

PINNED PATTERNS AND DENSITY THEOREMS IN $\mathbb{R}^d$

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ABSTRACT. For integers $k \geq 3, d \geq 2$, we consider the abundance property of pinned k -point patterns occurring in $E \subseteq \mathbb{R}^d$ with positive upper density $\delta(E)$. We show that for any fixed k -point pattern V , there is a set E with positive upper density such that E avoids all sufficiently large affine copies of V , with one vertex fixed at any point in E . However, we obtain a positive quantitative result, which states that for any fixed E with positive upper density, there exists a k -point pattern V , such that for any $x \in E$, the *pinned scaling factor set*

$$D_x^V(E) := \{r > 0 : \exists \text{ isometry } O \text{ such that } x + r \cdot O(V) \subseteq E\},$$

has upper density $\geq \tilde{\varepsilon} > 0$, where constant $\tilde{\varepsilon}$ depends on k, d and $\delta(E)$.

1. INTRODUCTION

1.1. Main results. Define the *upper density* of a set $A \subseteq \mathbb{R}^d$ as

$$\delta(A) := \limsup_{R \rightarrow \infty} \frac{\mathcal{L}^d(B(0, R) \cap A)}{R^d}$$

where $\mathcal{L}^d(\cdot)$ denotes the d -dimensional Lebesgue measure on $\mathbb{R}^d$.

In 1986, Bourgain proved the following variant of Szemerédi-type theorem for simplices.

Theorem A (Bourgain [3]). Suppose set $A \subseteq \mathbb{R}^d$ has positive upper density and $V = \{p_1, p_2, \dots, p_d\} \subseteq \mathbb{R}^d$ is a set of d points such that

$$\dim(\text{span} V) = d - 1.$$

Then there exists a number l such that for all $l' \geq l$, there is O in the d -dimensional orthogonal group $\mathcal{O}(d)$ and $x \in \mathbb{R}^d$ such that

$$x + l'O(V) \subseteq A.$$

The original proof of Theorem A is based on spherical operators [2]. A new proof for a stronger version of the theorem is given by Lyall and Magyar [9] on the basis of multilinear analysis.

Inspired by Bourgain's theorem A, C. Wang [12] considered a pinned variant and proved a result for 2-point patterns in all dimensions .

Theorem B. For $d \geq 2$, there is a constant $C_d > 0$ such that for all $A \subseteq \mathbb{R}^d$ with $\delta(A) > 0$ and all $x \in A$,

$$\delta(D_x(A)) = \limsup_{R \rightarrow \infty} \frac{\mathcal{L}^1(D_x(A) \cap (-R, R))}{R} \geq C_d \delta(A),$$

where the *pinned distance set* $D_x(A)$ is defined as

$$(1) \quad D_x(A) := \{|x - y| : y \in A\}.$$

The pinned distance set $D_x(A)$ collects all distances between a fixed point x and all other points in A .

As we mentioned, the above result is about k -point patterns where $k = 2$. In this case, there is only one type of patterns, which is distance. In this note, we follow this topic and study the case

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where $k \geq 3$. When $k \geq 3$, the patterns become more complicated and even the analogous result of Theorem B no longer holds. This is the following negative result.

Theorem 1. For $k \geq 3$ and any fixed k -point pattern $V = \{p_1, p_2, p_3, \dots, p_k\} \subseteq \mathbb{R}^d$ that avoids 3 collinear points, there is a set E with positive upper density satisfies that for any $x \in E$, there is $R(x) > 0$ such that

$$(R(x), \infty) \bigcap D_x^V(E) = \emptyset,$$

where

$$(2) \quad D_x^V(E) := \{r > 0 : \exists \text{ isometry } O \in \mathcal{O}(d) \text{ such that } x + rOV \subseteq E\}.$$

Let us give several remarks on the theorem.

- Without loss of generality, we can always assume that $p_1 = 0$ and $p_2 = (1, 0, \dots, 0)$ by translating and rescaling the pattern.
- Note that when $d \geq 3$, the condition here is weaker than Bourgain's original condition that the pattern V satisfies $\dim(\text{span} V) = d - 1$.
- In the case of k -point pattern ($k \geq 3$), we replace the pinned distance set $D_x(E)$ with *pinned scaling factor set* $D_x^V(E)$. Intuitively, the pinned scaling factor set consists of all scales at which a given pattern appears with one vertex fixed at x .

The proof of Theorem 1 is constructing an example and is provided in Example 1 in Section 2.

Theorem 1 confirms that for a fixed set with positive upper density, we cannot anticipate the abundance of pinned affine copies is true for all patterns when $k \geq 3$. However, there should be some patterns whose large pinned affine copies occur frequently since the set has positive upper density. This is the following result.

Theorem 2. For $d \geq 2$, $\varepsilon_0 > 0$ and $k \geq 3$, there exist a finite set $\mathcal{V} = \mathcal{V}(d, k, \varepsilon_0)$ of k -point patterns and a positive number $\tilde{\varepsilon}(\varepsilon_0, k, d)$ such that the following holds. For all $A \subseteq \mathbb{R}^d$ with $\delta(A) \geq \varepsilon_0$, there is a pattern $V \in \mathcal{V}$ such that for all $x \in A$, we have

$$\delta(D_x^V(A)) \geq \tilde{\varepsilon}(\varepsilon_0, k, d) > 0.$$

Remark 1. The number $\tilde{\varepsilon}(\varepsilon_0, k, d)$ can be written explicitly.

$$\tilde{\varepsilon}(\varepsilon_0, k, d) = \frac{\varepsilon(\varepsilon_0, k, d)}{M_d}.$$

In this expression, $M_d = 10^{10} \pi C_d^2$ for $k = 3$ and $10C_d$ for $k \geq 4$.

$$(3) \quad \varepsilon(\varepsilon_0, k, d) = \begin{cases} \varepsilon_0^2, & k = 3, \\ \exp[-\exp((\frac{20C_d\pi}{\varepsilon_0})^{1/c_{k-1}})], & k \geq 4. \end{cases}$$

C_d is a constant depending on d and c_{k-1} comes from Szemerédi's theorem and depends on k .

