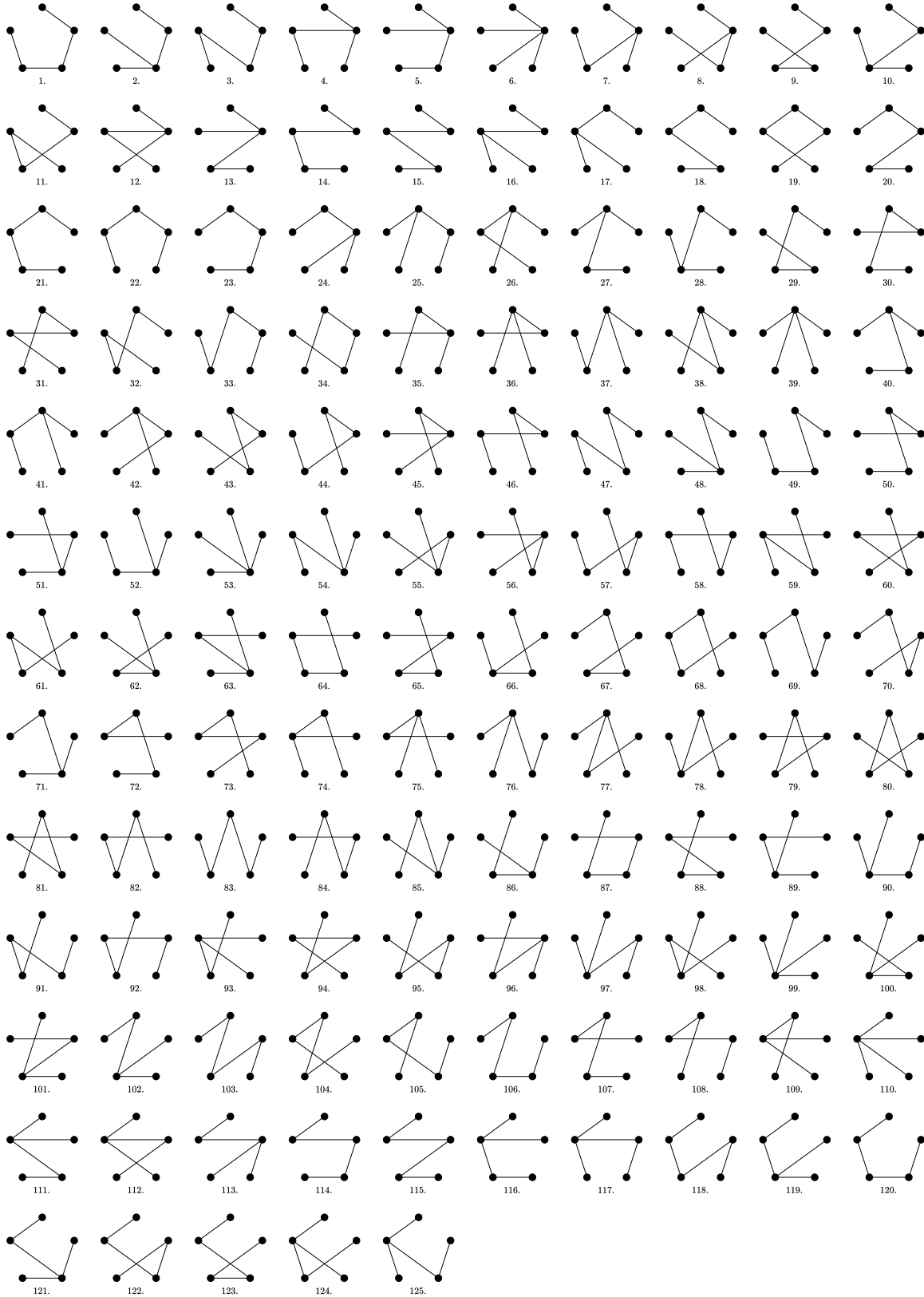
```

127
128 if __name__ == '__main__':
129     def str_tree(nodes, vmap=None):
130         res = ''
131         for nd in nodes:
132             res += '{}->{}; '.format(nd.val+1, nd.parent.val+1 if nd.parent is not None else -1)
133         return res
134
135     print('Enter type: #1 complete graph, #2 fan graph, #3 wheel graph, #4 petersen, #5 custom
graph')
136     i_type = int(input())
137
138     if i_type != 4:
139         print('Enter n:')
140         n = int(input())
141     else:
142         n = 10
143
144     if i_type == 1:
145         adj = [[1 for _ in range(n)] for _ in range(n)]
146
147     elif i_type == 2 or i_type == 3:
148         # fan graph
149         adj = np.zeros((n, n), dtype=int)
150         adj[0, :] = 1
151         adj[:, 0] = 1
152         for i in range(1, n):
153             adj[i, max(0, i - 1):min(n, i + 2)] = 1
154             adj[max(0, i - 1):min(n, i + 2), i] = 1
155
156         if i_type == 3:
157             # wheel graph
158             adj[1, -1] = 1
159             adj[-1, 1] = 1
160
161     elif i_type == 4:
162         # petersen graph
163         adj = np.array([
164             [0, 1, 0, 0, 1, 1, 0, 0, 0, 0],
165             [1, 0, 1, 0, 0, 0, 1, 0, 0, 0],
166             [0, 1, 0, 1, 0, 0, 0, 1, 0, 0],
167             [0, 0, 1, 0, 1, 0, 0, 0, 1, 0],
168             [1, 0, 0, 1, 0, 0, 0, 0, 0, 1],
169             [1, 0, 0, 0, 0, 0, 0, 1, 1, 0],
170             [0, 1, 0, 0, 0, 0, 0, 0, 1, 1],
171             [0, 0, 1, 0, 0, 1, 0, 0, 0, 1],
172             [0, 0, 0, 1, 0, 1, 1, 0, 0, 0],
173             [0, 0, 0, 0, 1, 0, 1, 1, 0, 0],
174         ])
175
176     elif i_type == 5:
177         print('Enter edges: e.g., 1,2; 2,3')
178         edge_str = input().replace(' ', '').split(';')
179         edge_list = []
180         for es in edge_str:
181             if es:
182                 edge_list.append(tuple(map(int, es.split(','))))
183         adj = np.zeros((n, n), dtype=int)
184         for edge in edge_list:
185             adj[edge[0]-1, edge[1]-1] = 1
186             adj[edge[1]-1, edge[0]-1] = 1
187
188     else:
189         raise Exception('Invalid input')

```

```
190 get_first_spanning_tree() # bfs
191
192
193 cnt = 0
194 for t in spanning_level(node_list, [1] + [0 for _ in range(len(adj)-1)]):
195     print(str_tree(t))
196     cnt += 1
197 print('Total: ' + str(cnt))
```

D Pivot Gray code of K_5 generated by GENS.



E Edge-exchange Gray code of the Petersen graph generated by GENG in compact representation

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