Partitions of complete twisted graphs into plane spanning trees

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Abstract

We characterize all partitions of the complete twisted graph T_{2n} into plane spanning trees. In the case of partitions of T_{2n} into isomorphic plane spanning trees, we show that all trees in these partitions must be balanced double stars. As a consequence of our results, any complete topological graph with n vertices contains a complete topological subgraph with $m \geq c \log^{1/8} n$ vertices that admits a partition into plane spanning trees.

Keywords. Twisted Graph. Plane Spanning Tree. Partition.

1 Introduction

Let P be a set of points in the plane. A *topological graph* with vertex set P is a graph G where edges are simple continuous arcs drawn in the plane in such a way that any two edges have at most one point in common. Two topological graphs G and F are called *weakly isomorphic* if there is a graph isomorphism between G and F such that two edges of G intersect if and only if the corresponding edges of F do. A *geometric graph* is a topological graph where all edges are straight line segments. A geometric graph G is *convex* if the vertices of G are in convex position. We denote the complete convex graph with n vertices by C_n.

The well-known so called "Happy Ending" theorem of Erdős and Szekeres [6, 7] can be reformulated as follows: every complete geometric graph with n vertices has a complete geometric subgraph which is weakly isomorphic to a complete convex graph C_m with $m \geq c \log n$. Harborth and Mengersen [9] introduced a class of topological graphs (later named *twisted graphs*) as examples of topological graphs without topological subgraphs weakly isomorphic to any complete convex geometric graph C_n with $n \geq 5$.

The complete twisted graph T_n is a complete topological graph with n vertices $v_1, v_2, \ldots, v_n$ in which two edges $v_i v_j$ $(i < j)$ and $v_s v_t$ $(s < t)$ cross each other if and only if either $i < s < t < j$ or $s < i < j < t$, see Fig. 1.

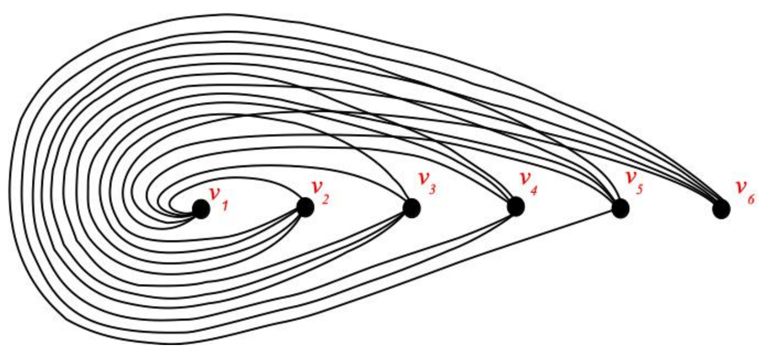

Figure 1: Complete twisted graph T_6.

Pach et al. [12] proved that every complete topological graph with n vertices contains a complete topological subgraph with $m \geq c \log^{1/8} n$ vertices which is weakly isomorphic to either the complete convex graph C_m or to the complete twisted graph T_m. Arroyo et al. [3] generalized this result to abstract rotational systems.

The above result by Pach et al. motivated the study of further properties of twisted graphs; in particular the study of properties shared by convex and twisted graphs. If a given property P is shared by complete convex graphs and complete twisted graphs, then any complete topological graph with n vertices contains a complete topological subgraph with $m \geq c \log^{1/8} n$ vertices that satisfies property P.

Given a topological graph G, a subgraph F of G is *plane* or *non self-crossing* if no two edges of F cross in G.

Omaña-Pulido and Rivera-Campo [11]; Ábrego et al. [1] and Figueroa and Fresán-Figueroa [8] studied some problems for which the corresponding versions for the complete convex graph C_n are known.

A *partition* of a complete graph K_{2n} into spanning trees is a collection of n edge disjoint spanning trees $S_1, S_2, \dots, S_n$ of K_{2n}. Since K_{2n} has $\binom{2n}{2} = n(2n-1)$ edges and each spanning tree of K_{2n} has $2n-1$ edges, $(E(S_1), E(S_2), \dots, E(S_n))$ is a partition of the set of edges of K_{2n}.