Let us pause to compare Theorem 1 and Theorem 2. While Theorem 1 says one can defeat any fixed pattern by constructing the set appropriately, Theorem 2 says that one cannot defeat them all at once in any fixed set with positive upper density. No matter how one choose a dense set A , there will always be some pattern V_i from a finite predetermined catalog that does appear frequently at every pin.

The proof is quantitative and relies on additive-combinatorial machinery, Szemerédi's theorem on arithmetic progressions combined with the spherical integral argument in [12]. This argument is useful when searching for some "isosceles" pattern. We will divide the proof into two parts, one is the case $d = 2$ in Section 3.3.1 and the other is the case $d \geq 3$ in Section 3.3.2.

Let us use the following Figure 1 to exhibit the difference between [12] and this note.

	[12]	This note
d : dimension of the ambient space	$d \geq 2$	$d \geq 2$
k : number of the points of the pattern	2	$k \geq 3$
Whether any pinned scaling factor set has positive upper density?	Yes	No
Whether there is a pinned scaling factor set (for a specific pattern) has positive upper density?	$\times$	Yes

FIGURE 1. difference between [12] and this note

Bourgain's result can be generalized in many directions. In [13], Ziegler considered all multi-point patterns on the plane and proved that all sufficiently large dilates of them can be contained in an arbitrarily small neighborhood of sets with positive upper density. Our results also give a partial answer for the pinned version of the Ziegler-type result in all dimensions.

1.2. Szemerédi's theorem. A key ingredient of our proof is the quantitative Szemerédi's theorem, which we record more details in Appendix A. By the technique in [12] that is used to search for a type of specific “isosceles” patterns, we can reduce the problem to a pattern avoidance problem over the torus $\mathbb{R}/2\pi\mathbb{Z}$. Szemerédi's theorem gives us a nice quantitative upper bound for a subset of $\mathbb{R}/2\pi\mathbb{Z}$ that forbids certain pattern. This is Proposition 1 in Section 3.1.

1.3. Structure of the note. The structure of the note is as follows: In Section 2, we will prove Proposition 1 with a counterexample. In Section 3, we will prove Theorem 2 where the whole section is divided into two cases: the planar case in Section 3.3.1 and the higher dimensional cases 3.3.2.

2. PROOF OF THEOREM 1

Recall the statement of Theorem 1. Its proof is based on the construction of an example. Heuristically, for any fixed k -point pattern V without three colinear point, we can take the smallest (positive) angle α formed by the points in V . Then one can always consider a thin cone satisfies the following conditions.

- The apex of the cone is the origin.
- The apex angle of the cone is smaller than the smallest angle α of V .
- The thin cone has positive upper density.

If the origin is chosen as one of the vertices of the dilate of V , then there is no dilate of V formed by the origin and other points in the thin cone, since any such shape has a smaller angle. When the scaling factor is sufficiently large, such an argument works for any fixed point in the thin cone, not only the origin. Now we make this rigorous.

Example 1 (thin cone). Let

$\alpha :=$ the smallest angle formed by the pattern V ,

$r_{\min} :=$ the shortest side-length of the pattern V .

By rescaling, we can assume $r_{\min} = 1$. Since the pattern avoids collinear points, we have $\alpha \in (0, \pi)$.

Let $\alpha' = \frac{\alpha}{2^{100}d} \ll 1$. Define the solid cone $C(\alpha')$ as

$$C(\alpha') := \{x \in \mathbb{R}^d : \angle \langle x/\|x\|, e_1 \rangle \leq \alpha'/2\}.$$

The figure for spatial case is depicted in Figure 2.

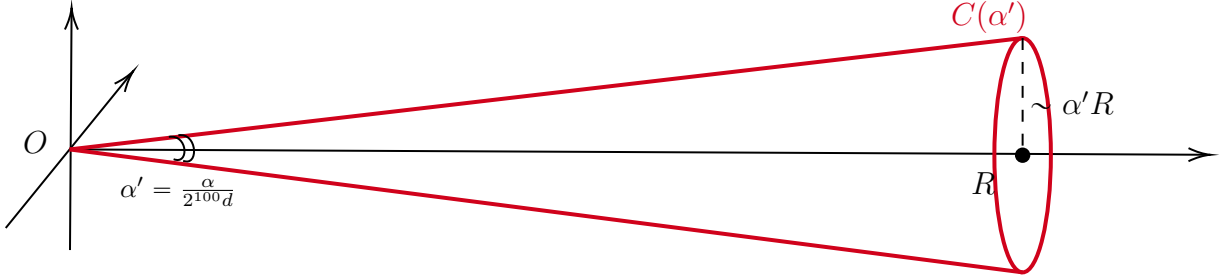


FIGURE 2. Spatial case of $C(\alpha')$

It can be checked that $\delta(C(\alpha')) > 0$. Indeed, for any $R > 0$, $C(\alpha') \cap B(0, R)$ is a “cone” with a cap from $R\mathbb{S}^{d-1}$ as its base. Therefore, by the volume formula of the $d - 1$ -dimensional cone,

$$\frac{\mathcal{L}^d(C(\alpha') \cap B(0, R))}{R^d} \geq C_d \frac{(\alpha' R)^{d-1} \cdot R}{R^d} = C_d \alpha'^{d-1} > 0.$$

Now it suffices to prove $C(\alpha')$ does not contain any sufficiently large pinned affine copy of V . We claim the following fact.

Lemma 1. *Fix any $x \in C(\alpha')$, there is $M(x, \alpha') > 0$ such that for all $y, y' \in C(\alpha')$ such that $M(x, \alpha') \leq |y - x| \leq |y' - x|$, we have*

$$\angle \langle y - x, y' - x \rangle \leq 2^{10} \alpha'.$$

We assume the lemma is true for the moment. For $x \in C(\alpha')$, let $R(x)$ be $2M(x, \alpha')$ in Lemma 1. We want to show for all $R > R(x) = 2M(x, \alpha')$, there is no R -dilated affine copy of V with x as one of its vertices. Assume by contradiction that there is a R -dilate copy $V^R \subseteq E$. Then the length of the shortest side is $r_{\min} R = R$. If x is one of its vertex, then for any other vertices $y, y' \in V^R \setminus \{x\}$,

$$|x - y| \geq R > R(x) = 2M(x, \alpha') \quad \text{and} \quad |x - y'| \geq R > R(x) = 2M(x, \alpha').$$

By Lemma 1,

$$\angle \langle y - x, y' - x \rangle \leq 2^{10} \alpha' = \frac{\alpha}{2^{90}d} < \alpha.$$

which contradicts the assumption that the smallest angle of V is α .

