Self-Generated Measures and the Centroid Rigidity of Power Laws

Vincent E. Coll, Jr.*[1] and Lee B. Whitt[2]
[1]Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA
[2]Northrop Grumman — Distinguished Technical Fellow (retired), San Diego, CA, lee.barlow.whitt@hotmail.com

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Abstract

We revisit a classical calculus computation—the centroid of the subgraph of a function on $[0, a]$—and show that it hides a rigidity theorem. Let $f : (0, \infty) \to (0, \infty)$ be C^2 with $f(0^+) = 0$, and set

$$\bar{x}(a) = \frac{\int_0^a x f(x)\,\mathrm{d}x}{\int_0^a f(x)\,\mathrm{d}x}, \qquad \bar{y}(a) = \frac{1}{2}\frac{\int_0^a f(x)^2\,\mathrm{d}x}{\int_0^a f(x)\,\mathrm{d}x}.$$

We prove that the *Geometric Scaling Property*

$$\bar{y}(a) = \lambda\, f(\bar{x}(a)) \qquad (\forall a > 0)$$

holds *if and only if* $f(x) = Ax^p$ with $A > 0$ and $p > 0$, and for power laws the optimal constant is

$$\lambda = \frac{p+1}{2(2p+1)}\left(\frac{p+2}{p+1}\right)^p.$$

After a scale-free normalization, the argument is probabilistic: under the self-generated probability measure on $[0, a]$ with density $f(x)\big/\int_0^a f(t)\,\mathrm{d}t$, one has $\bar{x}(a) = \mathbb{E}[X_a]$ and $\bar{y}(a) = \frac{1}{2}\mathbb{E}[f(X_a)]$, so GSP becomes an equality in expectation across all truncation scales. Differentiation in a yields a weighted-mean identity for the elasticity $E(x) = xf'(x)/f(x)$; a second differentiation forces a vanishing-variance principle that makes E constant, hence f a pure power and the stated λ follows. The proof uses no asymptotics and extends to $f \in C^1(0, \infty)$ with locally Lipschitz elasticity.

1. Introduction and motivation

Centroid formulas from first-year calculus suggest asking whether a persistent relation can hold between the centroid height and the function value sampled at the centroid abscissa (see, e.g., [1]). Precisely, for $f > 0$ with $f(0^+) = 0$, define

$$\bar{x}(a) = \frac{\int_0^a x f(x)\,\mathrm{d}x}{\int_0^a f(x)\,\mathrm{d}x}, \qquad \bar{y}(a) = \frac{1}{2}\frac{\int_0^a f(x)^2\,\mathrm{d}x}{\int_0^a f(x)\,\mathrm{d}x}.$$

We call

$$\bar{y}(a) = \lambda\, f(\bar{x}(a)) \qquad (\forall a > 0)$$

the *Geometric Scaling Property* (GSP). Despite its elementary appearance, GSP is rigid: we show it forces f to be a power law and compute λ explicitly. Our argument is scale-free and probabilistic.

Literature. Our WM/Var step can be viewed as an L^2 *equality–case* (variance–zero / Cauchy–Schwarz) phenomenon (cf. [6] for equality cases in quadratic forms) and as a functional–equation constraint on logarithmic scales (cf. [5]); see also [4] for the role of logarithmic derivatives in regular variation.

Functional-equation viewpoint. Equation (3.2) is a functional equation on logarithmic scales; our proof proceeds via an equality–case derivation.

*Corresponding author. Email: vec208@lehigh.edu. ORCID: 0000-0002-5775-5522

Provenance (and why this is trickier than it looks)

This note grew out of a first–year calculus discussion about centroids. We asked a naïve question: could the centroid height of the subgraph on $[0, a]$ be proportional to the function value sampled at the centroid abscissa, for *every* truncation length a? On the board this looks like harmless bookkeeping—two familiar centroid formulas and a single proportionality constant. The surprise is that this "calculus identity" quietly enforces a *global* structure on f: when written in scale–free form, the relation becomes an equality in expectation for a naturally self–generated probability measure. Differentiating with respect to the truncation scale a turns the question into one about the *elasticity* $E(x) = xf'(x)/f(x)$; a second differentiation reveals a *variance–zero (Cauchy–Schwarz equality)* case. The only way this can persist across all a is if the elasticity is constant—so f is a pure power law, and the proportionality constant is forced as well.

