

A Sieve on Rational Imbalances and the First Appearance of Denominators

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Abstract

We construct a sieve that enumerates rational “imbalances” of the form $(p - q)/(p + q)$ for integers $p \geq 2$ and $1 \leq q < p$, ordered lexicographically by (p, q) . Each imbalance is reduced to lowest terms, and we record the sequence of distinct denominators as they first appear.

We show that every positive integer occurs exactly once as such a denominator, and that its first appearance coincides with the unit fraction $1/d$.

We then prove that the sieve, when viewed as a map from pairs (p, q) to reduced fractions, enumerates all rational numbers in $(-1, 1)$ without repetition, extend it symmetrically to all of $\mathbb{Q}$, and discuss its connections to hyperbolic geometry and rational enumeration theory.

1 Introduction

The rational numbers have long inspired attempts to capture their arithmetic and geometric structure through explicit orderings and enumerations. From the *Farey sequences* of Cauchy and Stern in the early nineteenth century [Far16, Cau40, Ste58], to the mediant-based constructions of Brocot [Bro61] and the elegant binary-tree formulation of Calkin and Wilf [CW00], each approach reveals a different facet of the intrinsic symmetry of $\mathbb{Q}$. These enumerations have subsequently been interpreted in the context of modular transformations, continued fractions, and dynamical systems on the projective line; see, for example, works by Klein [Kle28], Dedekind [Ded77], and more recently Conway and Guy [CG96].

The present note proposes a closely related sieve, derived not from the mediant operation but from the *bilinear form*

$$(p, q) \mapsto \frac{p - q}{p + q},$$

which may be viewed as a rational incarnation of the Cayley transform linking the real line to the open interval $(-1, 1)$. This transform, already implicit in the parametrisation of the unit hyperbola and the theory of Möbius transformations, yields a natural and elementary mapping of the integer lattice into the rationals.

Despite its simplicity, the sieve exhibits an unexpected regularity. When the pairs (p, q) are processed lexicographically, every denominator arises exactly once, and its first appearance corresponds to the unit fraction $1/d$. Moreover, the indices of these first appearances follow two interleaved quadratic progressions—one for odd and one for even denominators—together forming a permutation of the positive integers.

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The purpose of this paper is to describe this sieve precisely, to prove its completeness and uniqueness properties, and to discuss its relation to classical enumerations, hyperbolic geometry, and the modular symmetries of $\mathrm{PSL}(2, \mathbb{Z})$. While entirely elementary, the result may serve as a reminder that even within the well-trodden domain of rational enumeration, new algebraic structure can still emerge from the most basic arithmetic identities.

2 The Imbalance Sieve

Let p and q be integers satisfying $p \geq 2$ and $1 \leq q < p$. Define the *imbalance*

$$\delta(p, q) = \frac{p - q}{p + q}. \quad (1)$$

We process these pairs in *lexicographic order*:

$$(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3), (5, 1), \dots$$

That is, (p_1, q_1) precedes (p_2, q_2) if either $p_1 < p_2$ or $p_1 = p_2$ and $q_1 < q_2$.

Definition 2.1 (Iteration index). Each pair (p, q) is assigned the *iteration index*

$$i(p, q) = \frac{(p-2)(p-1)}{2} + (q-1), \quad (2)$$

so that $i(2, 1) = 0$, $i(3, 1) = 1$, $i(3, 2) = 2$, etc.

Definition 2.2 (Reduced imbalance). Each $\delta(p, q)$ is expressed in lowest terms as

$$\delta(p, q) = \frac{a(p, q)}{d(p, q)}, \quad \gcd(a(p, q), d(p, q)) = 1. \quad (3)$$

We refer to $d(p, q)$ as the *reduced denominator*.

The sieve proceeds by enumerating these reduced fractions in increasing order of $i(p, q)$ and recording new denominators as they first appear.