A textbook exercise states that for each positive integer n there is a partition of the complete abstract graph K_{2n} into n spanning paths and a partition of K_{2n} into n spanning balanced double stars. Jaeger et al. [10] gave a generalization of this result concerning partitions of $(2k+1)$-regular graphs with perfect matchings into balanced double stars of order $2(k+1)$. For topological graphs a non self-crossing condition on each tree in the partition is required which changes the problem drastically from that of abstract graphs. For instance, as a corollary of Lemma 7 in Section 4, one can see that for $n \geq 3$ there is no partition of the complete twisted graph T_{2n} into n plane spanning paths.

Recently, the long-standing conjecture that any complete geometric graph on $2n$ vertices can be partitioned into n plane spanning trees was disproved by Aichholzer et al. [2]. For complete convex graphs, the existence of such a partition follows from a result by Bernhart and Kainen [4]; Bose et al. [5] gave a characterization of such partitions; as it turns out, for each partition of C_n into plane spanning trees, all trees in the partition must be pairwise isomorphic. In this paper we characterize partitions of the complete twisted graph T_{2n} into arbitrary plane spanning trees and partitions of T_{2n} into isomorphic plane spanning trees, see Fig. 2.

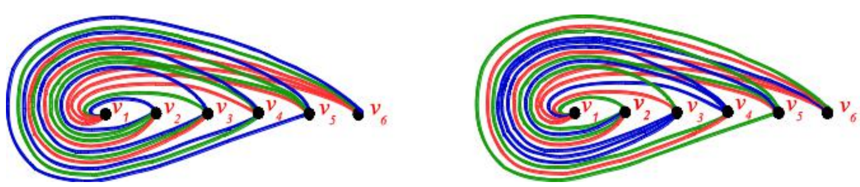

Figure 2: A partition of T_6 into non-isomorphic plane trees (left) and a partition of T_6 into isomorphic plane trees (right).

Let T_{2n} be a complete twisted graph with vertices $v_1, v_2, \dots, v_{2n}$. For the purpose of this article it is convenient to rename vertices $v_{n+1}, v_{n+2}, \dots, v_{2n}$ as follows: for $i = 1, 2, \dots, n$ let $w_i = v_{2n+1-i}$. From now onward the vertices of T_{2n} will be denoted by $v_1, v_2, \dots, v_n, w_n, w_{n-1}, \dots, w_1$ in that order.

Let n be a positive integer. A *spanning balanced double star* of T_{2n} with center edge $v_i w_i$ is a spanning tree of T_{2n} containing edge $v_i w_i$ and such that v_i is adjacent to half of the remaining vertices and w_i is adjacent to the other half. For example, the red tree in Fig. 2 (left) and all trees in Fig. 2 (right) are spanning balanced double stars of T_6.

2 Preliminary remarks

A spanning tree S of a complete twisted graph T_{2n} is *plane* or *non self-crossing* if no two edges of S cross in T_{2n}.

Let T_{2n} be a complete twisted graph with $2n$ vertices and $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a partition of T_{2n} into plane spanning trees. Since edges $v_1 w_1$, $v_2 w_2, \dots, v_n w_n$ are pairwise crossing in T_{2n}, they all must lie in different trees in $\mathcal{S}$. Without loss of generality, from now onward, we assume edge $v_i w_i$ lies in tree S_i for $i = 1, 2, \dots, n$.

Remark 1. *In any partition $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ of the complete twisted graph T_{2n} into plane spanning trees, tree S_1 is a balanced double star with center edge $v_1 w_1$.*

Proof. Vertices $v_2, v_3, \dots, v_n, w_n, w_{n-1}, \dots, w_2$ are independent in S_1 since all edges joining two of these vertices cross edge $v_1 w_1$ which is an edge of S_1. Therefore S_1 is a double star with center edge $v_1 w_1$.

