The Parity-Constrained Four-Peg Tower of Hanoi Problem and Its Associated Graph

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Abstract

We introduce and study a new four-peg variant of the Tower of Hanoi problem under *parity constraints*. Two pegs are neutral and allow arbitrary disc placements, while the remaining two pegs are restricted to discs of a prescribed parity: one for even-labelled discs and the other for odd-labelled discs. Within this constrained setting, we investigate four canonical optimization objectives according to distinct target configurations and derive for each the exact number of moves required to optimally transfer the discs.

We establish a coupled system of recursive relations governing the four optimal move functions and unfold them into higher-order recurrences and explicit closed forms. These formulas exhibit periodic growth patterns and reveal that all solutions grow exponentially, but at a significantly slower rate than the classical three-peg case. In particular, each optimal move sequence has an a half-exponential-like asymptotic order induced by the parity restriction.

In addition, we define the associated parity-constrained Hanoi graph, whose vertices represent all feasible states and whose edges represent legal moves. We determine its order, degrees, connectivity, planarity, diameter, Hamiltonicity, clique number, and chromatic properties, and show that it lies strictly between the classical three- and four-peg Hanoi graphs via the inclusion relation.

Keywords: Tower of Hanoi; Parity constraints; Recursive sequences; Asymptotic growth; State graphs; Hamiltonian graphs.

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1 Introduction

The classical Tower of Hanoi puzzle asks for the minimum number of moves required to transfer a stack of n discs from one peg to another, following the rules that only one disc may move at a time and no larger disc may rest upon a smaller one. In the three-peg setting, the problem admits a well-known exponential solution of $h_3^n = 2^n - 1$ moves. Although this version has a simple recursive solution that has been known for over a century, the optimal solution for the Tower of Hanoi problem with four pegs (also called *Reve's puzzle*) was only confirmed in 2014 by Bousch [2]. The optimal number of moves in this case is given by the recurrence

$$h_4^0 = 0, \qquad \forall n \geq 1 : \; h_4^n = \min\{\, 2h_4^k + h_4^{n-k} \mid 0 \leq k \leq n \,\}. \tag{1}$$

For five or more pegs, the Frame–Stewart algorithm [3,12] is believed to give the optimal solution, although a formal proof of optimality remains open; see [4, Chapter 5] for further background.

In this paper, we introduce a new variant of the four-peg Tower of Hanoi in which disc movements are subject to *parity constraints*. Among the four pegs, two are *neutral* and allow arbitrary discs, whereas the remaining two are restricted: one may only hold even-labelled discs, and the other only odd-labelled discs. These constraints significantly affect the recursive structure of the problem, producing new optimal strategies that blend classical transfer mechanisms with parity-induced separations.

We consider four canonical optimization objectives corresponding to different target configurations. For each objective, we derive an exact recurrence relation for the minimal number of moves and show that the resulting system forms a family of coupled recurrences. By further combining and simplifying these relations, we establish higher-order recurrence forms and obtain explicit closed expressions. These closed forms reveal periodic behavior and show that all four optimal move sequences grow exponentially, slower than the classical rate 2^n of the three-peg puzzle.

Beyond optimal move counts, we investigate the full *state graph* of this parity-constrained puzzle, denoted P_n, in which vertices represent feasible disc configurations, and edges represent legal moves. We derive recursive properties of this graph. Furthermore, we determine its number of edges, degree bounds, connectivity, planarity, diameter, Hamiltonicity, clique number, chromatic number, chromatic index, and relation to the classical Hanoi graphs H_3^n and H_4^n.

The remainder of the paper is organized as follows. Section 2 formalizes the model and introduces the four optimization objectives. Section 3 derives the coupled recurrences and analyzes the structure of the corresponding optimal solutions. Section 4 presents higher-order and explicit forms of the sequences. Section 5 provides a detailed asymptotic growth analysis. Section 6 develops the associated graph-theoretic properties. Finally, Section 7 summarizes the main results and proposes future research directions.

2 Problem Description

Fix an integer $n \geq 1$. Consider four pegs partitioned as follows:

$$P = \{N_1, N_2\} \cup \{O, E\},$$

where N_1 and N_2 are *neutral* pegs that may hold discs of any label, O is the *odd* peg that may only hold discs with odd labels, and E is the *even* peg that may only hold discs with even labels.

The discs are labeled from top to bottom as

$$[n] = \{1, 2, \dots, n\},$$

with disc 1 being the smallest. A *state* is a placement of all n discs on the four pegs such that:

(i) the discs on every peg form a strictly increasing sequence from top to bottom (no larger disc lies on a smaller one);

(ii) every disc occupies a peg compatible with its parity.

Set–theoretic representation. For $k \geq 1$, define

$$[k] = \{1, 2, \dots, k\}, \qquad [k]_0 = \{d \in [k] \mid d \text{ is even}\}, \qquad [k]_1 = \{d \in [k] \mid d \text{ is odd}\}.$$

Any feasible state can then be expressed as an ordered 4-partition of $[n]$:

$$S = (N_1, E, O, N_2),$$

where each component denotes the (stacked) set of discs assigned to the corresponding peg. This representation is a parity-restricted adaptation of the classical encoding used in [11].

Legal moves. Given a state S and a disc $d \in [n]$ currently on peg $p \in P$, a move

$$p \longrightarrow q, \qquad q \in P,$$

is *legal* if and only if:

(a) *(classical rule)* disc d is the topmost disc on p, and q is either empty or its top disc is larger than d;

(b) *(parity rule)* both p and q permit the parity of d, that is,

$$d \text{ odd} \Rightarrow p, q \in \{N_1, N_2, O\}, \qquad d \text{ even} \Rightarrow p, q \in \{N_1, N_2, E\}.$$

Optimization objectives. Starting from the *initial state*

$$S_0 = \big([n], \varnothing, \varnothing, \varnothing\big),$$

we consider four canonical target configurations, each defining an optimal move-count function.

(a) **Full-tower transfer.** Move all discs from peg N_1 to peg N_2. The optimum number of moves is denoted by a_n:
$$([n], \varnothing, \varnothing, \varnothing) \longrightarrow (\varnothing, \varnothing, \varnothing, [n]).$$

(b) **Parity separation.** Move all even-labelled discs to peg E and all odd-labelled discs to peg O. Denote the optimum number by b_n:

$$([n], \varnothing, \varnothing, \varnothing) \longrightarrow (\varnothing, [n]_0, [n]_1, \varnothing).$$

(c) **Odd to O, even to N_2.** Transfer all odd discs to peg O and all even discs to peg N_2. Denote the optimum number by c_n:

$$([n], \varnothing, \varnothing, \varnothing) \longrightarrow (\varnothing, \varnothing, [n]_1, [n]_0).$$

(d) **Odd to N_2, even to E.** Transfer all odd discs to peg N_2 and all even discs to peg E. Denote the optimum number by d_n:

$$([n], \varnothing, \varnothing, \varnothing) \longrightarrow (\varnothing, [n]_0, \varnothing, [n]_1).$$

Perfect states. A state is called *perfect* if all discs are stacked on the same peg in increasing order. Under the given parity constraints, only two such perfect states exist for $n \geq 2$:

$$([n], \varnothing, \varnothing, \varnothing) \quad \text{and} \quad (\varnothing, \varnothing, \varnothing, [n]),$$

which correspond to the initial and final configurations of objective (a).

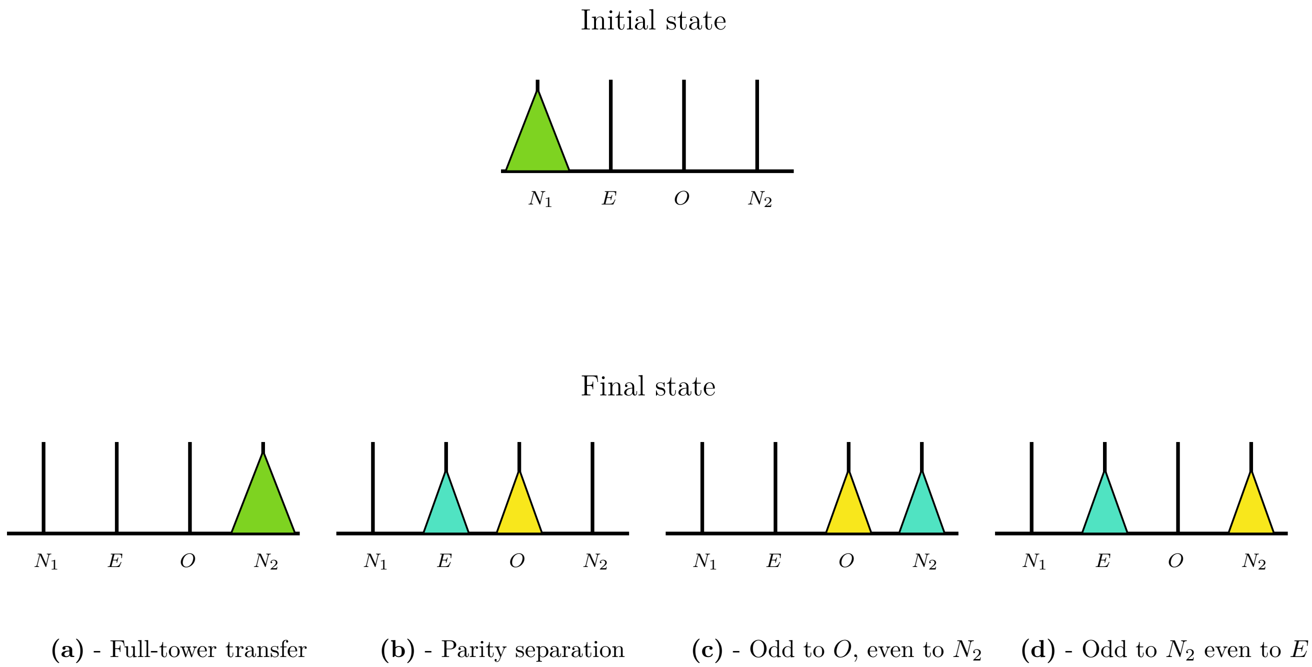

Figure 1: Illustration of the four main objectives (a)–(d). Green: entire tower; blue: even sub-tower; yellow: odd sub-tower.