In short, a classroom computation leads to a rigidity theorem: an apparently benign proportionality at the *centroid* pins down the *shape* of the entire function. The argument stays elementary, but the difficulty is conceptual—one is comparing quantities formed at different places (the integrals over $[0, a]$) and at different scales (the function sampled at the moving point $\bar{x}(a)$), and only after recasting the problem probabilistically does the hidden symmetry become visible.

2. Setup and notation

Let $f \in C^2(0, \infty)$ with $f(x) > 0$ for $x > 0$ and $f(0^+) = 0$. Define

$$F(a) = \int_0^a f(x)\,\mathrm{d}x, \qquad H(a) = \int_0^a x\,f(x)\,\mathrm{d}x, \qquad G(a) = \int_0^a f(x)^2\,\mathrm{d}x.$$

The centroid of the subgraph $\{(x, y) : 0 \le x \le a,\ 0 \le y \le f(x)\}$ is

$$\bar{x}(a) = \frac{H(a)}{F(a)}, \qquad \bar{y}(a) = \frac{G(a)}{2F(a)}.$$

Notation (all symbols used)

- $f : (0, \infty) \to (0, \infty)$: base function; $f(0^+) = 0$; smoothness as stated.
- F, H, G: primitives $F(a) = \int_0^a f$, $H(a) = \int_0^a xf$, $G(a) = \int_0^a f^2$.
- $\bar{x}(a), \bar{y}(a)$: centroid coordinates of the subgraph on $[0, a]$.
- $g_a(s) := \dfrac{f(as)}{f(a)}$ for $s \in [0, 1]$ (shape profile at scale a).
- $A(a) := \dfrac{F(a)}{af(a)}$, $B(a) := \dfrac{H(a)}{a^2 f(a)}$, $C(a) := \dfrac{G(a)}{af(a)^2}$.
- $\theta(a) := \dfrac{B(a)}{A(a)} \in (0, 1)$ (scale-free centroid abscissa).
- $E(x) := \dfrac{x\,f'(x)}{f(x)}$ (elasticity; logarithmic derivative of f).
- μ_a: self-generated probability on $[0, a]$, $\mu_a(\mathrm{d}x) = \dfrac{f(x)}{F(a)}\mathbf{1}_{[0,a]}(x)\,\mathrm{d}x$.
- $X_a \sim \mu_a$: random variable with $\mathbb{E}[X_a] = \bar{x}(a)$ and $\mathbb{E}[f(X_a)] = 2\bar{y}(a)$.
- $D(a) := \int_0^1 (s - \theta(a))^2 g_a(s)\,\mathrm{d}s > 0$ (quadratic weight normalizer).

Fix $a > 0$ and define the *rescaled shape*

$$g_a(s) := \frac{f(as)}{f(a)} \qquad (0 \le s \le 1),$$

and the *elasticity*

$$E(x) := \frac{x\,f'(x)}{f(x)}.$$

The function $E(x)$ is called the *elasticity* of f.[1]

A change of variables $x = as$ gives the normalized moments

$$A(a) := \int_0^1 g_a(s)\,\mathrm{d}s = \frac{F(a)}{af(a)}, \quad B(a) := \int_0^1 s\,g_a(s)\,\mathrm{d}s = \frac{H(a)}{a^2 f(a)}, \quad C(a) := \int_0^1 g_a(s)^2\,\mathrm{d}s = \frac{G(a)}{af(a)^2}. \tag{2.1}$$

Set

$$\theta(a) := \frac{B(a)}{A(a)} \in (0,1), \qquad \bar{x}(a) = a\,\theta(a).$$

Two basic differential identities (chain and quotient rules), valid for $s \in (0,1]$, are

$$g_a(1) = 1, \qquad g_a'(s) = \frac{E(as)}{s}\,g_a(s), \qquad \partial_a g_a(s) = \frac{g_a(s)}{a}\big(E(as) - E(a)\big). \tag{2.2}$$

These identities are intended for $s \in (0,1]$; the factor $1/s$ in $g_a'(s)$ precludes evaluation at $s = 0$.