Clearly $d_{S_i}(v_1) \geq 1$ for $i = 2, 3, \dots, n$, therefore vertex v_1 cannot be adjacent to more than n vertices in S_1. Analogously vertex w_1 is adjacent to at most n in S_1. This implies $d_{S_1}(v_1) = d_{S_1}(w_1) = n$ and therefore S_1 is balanced. $\square$

Remark 2. *Let $n \geq 2$ be an integer. If $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ is a partition of the complete twisted graph T_{2n} into plane spanning trees, then $S_2 - \{v_1, w_1\}, S_3 - \{v_1, w_1\}, \dots, S_n - \{v_1, w_1\}$ is a partition of the complete twisted graph with vertices $v_2, v_3, \dots, v_n, w_n, w_{n-1}, \dots, w_2$ into plane spanning trees.*

Proof. By Remark 1, $d_{S_1}(v_1) = d_{S_1}(w_1) = n$. This implies that vertices v_1 and w_1 have degree 1 in trees $S_2, S_3, \dots, S_n$. Therefore $S_i - \{v_1, w_1\}$ is a plane tree for $i = 2, 3, \dots, n$. Since $v_2, v_3, \dots v_n, w_n, w_{n-1}, \dots, w_2$ are independent in S_1, all edges joining two of these vertices lie in one of the trees $S_2 - \{v_1, w_1\}, S_3 - \{v_1, w_1\}, \ldots, S_n - \{v_1, w_1\}$. $\square$

3 Arbitrary partitions

In this section we study partitions of T_{2n} into arbitrary plane spanning trees.

Lemma 3. *Let $n \geq 2$ be an integer. If $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ is a partition of the complete twisted graph T_{2n} into plane spanning trees, then for $i = 1, 2, \ldots, n-1$, the subgraph $S_i[V_i]$ of S_i induced by the set of vertices $V_i = \{v_i, v_{i+1}, \ldots, v_n, w_n, w_{n-1}, \ldots, w_i\}$ is a balanced double star with center edge v_iw_i. Moreover, tree $S_i[V_i]$ contains edges $v_iw_{n-1}, v_iw_{n-2}, \ldots, v_iw_{i+1}$, edges $w_iv_{n-1}, w_iv_{n-2}, \ldots, w_iv_{i+1}$ and one of the following pairs of edges $\{v_iw_n, w_iv_n\}$ or $\{v_iv_n, w_iw_n\}$.*

Proof. If $n = 2$, then $\mathcal{S} = \{S_1, S_2\}$. By Remark 1, tree S_1 is a balanced double star with center edge v_1w_1 and edge set $\{v_1w_2, v_1w_1, w_1v_2\}$ or $\{v_1v_2, v_1w_1, w_1w_2\}$.

We proceed by induction assuming the result holds for $n = k$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{k+1}\}$ be a partition of the complete twisted graph with vertices $v_1, v_2, \ldots, v_{k+1}, w_{k+1}, w_k, \ldots, w_1$ into plane spanning trees.

For $i = 2, 3, \ldots, k+1$ let $S'_i = S_i - \{v_1, w_1\}$. By Remark 2, $\mathcal{S}' = \{S'_2, S'_3, \ldots, S'_{k+1}\}$ is a partition of the complete twisted graph with vertices $v_2, v_3, \ldots, v_{k+1}, w_{k+1}, w_k, \ldots, w_2$ into plane spanning trees.

By the induction hypothesis, for $i = 2, 3, \ldots, k$, the subgraph $S'[V_i]$ of S'_i is a balanced double star with center edge v_iw_i that contains edges $v_iw_k, v_iw_{k-1}, \ldots, v_iw_{i+1}$, edges $w_iv_k, w_iv_{k-1}, \ldots, w_iv_{i+1}$ and one of the pairs of edges $\{v_iw_{k+1}, w_iv_{k+1}\}$ or $\{v_iv_{k+1}, w_iw_{k+1}\}$. Notice that $S'[V_i] = S[V_i]$ for $i = 2, 3, \ldots, k$.

Finally, by Remark 1, tree S_1 is a balanced double star with center edge v_1w_1. Therefore each vertex $v_2, v_3, \ldots, v_{k+1}, w_{k+1}, w_k, \ldots, w_2$ is adjacent to either v_1 or w_1. Let j be an integer with $2 \leq j \leq k$. Notice that for $i = 2, 3, \ldots, k$, edge v_1w_j crosses edges v_iw_{k+1} and v_iv_{k+1} of which one is an edge of S_i, therefore for $i = 2, 3, \ldots, k$, edge v_1w_j is not an edge of S_i. Since v_1w_j also crosses edge $v_{k+1}w_{k+1}$ which is an edge of S_{k+1}, edge v_1w_j is not an edge of S_{k+1} either. Therefore for $j = 2, 3, \ldots, k$, edge v_1w_j must be an edge of S_1. Analogously $w_1v_j \in E(S_1)$ for $j = 2, 3, \ldots, k$.