This concludes our construction for the counterexample.

Now it remains to prove Lemma 1.

Proof of Lemma 1. Let $x \in C(\alpha')$ be fixed. We first parameterize points $y, y' \in C(\alpha')$ via spherical coordinates with e_1 as the axis:

$$\begin{aligned} y &= \|y\| (\cos \theta_y e_1 + \sin \theta_y v_y), \\ y' &= \|y'\| (\cos \theta_{y'} e_1 + \sin \theta_{y'} v_{y'}), \end{aligned}$$

where $\theta_y, \theta_{y'} \leq \alpha'/2$, $v_y, v_{y'} \perp e_1$, and $\|v_y\| = \|v_{y'}\| = 1$. Similarly, decompose x as:

$$x = \|x\| (\cos \theta_x e_1 + \sin \theta_x v_x), \quad \theta_x \leq \alpha'/2.$$

As a result,

$$(4) \quad \begin{aligned} y - x &= (\|y\| \cos \theta_y - \|x\| \cos \theta_x) e_1 + (\|y\| \sin \theta_y v_y - \|x\| \sin \theta_x v_x), \\ y' - x &= (\|y'\| \cos \theta_{y'} - \|x\| \cos \theta_x) e_1 + (\|y'\| \sin \theta_{y'} v_{y'} - \|x\| \sin \theta_x v_x). \end{aligned}$$

If we denote $\phi = \angle \langle y' - x, y - x \rangle$, then

$$(5) \quad \cos \phi = \frac{(y' - x) \cdot (y - x)}{\|y' - x\| \cdot \|y - x\|}.$$

Plug equation (4) to the numerator of equation (5). The coefficient of the e_1 term,

$$(\|y\| \cos \theta_y - \|x\| \cos \theta_x) (\|y'\| \cos \theta_{y'} - \|x\| \cos \theta_x)$$

is dominated by $\|y\| \|y'\| \cos \theta_y \cos \theta_{y'}$ when $\|y\|, \|y'\| \gg \|x\|$. For the remaining terms,

$$\begin{aligned} &|(\|y\| \sin \theta_y v_y - \|x\| \sin \theta_x v_x) \cdot (\|y'\| \sin \theta_{y'} v_{y'} - \|x\| \sin \theta_x v_x)| \\ &\leq \|y\| \|y'\| \sin \theta_y \sin \theta_{y'} + O(\|x\|(\|y\| + \|y'\|)). \end{aligned}$$

When $\|y\|$ and $\|y'\|$ are sufficiently large (depending on x and α'), the term $\|y\| \|y'\| \sin \theta_y \sin \theta_{y'}$ dominates.

Combining the terms in equation (5), we have

$$\cos \phi \geq \frac{\|y\| \|y'\| \cos(\theta_y + \theta_{y'}) - O(\|x\|(\|y\| + \|y'\|))}{\|y\| \|y'\|}.$$

Since $\theta_y + \theta_{y'} \leq \alpha'$ and the continuity and monotonicity of $\cos$ function, when $\|y\|, \|y'\| \geq M(x, \alpha') \gg \|x\|$:

$$\cos \phi \geq \cos \alpha' \quad \Rightarrow \quad \phi \leq \alpha' < 2^{10} \alpha'. \quad \square$$

3. PROOF OF THEOREM 2

We prove Theorem 2 in the following way. First, the arbitrarily chosen fixed base point x can be assumed to be the origin. Therefore, it suffices to prove a “pinned at the origin” version of Theorem 2, which is the following Proposition 1. To prove Proposition 1, we need to use Szemerédi’s theorem via Gowers bounds. We reformulate the theorem to fit our setting in Lemma 3. With this lemma in hand, we split the case $d \geq 2$ in Proposition 1 into two sub-cases: one is $d = 2$ and the other one is $d \geq 3$. For the case of $d = 2$, we reduce the problem to a pattern avoidance problem, where Lemma 3 can be applied. For the case of $d = 3$, we use a polar coordinate argument to lower the dimension to 2 and repeat our reasoning in the planar case.

3.1. Reduction to “pinned at the origin” version. As observed in [12], due to the definition of upper density, the main theorem is equivalent to the following “pinned at the origin” version.

Proposition 1 (main lemma). *For $d \geq 2$, $\varepsilon_0 > 0$ and $k \geq 3$, there exist a finite set $\mathcal{V} = \mathcal{V}(d, k, \varepsilon_0)$ of k -point patterns and a positive number $\tilde{\varepsilon}(\varepsilon_0, k, d)$ such that the following holds. For all $A \subseteq \mathbb{R}^d$ with $\delta(A) \geq \varepsilon_0$, there is a pattern $V \in \mathcal{V}$ such that*

$$\delta(D_0^V(A)) \geq \tilde{\varepsilon}(\varepsilon_0, k, d) > 0.$$

To see this, we record the following translation invariance lemma from [12], whose proof is direct.

Lemma 2 (translation invariance). *For all $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, $\delta(A - x) = \delta(A)$.*

Proof of Theorem 2 assuming Proposition 1. By the conditions of Theorem 2, $\delta(A) \geq \varepsilon_0$ and $x \in A$. By Lemma 2, $\delta(A - x) = \delta(A) \geq \varepsilon_0$. Then the set $A - x$ satisfies the condition of Proposition 1. Applying Proposition 1 to $A - x$, we obtain that there is a pattern V such that

$$\delta(D_0^V(A - x)) \geq \tilde{\varepsilon}(\varepsilon_0, k, d) > 0.$$

This concludes the proof since $D_0^V(A - x) = D_x^V(A)$. $\square$

To prove Proposition 1, we adapt Szemerédi's theorem to our setting.