3 Optimal Solutions

In this section, we establish recursive relations governing the optimal number of moves for the four objectives introduced in Section 2. The reasoning follows the same inductive principle as in the classical Tower of Hanoi problem: to move the largest disc n, all smaller discs must first be relocated to auxiliary pegs in such a way that parity constraints are respected and the target peg for disc n is made available.

Theorem 1. *For all $n \geq 1$, the optimal move counts (a_n, b_n, c_n, d_n) satisfy the coupled system*

$$a_n = 2b_{n-1} + 1, \tag{2}$$

$$b_n = \begin{cases} c_{n-1} + h_3^{\lfloor \frac{n-1}{2} \rfloor} + 1, & n \ odd, \\ d_{n-1} + h_3^{\lfloor \frac{n-1}{2} \rfloor} + 1, & n \ odd, \end{cases} \tag{3}$$

$$c_n = \begin{cases} b_{n-1} + h_3^{\lfloor \frac{n-1}{2} \rfloor} + 1, & n \ odd, \\ b_{n-2} + 2h_3^{\lfloor \frac{n-1}{2} \rfloor} + h_3^{\lfloor \frac{n-2}{2} \rfloor} + 2, & n \ odd, \end{cases} \tag{4}$$

$$d_n = \begin{cases} b_{n-2} + 3h_3^{\lfloor \frac{n-2}{2} \rfloor} + 2, & n \ odd, \\ b_{n-1} + h_3^{\lfloor \frac{n-1}{2} \rfloor} + 1, & n \ odd, \end{cases} \tag{5}$$

with initial conditions

$$a_0 = b_0 = c_0 = d_0 = 0, \qquad a_1 = b_1 = c_1 = d_1 = 1.$$

Proof. We proceed by induction on n. The base cases $n = 0, 1$, are trivial. Assume the recurrences hold for all $k < n$. We now justify each relation by analyzing the movement of disc n (or discs n and $n-1$) in an optimal strategy.

(a) Full-tower transfer. Disc n must eventually move from peg N_1 to peg N_2. Before this move, all smaller discs must be split by parity between pegs E and O, which requires b_{n-1} moves. After disc n is moved once, the configuration is symmetric and the same number of moves is needed to transfer the smaller discs onto peg N_2. Hence,

$$a_n = 2b_{n-1} + 1.$$

(b) Parity separation. Assume n is even (the odd case is symmetric with E and O swapped). To place disc n on peg E, that peg must first be emptied. Therefore, the smaller discs must be transferred so that even discs lie on N_2 and odd discs on O, which takes c_{n-1} moves (because in the final configuration, even and odd discs are separated by parity. Therefore, having all odd discs already stacked on peg O at this stage results in fewer moves than if some odd discs were still on peg N_2 together with even discs at this stage). Once disc n is placed on E, the remaining $\left\lfloor \frac{n-1}{2} \right\rfloor$ even discs join it via a classical three-peg transfer, requiring $h_3^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ moves. This yields the first case of b_n; the odd case is analogous with d_{n-1} replacing c_{n-1}.

(c) Odd to O, even to N_2. n even, disc n must end on N_2, so smaller discs are first parity-separated between E and O using b_{n-1} moves. Then disc n moves to N_2, and the even sub-tower joins it using $h_3^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ moves. This gives the even case of c_n.
n odd, disc n must end on O, which must be cleared first. Moving disc $n-1$ (which is even) to N_2 forces the $n-2$ smaller discs to be separated between E and O, requiring b_{n-2} moves. After disc $n-1$ moves to N_2, the necessary rearrangements of odd and even sub-towers proceed using classical three-peg transfers of sizes $\left\lfloor \frac{n-1}{2} \right\rfloor$ and $\left\lfloor \frac{n-2}{2} \right\rfloor$, leading to the second case of c_n.

(d) Odd to N_2, even to E. This case is analogous to part (c), with the roles of odd and even discs exchanged, which yields the cases of d_n.

This completes the induction and proves the theorem. $\square$

Corollary 1 (Uniqueness of optimal solutions)**.** *For each objective (a)–(d) defined in Section 2, the optimal solution is unique. Equivalently, the corresponding move sequence* $\mathrm{Sol}(x)$*, where* $x \in \{a, b, c, d\}$*, is uniquely determined by the recurrence system in Theorem 1.*

Proof. The recursive structure in Theorem 1 specifies, at each stage, a unique order in which the largest disc is moved: all smaller discs must first reach a parity-feasible configuration, disc n then moves exactly once, and the remaining configuration is determined uniquely by the inductive hypothesis. Since no alternative sequence can satisfy these constraints with the same minimal number of moves, the optimal solution is unique. □

Proposition 1 (Structure of optimal solutions)**.** *For each objective (a)–(d), the optimal sequence of configurations is uniquely determined and follows a recursive structure. The transitions below describe the evolution of configurations in terms of parity-separated subproblems.*

$$Sol(a) = ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{b_{n-1}} (\{n\}, [n-1]^0, [n-1]^1, \emptyset) \xrightarrow{1} (\emptyset, [n-1]^0, [n-1]^1, \{n\}) \xrightarrow{b_{n-1}} (\emptyset, \emptyset, \emptyset, [n]). \tag{6}$$

$$Sol(b) = \begin{cases} ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{c_{n-1}} (\{n\}, \emptyset, [n-1]^1, [n-1]^0) \xrightarrow{1} (\emptyset, \{n\}, [n-1]^0, [n-1]^1) & \\ \xrightarrow{h_3^{\lfloor \frac{n-1}{2} \rfloor}} (\emptyset, [n]^0, [n]^1, \emptyset), & n \textit{ even;} \\ ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{d_{n-1}} (\{n\}, [n-1]^0, \emptyset, [n-1]^1) \xrightarrow{1} (\emptyset, [n-1]^0, \{n\}, [n-1]^1) & \\ \xrightarrow{h_3^{\lfloor \frac{n-1}{2} \rfloor}} (\emptyset, [n]^0, [n]^1, \emptyset), & n \textit{ odd;} \end{cases} \tag{7}$$

$$Sol(c) = \begin{cases} ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{b_{n-1}} (\{n\}, [n-1]^0, [n-1]^1, \emptyset) \xrightarrow{1} (\emptyset, [n-1]^0, [n-1]^1, \{n\}) & \\ \xrightarrow{h_3^{\lfloor \frac{n-1}{2} \rfloor}} (\emptyset, \emptyset, [n]^1, [n]^0), & n \textit{ even;} \\ ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{b_{n-2}} (\{n, n-1\}, [n-2]^0, [n-2]^1, \emptyset) \xrightarrow{1} (\{n\}, [n-2]^0, [n-2]^1, \{n-1\}) & \\ \xrightarrow{h_3^{\lfloor \frac{n-1}{2} \rfloor}} (\{n\}, [n-2]^0, \emptyset, [n-2]^1 \cup \{n-1\}) \xrightarrow{1} (\emptyset, [n-2]^0, \{n\}, [n-2]^1 \cup \{n-1\}) & n \textit{ odd;} \\ \xrightarrow{h_3^{\lfloor \frac{n-1}{2} \rfloor}} (\emptyset, [n-2]^0, [n]^1, \{n-1\}) \xrightarrow{h_3^{\lfloor \frac{n-2}{2} \rfloor}} (\emptyset, [n]^0, [n]^1, \emptyset), & \end{cases} \tag{8}$$

$$Sol(d) = \begin{cases} ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{b_{n-2}} (\{n, n-1\}, [n-2]^0, [n-2]^1, \emptyset) \xrightarrow{1} (\{n\}, [n-2]^0, [n-2]^1, \{n-1\}) & \\ \xrightarrow{h_3^{\lfloor \frac{n-2}{2} \rfloor}} (\{n\}, \emptyset, [n-2]^1, [n-2]^0 \cup \{n-1\}) \xrightarrow{1} (\emptyset, \{n\}, [n-2]^1, [n-2]^0 \cup \{n-1\}) & n \textit{ even;} \\ \xrightarrow{h_3^{\lfloor \frac{n-2}{2} \rfloor}} (\emptyset, [n]^0, [n-2]^1, \{n-1\}) \xrightarrow{h_3^{\lfloor \frac{n-2}{2} \rfloor}} (\emptyset, [n]^0, \emptyset, [n]^1), & \\ ([n], \emptyset, \emptyset, \emptyset) \xrightarrow{b_{n-1}} (\{n\}, [n-1]^0, [n-1]^1, \emptyset) \xrightarrow{1} (\emptyset, [n-1]^0, [n-1]^1, \{n\}) & \\ \xrightarrow{h_3^{\lfloor \frac{n-1}{2} \rfloor}} (\emptyset, [n]^0, \emptyset, [n]^1), & n \textit{ odd;} \end{cases} \tag{9}$$