Since S_1 is a balanced double star with center edge v_1w_1, either v_1w_{k+1}, $w_1v_{k+1} \in E(S_1)$ or $v_1v_{k+1}, w_1w_{k+1} \in E(S_1)$. Thus the result also holds for $n = k + 1$. By induction it holds for all integers $n \geq 2$.

$\square$

Remark 4. *Let $n \geq 2$ be an integer. If $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ is a partition of the complete twisted graph T_{2n} into plane spanning trees, then edges $v_1v_2, v_1v_3, \ldots, v_1v_n, v_1w_n$ lie in different trees and edges $w_1w_2, w_1w_3, \ldots, w_1w_n, w_1v_n$ also lie in different trees.*

Proof. By Lemma 3, S_1 is a balanced double star with center edge v_1w_1 that contains edges $w_1v_2, w_1v_3, \ldots, w_1v_{n-1}$ and edges $v_1w_2, v_1w_3, \ldots, v_1w_{n-1}$. Notice that in both cases edges $v_1v_2, v_1v_3, \ldots, v_1v_{n-1}$ and $w_1w_2, w_1w_3, \ldots, w_1w_{n-1}$ are not edges of S_1; since $\mathcal{S}$ is a partition of T_{2n}, all of these edges must lie in some tree S_i with $i \geq 2$.

Also by Lemma 3, tree S_1 contains one of the following pairs of edges $\{v_1w_n, w_1v_n\}$ or $\{v_1v_n, w_1w_n\}$. In the first case $v_1w_n, w_1v_n \in E(S_1)$, $v_1v_n \in E(S_i)$ for some $i \geq 2$ and $w_1w_n \in E(S_j)$ for some $j \geq 2$ and, in the second case $v_1v_n, w_1w_n \in E(S_1)$, $v_1w_n \in E(S_r)$ for some $r \geq 2$ and $w_1v_n \in E(S_t)$ for some $t \geq 2$.

From the proof of Remark 2, $d_{S_i}(v_1) = d_{S_i}(w_1) = 1$ for $i = 2, 3, \ldots, n$. Therefore edges $v_1v_2, v_1v_3, \ldots, v_1v_n, v_1w_n$ lie in different trees and $w_1w_2, w_1w_3, \ldots, w_1w_n, w_1v_n$ lie in different trees. $\square$

Next we give a recursive formula for the number of partitions of T_{2n} into arbitrary plane spanning trees.

Theorem 5. *The number p_n of partitions of the complete twisted graph T_{2n} into arbitrary plane spanning trees satisfies the recursive formula $p_{n+1} = 5(2^{2(n-2)})p_n$ for $n \geq 2$.*

Proof. Let $n \geq 2$ be an integer and $\mathcal{S}' = \{S'_2, S'_3, \ldots, S'_{n+1}\}$ be a partition of the complete twisted graph with vertices $v_2, v_3, \ldots, v_{n+1}, w_{n+1}, w_n, \ldots, w_2$ into plane spanning trees. We show that there are $5(2^{2(n-2)})$ partitions $\mathcal{S} = \{S_1, S_2, \ldots, S_{n+1}\}$ of $T_{2(n+1)}$ where $S_i - \{v_1, w_1\} = S'_i$ for $i = 2, 3, \ldots, n+1$. By Lemma 3, for all such partitions, tree S_1 is a balanced double star with center edge v_1w_1, containing all edges v_1w_i and w_1v_i for $i = 2, 3, \ldots, n$ and one of the following pairs of edges $\{v_1w_{n+1}, w_1v_{n+1}\}$ or $\{v_1v_{n+1}, w_1w_{n+1}\}$.