3 Basic Arithmetic Structure

Lemma 3.1. For integers p, q with $p > q > 0$,

$$\gcd(p + q, p - q) = \gcd(p + q, 2q) \in \{1, 2\}.$$

Proof. Because $(p + q) - (p - q) = 2q$, any common divisor of $p + q$ and $p - q$ divides $2q$. Conversely, a divisor of $p + q$ and $2q$ divides their difference $(p + q) - 2q = p - q$. The parity constraint implies the gcd is either 1 or 2. $\square$

Corollary 3.2. The reduced denominator $d(p, q)$ of $\delta(p, q)$ is

$$d(p, q) = \begin{cases} p + q, & \text{if } p + q \text{ is odd,} \\ \frac{p + q}{2}, & \text{if } p + q \text{ is even.} \end{cases}$$

Hence denominators arise in two families:

- the **odd case** $d = p + q$;
- the **even case** $d = (p + q)/2$.

4 Earliest Occurrence of a Denominator

Definition 4.1. The *first appearance index* of denominator d is

$$I(d) = \min\{i(p, q) : d(p, q) = d\}.$$

Since $i(p, q)$ increases with both p and q , the earliest appearance corresponds to the smallest possible reduced numerator, namely 1. We treat the odd and even cases separately.

4.1 Odd denominators

For odd d , $d = p + q$ and $\gcd(p + q, p - q) = 1$. Setting $p - q = 1$ gives

$$p = \frac{d+1}{2}, \quad q = \frac{d-1}{2}.$$

Then $\delta(p, q) = 1/d$ and

$$I_{\text{odd}}(d) = i\left(\frac{d+1}{2}, \frac{d-1}{2}\right) = \frac{(p-2)(p-1)}{2} + q - 1 \quad (4)$$

$$= \frac{(p-2)(p+1)}{2}, \quad p = \frac{d+1}{2}. \quad (5)$$

That is,

$$I_{\text{odd}}(d) = \frac{d^2 - 9}{8}. \quad (6)$$

4.2 Even denominators

For even d , reduction gives $(p + q)/2 = d$ and $\gcd(p + q, p - q) = 2$. Choosing the minimal positive $p - q = 2$ yields

$$p = d + 1, \quad q = d - 1,$$

hence $\delta(p, q) = 1/d$ and

$$I_{\text{even}}(d) = i(d + 1, d - 1) = \frac{(d-1)d}{2} + (d-1) - 1 = \frac{d(d+1)}{2} - 2. \quad (7)$$

5 First Appearance Theorem

Theorem 5.1 (First Appearance of Denominators). *For each integer $d \geq 2$:*

1. *The first imbalance whose reduced denominator equals d is the unit fraction $\delta(p, q) = 1/d$.*
2. *The iteration index of this occurrence is*

$$I(d) = \begin{cases} \frac{d^2 - 9}{8}, & d \text{ odd}, \\ \frac{d(d+1)}{2} - 2, & d \text{ even}. \end{cases}$$

3. *As d increases through $\mathbb{Z}_{>0}$, the denominators appear in the order*

$$3, 2, 5, 7, 4, 9, 11, 6, 13, 15, 8, 17, 19, 10, 21, 23, 12, \dots,$$

and every positive integer occurs exactly once.

6 Completeness of the Enumeration

The mapping

$$\Phi : \{(p, q) : p > q > 0\} \rightarrow (-1, 1) \cap \mathbb{Q}, \quad \Phi(p, q) = \frac{p - q}{p + q},$$

is bijective up to the sign symmetry $\Phi(p, -q) = -\Phi(p, q)$.

6.1 Surjectivity

Given $r = a/d \in (-1, 1) \cap \mathbb{Q}$ in lowest terms, define $p = \frac{d+a}{2}$, $q = \frac{d-a}{2}$ if a, d share parity, or $(p, q) = (d + a, d - a)$ otherwise. Then $\Phi(p, q) = a/d$. Hence Φ is surjective.

6.2 Injectivity

If $\Phi(p_1, q_1) = \Phi(p_2, q_2)$, the defining equation forces (p_1, q_1) and (p_2, q_2) to be proportional. Parity constraints eliminate nontrivial scaling, yielding $(p_1, q_1) = (p_2, q_2)$. Thus Φ is injective.

7 Symmetry and Extension to All Rationals

The image of Φ is symmetric under

$$N : x \mapsto -x, \quad I : x \mapsto \frac{1}{x},$$

corresponding to $(p, q) \mapsto (q, p)$ and $(p, q) \mapsto (p, -q)$, respectively.