3.2. Adaption of Szemerédi's theorem. The main result of this section is the following.

Lemma 3 (size of sets avoiding $(k - 1)$ -AP). *Fix $k \geq 3$. Let $n + 1$ be a large prime number such that $n \gg k$. If $E \subseteq \mathbb{R}/2\pi\mathbb{Z} = [0, 2\pi)$ satisfies for any fixed $x \in \mathbb{R}/2\pi\mathbb{Z}$ and $i \in \{1, 2, \dots, n\}$,*

$$(6) \quad \left\{ x, x + \frac{2\pi i}{n+1}, x + 2 \cdot \frac{2\pi i}{n+1}, \dots, x + (k-2) \cdot \frac{2\pi i}{n+1} \right\} \not\subseteq E.$$

Then

$$(7) \quad \mathcal{L}^1(E) \leq \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4, \end{cases}$$

where c_{k-1} is defined in (26).

Let us pause to explain the lemma.

- Since we view each number from the set of left hand side of (6) as an element in $\mathbb{R}/2\pi\mathbb{Z}$. Naturally, $x + j \cdot \frac{2\pi i}{n+1} = x + j \cdot \frac{2\pi i}{n+1} \bmod 2\pi$ for each $i \in \{1, \dots, n\}$ and $j \in \{0, \dots, k-2\}$.
- For each fixed k , the set of left hand side of (6) is exactly a $(k-1)$ -term arithmetic progression, with common difference $\frac{2\pi i}{n+1}$. As i ranges over all $\{1, \dots, n\}$, the set ranges over all possible “ $(k-1)$ -term APs”. Therefore, roughly speaking, the condition says that E avoids all “ $(k-1)$ -term APs”. And the conclusion of the lemma gives a quantitative upper bound for the Lebesgue measure of such a set.

In addition, note that since $n+1$ is a prime number, fix $i \in \{1, \dots, n\}$, $x + j \cdot \frac{2\pi i}{n+1}$, $j = 0, 1, \dots, (k-2)$ are distinct in $\mathbb{R}/2\pi\mathbb{Z}$.

Proof. By change of variable,

$$(8) \quad \begin{aligned} \mathcal{L}^1(E) &= \int_{\mathbb{R}/2\pi\mathbb{Z}} \chi_E(x) dx \\ &= \int_0^{\frac{2\pi}{n+1}} \sum_{\tau=0}^n \chi_E\left(x + \frac{\tau 2\pi}{n+1}\right) dx \\ &= \int_0^{\frac{2\pi}{n+1}} \sum_{\tau=0}^n \chi_{E-x}\left(\frac{\tau 2\pi}{n+1}\right) dx \end{aligned}$$

Fix each $x \in (0, \frac{2\pi}{n+1})$. Since $\{\frac{\tau 2\pi}{n+1}, \tau \in \{0, 1, \dots, n\}\} \cong \mathbb{Z}/(n+1)\mathbb{Z}$. The set $E - x$ can be viewed as a subset of $\mathbb{Z}/(n+1)\mathbb{Z}$ under the canonical correspondence. Therefore,

$$(9) \quad \begin{aligned} \mathcal{L}^1(E) &= \int_0^{\frac{2\pi}{n+1}} \sum_{\tau=0}^n \chi_{E-x}\left(\frac{\tau 2\pi}{n+1}\right) dx \\ &= \int_0^{\frac{2\pi}{n+1}} \left(\int_{\mathbb{Z}/(n+1)\mathbb{Z}} \chi_{E-x}\left(\frac{\tau 2\pi}{n+1}\right) d\#\tau \right) dx. \end{aligned}$$

In the above expression, τ is viewed as an element in $\mathbb{Z}/(n+1)\mathbb{Z}$. We claim that the set

$$B := \left\{ \tau \in \mathbb{Z}/(n+1)\mathbb{Z} : \frac{2\pi\tau}{n+1} \in E - x \right\}$$

is a subset of $\mathbb{Z}/(n+1)\mathbb{Z}$ without $(k-1)$ -A.P.. If not, by our Definition 1, there are distinct

$$\tau_1, \tau_2, \dots, \tau_{k-1} \in \mathbb{Z}/(n+1)\mathbb{Z}, \quad \tau_{i+1} - \tau_i = d \pmod{n+1}$$

such that

$$(10) \quad \frac{2\pi\tau_j}{n+1} \in E - x, \quad \forall j = 1, 2, \dots, k-1.$$

Note that in $\mathbb{Z}/(n+1)\mathbb{Z}$,

$$\tau_j = \tau_1 + (j-1)d.$$

Therefore, in $\mathbb{R}/2\pi\mathbb{Z}$,

$$2\pi\tau_i = 2\pi(\tau_1 + (j-1)d) \Rightarrow \frac{2\pi\tau_j}{n+1} = \frac{2\pi(\tau_1 + (j-1)d)}{n+1}.$$

Hence (10) implies that for all $j = 1, 2, \dots, k-1$,

$$\mathbb{R}/2\pi\mathbb{Z} \supseteq E \ni x + \frac{2\pi\tau_j}{n+1} = x + \frac{2\pi(\tau_1 + (j-1)d)}{n+1} = \left(x + \frac{2\pi\tau_1}{n+1}\right) + \frac{(j-1)d}{n+1},$$

This means that

$$\left\{ \left(x + \frac{2\pi\tau_1}{n+1}\right) + \frac{(j-1)d}{n+1} \right\}_{j=1}^{k-1} \subseteq E$$

which contradicts (6) with $x = x + \frac{2\pi\tau_1}{n+1}$, $i = d$.

Therefore, return to our computation (9),

$$\begin{aligned} \mathcal{L}^1(E) &= \int_0^{\frac{2\pi}{n+1}} \left(\int_{\mathbb{Z}/(n+1)\mathbb{Z}} \chi_{E-x}\left(\frac{\tau 2\pi}{n+1}\right) d\#\tau \right) dx \\ &= \int_0^{\frac{2\pi}{n+1}} \left(\int_{\mathbb{Z}/(n+1)\mathbb{Z}} \chi_B(\tau) d\#\tau \right) dx \\ &= \int_0^{\frac{2\pi}{n+1}} \#B dx \end{aligned}$$

Apply Theorem D to bound $\#B$,

$$\begin{aligned} \mathcal{L}^1(E) &\leq \int_0^{\frac{2\pi}{n+1}} r_{k-1}(\mathbb{Z}/(n+1)\mathbb{Z}) dx \\ &\leq \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4. \end{cases} \end{aligned}$$

□

3.3. Proof of Proposition 1. Now we start to prove Proposition 1. We first prove it in the planar case. The higher dimensional case can be deduced with a little additional effort. In what following, we are actually proving the following result.