We show first that there are $2^{2(n-1)}$ partitions where tree S_1 contains edges v_1w_{n+1} and w_1v_{n+1}. Notice that in such cases the edges of $T_{2(n+1)}$

not contained in $E(S_1) \cup E(S'_2) \cup \cdots \cup E(S'_{n+1})$ are edges v_1v_i and w_1w_i with $i = 2, 3, \ldots, n+1$. We claim that there are 2^{n-1} ways to attach edges v_1v_i, with $i = 2, 3, \ldots, n+1$, and 2^{n-1} ways to attach edges w_1w_i, with $i = 2, 3, \ldots, n+1$, to $S'_2, S'_3, \ldots, S'_{n+1}$ to obtain a partition $S_1, S_2, \ldots, S_{n+1}$ of $T_{2(n+1)}$.

By Remark 4, each tree in $\mathcal{S}'$ contains one of the edges $v_2v_3, v_2v_4, \ldots, v_2v_{n+1}$, v_2w_{n+1}. For $j = 3, 4, \ldots, n+1$, let i_j be an integer such that v_2v_j is an edge of S'_{i_j} and let i_{n+2} be an integer such that edge v_2w_{n+1} lies in $S'_{i_{n+2}}$.

Notice that for $j = 3, 4, \ldots, n+1$ any single edge in $\{v_1v_2, v_1v_3, \ldots, v_1v_j\}$ can be attached to S'_{i_j} and any single edge in $\{v_1v_2, v_1v_3, \ldots, v_1v_{n+1}\}$ can be attached to $S'_{i_{n+2}}$ obtaining a plane tree. We can choose first any edge in $\{v_1v_2, v_1v_3\}$ to attach to S'_{i_3}, then choose any remaining edge in $\{v_1v_2, v_1v_3, v_1v_4\}$ to attach to S'_{i_4} and so on. For $S'_{i_3}, S'_{i_4}, \ldots, S'_{i_{n+1}}$ there are two remaining edges and for $S'_{i_{n+2}}$ only one possible edge remains. This gives 2^{n-1} ways to attach edges v_1v_i, with $i = 2, 3, \ldots, n+1$ to $S'_2, S'_3, \ldots, S'_{n+1}$.

Analogously, there are also 2^{n-1} ways to attach edges w_1w_i, with $i = 2, 3, \ldots, n+1$ to $S'_2, S'_3, \ldots, S'_{n+1}$. Since the ways of attaching edges v_1v_i and edges w_1w_i, with $i = 2, 3, \ldots, n+1$ are independent, there are $2^{2(n-1)}$ ways to obtain a partition $\{S_1, S_2, \ldots, S_{n+1}\}$ of $T_{2(n+1)}$ from $\{S'_2, S'_3, \ldots, S'_{n+1}\}$ where S_1 contains edges v_1w_{n+1}, w_1v_{n+1} and $S_i - \{v_1, w_1\} = S'_i$ for $i = 2, 3, \ldots, n+1$.

With a similar argument we can prove that there are $2^{2(n-2)}$ ways to obtain a partition $\{S_1, S_2, \ldots, S_{n+1}\}$ of $T_{2(n+1)}$ from $\{S'_2, S'_3, \ldots, S'_{n+1}\}$ where S_1 contains edges v_1v_{n+1} and w_1w_{n+1} and $S_i - \{v_1, w_1\} = S'_i$ for $i = 2, 3, \ldots, n+1$. Therefore $p_{n+1} = (2^{2(n-1)} + 2^{2(n-2)})p_n = 5(2^{2(n-2)})p_n$. □

Corollary 6. *For $n \geq 2$ there are $(5^{n-2})(2^{1+(n-2)(n-3)})$ partitions of the complete twisted graph T_{2n} into plane spanning trees.*

Proof. By Theorem 5, the number p_n of partitions of the complete twisted graph T_{2n} into plane spanning trees satisfies the recursive formula $p_{n+1} = 5(2^{2(n-2)})p_n$.

A simple proof by induction shows that for $p_n = (5^{n-2})(2^{1+(n-2)(n-3)})$ for $n \geq 2$, since T_4 has $p_2 = 2$ such partitions. □

4 Partitions into isomorphic trees

In this section we give a formula for the number of partitions of the complete twisted graph T_{2n} into isomorphic plane spanning trees $S_1, S_2, \ldots, S_n$.