3. Probabilistic reformulation

For each $a > 0$, define a probability measure μ_a on $[0,a]$ by

$$\mu_a(\mathrm{d}x) = \frac{f(x)}{F(a)}\,\mathbf{1}_{[0,a]}(x)\,\mathrm{d}x, \qquad F(a) = \int_0^a f(x)\,\mathrm{d}x,$$

and let $X_a \sim \mu_a$. Then

$$\bar{x}(a) = \mathbb{E}[X_a], \qquad \bar{y}(a) = \tfrac{1}{2}\,\mathbb{E}[f(X_a)].$$

Thus GSP is the equality-in-expectation

$$\tfrac{1}{2}\,\mathbb{E}[f(X_a)] \;=\; \lambda\, f\big(\mathbb{E}[X_a]\big) \qquad (\forall a > 0). \tag{3.1}$$

Passing to scale-free variables $s = x/a$, set

$$g_a(s) := \frac{f(as)}{f(a)}, \qquad \nu_a(\mathrm{d}s) := \frac{g_a(s)}{A(a)}\,\mathbf{1}_{[0,1]}(s)\,\mathrm{d}s,$$

so that if $S_a \sim \nu_a$ then $\mathbb{E}[S_a] = \theta(a)$ and $\mathbb{E}[g_a(S_a)] = C(a)/A(a)$. In these terms, (3.1) is equivalent to the scale-free identity

$$\frac{1}{2}\,\frac{C(a)}{A(a)} = \lambda\, g_a\big(\theta(a)\big) \qquad (\forall a > 0). \tag{3.2}$$

4. Main theorem

Theorem 4.1. *Assume the* Geometric Scaling Property (3.2). *Then there exist $A > 0$ and $p > 0$ such that $f(x) = A\,x^p$ for all $x > 0$, and*

$$\lambda = \frac{p+1}{2(2p+1)}\left(\frac{p+2}{p+1}\right)^p.$$

Proof. We begin by differentiating the scale-free centroid identity, which yields a weighted-mean formula for the elasticity $E(x) = xf'(x)/f(x)$. Differentiating once more converts this into a vanishing-variance identity, forcing E to be constant. Constancy of elasticity implies that f is a pure power law, and the explicit form of λ follows by direct substitution.

Step 1 (Closed forms for scale-derivative integrals). A change of variables $u = as$ and integration by parts, using $f(0^+) = 0$, yield

$$\int_0^1 g_a(s)\,E(as)\,\mathrm{d}s = \frac{1}{af(a)}\int_0^a u f'(u)\,\mathrm{d}u = \frac{af(a) - F(a)}{af(a)} = 1 - A(a), \tag{4.1}$$

$$\int_0^1 s\,g_a(s)\,E(as)\,\mathrm{d}s = \frac{1}{a^2 f(a)}\int_0^a u^2 f'(u)\,\mathrm{d}u = \frac{a^2 f(a) - 2H(a)}{a^2 f(a)} = 1 - 2B(a), \tag{4.2}$$

$$\int_0^1 g_a(s)^2\,E(as)\,\mathrm{d}s = \frac{1}{af(a)^2}\int_0^a u f(u) f'(u)\,\mathrm{d}u = \frac{af(a)^2 - G(a)}{2af(a)^2} = \frac{1 - C(a)}{2}. \tag{4.3}$$

[1] Elasticity is familiar in economics as the percentage change in f per percentage change in x (cf. [2]); analytically it is the logarithmic derivative, as used in regular variation [4].