Corollary 2 (Simplified coupled system)**.** *The optimal move sequences also satisfy the following* *equivalent parity-dependent relations:*

$$b_n = \begin{cases} c_{n-1} + 2^{\frac{n-2}{2}}, & n \text{ even}, \\ d_{n-1} + 2^{\frac{n-1}{2}}, & n \text{ odd}, \end{cases} \tag{10}$$

$$c_n = \begin{cases} b_{n-1} + 2^{\frac{n-2}{2}}, & n \text{ even}, \\ b_{n-2} + 5 \cdot 2^{\frac{n-3}{2}} - 1, & n \text{ odd}, \end{cases} \tag{11}$$

$$d_n = \begin{cases} b_{n-2} + 3 \cdot 2^{\frac{n-2}{2}} - 1, & n \text{ even}, \\ b_{n-1} + 2^{\frac{n-1}{2}}, & n \text{ odd}. \end{cases} \tag{12}$$

Proof. These identities follow from substituting the classical three-peg solution $h_3^m = 2^m - 1$ into the recurrences of Theorem 1 and simplifying. The simplifications rely on the fact that for all integers $n, t \geq 0$,

$$\left\lfloor \frac{n-2t}{2} \right\rfloor = \left\lfloor \frac{n-(2t+1)}{2} \right\rfloor \quad n \text{ odd}, \qquad \left\lfloor \frac{n-(2t+1)}{2} \right\rfloor = \left\lfloor \frac{n-(2t+2)}{2} \right\rfloor \quad n \text{ even}. \qquad \square$$

4 Derived Recurrences and Explicit Formulas

The coupled system of recursive relations in Theorem 1 can be algebraically combined to yield single higher-order recurrences for each of the four optimal move sequences a_n, b_n, c_n, and d_n. In this section, we derive these recurrences and list their initial values. We then provide explicit closed-form expressions for each sequence.

Proposition 2 (Higher-order recurrences)**.** *The optimal move sequences a_n, b_n, c_n, d_n defined in Theorem 1 satisfy the following higher-order recurrences:*

$$\begin{array}{ll} a_0 = 0,\ a_1 = 1,\ a_2 = 3, \\ a_3 = 4, \end{array} \qquad \forall n \geq 4 : a_n = \begin{cases} a_{n-3} + 5 \cdot 2^{\frac{n-2}{2}} - 2, & n \text{ even}, \\ a_{n-3} + 7 \cdot 2^{\frac{n-3}{2}} - 2, & n \text{ odd}; \end{cases} \tag{13}$$

$$b_0 = 0,\ b_1 = 1,\ b_2 = 2, \qquad \forall n \geq 3 : b_n = \begin{cases} b_{n-3} + 7 \cdot 2^{\frac{n-4}{2}} - 1, & n \text{ even}, \\ b_{n-3} + 5 \cdot 2^{\frac{n-3}{2}} - 1, & n \text{ odd}; \end{cases} \tag{14}$$

$$\begin{array}{ll} c_0 = 0,\ c_1 = 1,\ c_2 = 2, \\ c_3 = 5,\ c_4 = 6,\ c_5 = 13, \end{array} \qquad \forall n \geq 6 : c_n = \begin{cases} c_{n-5} + 15 \cdot 2^{\frac{n-6}{2}} - 1, & n \text{ even}, \\ c_{n-6} + 31 \cdot 2^{\frac{n-7}{2}} - 2, & n \text{ odd}; \end{cases} \tag{15}$$

$$\begin{array}{ll} d_0 = 0,\ d_1 = 1,\ d_2 = 2, \\ d_3 = 4,\ d_4 = 7,\ d_5 = 11, \end{array} \qquad \forall n \geq 6 : d_n = \begin{cases} d_{n-6} + 5 \cdot 2^{\frac{n-2}{2}} - 2, & n \text{ even}, \\ d_{n-5} + 3 \cdot 2^{\frac{n-1}{2}} - 1, & n \text{ odd}. \end{cases} \tag{16}$$

Proof. We first derive the recurrence for b_n using the simplified coupled system of Corollary 2:

$$b_n = \begin{cases} c_{n-1} + 2^{\frac{n-2}{2}}, & n \text{ even}, \\ d_{n-1} + 2^{\frac{n-1}{2}}, & n \text{ odd}. \end{cases}$$

Substituting the expressions of c_{n-1} and d_{n-1} from the same corollary and applying the parity-dependent identities for floor functions from its proof yield

$$b_n = \begin{cases} b_{n-3} + 7 \cdot 2^{\frac{n-4}{2}} - 1, & n \text{ even}, \\ b_{n-3} + 5 \cdot 2^{\frac{n-3}{2}} - 1, & n \text{ odd}, \end{cases}$$

which establishes (14). The initial conditions follow directly from Corollary 2.
Having determined b_n, we derive the recurrence for a_n from

$$a_n = 2b_{n-1} + 1.$$

By applying (14) to b_{n-1} and noting that $n-1$ has opposite parity to n, we obtain

$$a_n = \begin{cases} a_{n-3} + 5 \cdot 2^{\frac{n-2}{2}} - 2, & n \text{ even}, \\ a_{n-3} + 7 \cdot 2^{\frac{n-3}{2}} - 2, & n \text{ odd}, \end{cases}$$

using $a_{n-3} = 2b_{n-4} + 1$ to eliminate b_{n-4}. This proves (13).
The recurrences for c_n and d_n are obtained analogously by substituting from Corollary 2 and then eliminating shifted b-terms via (14). Their initial values follow from explicit computation for $n \leq 5$.
Hence, all four higher-order recurrences are established. $\square$

The first fifteen terms of each of the six sequences h_3^n, h_4^n, a_n, b_n, c_n, and d_n are listed in Table 1. These values confirm the consistency of the derived recurrences with known results for the classical three-peg and four-peg Tower of Hanoi problems.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
h_3^n	0	1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	16383
h_4^n	0	1	3	5	9	13	17	25	33	41	49	65	81	97	113
a_n	0	1	3	5	9	15	23	35	53	77	113	163	235	335	481
b_n	0	1	2	4	7	11	17	26	38	56	81	117	167	240	340
c_n	0	1	2	5	6	13	15	30	34	65	72	135	149	276	304
d_n	0	1	2	4	7	11	18	25	40	54	85	113	176	231	358

Table 1: First fifteen terms of the sequences h_3^n, h_4^n, a_n, b_n, c_n, and d_n.

Proposition 3 (Closed-form expressions). *For all integers $n \geq 0$, let*

$$\rho(n) := n \bmod 3, \qquad \theta(n) := \frac{n - \rho(n)}{3}.$$

Then the four sequences admit the following closed forms:

$$a_n = \begin{cases} a_{\rho(n)} - 2\,\theta(n) \;+\; 2^{\frac{n}{2}} \left(\left(1 - 2^{-3\left\lfloor \frac{\theta(n)}{2} \right\rfloor}\right) \;+\; \frac{20}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n)-1}{2} \right\rfloor + 1}\right)\right), & n \text{ even}, \\ a_{\rho(n)} - 2\,\theta(n) \;+\; 2^{\frac{n+1}{2}} \left(\frac{5}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n)}{2} \right\rfloor}\right) \;+\; 2\left(1 - 2^{-3\left\lfloor \frac{\theta(n)-1}{2} \right\rfloor + 1}\right)\right), & n \text{ odd}, \end{cases} \tag{17}$$

$$b_n = \begin{cases} \frac{a_{\rho(n+1)} - 1}{2} - \theta(n+1) \;+\; 2^{\frac{n}{2}} \left(\frac{5}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n+1)}{2} \right\rfloor}\right) \;+\; 2\left(1 - 2^{-3\left\lfloor \frac{\theta(n+1)-1}{2} \right\rfloor + 1}\right)\right), & n \text{ even}, \\ \frac{a_{\rho(n+1)} - 1}{2} - \theta(n+1) \;+\; 2^{\frac{n-1}{2}} \left(\left(1 - 2^{-3\left\lfloor \frac{\theta(n+1)}{2} \right\rfloor}\right) \;+\; \frac{20}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n+1)-1}{2} \right\rfloor + 1}\right)\right), & n \text{ odd}, \end{cases} \tag{18}$$

$$c_n = \begin{cases} \frac{a_{\rho(n)} - 1}{2} - \theta(n) \;+\; 2^{\frac{n-2}{2}} \left(2 - 2^{-3\left\lfloor \frac{\theta(n)}{2} \right\rfloor} \;+\; \frac{20}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n)-1}{2} \right\rfloor + 1}\right)\right), & n \text{ even}, \\ \frac{a_{\rho(n-1)} - 1}{2} - \theta(n-1) \;+\; 2^{\frac{n-3}{2}} \left(6 - 2^{-3\left\lfloor \frac{\theta(n-1)}{2} \right\rfloor} \;+\; \frac{20}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n-1)-1}{2} \right\rfloor + 1}\right)\right) - 1, & n \text{ odd}, \end{cases} \tag{19}$$

$$d_n = \begin{cases} \frac{a_{\rho(n-1)} - 1}{2} - \theta(n-1) \;+\; 2^{\frac{n-2}{2}} \left(\frac{5}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n-1)}{2} \right\rfloor}\right) \;+\; 2\left(1 - 2^{-3\left\lfloor \frac{\theta(n-1)-1}{2} \right\rfloor + 1}\right) + 3\right) - 1, & n \text{ even}, \\ \frac{a_{\rho(n)} - 1}{2} - \theta(n) \;+\; 2^{\frac{n-1}{2}} \left(\frac{5}{7}\left(1 - 2^{-3\left\lfloor \frac{\theta(n)}{2} \right\rfloor}\right) \;+\; 2\left(1 - 2^{-3\left\lfloor \frac{\theta(n)-1}{2} \right\rfloor + 1}\right) + 1\right), & n \text{ odd}. \end{cases} \tag{20}$$

Proof. The result can be proved by induction on n, using the higher-order recurrences in Proposition 2 and the definitions of $\rho(n)$ and $\theta(n)$. □

5 Asymptotic Growth Analysis

We now study the relative growth behavior of the four optimal move sequences a_n, b_n, c_n, and d_n, and compare them with the classical three-peg and four-peg Tower of Hanoi sequences h_3^n and h_4^n, respectively.