Lemma 7. *Let $n \geq 2$ be an integer and $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ be a partition of the complete twisted graph T_{2n} into isomorphic plane spanning trees. Then for $i = 1, 2, \dots, n$ tree S_i is a balanced double star with center edge $v_i w_i$.*

Proof. By Remark 1, tree S_1 is a balanced double star with center edge $v_1 v_{2n}$. Since S_i is isomorphic to S_1 for $i = 2, 3, \dots, n$, all trees in $\mathcal{S}$ are balanced double stars. By Lemma 3, for $i = 2, 3, \dots, n-1$, tree $S_i[V_i]$ is a balanced double star with center edge $v_i w_i$, therefore $v_i w_i$ is also the center edge of S_i since $S_i[V_i]]$ is a balanced double star contained in balanced double star S_i.

Finally, since we have shown that for $i = 1, 2, \dots n-1$ tree S_i is a balanced double star with center edge $v_i w_i$, we have $d_{S_i}(v_n) = d_{S_i}(w_n) = 1$ for $i = 1, 2, \dots n-1$. This implies $d_{S_n}(v_n) = d_{S_n}(w_n) = n$ and therefore edge $v_n w_n$ must be the center edge of S_n. □

For any vertex u of a graph G, let $N_G(u)$ be the set of vertices of G which are adjacent to u in G.

Let $n \geq 2$ be an integer. For $k = 1, 2, \dots, n$, denote by A_k be the spanning balanced double star of T_{2n} with center edge $v_n w_n$ and such that $N_{A_k}(v_n) = \{v_1, v_2, \dots, v_{n-k}\} \cup \{w_n, w_{n-1}, \dots, w_{n-(k-1)}\}$ for $k \leq n-1$ and $N_{A_k}(v_n) = \{w_n, w_{n-1}, \dots, w_1\}$ for $k = n$. Since A_k is a balanced double star with center edge $v_n w_n$, $N_{A_k}(w_n) = \{w_1, w_2, \dots, w_{n-k}\} \cup \{v_n, v_{n-1}, \dots, v_{n-(k-1)}\}$ for $k \leq n-1$ and $N_{A_k}(w_n) = \{v_n, v_{n-1}, \dots, v_1\}$ for $k = n$.

Theorem 8. *Let $n \geq 2$ be an integer. If $\mathcal{S} = \{S_1, S_2, \dots, S_n\}$ is a partition of the complete twisted graph T_{2n} into isomorphic plane spanning trees, then there is an integer k with $1 \leq k \leq n$ such that $S_n = A_k$.*

Proof. By Lemma 7, tree S_n is a balanced double star with center edge $v_n w_n$; let $k = \max\{j : v_n w_n, v_n w_{n-1}, \dots, v_n w_{n-j+1} \in E(S_n)\}$.

If $k = n$, then vertex v_n is adjacent in S_n to all vertices $w_n, w_{n-1}, \dots, w_1$. In this case $S_n = A_n$ since S_n is a balanced double star with center edge $v_n w_n$.

If $k < n$, then $w_n w_{n-k} \in E(S_n)$; this implies $w_n w_{n-k-1}, w_n w_{n-k-2}, \dots, w_n w_1 \in E(S_n)$, otherwise $v_n w_{n-k-j}$ is an edge of S_n for some integer j with $1 \leq j \leq n-k-1$, which is not possible since edges $w_n w_{n-k}$ and $v_n w_{n-k-j}$ cross each other. Therefore $S_n = A_k$. □

Lemma 9. *For $i = 2, 3, \dots, n-1$, tree S_i contains edges $v_1 v_i, v_2 v_i, \dots, v_{i-1} v_i$ and edges $w_1 w_i, w_2 w_i, \dots, w_{i-1} w_i$.*

Proof. Let $i = 2, 3, \ldots, n-1$ and $t = 1, 2, \ldots, i-1$. As S_i is a balanced double star with center edge $v_i w_i$, vertex v_t is adjacent either to v_i or to w_i in S_i. By Lemma 3, tree S_i contains one of the edges $v_i v_n$ or $v_i w_n$. Notice that edge $v_t w_i$ is not an edge of S_i as it crosses both edges $v_i v_n$ and $v_i w_n$ in T_{2n}. This implies v_t is adjacent to v_i in S_i. Analogously w_t is adjacent to w_i in S_i.