Proposition 1'. *For $d \geq 2$, $\varepsilon_0 > 0$ and $k \geq 3$, there is a finite set of k -point patterns $\mathcal{V} = \mathcal{V}(d, k, \varepsilon_0)$ and a constant M_d depending on d and k such that the following holds. Suppose $A \subseteq \mathbb{R}^d$ with $\delta(A) \geq \varepsilon_0$ and $0 \in A$. Then there is a pattern $V \in \mathcal{V}$ such that the upper density of the pinned scaling factor set satisfies*

$$(11) \quad \delta(D_0^V(A)) \geq \frac{\varepsilon(\varepsilon_0, k, d)}{M_d}.$$

We first record the following elementary result in linear algebra.

Lemma 4. *$d \geq 2$, $m \geq 1$ are two integers. For two groups of coplanar vectors of the same length ℓ , $U = \{u_1, \dots, u_m\} \subseteq \ell\mathbb{S}^{d-1}$ and $W = \{w_1, \dots, w_m\} \subseteq \ell\mathbb{S}^{d-1}$. Assume*

$$(12) \quad u_i = \ell(\cos \alpha_i, \sin \alpha_i, 0, \dots, 0), \quad w_i = \ell(\cos \beta_i, \sin \beta_i, 0, \dots, 0),$$

$\alpha_i < \alpha_{i+1}, \beta_i < \beta_{i+1}$ for all i . If

$$\alpha_{i+1} - \alpha_i = \beta_{i+1} - \beta_i,$$

then there is isometry $O \in \mathcal{O}(d)$ such that

$$O(U) = W.$$

Proof. Swapping U and V if necessary, we can assume $\alpha_1 \leq \beta_1$. It can be directly checked that the orthogonal matrix

$$O' = \begin{bmatrix} \cos(\beta_1 - \alpha_1) & -\sin(\beta_1 - \alpha_1) & 0 & \cdots & 0 \\ \sin(\beta_1 - \alpha_1) & \cos(\beta_1 - \alpha_1) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

sends U to W . □

3.3.1. Case $d = 2$. Fix $k \geq 3$. For $d = 2$, assume the conclusion is false, which means there is a set A with $\delta(A) \geq \varepsilon_0$ such that for all finite sets of patterns $\mathcal{V}$ and all $V \in \mathcal{V}$, equation (11) does not hold, which is

$$\delta(D_0^V(A)) < \frac{\varepsilon(\varepsilon_0, k, d)}{M_d}$$

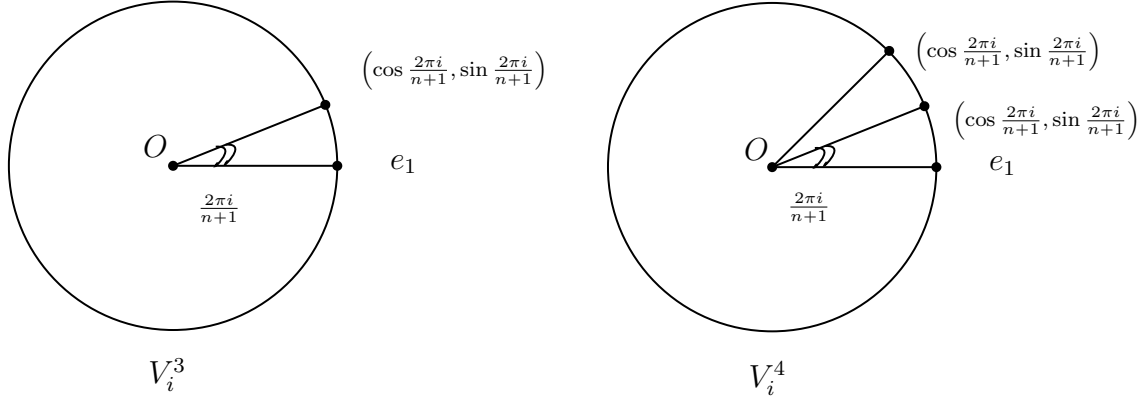
Let $n = n(d, k, \varepsilon_0) \gg 1$ be a prime integer to be determined later. We choose $\mathcal{V} = \{V_i^k, i = 1, 2, \dots, n\}$ as follows

$$V_i^k = \left\{ 0, e_1, \left(\cos \frac{2\pi i}{n+1}, \sin \frac{2\pi i}{n+1} \right), \dots, \left(\cos \frac{(k-2)2\pi i}{n+1}, \sin \frac{(k-2)2\pi i}{n+1} \right) \right\} \subseteq \mathbb{R}^d, \quad i = 1, \dots, n$$

and they satisfy that

$$(13) \quad \delta(D_0^{V_i^k}(A)) < \frac{\varepsilon(\varepsilon_0, k, d)}{M_d}, \quad \forall i = 1, 2, \dots, n.$$

The pattern for $k = 3$ and 4 can be found in Figure 3.

FIGURE 3. Pattern in xOy -plane

By the definition of upper density and limit superior, equation (13) means that for any $\eta > 0$, there is $R(\eta, n, i) > 0$ such that for all $R \geq R(\eta, n, i)$,

$$(14) \quad \frac{\mathcal{L}^1(D_0^{V_i^k}(A) \cap [0, R])}{R} < \frac{\eta}{n} + \frac{\varepsilon(\varepsilon_0, k, d)}{M_d}, \quad \forall i = 1, 2, \dots, n,$$

which is equivalent to

$$(15) \quad \mathcal{L}^1(D_0^{V_i^k}(A) \cap [0, R]) < \frac{\eta R}{n} + \frac{\varepsilon_0 \varepsilon(\varepsilon_0, k, d) R}{M_d}, \quad \forall i = 1, 2, \dots, n.$$

Denote $D_0^{V_i^k}(A)$ as $D_0^{i,k}(A)$ and define the union of the pinned scaling factor sets as

$$D_0(n, k) := \bigcup_{i=1}^n D_0^{i,k}(A).$$

Then for all $R \geq R(\eta, n) := \max\{R(\eta, n, i) : i = 1, 2, \dots, n\}$,

$$(16) \quad \begin{aligned} \mathcal{L}^1(D_0(n, k) \cap [0, R]) &= \mathcal{L}^1\left(\bigcup_{i=1}^n D_0^{i,k}(A) \cap [0, R]\right) \\ &\leq \sum_{i=1}^n \mathcal{L}^1(D_0^{i,k}(A) \cap [0, R]) \\ &< \eta R + \frac{n \varepsilon(\varepsilon_0, k, d) R}{M_d}. \end{aligned}$$

In the last inequality, we apply (15).