Proposition 4 (Comparative bounds)**.** *For all $n \geq 0$, the following hold:*

1. *For all $n \geq 0$,*
$$h_4^n \leq a_n \leq h_3^n. \tag{21}$$

2. *For all $n \geq 0$,*
$$\max\{b_n, c_n, d_n\} \leq a_n. \tag{22}$$

3. *Moreover, the sequences b_n, c_n, d_n remain close to one another, and none dominates the others for all $n \geq 0$.*

Proof.

1. The inequality $a_n \leq h_3^n$ can be proved by induction on n using the recurrences in Proposition 2. Moreover, since the parity constraints can only reduce the set of feasible moves compared with the unconstrained four-peg problem, we also have $a_n \geq h_4^n$.

2. The inequality $\max\{b_n, c_n, d_n\} \leq a_n$ can be proved by checking the sign of the difference $a_n - x_n$, for each $x \in \{b, c, d\}$, using the explicit formulas in Proposition 3.

3. The non-domination follows from inspection of Table 1; e.g., for $n = 6$ we have $d_6 > b_6 > c_6$, while for $n = 7$ we have $c_7 > b_7 > d_7$. $\square$

Remark 1. *Numerical evidence (see Table 1) suggests that*

$$\min\{b_n, c_n, d_n\} \geq h_4^n \qquad \text{for all } n \geq 7.$$

A formal proof of this observation lies beyond the scope of this paper and is left as a future research direction.

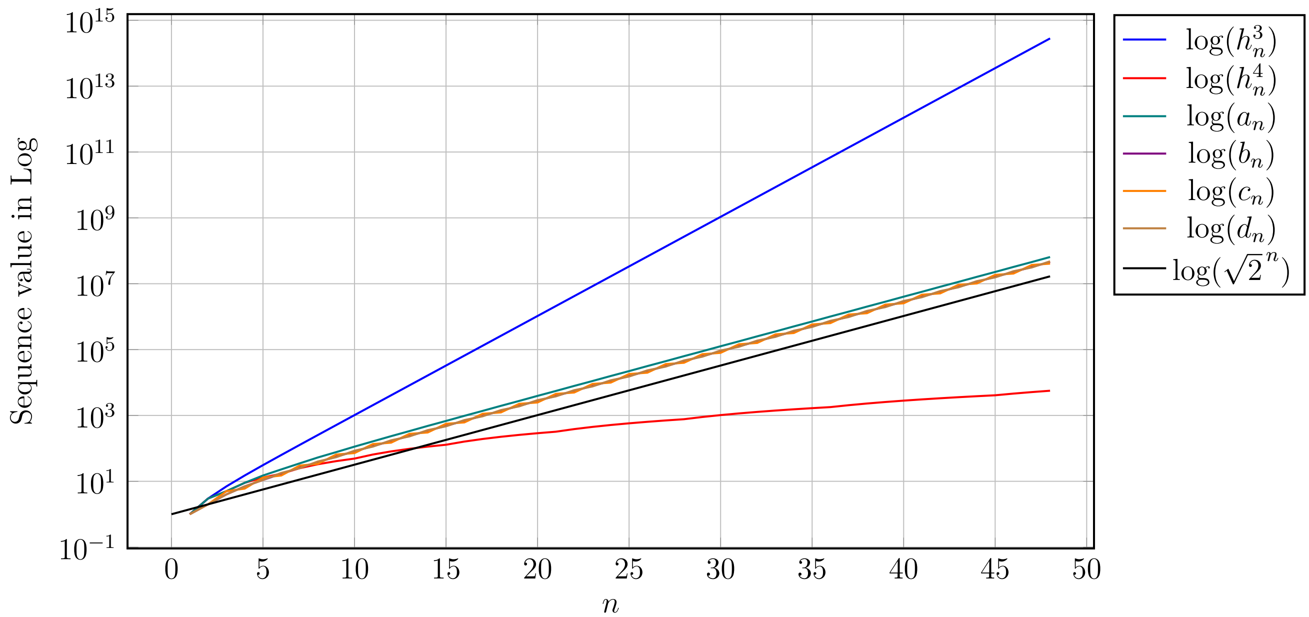

Figure 2: Comparison of the parity-constrained sequences a_n, b_n, c_n, d_n with the classical three-peg and four-peg optima h_3^n and h_4^n, and the sequence $(\sqrt{2}^n)_{n\geq 0}$.

Figure 2 displays the logarithmic growth of a_n, b_n, c_n, and d_n for $n \geq 7$, alongside the classical optima h_3^n and h_4^n. The four parity-constrained sequences lie strictly between h_3^n and h_4^n, in accordance with Proposition 4 and Remark 1.
Furthermore, the slopes of the logarithmic curves of a_n, b_n, c_n, and d_n are nearly parallel, indicating that all four sequences exhibit the same exponential order of growth. This visual trend is reinforced by plotting the auxiliary function $y = (\sqrt{2})^n$, whose logarithmic slope closely matches those of the four sequences. This suggests a half-exponential asymptotic behavior of the form

$$a_n,\ b_n,\ c_n,\ d_n = \Theta\Big((\sqrt{2})^n\Big).$$

To make this observation precise, we next analyze subsequence-dependent growth ratios and show that all four sequences double asymptotically over two-step increments.

Theorem 2 (Asymptotic subsequence ratios)**.** *Each of the four sequences a_n, b_n, c_n, and d_n*

admits two asymptotically distinct subsequences whose ratios converge to fixed constants.

$$\lim_{k\to\infty} \frac{a_{2k}}{a_{2k-1}} = \frac{27}{19}, \qquad \lim_{k\to\infty} \frac{a_{2k+1}}{a_{2k}} = \frac{38}{27},$$

$$\lim_{k\to\infty} \frac{b_{2k}}{b_{2k-1}} = \frac{38}{27}, \qquad \lim_{k\to\infty} \frac{b_{2k+1}}{b_{2k}} = \frac{27}{19},$$

$$\lim_{k\to\infty} \frac{c_{2k}}{c_{2k-1}} = \frac{34}{31}, \qquad \lim_{k\to\infty} \frac{c_{2k+1}}{c_{2k}} = \frac{62}{34},$$

$$\lim_{k\to\infty} \frac{d_{2k}}{d_{2k-1}} = \frac{20}{13}, \qquad \lim_{k\to\infty} \frac{d_{2k+1}}{d_{2k}} = \frac{26}{20}.$$

Proof sketch. The limits follow by expressing each sequence using the closed-form expressions in Proposition 3 and applying dominant-term asymptotics as $k \to \infty$. For instance, for a_n the leading exponential behavior alternates with parity, giving the stated ratios. The remaining limits are obtained similarly. □

Proposition 5 (Two-step growth behavior)**.** *For each sequence $x_n \in \{a_n, b_n, c_n, d_n\}$, the two-step growth ratio satisfies*

$$\lim_{n\to\infty} \frac{x_n}{x_{n-2}} = 2. \tag{23}$$

Proof sketch. From Theorem 2, each sequence has two parity-dependent subsequences whose successive ratios converge to distinct constants $\alpha > 1$ and $\beta > 1$. Multiplying these constants shows

$$\frac{x_n}{x_{n-2}} \sim \alpha \cdot \beta = 2.$$

The same behavior follows directly from the leading exponential terms in the closed forms of Proposition 3. □

Theorem 3 (Half-exponential asymptotic growth)**.** *For each of the four sequences $x_n \in \{a_n, b_n, c_n, d_n\}$, we have*

$$x_n = \Theta\Big((\sqrt{2})^n\Big). \tag{24}$$

Proof. From Proposition 5, we have

$$\frac{x_n}{x_{n-2}} \to 2,$$

which implies an asymptotic doubling every two steps. Hence, for sufficiently large n,

$$x_n \sim C \cdot (\sqrt{2})^n$$

for some constant $C > 0$, establishing the stated order of growth. □

6 Associated Graph

We now introduce the graph naturally induced by the parity-constrained Tower of Hanoi problem with n discs. This graph encodes all feasible configurations and legal transitions under the rules of Section 2.
Analogously to the classical Hanoi graph H_m^n (the state graph of the m-peg Tower of Hanoi with n discs; see [4, Section 5.6]), the four-peg parity-constrained problem gives rise to a family of graphs, which we call the *parity-constrained Hanoi graphs.*

Peg coding and parity map. We code the four pegs by

$$N_1 = 0, \qquad E = 1, \qquad O = 2, \qquad N_2 = 3,$$

and denote by

$$p(d) := \begin{cases} 1, & \text{if } d \text{ is even,} \\ 2, & \text{if } d \text{ is odd,} \end{cases} \qquad \overline{p}(d) := 3 - p(d)$$

the *allowed parity peg* and the *forbidden parity peg* of disc d, respectively. Thus an even disc may not occupy peg 2, and an odd disc may not occupy peg 1.