□

Theorem 10. *Let $n \geq 2$ be an integer. For $k = 1, 2, \ldots, n$ there exists a unique partition $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of the complete twisted graph T_{2n} into isomorphic plane spanning trees such that $S_n = A_k$.*

Proof. There are two partitions of T_4 into isomorphic plane spanning trees, one contains A_1 and the other contains A_2, see Fig. 3.

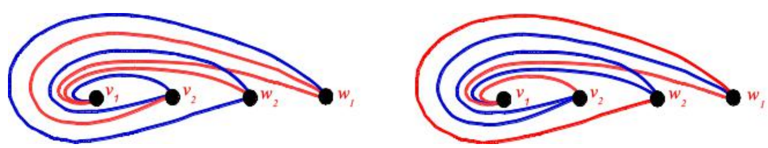

Figure 3: Two partitions of T_4. $S_2 = A_1$ on the left and $S_2 = A_2$ on the right.

Assume $n \geq 3$. By Lemma 3 and Lemma 9, for $i = 1, 2, \ldots, n-1$ the only edges of S_i that are not fixed in any partition $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ of T_{2n} into isomorphic plane spanning trees are those incident to v_n and to w_n. Let $k = 1, 2, \ldots, n$ and let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a such a partition of T_{2n} with $S_n = A_k$. Notice that since each tree S_i is a balanced double star with center edge $v_i w_i$, in order to prove that tree S_i is uniquely determined by A_k, it suffices to show which of the vertices v_i or w_i is adjacent to v_n in S_i for $i = 1, 2, \ldots, n-1$.

We first look at the case $k = n$. Then $S_n = A_n$ contains all edges $v_n w_i$ with $i = 1, 2, \ldots, n-1$. In this case v_i must be adjacent to v_n in S_i for $i = 1, 2, \ldots n-1$.

Assume now $k < n$. If $1 \leq i \leq n-k$, then $v_n v_i$ is an edge of $A_k = S_n$ and therefore cannot be an edge of S_i. This implies that w_i is adjacent to v_n in S_i for $1 \leq i \leq n-k$. See tree S_1 in Fig. 4.

Analogously, if $i > n-k$, then $w_n v_i$ is an edge of S_n and therefore cannot be an edge of S_i. This implies that v_i is adjacent to v_n in S_i for $i > n-k$. See tree S_2 in Fig. 4.

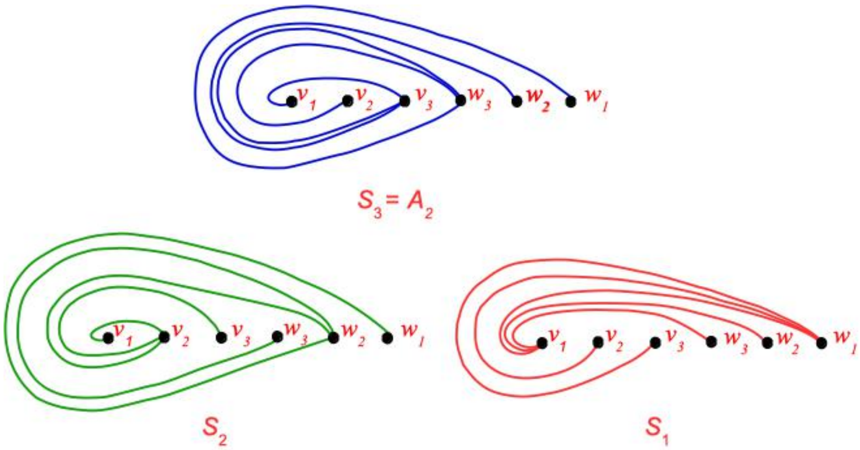

Figure 4: Partition of T_6 into balanced double stars S_1, S_2, S_3 with $S_3 = A_2$. Here $v_3v_1 \in E(S_3)$ implies $v_3w_1 \in E(S_1)$ and $w_3v_2 \in E(S_3)$ implies $v_3v_2 \in E(S_2)$.

□

As an immediate consequence of theorems 8 and 10, we have the following.

Corollary 11. *Let $n \geq 2$ be an integer. There are n partitions of T_{2n} into isomorphic plane spanning trees.*

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