On the other hand, for such R , we estimate the nominator of the upper density $\delta(A)$.

$$(17) \quad \begin{aligned} \mathcal{L}^d(A \cap B(0, R)) &= \int_0^R \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) dr \\ &= \left(\int_{D_0(n, k) \cap [0, R]} + \int_{[0, R] \setminus D_0(n, k)} \right) \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) dr \\ &= I_1 + I_2, \end{aligned}$$

where in the first line, the area measure σ_r^{d-1} on $r\mathbb{S}^{d-1}$ is defined as

$$(18) \quad \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) := \int_{\mathbb{R}/2\pi\mathbb{Z}} \chi_{r\mathbb{S}^{d-1} \cap A}(r \cos \theta, r \sin \theta) d\theta r^{d-1}.$$

We first address I_1 . By (16),

$$\begin{aligned}
 I_1 &= \int_{D_0(n,k) \cap [0,R]} \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) dr \\
 &\leq \mathcal{L}^1 \left(D_0(n,k) \cap [0,R] \right) \cdot \sigma_r^{d-1}(r\mathbb{S}^{d-1}) \\
 &< C_d R^{d-1} \cdot \left(\eta R + \frac{n\varepsilon(\varepsilon_0, k, d)R}{M_d} \right) \\
 &= C_d R^d \left(\eta + \frac{n\varepsilon(\varepsilon_0, k, d)}{M_d} \right).
 \end{aligned}
 \tag{19}$$

For I_2 ,

$$I_2 = \int_{[0,R] \setminus D_0(n,k)} \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) dr,$$

we can apply Lemma 3.

Assume $r \notin D_0(n,k)$. Define Lipschitz map

$$\begin{aligned}
 \varphi : \mathbb{R}/2\pi\mathbb{Z} &\rightarrow r\mathbb{S}^{d-1} \\
 \theta &\mapsto r(\cos \theta, \sin \theta)
 \end{aligned}$$

This is an isomorphism. By (18) and change of variable, the integrand of I_2

$$\sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) = r^{d-1} \mathcal{L}^1(\varphi^{-1}(r\mathbb{S}^{d-1} \cap A)). \tag{20}$$

It suffices to estimate the right hand side. Denote

$$\varphi^{-1}(r\mathbb{S}^{d-1} \cap A) \triangleq E_r \subseteq \mathbb{R}/2\pi\mathbb{Z}.$$

We claim E_r satisfies the condition (6) of Lemma 3. If not, there is $d \in \{1, 2, \dots, n\}$ such that distinct numbers

$$\left\{ x, x + \frac{2\pi d}{n+1}, x + 2 \cdot \frac{2\pi d}{n+1}, \dots, x + (k-2) \cdot \frac{2\pi d}{n+1} \right\} \subseteq E_r.$$

This implies their images S under φ ,

$$S := \left\{ r \left(\cos \left(x + j \cdot \frac{2\pi d}{n+1} \right), \sin \left(x + j \cdot \frac{2\pi d}{n+1} \right) \right) : j = 0, 1, \dots, k-2 \right\} \subseteq r\mathbb{S}^{d-1} \cap A.$$

It can be checked that $W = S$ and

$$U = rV_d^k \setminus \{0\} = \left\{ re_1, r \left(\cos \frac{2\pi d}{n+1}, \sin \frac{2\pi d}{n+1} \right), \dots, r \left(\cos \frac{(k-2)2\pi d}{n+1}, \sin \frac{(k-2)2\pi d}{n+1} \right) \right\}$$

satisfy the condition of Lemma (4) (switching U, W if necessary). Therefore, there is $O \in \mathcal{O}(d)$ such that

$$O(U) = W.$$

Combining this with $O(0) = 0$, $0 \in A$ and $\varphi(E_r) = r\mathbb{S}^{d-1} \cap A$, we obtain

$$rO(V_d^k) = O(rV_d^k) = S \cup \{0\} \subseteq A. \tag{21}$$

By definition, this implies $r \in D_0^{d,k}(A) \subseteq D_0(n,k)$ which contradicts with $r \notin D_0(n,k)$.

Return to our estimate (20) to the integrand of I_2 . This means we can apply Lemma 3 to $\mathcal{L}^1(E_r)$. Therefore,

$$\sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) = r^{d-1} \mathcal{L}^1(E_r) \leq R^{d-1} \cdot \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4. \end{cases}$$

Plug this back to I_2

$$(22) \quad I_2 = \int_{[0,R] \setminus D_0(n,k)} \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) dr \leq R^d \cdot \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4. \end{cases}$$

Combining (19) and (22), we obtain for (17)

$$\mathcal{L}^d(A \cap B(0, R)) = I_1 + I_2 \leq C_d R^d \left(\eta + \frac{n\varepsilon(\varepsilon_0, k, d)}{M_d} \right) + R^d \cdot \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4, \end{cases}$$

which means

$$\frac{\mathcal{L}^d(A \cap B(0, R))}{R^d} \leq C_d \left[\eta + \frac{n\varepsilon(\varepsilon_0, k, d)}{M_d} + \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4, \end{cases} \right].$$

In the computations, C_d may change from line to line. The one in the definition of $\varepsilon(\varepsilon_0, k, d)$ is the final C_d .