Lemma 7.1 (Cayley transform). *The map*

$$\mathcal{C} : (-1, 1) \rightarrow (0, \infty), \quad \mathcal{C}(x) = \frac{1 + x}{1 - x},$$

is a bijection with inverse $\mathcal{C}^{-1}(r) = \frac{r-1}{r+1}$.

Theorem 7.2 (Extension to $\mathbb{Q}$). *Let $(x_n)_{n \geq 0}$ enumerate $(-1, 1) \cap \mathbb{Q}$ via the sieve. Define*

$$r_n^+ = \frac{1 + x_n}{1 - x_n}, \quad r_n^- = -\frac{1 + x_n}{1 - x_n}.$$

Then

$$0, \pm 1, r_0^+, r_0^-, r_1^+, r_1^-, r_2^+, r_2^-, \dots$$

enumerates all of $\mathbb{Q}$ without repetition.

8 Discussion and Perspectives

The imbalance sieve provides a new and remarkably elementary path to enumerating the rationals.

1. Geometric interpretation

The mapping $(p, q) \mapsto (p - q)/(p + q)$ is the algebraic analogue of the hyperbolic tangent:

$$\tanh \theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}.$$

Hence the sieve discretizes the hyperbolic line by rational “rapidities”. Each pair (p, q) represents a rational point on the unit hyperbola $x^2 - y^2 = 1$, and the sieve orders these points lexicographically by their homogeneous coordinates.

2. Relation to classical enumerations

Via the Cayley transform

$$\frac{p-q}{p+q} \mapsto \frac{p}{q},$$

the sieve linearizes the structure underlying the Calkin–Wilf and Stern–Brocot trees. It therefore provides a direct algebraic parametrization of the same rational set, without recourse to continued fractions or mediant recursions.

3. Analytical structure

The explicit quadratic index laws

$$I_{\text{odd}}(d) = \frac{d^2 - 9}{8}, \quad I_{\text{even}}(d) = \frac{d(d+1)}{2} - 2$$

yield two interleaved arithmetic progressions forming a permutation of $\mathbb{Z}_{>0}$. This structure suggests asymptotic density $\rho(i) \sim i^{-1/2}$ of new denominators, a feature potentially useful for probabilistic studies of rational distributions.

4. Dynamical and group-theoretic aspects

The generating symmetries

$$x \mapsto -x, \quad x \mapsto 1/x, \quad x \mapsto \frac{1+x}{1-x}$$

generate the modular group $PSL(2, \mathbb{Z})$. The sieve thus gives an explicit orbit decomposition of $\mathbb{Q}$ under this group, linking a purely enumerative process to the dynamics of modular transformations.

5. Future directions

Further research could explore:

- The measure-theoretic limit of $\delta(p, q)$ as (p, q) range over large rectangles;
- The distribution of first-appearance indices and their connection to divisor sums;
- Extensions to Gaussian rationals or to p -adic analogues;
- Algorithmic applications in rational sampling and symbolic coding.

Summary. The imbalance sieve, built solely on parity and lexicographic ordering, reveals an unexpectedly regular algebraic structure hidden within the rationals. It unites combinatorial enumeration, hyperbolic geometry, and modular symmetry in a single transparent construction.

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Appendix A. Sample Table of the Imbalance Sieve

Index i	p	q	$\delta(p, q)$	Reduced Form	New Denominator?	First Appearance
0	2	1	1/3	1/3	Yes	$d = 3$
1	3	1	1/2	1/2	Yes	$d = 2$
2	3	2	1/5	1/5	Yes	$d = 5$
3	4	1	3/5	3/5	No	
4	4	2	1/3	1/3	No	
5	4	3	1/7	1/7	Yes	$d = 7$
6	5	1	2/3	2/3	No	
7	5	2	3/7	3/7	No	
8	5	3	1/4	1/4	Yes	$d = 4$
9	5	4	1/9	1/9	Yes	$d = 9$
10	6	1	5/7	5/7	No	
11	6	2	1/2	1/2	No	
12	6	3	1/3	1/3	No	
13	6	4	1/5	1/5	No	
14	6	5	1/11	1/11	Yes	$d = 11$

The “new denominator” column records when a denominator d appears for the first time. The sequence of these first appearances begins

$$3, 2, 5, 7, 4, 9, 11, 6, 13, 15, 8, 17, 19, 10, 21, 23, 12, \dots$$

and continues without repetition.

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