Vertices. Let $Q = \{0, 1, 2, 3\}$, and $Q^d = \{0, p(d), 3\}$. A state is encoded by the word $s = s_n \cdots s_1 \in Q^n$, where s_d is the peg occupied by disc d. The vertex set of the *parity-constrained Hanoi graph* P^n is

$$V(P^n) = \Big\{ s \in Q^n \;\Big|\; \big(d \text{ even} \Rightarrow s_d \neq 2\big) \;\wedge\; \big(d \text{ odd} \Rightarrow s_d \neq 1\big) \Big\}. \tag{25}$$

Edges. Two states are adjacent if they differ in exactly one coordinate corresponding to a legal move. Formally,

$$E(P^n) = \Big\{ \{\underline{s}\, i\, \overline{s},\ \underline{s}\, j\, \overline{s}\} \;\Big|\; \underline{s}\, i\, \overline{s},\ \underline{s}\, j\, \overline{s} \in V(P^n),\ d \in [n],\ i \neq j,\ i, j \in Q^d,\ \overline{s}_k \notin \{i, j\}\ \forall k < d \Big\}, \tag{26}$$

i.e., disc d moves between two allowed pegs i and j while no smaller disc $k < d$ occupies either of those pegs. (This is the parity-restricted analogue of [4, Eq. (5.44)].)
Figure 3 illustrates P^1, P^2, and P^3. Red edges indicate moves of the largest disc, the red vertex marks 0^n (the initial state), green vertices mark the four targets of objectives (a)–(d), and dashed edges depict an optimal path for objective (a).

Recursive structure. The graph P^n decomposes into three copies of P^{n-1} according to the peg occupied by the largest disc n:

$$P^n \cong 0P^{n-1} \;\cup\; p(n)P^{n-1} \;\cup\; 3P^{n-1}, \tag{27}$$

where iP^{n-1} denotes the induced subgraph on states with $s_n = i \in Q^d$. Thus

$$V(P^n) = V(0P^{n-1}) \;\cup\; V(p(n)P^{n-1}) \;\cup\; V(3P^{n-1}). \tag{28}$$

Edges are of two types: (i) *internal* edges inside each copy iP^{n-1} (moves of a disc $< n$), and (ii) *bridges* between distinct copies (moves of disc n). Therefore, we have the following recursive definition of $E(P^n)$.

$$\begin{aligned} E(P^0) = \varnothing,\ \forall n \geq 1:\ E(P^n) = &\{\{is, it\} \mid i \in Q^d, \{s, t\} \in E(P^{n-1})\} \\ &\cup \{\{ir, jr\} \mid i, j \in Q^d, i \neq j, r \in V(P^{n-1}), r_k \neq i, j\ \forall k < n\} \end{aligned} \tag{29}$$

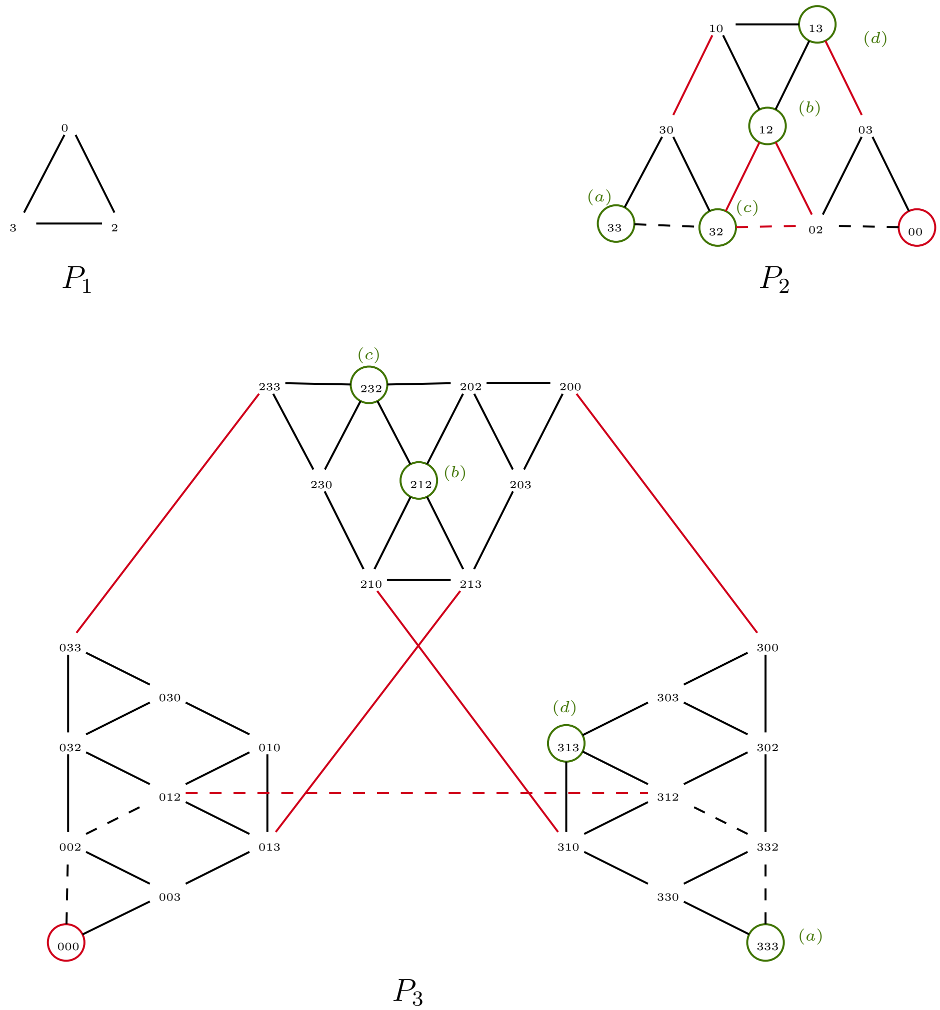

Figure 3: The graphs P^1, P^2, and P^3. Red edges represent moves of the largest disc. Dashed edges show the optimal path for objective (a). The red circle marks the initial state; green circles mark the final states of objectives (a)–(d).

Counting vertices and edges

Proposition 6 (Vertex count)**.** *For all $n \geq 0$,*

$$|V(P^n)| = 3^n. \tag{30}$$

Proof. Each disc d may occupy exactly three pegs: 0, $p(d)$, or 3. Hence there are 3^n admissible words $s \in Q^n$, one for each state. □

Proposition 7 (Edge-count recurrence)**.** *The number of edges in P^n, satisfy the following recurrence.*

$$|E(P^0)| = 0, \quad \forall n \geq 1 : |E(P^n)| \ = \ 3\,|E(P^{n-1})| \ + \ 2^{\left\lfloor \frac{n}{2} \right\rfloor + 1} \ + \ 1. \tag{31}$$

Proof. Internal edges: for each fixed position of disc n there is an induced copy of P^{n-1}, contributing $|E(P^{n-1})|$ edges; three such copies give $3\,|E(P^{n-1})|$.
Bridges (moves of disc n): disc n may move between any two of its three allowed pegs Q^d, provided no smaller disc lies on either of those two pegs.

- Between 0 and $p(n)$: peg $p(n)$ must be empty. All smaller discs of the *same* parity as n must therefore be on peg 3, while the opposite-parity smaller discs may be anywhere among their two allowed pegs; this yields $2^{\lfloor \frac{n}{2} \rfloor}$ bridge edges.
- Between $p(n)$ and 3: symmetrically, peg 3 must be empty; again we obtain $2^{\lfloor \frac{n}{2} \rfloor}$ bridge edges.
- Between 0 and 3: both pegs 0 and 3 must be free of smaller discs, forcing a unique parity-separated arrangement of the $n-1$ smaller discs on pegs 1 and 2; hence exactly one bridge edge.

Summing gives $2 \cdot 2^{\lfloor \frac{n}{2} \rfloor} + 1 = 2^{\lfloor \frac{n}{2} \rfloor + 1} + 1$ bridges, proving the recurrence. □

Proposition 8 (Edge-count closed form)**.** *For all $n \geq 0$,*

$$|E(P^n)| = \begin{cases} \dfrac{3^{n+3} - 20 \cdot 2^{\frac{n}{2}} - 7}{14}, & n \text{ even}, \\ \dfrac{3^{n+3} - 32 \cdot 2^{\frac{n-1}{2}} - 7}{14}, & n \text{ odd}. \end{cases}$$

Proof. Unfold the linear inhomogeneous recurrence of Proposition 7 separately for even and odd n, using $|E(P^0)| = 0$ and the geometric sums $\sum 3^k$ and $\sum 2^{\lfloor \frac{k}{2} \rfloor}$. A routine calculation yields the stated expressions. □

The number of edges in Hanoi graphs H_m^n has been computed using various approaches in [1,6,9,10]. In Proposition 9, we compare the number of edges in P^n with those of the three-peg and four-peg Hanoi graphs.

Proposition 9 (Edges number comparison)**.** *For all $n \geq 0$,*

$$|E(H_3^n)| \;\leq\; |E(P^n)| \;\leq\; |E(H_4^n)|.$$

Proof. The left inequality follows by induction on n, using that in P^n each disc may occupy three pegs, whereas in H_3^n it may occupy only two. The right inequality holds since P^n is obtained from H_4^n by removing all states violating the parity constraints (see Theorem 7). □

Table 2 lists the first values of the three edge-count sequences.

n	0	1	2	3	4	5	6	7	8	9	10
$\lvert E(H_3^n)\rvert$	0	3	12	39	120	363	1092	3279	9840	29523	88572
$\lvert E(H_4^n)\rvert$	0	6	36	168	720	2976	12096	48768	195840	784896	3142656
$\lvert E(P^n)\rvert$	0	3	14	47	150	459	1394	4199	12630	37923	113834

Table 2: First eleven values of $|E(H_3^n)|$, $|E(H_4^n)|$, and $|E(P^n)|$.