Consequently, substituting (2.2) and (4.1)–(4.3) into

$$A' = \int_0^1 \partial_a g_a(s)\,\mathrm{d}s, \quad B' = \int_0^1 s\,\partial_a g_a(s)\,\mathrm{d}s, \quad C' = 2\int_0^1 g_a(s)\,\partial_a g_a(s)\,\mathrm{d}s$$

gives

$$A' = \frac{1}{a}\big(1-(1+E(a))A\big), \;\; B' = \frac{1}{a}\big(1-(2+E(a))B\big), \;\; C' = \frac{1}{a}\big(1-(1+2E(a))C\big), \tag{4.4}$$

and, by differentiating $\theta = B/A$,

$$\theta' = \frac{B'A - BA'}{A^2} = \frac{1}{aA}\int_0^1 (s-\theta)\,g_a(s)\,E(as)\,\mathrm{d}s. \tag{4.5}$$

By Lemma 4.5 (Supplement) we obtain (4.4) and (4.5) rigorously.

Step 2 (Weighted-mean identity, by citation). By Lemma 4.6 in the Supplement, differentiating (3.2) yields the *weighted–mean identity*

$$\frac{\displaystyle\int_0^1 (s-\theta)^2 g_a(s)\,E(as)\,\mathrm{d}s}{\displaystyle\int_0^1 (s-\theta)^2 g_a(s)\,\mathrm{d}s} \;=\; E(a\theta) \qquad (\theta = \theta(a)). \tag{4.6}$$

The denominator is strictly positive since $g_a > 0$ a.e. on $(0,1]$ and $(s-\theta)^2 \not\equiv 0$.

Step 3 (Vanishing variance, by citation). Differentiating (4.6) and invoking Lemma 4.6 again gives

$$\int_0^1 (s-\theta(a))^2\,g_a(s)\,\big(E(as) - E(a\theta(a))\big)^2\,\mathrm{d}s \;=\; 0 \qquad (\forall a > 0). \tag{4.7}$$

As a non-example, the oscillatory perturbation $f(x) = x^p\big(1 + \varepsilon\sin\log x\big)$ violates (4.6)–(4.7) for any $\varepsilon \neq 0$, since $E(x) = p + \varepsilon\cos\log x + O(\varepsilon^2)$ is non-constant.

Step 4 (Constancy of elasticity and conclusion). Since the weight $(s-\theta(a))^2 g_a(s)$ is nonnegative on $[0,1]$ and positive on every subinterval of $(0,1]\setminus\{\theta(a)\}$ (because $g_a > 0$ on $(0,1]$), (4.7) implies

$$E(as) \equiv E(a\theta(a)) \quad \text{for all } s \in [0,1].$$

Hence for each $a > 0$, E is constant on $(0,a]$. If $0 < a_1 < a_2$, then $E \equiv c(a_1)$ on $(0,a_1]$ and $E \equiv c(a_2)$ on $(0,a_2]$, so $c(a_1) = c(a_2)$ by overlap. Therefore $E(x) \equiv p$ on $(0,\infty)$ for some constant $p \in \mathbb{R}$. The ODE $xf'(x) = p\,f(x)$ integrates to $f(x) = A\,x^p$ with $A > 0$. Substituting into the centroid formulas yields

$$\bar{x}(a) = \frac{p+1}{p+2}\,a, \qquad \bar{y}(a) = \frac{p+1}{2(2p+1)}\,A\,a^p,$$

so

$$\bar{y}(a) = \frac{p+1}{2(2p+1)}\left(\frac{p+2}{p+1}\right)^p f(\bar{x}(a)),$$

identifying $\lambda = \dfrac{p+1}{2(2p+1)}\big(\frac{p+2}{p+1}\big)^p$. Conversely, direct substitution of $f(x) = Ax^p$ into the centroid formulas shows (3.2) holds with this λ. Finally, since $f(0^+) = 0$ and $f(x) = Ax^p > 0$ for $x > 0$, we must have $p > 0$. Indeed, if $0 < a_1 < a_2$, constancy on $(0,a_2]$ implies constancy on $(0,a_1]$, so the constants coincide. □