For $R_i \geq R(\eta, n)$ where $\{R_i\}$ is the subsequence of R that attains the limit superior in $\delta(A)$,

$$\begin{aligned} \varepsilon_0 \leq \delta(A) &= \lim_{i \rightarrow \infty} \frac{\mathcal{L}^d(A \cap B(0, R_i))}{R_i^d} \\ &\leq C_d \left[\eta + \frac{n\varepsilon(\varepsilon_0, k, d)}{M_d} + \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_{k-1}}}, & k \geq 4, \end{cases} \right] \\ &< \frac{\varepsilon_0}{2} < \varepsilon_0, \end{aligned}$$

if we choose

$$k = 3 : M_d = 10^{10} \pi C_d^2, \eta < \frac{\varepsilon_0}{10C_d}, \text{ and prime number } n+1 = n(\varepsilon_0, k, d) + 1 \in \left(\frac{20\pi C_d}{\varepsilon_0}, \frac{60\pi C_d}{\varepsilon_0} \right).$$

According to Bertrand–Chebyshev theorem, such a prime exists.

$$k \geq 4 : M_d = 10C_d, \eta < \frac{\varepsilon_0}{10C_d}, \text{ and prime number } n+1 \text{ is contained in}$$

$$\left(\frac{1}{3} \cdot \exp \exp \left((20C_d \pi / \varepsilon_0)^{1/c_{k-1}} \right), \exp \exp \left((20C_d \pi / \varepsilon_0)^{1/c_{k-1}} \right) \right).$$

This is a contradiction hence concludes the proof for $d = 2$.

3.3.2. Case $d \geq 3$. The method can be generalized to higher dimensions by combining a polar coordinate argument. Still analyzing by contradiction, we redefine V_i^k as

$$V_i^k = \left\{ 0, e_1, \left(\cos \frac{2\pi i}{n+1}, \sin \frac{2\pi i}{n+1}, 0 \right), \dots, \left(\cos \frac{(k-2)2\pi i}{n+1}, \sin \frac{(k-2)2\pi i}{n+1}, 0 \right) \right\} \subseteq \mathbb{R}^d, \quad i = 1, \dots, n.$$

Proceeding to the analysis of I_1 and I_2 , we need to change the argument of estimating I_2 since φ is not well-defined when we are in higher dimensions.

Assume $r \notin D_0(n, k)$. We apply the repeated polar coordinate or the change of variable formula to $r\mathbb{S}^{d-1}$. We parametrize the sphere by $r\omega$ where

$$\omega = \omega(\theta_1, \dots, \theta_{d-2}, \phi) = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \cos \theta_2 \\ \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ \vdots \\ \sin \theta_1 \cdots \sin \theta_{d-3} \cos \theta_{d-2} \\ \sin \theta_1 \cdots \sin \theta_{d-3} \sin \theta_{d-2} \cos \phi \\ \sin \theta_1 \cdots \sin \theta_{d-3} \sin \theta_{d-2} \sin \phi \end{pmatrix} \in \mathbb{S}^{d-1},$$

$\theta_1 \in (0, 2\pi)$, $\theta_i, \phi \in (0, \pi)$, $i = 2, 3, \dots, d-2$.

One can check that the Jacobian determinant of this change of variable is

$$r^{d-1} \prod_{j=1}^{d-2} \sin^{d-j-1} \theta_j.$$

This means that under this change of variable, the area $\sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A)$ can be written as an integral over angles $\theta_1, \dots, \theta_{d-2}, \phi$, which is

(23)

$$\begin{aligned} \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) &= \int_{r\mathbb{S}^{d-1}} \chi_A(\omega') d\sigma_r^{d-1}(\omega') \\ &= r^{d-1} \int_{\phi=0}^{\pi} \int_{\theta_{d-2}=0}^{\pi} \dots \int_{\theta_1=0}^{2\pi} \chi_A(r\omega(\theta_1, \dots, \theta_{d-2}, \phi)) \left(\prod_{j=1}^{d-2} \sin^{d-j-1} \theta_j \right) d\theta_1 \dots d\theta_{d-2} d\phi \\ &\leq r^{d-1} \int_{\phi=0}^{\pi} \int_{\theta_{d-2}=0}^{\pi} \dots \int_{\theta_1=0}^{2\pi} \chi_A(r\omega(\theta_1, \dots, \theta_{d-2}, \phi)) d\theta_1 \dots d\theta_{d-2} d\phi. \end{aligned}$$

Note that in equation (23), for any fixed $\alpha = (\theta_2, \dots, \theta_{d-2}, \phi)$, $\{r\omega(\theta_1, \alpha) : \theta_1 \in (0, 2\pi)\}$ forms a circle with radius r contained in $r\mathbb{S}^{d-1}$. In fact, to prove this, it suffices to show that $\{\omega(\theta_1, \alpha) : \theta_1 \in (0, 2\pi)\}$ forms a unit circle.

Fix all angles except θ_1 , that is, fix $\theta_2, \dots, \theta_{d-2}, \phi$. Define the $(d-1)$ -dimensional vector:

$$\vec{\beta} = \begin{pmatrix} \cos \theta_2 \\ \sin \theta_2 \cos \theta_3 \\ \vdots \\ \sin \theta_2 \cdots \sin \theta_{d-2} \cos \phi \\ \sin \theta_2 \cdots \sin \theta_{d-2} \sin \phi \end{pmatrix} \in \mathbb{R}^{d-1}.$$

It is directly to check that $\vec{\beta}$ is a unit vector. Then the $\omega(\theta_1, \alpha)$ can be rewritten as:

$$\omega(\theta_1, \alpha) = \begin{pmatrix} \cos \theta_1 \\ \sin \theta_1 \cdot \vec{\beta} \end{pmatrix} = \cos \theta_1 \cdot u + \sin \theta_1 \cdot v,$$

where

$$u = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ \vec{\beta} \end{pmatrix} \in \mathbb{R}^d.$$

Since $\|u\| = \|v\| = 1$ and $u \perp v$, the trajectory $\omega(\theta_1, \alpha)$ lies entirely in the 2-dimensional plane spanned by u and v , and moves along the unit circle in that plane.

Hence, as θ_1 varies, the point $\omega(\theta_1, \alpha)$ traces out the intersection of this 2-dimensional plane, which passes through the origin with the unit sphere $\mathbb{S}^{d-1}$ —that is, a unit circle.