Degrees

Proposition 10 (Minimum and maximum degrees)**.** *For all $n \geq 1$,*

$$\delta(P^n) = 2, \qquad \Delta(P^n) = \begin{cases} 2, & n = 1, \\ 4, & n = 2, \\ 5, & n \geq 3. \end{cases}$$

Proof.

- *Minimum.* Clearly $\delta(P^n) \neq 1$. On the other hand, in any perfect state (all discs on a single peg), the only movable disc is 1. By parity, disc 1 cannot use peg E (peg 1), so it has exactly two legal moves. Hence $\delta(P^n) = 2$.

- *Maximum.* For $n = 1$, the graph is a 3-cycle, so $\Delta(P^1) = 2$. For $n = 2$, one checks directly that the maximum degree is $\Delta(P^2) = 4$ from Figure 3. For $n \geq 3$, consider a configuration where discs $1, 2, 3$ are simultaneously unblocked and occupy distinct allowed pegs; in H_4^n this gives $2 + 2 + 2 = 6$ moves, but the parity restriction removes one move for disc 1, namely, the move between pegs 1 and 2, leaving $2 + 2 + 1 = 5$ total allowed moves in a such state. Such a state exists, so $\Delta(P^n) = 5$ for $n \geq 3$. $\square$

Corollary 3. *For all $n \geq 1$,*

$$\delta(P^n) = \delta(H_3^n), \qquad \Delta(P^n) < \Delta(H_4^n).$$

Proposition 11 (Average degree)**.** *Let $\overline{d}(P^n)$ denote the average degree. Then, using $|V(P^n)| = 3^n$ and Proposition 8,*

$$\overline{d}(P^n) = \frac{2\,|E(P^n)|}{|V(P^n)|} = \begin{cases} \dfrac{27}{7} - \dfrac{20}{7}\,\dfrac{2^{\frac{n}{2}}}{3^n} - \dfrac{1}{3^n}, & n \text{ even}, \\[2ex] \dfrac{27}{7} - \dfrac{32}{7}\,\dfrac{2^{\frac{n-1}{2}}}{3^n} - \dfrac{1}{3^n}, & n \text{ odd}. \end{cases} \tag{32}$$

In particular,

$$\lim_{n\to\infty} \overline{d}(P^n) = \frac{27}{7} \approx 3.8571, \qquad 2 = \delta(P^n) \;\leq\; \overline{d}(P^n) \;<\; 5 = \Delta(P^n) \;\; \text{for } n \geq 3.$$

Proof. A straightforward computation using the closed form of $|E(P^n)|$ from Proposition 8, together with $|V(P^n)| = 3^n$, yields the claim. $\square$

Planarity

It is known that the only planar Hanoi graphs H_m^n are H_3^n (for all $n \geq 0$), together with H_4^0, H_4^1, and H_4^2; whereas H_4^n is non-planar for all $n \geq 3$ [7]. In Theorem 4, we show that, similarly to the case of H_4^n, the graphs P^n are planar only for $n \leq 2$, and become non-planar for all $n \geq 3$.

Theorem 4 (Planarity)**.** *The graphs P^0, P^1, and P^2 are planar, whereas P^n is non-planar for all $n \geq 3$.*

Proof. Figure 3 shows planar embeddings of P^1 and P^2, and the planarity of P^0 is trivial.
For $n \geq 3$, observe that $P^3 \subset P^n$; thus, it suffices to prove that P^3 is non-planar. Consider the subgraph of P^3 induced by the vertex set

$$\{032, 012, 013, 002, 010, 213, 003, 033, 030, 210, 212, 232, 233, 310, 312\}.$$

After deleting the four edges $\{210, 212\}$, $\{013, 012\}$, $\{012, 032\}$, and $\{030, 033\}$, the resulting subgraph contains a subdivision of the complete bipartite graph $K_{3,3}$. Figure 4 illustrates two embeddings of this subdivision: one aligned with the vertex positions used in Figure 3, and another using a standard layout of $K_{3,3}$. The red and blue vertices represent the two partite sets of $K_{3,3}$, while the green vertices correspond to subdivision vertices. $\square$

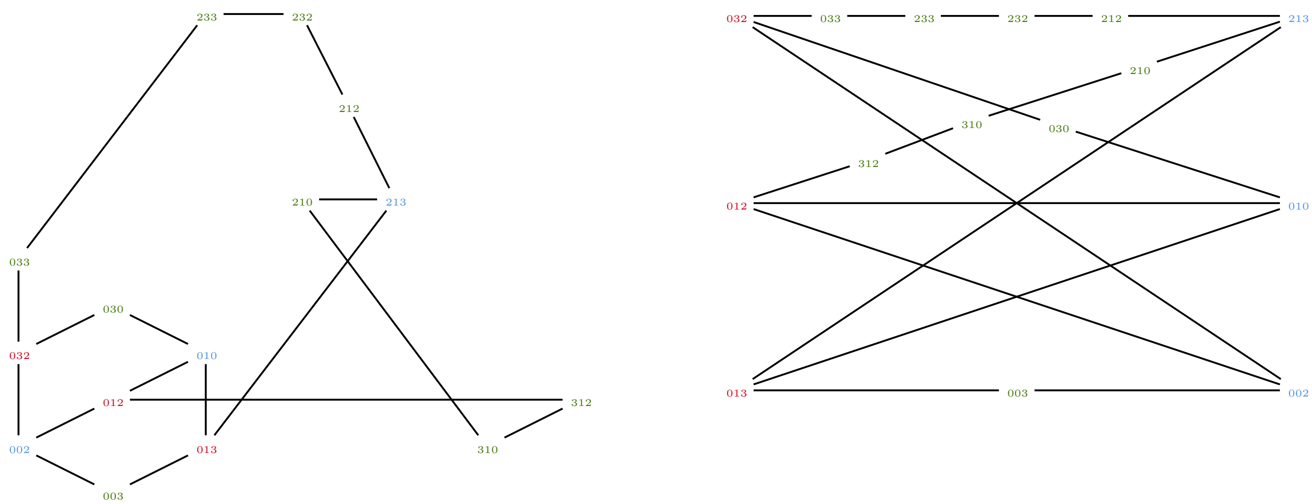

Figure 4: Two embeddings of a subdivision of $K_{3,3}$ contained in P_3.

Connectivity

Many properties of the parity-constrained Hanoi graphs P^n can be deduced from their recursive structure, such as their connectedness or, more precisely, their 2-connectedness.

Theorem 5 (Vertex connectivity)**.** *For all $n \geq 1$, the vertex connectivity of P^n satisfies*

$$\kappa(P^n) = 2. \tag{33}$$

Proof. The two perfect-state vertices (where all discs lie on a single peg) each have two neighbors, each of them serves as an articulation point whose removal disconnects the graph. Hence $\kappa(P^n) \leq 2$.

To show that the deletion of only one vertex is not sufficient to disconnect the graph, we proceed by induction on n. For $n = 1$, the graph $P^1 \cong K_3$, which is 2-connected.

Assume now that P^{n-1} is 2-connected for some $n \geq 2$. In the recursive construction of P^n, the graph is composed of three copies $0P^{n-1}$, $p(n)P^{n-1}$, and $3P^{n-1}$, which are interconnected by edges corresponding to moves of the largest disc n, called a *bridge*.

Removing a single vertex that has all its neighbors within the same copy of P^{n-1} does not disconnect that copy by the induction hypothesis. Thus, it remains to consider the removal of a *frontier vertex*, i.e., a vertex that has at least one neighbor in another copy.

Between any two adjacent copies, say $0P^{n-1}$ and $p(n)P^{n-1}$, there exist at least two bridges: one where peg $\overline{p}(n)$ is empty, and another where peg $\overline{p}(n)$ is not empty (the only exception is the case $n = 1$, which has already been handled as the base case). A similar argument applies to the pair $p(n)P^{n-1}$ and $3P^{n-1}$.

Consequently, removing a frontier vertex corresponds to deleting exactly one bridge, which does not disconnect the three copies. Hence, the graph P^n remains connected after the removal of any single vertex.

Therefore, no single vertex removal can disconnect the graph, and we conclude that $\kappa(P^n) \geq 2$. Combining both inequalities yields $\kappa(P^n) = 2$. □

Theorem 6 (Edge connectivity)**.** *For all $n \geq 1$, the edge connectivity of P^n satisfies*

$$\lambda(P^n) = 2. \tag{34}$$

Proof. From Proposition 10, we know that $\delta(P^n) = 2$, and by Theorem 5, we have $\kappa(P^n) = 2$. Since, for any graph G, it holds that

$$\kappa(G) \leq \lambda(G) \leq \delta(G),$$

we obtain

$$2 = \kappa(P^n) \leq \lambda(P^n) \leq \delta(P^n) = 2.$$

Hence, $\lambda(P^n) = 2$. □

Since $\kappa(P^n) = \lambda(P^n) = \delta(P^n)$, we may state the following result.

Proposition 12. *For all $n \geq 1$, the parity-constrained Hanoi graph P^n is maximally connected; that is, it is both maximally vertex-connected and maximally edge-connected.*

Subgraph relations to Hanoi graphs

Theorem 7 establishes the subgraph relationships between P^n and the classical Hanoi graphs on three and four pegs.