Remark 4.2 (Regularity and finiteness). The assumptions $f \in C^2$, $f > 0$, $f(0^+) = 0$ ensure that the integrals in (2.1) are finite for all $a > 0$. Differentiation under the integral signs in A', B', C' is justified by dominated convergence on the compact s–interval together with the reductions (4.1)–(4.3), which convert the potentially singular factors into integrals of $uf'(u)$, $u^2f'(u)$, and $uf(u)f'(u)$ that are finite by the fundamental theorem of calculus (see, e.g., [3, Ch. 2]). Moreover, $f \in C^2$ implies g_a is C^1 in (a,s) for $s > 0$, and θ is C^1; thus the chain rule justifies the differentiation $\partial_a g_a(\theta) + g_a'(\theta)\,\theta'$ used in Step 2.

Remark 4.3 (Weaker hypotheses). The C^2 assumption in the Main Theorem can be weakened. It suffices to assume that $f : (0, \infty) \to (0, \infty)$ is C^1, with $f(0^+) = 0$, and that its elasticity

$$E(x) = \frac{x f'(x)}{f(x)}$$

is *locally Lipschitz* on $(0, \infty)$ (and, with minor routine changes, one can relax to bounded variation). Under these hypotheses:

- The identities in (2.2) hold for a.e. $s \in (0, 1]$ (Rademacher + chain rule), and A, B, C are absolutely continuous in a with derivatives given by (4.4) for a.e. a¿0 (differentiate the closed forms in (2.1)).
- The weighted-mean identity (4.6) holds for a.e. $a > 0$. Moreover, $a \mapsto A(a), B(a), C(a)$ and $a \mapsto g_a(\theta(a))$ are continuous on $(0, \infty)$, so (4.6) holds for a dense set of a and therefore for all $a > 0$ by continuity of both sides.
- Differentiating (4.6) is legitimate for a.e. a because E is locally Lipschitz, hence a.e. differentiable and locally bounded; the same cancellations yield (4.7) for a.e. a, and by continuity in a (of the integrand and weights) the identity extends to all $a > 0$.

As in Step 4, (4.7) forces E to be constant on every $(0, a]$, hence constant on $(0, \infty)$ by overlap; integrating $x f'(x) = p f(x)$ gives $f(x) = Ax^p$ with $A > 0$, $p > 0$. Thus $f \in C^1$, $f > 0$, $f(0^+) = 0$, and E locally Lipschitz already suffice for the theorem.

Appendix: Derivative identities (expanded algebra)

We record the computations used above.

Lemma 4.4. *With A, B, C, θ as in (2.1) and g_a as in (2.2), one has:*

(a) Integral reductions and derivative formulas. *The identities (4.1)–(4.3) hold. Consequently,*

$$A' = \frac{1 - (1 + E)A}{a}, \quad B' = \frac{1 - (2 + E)B}{a}, \quad C' = \frac{1 - (1 + 2E)C}{a}, \quad \theta' = \frac{1}{aA} \int_0^1 (s - \theta)\, g_a(s)\, E(as)\, \mathrm{d}s.$$

(a) (For WM/Var, see Lemma 4.6 in the Supplement.)

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Supplementary Note: Regularity and Differentiation Details

Lemma 4.5 (Regularity of A, B, C, θ and closed forms)**.** *Assume $f \in C^2(0, \infty)$, $f(0^+) = 0$, and $f(x) > 0$ for $x > 0$. Define*

$$F(a) = \int_0^a f(x)\, \mathrm{d}x, \quad H(a) = \int_0^a x f(x)\, \mathrm{d}x, \quad G(a) = \int_0^a f(x)^2\, \mathrm{d}x,$$

$$A(a) = \frac{F(a)}{a f(a)}, \quad B(a) = \frac{H(a)}{a^2 f(a)}, \quad C(a) = \frac{G(a)}{a f(a)^2}, \quad \theta(a) = \frac{B(a)}{A(a)}, \quad E(x) = \frac{x f'(x)}{f(x)}.$$