Denote the circle $\{r\omega(\theta_1, \alpha) : \theta_1 \in (0, 2\pi)\}$ as $\mathbf{S}_r(\alpha)$. It corresponds to the inner integral over θ_1

$$(24) \quad \int_{\theta_1=0}^{2\pi} \chi_A(r\omega(\theta_1, \dots, \theta_{d-2}, \phi)) d\theta_1.$$

For fixed $\mathbf{S}_r(\alpha)$, our goal is still deducing a contradiction of form (21) and apply Lemma 3. We first apply rotation O_α , such that $O_\alpha(\mathbf{S}_r(\alpha)) = r\mathbb{S}^1 \times \{0\} \subseteq \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^d$ and then repeat our

planar argument. Slightly abusing the notation, we redefine

$$\begin{aligned}\varphi : \mathbb{R}/2\pi\mathbb{Z} &\rightarrow r\mathbb{S}^1 \times \{0\} \\ \theta_1 &\mapsto r(\cos \theta_1, \sin \theta_1, 0),\end{aligned}$$

and define

$$E_r^\alpha := \{\theta_1 \in \mathbb{R}/2\pi\mathbb{Z} : \theta_1 \in \varphi^{-1}(O_\alpha(A \cap \mathbf{S}_r(\alpha)))\}.$$

Then the inner integral (24) can be rewritten as

$$(25) \quad \int_{\theta_1=0}^{2\pi} \chi_{E_r^\alpha}(\theta_1) d\theta_1 = \mathcal{L}^1(E_r^\alpha).$$

Similarly, we claim the set E_r satisfies the condition of Lemma 3. If not, the argument is exactly the same as the planar discussion. At last, we can find an orthogonal map $O_\alpha^{-1} \circ O$ that sends certain rV_d^k to A , which contradicts $r \notin D_0(n, k)$.

The claim allows us to apply Lemma 3 to the integral (25) over θ_1 . Therefore,

$$\begin{aligned}\int_{\theta_1=0}^{2\pi} \chi_A(r\omega(\theta_1, \dots, \theta_{d-2}, \phi)) d\theta_1 &= \int_{\theta_1=0}^{2\pi} \chi_{E_r^\alpha}(\theta_1) d\theta_1 = \mathcal{L}^1(E_r^\alpha) \\ &\leq \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_k-1}}, & k \geq 4 \end{cases}.\end{aligned}$$

Combining it with (23), we obtain

$$\text{RHS of (23)} \leq \pi^{d-2} R^{d-1} \cdot \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_k-1}}, & k \geq 4 \end{cases} = C_d R^{d-1} \cdot \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_k-1}}, & k \geq 4 \end{cases}.$$

Plug this back to I_2 ,

$$\begin{aligned}I_2 &= \int_{[0, R] \setminus D_0(n, k)} \sigma_r^{d-1}(r\mathbb{S}^{d-1} \cap A) dr \\ &\leq C_d R^d \cdot \begin{cases} \frac{2\pi}{n+1}, & k = 3, \\ \frac{2\pi}{(\log \log(n+1))^{c_k-1}}, & k \geq 4 \end{cases}.\end{aligned}$$

This is the higher dimensional version of our estimate for I_2 in (22). The rest of the proof is identical to the planar case so we omit the details.

Finally, we conclude the proof for Proposition 1.

4. FURTHER DIRECTIONS

We discuss possible further directions in this section.

Remark 2. In the proof, what we essentially work with is the scaling factors associated to the dilated pattern of V where x is fixed as $0 \in V$. We do not know if other types of change of variable can be used to test other non-isosceles patterns.

Remark 3. If we denote

$$m := \text{span}(V),$$

then the pattern we found satisfies $m = 2$. One can also consider higher dimensional patterns which correspond to more complicated avoidance problems. For example, if we assume $m = 3$, one possible pattern we can consider is “equilateral triangle” on $\mathbb{S}^{d-1}$. In the last integration we may leave θ_1 and ϕ as variables and ask: If $E \subseteq \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ avoids all equilateral triangle with side-length i/n . What is the quantitative upper bound (depending on n) of $\mathcal{L}^2(E)$? In this case, results in [7] may be applied.

APPENDIX A. SZEMERÉDI'S THEOREM

We record the following prerequisites related to Szemerédi's theorem.

Definition 1 (m term arithmetic progression, m -A.P.). For $m \geq 3$ and $N \gg m \geq 3$, a sequence of m elements $a_1, a_2, \dots, a_m \in \mathbb{Z}/N\mathbb{Z}$ is called an m term arithmetic progression (m -A.P.) with common difference d if

- $a_{i+1} - a_i = d \pmod{N}$, where $d \in \{1, 2, \dots, N-1\}$ for all i .
- $a_i \neq a_j$, if $i \neq j$.

For example, the common difference d of 3-A.P. 6, 1, 3 in $\mathbb{Z}/7\mathbb{Z}$ is 2. We require the common difference is a number between 1, 2, ..., $N-1$.

Interestingly, such pattern existence or abundance problem can be linked to Szemerédi's theorem in avoidance problem. For our purpose, the following quantitative version is needed.

Theorem C (Gowers [4]). Define

$$r_m(\mathbb{Z}/N\mathbb{Z}) := \text{the cardinality of maximal subsets of } \mathbb{Z}/N\mathbb{Z} \text{ without } m\text{-A.P..}$$

Then

$$(26) \quad r_m(\mathbb{Z}/N\mathbb{Z}) \leq \frac{N}{(\log \log N)^{c_m}}, \quad \text{where } c_m = 1/2^{2^{m+9}}.$$

This result does not appear explicitly in Gowers' original paper [4], whereas his method with Gowers' norm works well for more general groups. One can find equation (26) in Tao and Vu's book [11, Proposition 11.12]. More recent results about Szemerédi's theorem such as [1, 5, 6, 8, 10] can be applied and a tiny improvement in the quantitative bounds in Theorem 2 can be obtained. We do not do this for computational simplicity.

If we also define 2-A.P. to be an ordered pair $(a, b) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, $\bar{a} \neq \bar{b}$, then trivially,

$$r_2(\mathbb{Z}/N\mathbb{Z}) \leq 1.$$

Combining this with Gowers' result (26), we have

Theorem D (Szemerédi's Theorem).

$$(27) \quad r_m(\mathbb{Z}/N\mathbb{Z}) \leq \begin{cases} \frac{N}{(\log \log N)^{c_m}}, & m \geq 3, \\ 1, & m = 2. \end{cases}$$

We will apply Theorem D to prove Theorem 2.

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