Theorem 7 (Subgraph inclusion)**.** *For all $n \geq 0$,*

$$H_3^{\left\lceil \frac{n}{2} \right\rceil} \;\subseteq\; P^n \;\subseteq\; H_4^n. \tag{35}$$

Proof. The right inclusion is immediate: P^n is obtained from H_4^n by deleting states and edges that violate parity constraints, hence P^n is a subgraph of H_4^n.
For the left inclusion, fix the majority parity among $\{1, \ldots, n\}$; there are $\left\lceil \frac{n}{2} \right\rceil$ discs of that parity. These discs may move freely among the three pegs $\{0, 3, p\}$ with $p \in \{1, 2\}$ matching that parity. The induced subgraph on these discs (with the others kept fixed) is isomorphic to $H_3^{\left\lceil \frac{n}{2} \right\rceil}$. □

Corollary 4 (Largest embedded 3-peg sub-Hanoi graph)**.** *For $n \geq 1$, a largest subgraph of P^n isomorphic to $H_3^{\left\lceil \frac{n}{2} \right\rceil}$ is given by*

$$V\left(H_3^{\left\lceil \frac{n}{2} \right\rceil}\right) = \begin{cases} \{\, s \in V(P^n) \mid s_d = 2 \textit{ for all odd } d \,\}, & n \textit{ even} \\ \{\, s \in V(P^n) \mid s_d = 1 \textit{ for all even } d \,\}, & n \textit{ odd}, . \end{cases} \tag{36}$$

Figure 5 illustrates the largest Hanoi subgraph of P^3 that is isomorphic to H_3^2.

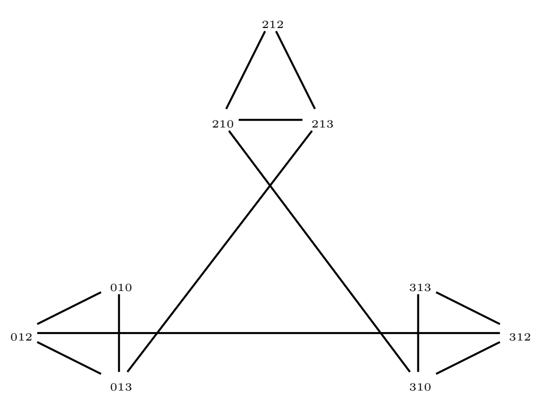

Figure 5: Largest Hanoi subgraph (isomorphic to $H_3^{\lceil n/2 \rceil}$) inside P^3.

Proposition 13 (Disconnection after removal)**.** *If $n \geq 4$, then the graph $P^n \backslash H_3^{\lceil \frac{n}{2} \rceil}$ is disconnected; otherwise, it is connected.*

Proof. The claim is trivial for $n = 0$ and can be checked directly for $n = 1, 2, 3$ (see Figure 3). Assume $n \geq 4$. By Corollary 4, removing $H_3^{\lceil \frac{n}{2} \rceil}$ forbids all states in which any disc of the majority parity sits on its parity peg. Consequently, such discs can only move between the two neutral pegs $\{0, 3\}$ in the remaining graph.
For n even, fix any $(n-3)$-prefix s and consider the six states

$$s000,\ s003,\ s013,\ s010,\ s030,\ s033.$$

In $P^n \setminus H_3^{\lceil \frac{n}{2} \rceil}$ (with n even, odd discs cannot use peg 2), these form an induced path with no admissible continuation. Hence the graph is disconnected.
For n odd, a similar argument uses parity reversal: even discs cannot use peg 1, so there is no path connecting $s0000$ to $s3333$ once $H_3^{\lceil \frac{n}{2} \rceil}$ is removed, as moving disc 4 requires two intermediate stacks, stack including only disc 2 on peg 1, and another including discs 1 and 3 on peg 2, which is impossible, sine disc 2 is forbidden to use peg 1. Thus $P^n \setminus H_3^{\lceil \frac{n}{2} \rceil}$ is disconnected for all $n \geq 4$. □

Proposition 14 (Number of largest sub-Hanoi graphs in P^n)**.** *For n even, the graph P^n contains two distinct subgraphs isomorphic to $H_3^{\frac{n}{2}}$, each corresponding to configurations where exactly $\frac{n}{2}$ discs (either all even or all odd) are free to move under unconstrained three-peg rules.*
For n odd, the graph P^n contains one subgraph isomorphic to $H_3^{\frac{n+1}{2}}$, corresponding to states where all even discs are placed on their parity peg 1, and another subgraph isomorphic to $H_3^{\frac{n-1}{2}}$, corresponding to states where all odd discs are placed on their parity peg 2.
In both cases, these subgraphs share a unique common vertex, namely the configuration in which all discs are placed on their respective parity pegs and are fully separated by parity.

Proof. The number of discs represented in each subgraph follows immediately from Corollary 4. The uniqueness of the shared vertex is also clear: this state is the only configuration belonging to both vertex sets identified in Corollary 4, as it is the sole configuration in which full parity separation occurs, allowing it to lie simultaneously at the base of both sub-Hanoi structures. □

Diameter

Using Theorem 7, we can derive bounds on the diameter of the graphs P^n, as stated in Proposition 15.

Proposition 15 (Diameter bounds)**.** *For all $n \geq 1$,*

$$\operatorname{diam}(H_4^n) \ \leq \ \operatorname{diam}(P^n) \ \leq \ \operatorname{diam}(H_3^{\lceil \frac{n}{2} \rceil}). \tag{37}$$

Proof. As P^n is a subgraph of H_4^n but contains a copy of $H_3^{\lceil \frac{n}{2} \rceil}$ (Theorem 7), all distances between two vertices in P^n are at least as large as those in H_4^n and at most as large as those in $H_3^{\lceil \frac{n}{2} \rceil}$. □

Theorem 8 (Explicit diameter estimate)**.** *For all $n \geq 1$,*

$$4n - 7 \ \leq \ \operatorname{diam}(P^n) \ \leq \ 2^{\lceil \frac{n}{2} \rceil} - 1. \tag{38}$$

Proof. From [4, Prop. 5.20], $\operatorname{diam}(H_4^n) \geq 4n-7$, and from [4, Prop. 5.36], $\operatorname{diam}(H_3^k) \leq 2^k - 1$. Since $H_3^{\lceil \frac{n}{2} \rceil} \subseteq P^n$, the result follows from Proposition 15. □

Automorphism

Proposition 16 (Automorphism group)**.** *For all $n \geq 1$, the automorphism group of P^n is isomorphic to S_2, generated by the swap of the two neutral pegs $0 \leftrightarrow 3$.*

Proof. Swapping 0 and 3 preserves both legality of moves and parity constraints, yielding a nontrivial automorphism. No other peg permutation is permitted: swapping 1 and 2 breaks parity feasibility, while a neutral peg cannot be exchanged with a parity-restricted one. Hence $\operatorname{Aut}(P^n) \cong S_2$. □

Hamiltonicity

It is known that the Hanoi graphs H_m^n are Hamiltonian for all $m \geq 3$ and $n \geq 1$ [7]. In the next three theorems, namely Theorems 9, 10, and 11, we establish Hamiltonicity properties of the parity-constrained Hanoi graphs P^n.

Theorem 9 (Hamiltonian path between perfect states)**.** *For all $n \geq 1$, P^n has a Hamiltonian path between 0^n and 3^n.*

Proof. By induction on n. For $n = 1$, P^1 is a 3-cycle. Suppose the result holds for $n-1$. Then the copies $0P^{n-1}$, $p(n)P^{n-1}$, and $3P^{n-1}$ are linked by moves of disc n, allowing concatenation of Hamiltonian paths:

$$0^n \rightsquigarrow 0\,3^{n-1} \to p(n)\,3^{n-1} \rightsquigarrow p(n)\,0^{n-1} \to 3\,0^{n-1} \rightsquigarrow 3^n.$$

This produces a Hamiltonian path. □

Theorem 10 (Hamiltonian path from perfect state to parity-separated configuration)**.** *For all $n \geq 1$, there is a Hamiltonian path from i^n, where $i \in \{0, 3\}$, to the parity-separated state in which all even discs lie on peg 1 and all odd discs on peg 2.*

Proof. Without loss of generality, consider the initial and final states I and F, respectively, of objective (b), as illustrated in Figure 1. Let $\bar{b}_n$ denote the maximum number of moves required to reach F from I without visiting any state more than once. The goal is to construct an elementary path from I to F of maximum possible length, namely $|V(P^n)| - 1 = 3^n - 1$.
We present a feasible solution based on two moves of the largest disc n. The steps are as follows: 1. Move the $n-1$ smaller discs to pegs 1 and 2, forming two subtowers separated by parity. This requires $\bar{b}_{n-1}$ moves. 2. Move disc n from peg 0 to peg 3. 3. Move the $n-1$ smaller discs from pegs 1 and 2 to peg 0, again using $\bar{b}_{n-1}$ moves. 4. Move disc n from peg 3 to the parity peg $p(n)$. 5. Finally, move the $n-1$ discs from peg 0 to pegs 1 and 2 in parity-separated configuration, using another $\bar{b}_{n-1}$ moves.
For n even, this sequence may be represented as

$$\begin{aligned}([n], \emptyset, \emptyset, \emptyset) &\xrightarrow[\bar{b}_{n-1}]{} (\{n\}, [n-1]^0, [n-1]^1, \emptyset) \xrightarrow[1]{} (\emptyset, [n-1]^0, [n-1]^1, \{n\}) \\ &\xrightarrow[\bar{b}_{n-1}]{} ([n-1], \emptyset, \emptyset, \{n\}) \xrightarrow[1]{} ([n-1], \{n\}, \emptyset, \emptyset) \xrightarrow[\bar{b}_{n-1}]{} (\emptyset, [n]^0, [n]^1, \emptyset).\end{aligned}$$