Then $A, B, C, \theta \in C^1(0, \infty)$ *and, for all* $a > 0$,

$$A' = \frac{1 - (1 + E(a))A}{a}, \qquad B' = \frac{1 - (2 + E(a))B}{a}, \qquad C' = \frac{1 - (1 + 2E(a))C}{a}, \qquad \theta' = \frac{B'A - BA'}{A^2}.$$

Proof. *Use* $F'(a) = f(a)$ *and* $H'(a) = af(a)$ *by the fundamental theorem of calculus. Integration by parts yields*

$$\int_0^a uf'(u)\,\mathrm{d}u = af(a) - F(a), \qquad \int_0^a u^2 f'(u)\,\mathrm{d}u = a^2 f(a) - 2H(a),$$

$$\int_0^a uf(u)f'(u)\,\mathrm{d}u = \tfrac{1}{2}\big(af(a)^2 - G(a)\big),$$

all finite on $[0, a]$ *since* $f \in C^2$ *and* $f(0^+) = 0$*. Differentiate* A, B, C *as quotients of* C^1 *functions and simplify using the three identities;* $\theta' = (B'A - BA')/A^2$ *follows.*

Lemma 4.6 (Weighted-mean and variance identities)**.** *Let* $g_a(s) = f(as)/f(a)$ *on* $[0, 1]$*, and set*

$$D(a) := \int_0^1 (s - \theta(a))^2 g_a(s)\,\mathrm{d}s \quad (> 0).$$

Assume the scale-free GSP identity $\frac{1}{2}\frac{C(a)}{A(a)} = \lambda\, g_a(\theta(a))$ *for all* $a > 0$*. Then, for all* $a > 0$,

$$E\big(a\theta(a)\big) = \frac{\int_0^1 (s - \theta(a))^2 g_a(s)\, E(as)\,\mathrm{d}s}{D(a)}. \tag{4.8}$$

Differentiating (4.8) *in* a *yields*

$$\int_0^1 (s - \theta(a))^2\, g_a(s)\, \big(E(as) - E(a\theta(a))\big)^2\,\mathrm{d}s = 0. \tag{4.9}$$

Proof. *Differentiate* $\frac{1}{2}(C/A) = \lambda g_a(\theta)$ *using Lemma 4.5 for* $(C/A)'$ *and the chain/quotient rules. Express all* a*–derivatives of* A, B, C *via Lemma 4.5. Rearranging gives* (4.8)*. Differentiating* (4.8)*, the terms linear in* $E(as) - E(a\theta)$ *cancel by* (4.8)*, leaving* (4.9)*. Since* $(s-\theta)^2 g_a \geq 0$ *and not a.e. zero,* (4.9) *implies* $E(as) \equiv E(a\theta)$ *on [0,1].* Identity (4.9) is exactly the equality case of the Cauchy–Schwarz inequality for the weighted L^2 norm with weight $(s - \theta(a))^2 g_a(s)$; no general convexity assumption is needed beyond this quadratic form.

Remark 4.7 (Converse for power laws). For $f(x) = Ax^p$ with $A > 0$, $p > 0$, one computes

$$\bar{x}(a) = \frac{p+1}{p+2}a, \qquad \bar{y}(a) = \frac{p+1}{2(2p+1)}Aa^p,$$

hence $\bar{y}(a) = \lambda f(\bar{x}(a))$ with

$$\lambda = \frac{p+1}{2(2p+1)}\left(\frac{p+2}{p+1}\right)^p.$$

Thus the "if and only if" in Theorem 4.1 is immediate.

Scope note. Throughout we assume GSP holds for all $a > 0$ and $f \in C^2(0, \infty)$ with $f > 0$, $f(0^+) = 0$. The proof is elementary and self-contained.