For n odd, the fourth configuration changes to $([n-1], \emptyset, \{n\}, \emptyset)$.
Thus, the total number of moves in this construction is

$$\bar{b}_n = 3\bar{b}_{n-1} + 2.$$

Since $\bar{b}_0 = 0$, solving the recurrence yields $\bar{b}_n = 3^n - 1$.
Moreover, this construction is elementary: no state is repeated at any stage, because each subprocedure recursively performs an elementary traversal over a distinct P^{n-1} configuration space, and transitions involving disc n always lead to new state classes distinguished by the position of the largest disc.
Since $|V(P^n)| = 3^n$, the path constructed has maximum possible length $3^n - 1$, proving that it is Hamiltonian. Therefore, this procedure provides a Hamiltonian path between any two states of the form $s \in V(P^n)$ such that $s_0 \in \{0, 3\}$, $s_d = 1$ for even d, and $s_d = 2$ for odd d. ☐

Theorem 11 (Hamiltonicity)**.** *For all $n \geq 1$, the graph P^n contains a Hamiltonian cycle.*

Proof. Consider the vertex

$$0s \in V(P^n), \quad \text{where } s \in V(P^{n-1}) \text{ is defined by } s_d = \begin{cases} 1, & d \text{ even}, \\ 2, & d \text{ odd}, \end{cases}$$

which represents the configuration where disc n is on peg 0 and the smaller discs are distributed on pegs 1 and 2 according to parity. Define similarly

$$3s \in V(P^n)$$

as the state where disc n is on peg 3 and the smaller discs remain in the same configuration.
From Theorem 10, there exists a Hamiltonian path in $0P^{n-1}$ from $0s$ to 03^{n-1}. Similarly, by Theorem 9, there exists a Hamiltonian path in $p(n)P^{n-1}$ from $p(n)3^{n-1}$ to $p(n)0^{n-1}$, and again by Theorem 10, there exists a Hamiltonian path in $3P^{n-1}$ from 30^{n-1} to $3s$.
These three Hamiltonian paths can be concatenated via the three edges

$$\{03^{n-1}, p(n)3^{n-1}\}, \quad \{p(n)0^{n-1}, 30^{n-1}\}, \quad \{3s, 0s\},$$

which correspond to legal moves of disc n between the three subgraphs $0P^{n-1}$, $p(n)P^{n-1}$, and $3P^{n-1}$. This yields a closed walk that visits every vertex exactly once, that is, a Hamiltonian cycle in P^n. ☐

Maximum clique

The clique number of Hanoi graphs H_m^n was established in [5], namely $\omega(H_m^n) = m$. In Theorem 12, we show that $\omega(P^n) = \omega(H_3^n) = 3 < \omega(H_4^n)$.

Theorem 12. *Every complete subgraph of P^n, $n \geq 1$, is induced by edges corresponding to moves of a single disc, and this disc is necessarily either disc 1 or disc 2. In particular,*

$$\omega(P^n) = 3. \tag{39}$$

Proof. Consider a vertex s that is adjacent to two vertices s' and s'' via moves of two different discs. Then the positions of these two discs differ between s' and s''. Since two vertices in P^n can only be adjacent if they differ in exactly one coordinate (i.e., in the position of a single disc), it follows that s' and s'' cannot be adjacent. Hence, any complete subgraph must consist of vertices obtained by successive moves of the same disc. This proves the first claim.
We now show that only discs 1 and 2 can generate a maximal clique. In any state $s^{(1)}$ of P^n, disc 1 can move to two different pegs, giving rise to three mutually adjacent vertices (including $s^{(1)}$), forming a 3-clique. Similarly, in any state $s^{(2)}$ where disc 1 is on peg 2, disc 2 can move to two different pegs, again forming a 3-clique.
However, no disc $d \geq 3$ can generate a 3-clique. Indeed, although disc d may be placed on three pegs in the underlying unconstrained four-peg Hanoi graph, in order for disc d to move freely among its two possible target pegs, both of these pegs must be free of all smaller discs. This would require all discs smaller than d to be placed on the forbidden parity peg for disc d (the peg whose parity is opposite to that of disc d). But this is impossible under the parity constraint, since discs 1 and 2 cannot be simultaneously located on the same (forbidden) parity peg. Therefore, no disc $d \geq 3$ can generate a 3-clique.
Consequently, the only maximal complete subgraphs arise from moves of disc 1 or disc 2, each giving a triangle. Hence, $\omega(P^n) = 3$. □

Coloring

Next, in Theorems 13 and 14, we present coloring results, namely the chromatic number and the chromatic index of P^n. The exact values of these two parameters for the Hanoi graphs H_m^n were established in [1, 8].

Theorem 13 (Chromatic number)**.** *For all $n \geq 1$, the chromatic number is*

$$\chi(P^n) = 3. \tag{40}$$

Proof. It is known that $\chi(H_4^n) = 4$ (see [8]). Consider such a proper 4-coloring of H_4^n. Since P^n is obtained from H_4^n by forbidding moves of discs between the two parity pegs 1 and 2, the states connected by such forbidden moves form disjoint pairs that can be merged into a single color class without introducing any monochromatic edges. By identifying the colors of each such pair of states, we obtain a valid 3-coloring of P^n. Hence, $\chi(P^n) \leq 3$.
On the other hand, from Theorem 12, we know that P^n contains a 3-clique. Therefore, $\chi(P^n) \geq 3$. Combining both bounds gives $\chi(P^n) = 3$. □

Since $\omega(P^n) = \chi(P^n)$, then we have the following.

Proposition 17 (Perfect graph)**.** *For all $n \geq 0$, the graph P^n is perfect.*

Theorem 14 (Chromatic index)**.** *For all $n \geq 2$, we have*

$$\chi'(P^n) = \Delta(P^n) = 5. \tag{41}$$

Proof. We know that, for any graph G, we have

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Thus, it suffices to exhibit a proper edge coloring of P^n using exactly $\Delta(P^n) = 5$ colors (see Proposition 10) for $n \geq 3$.
The case $n = 2$ is easily verified directly, as P^2 is 4-regular and $\chi'(P^2) = 4$.
Now assume $n \geq 3$. We define a partition of the edge set as follows:

$$E(P^n) = E^1(P^n) \cup E^2(P^n) \cup E^3(P^n) \cup E^4(P^n) \cup E^5(P^n),$$

where the classes are pairwise disjoint and given by:

$$\begin{aligned}
E^1(P^n) &= \{e \in E(P^n) \mid e \text{ corresponds to a move between pegs 0 and 1}\},\\
E^2(P^n) &= \{e \in E(P^n) \mid e \text{ corresponds to a move between pegs 0 and 2}\},\\
E^3(P^n) &= \{e \in E(P^n) \mid e \text{ corresponds to a move between pegs 0 and 3}\},\\
E^4(P^n) &= \{e \in E(P^n) \mid e \text{ corresponds to a move between pegs 1 and 3}\},\\
E^5(P^n) &= \{e \in E(P^n) \mid e \text{ corresponds to a move between pegs 2 and 3}\}.
\end{aligned}$$

Note that, due to the parity constraint, no edge in P^n corresponds to a move between pegs 1 and 2.
We now claim that each set $E^f(P^n)$, $f \in [5]$, is an induced matching; i.e., no two distinct edges in $E^f(P^n)$ are adjacent. Indeed, suppose two distinct edges $e, e' \in E^f(P^n)$ were adjacent. Then they would both correspond to moves of the same disc between the same two pegs, implying $e = e'$, a contradiction. Thus, the edges in each $E^f(P^n)$ form a matching.
Therefore, we may assign a single color to each $E^f(P^n)$, for $f \in [5]$. This yields a proper edge coloring of P^n using exactly 5 colors.
Since $\Delta(P^n) = 5$ for $n \geq 3$ (Proposition 10), we conclude that $\chi'(P^n) = \Delta(P^n)$. Combined with the case $n = 2$, the result follows for all $n \geq 2$. $\square$

7 Conclusion and Discussion

In this work, we introduced the parity-constrained Tower of Hanoi problem on four pegs and developed four optimal move sequences corresponding to distinct objectives. We established a coupled recursive framework, derived higher-order recurrences, and provided explicit closed-form expressions. Our asymptotic analysis showed that all sequences exhibit a half-exponential growth of order $\Theta((\sqrt{2})^n)$, placing the problem strictly between the classical three-peg and four-peg versions. We further associated the problem with a new family of graphs P^n, and fully characterized their structural properties, including vertex count, edge count, degrees, connectivity, planarity, Hamiltonicity, clique number, and chromatic parameters.
Several extensions of this work are possible. From a graph-theoretic perspective, the family $(P^n)_{n\geq 0}$ invites further investigation of structural invariants such as spectral parameters, distance-based indices (e.g., Wiener, Hosoya, and Harary indices), domination numbers, and metric dimensions. From a generalization viewpoint, one may impose more elaborate constraint classes, such as modular restrictions beyond parity or multiple grouped parity conditions. A broader direction is to extend the model to p pegs with k restricted clusters of discs, investigating how the number and configuration of constrained pegs affect optimality and graph topology. Such generalizations promise to deepen the connection between recursive optimization, constrained state transitions, and graph-structural properties.

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