TETRAGONAL MODULAR QUOTIENTS $X_0^*(N)$

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ABSTRACT. Let N be a positive integer. For every $d \mid N$ such that $(d, N/d) = 1$ there exists an Atkin-Lehner involution w_d of the modular curve $X_0(N)$. The curve $X_0^*(N)$ is a quotient curve of $X_0(N)$ by $B(N)$, the group of all involutions w_d. In this paper we determine all quotient curves $X_0^*(N)$ whose $\mathbb{C}$-gonality is equal to 4. We also determine all curves $X_0^*(N)$ whose $\mathbb{Q}$-gonality is equal to 4 with the exception of level $N = 378$.

1. INTRODUCTION

Let C be a smooth projective curve over a field k. Studying k-rational points on C and points of small degree over k is an interesting subject in arithmetic geometry. Morphisms $C \to Y$ defined over k are a source of such points. This is one of the reasons to study morphisms from C.

The k-gonality of C, denoted by $\mathrm{gon}_k C$, is the least degree of a non-constant k-rational morphism $f : C \to \mathbb{P}^1$. For curves of genus $g \geq 2$ there exists an upper bound for $\mathrm{gon}_k C$, linear in terms of the genus, see Proposition 2.1.

When C is a modular curve, there also exists a linear lower bound for the $\mathbb{C}$-gonality. This was first proved by Zograf [29]. The constant was later improved by Abramovich [1] and by Kim and Sarnak in Appendix 2 to [19].

The gonality of the modular curve $X_0(N)$ and its quotients has been extensively studied over the years. Ogg [22] determined all hyperelliptic curves $X_0(N)$, Bars [2] determined all bielliptic curves $X_0(N)$, Hasegawa and Shimura determined all trigonal curves $X_0(N)$ over $\mathbb{C}$ and $\mathbb{Q}$, Jeon and Park determined all tetragonal curves $X_0(N)$ over $\mathbb{C}$, and Najman and Orlić [21] determined all curves $X_0(N)$ with $\mathbb{Q}$-gonality equal to 4, 5, or 6, and also determined the $\mathbb{Q}$ and $\mathbb{C}$-gonality for many other curves $X_0(N)$.

Regarding the gonality of the quotients of the curve $X_0(N)$, Furumoto and Hasegawa [8] determined all hyperelliptic quotients, and Hasegawa and Shimura [13, 14, 15] determined all trigonal quotients over $\mathbb{C}$. Jeon [16] determined all bielliptic curves $X_0^+(N)$, Bars, Gonzalez, and Kamel [3] determined all bielliptic quotients of $X_0(N)$ for squarefree levels N, Bars and Gonzalez determine all bielliptic curves $X_0^*(N)$, and Bars, Kamel, and Schweizer [4] determined all bielliptic quotients of $X_0(N)$ for non-squarefree levels N, completing the classification of bielliptic quotients.

The next logical step is to determine all tetragonal quotients of $X_0(N)$. All tetragonal quotients $X_0^{+d}(N) = X_0(N)/\langle w_d \rangle$ over $\mathbb{C}$ and $\mathbb{Q}$ were determined in [24, 23]. Here, we will do the same for the curves $X_0^*(N)$. We also determine all curves $X_0^*(N)$ of genus 4 that are trigonal over $\mathbb{Q}$, thus completing the classification of all $\mathbb{Q}$-trigonal curves $X_0^*(N)$, which has already mostly been done by Hasegawa and Shimura [14].

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Our main results are the following theorems.

Theorem 1.1. *The curve $X_0^*(N) := X_0(N)/w_d$ is of genus 4 and has $\mathbb{Q}$-gonality equal to 3 if and only if*

$$N \in \{137, 148, 172, 201, 202, 214, 219, 242, 253, 254, 260,$$
$$262, 302, 305, 319, 323, 345, 351, 395, 434, 555\}.$$

Theorem 1.2. *Suppose that $N \neq 378$. The curve $X_0^*(N)$ has $\mathbb{Q}$-gonality equal to 4 if and only if N is one of the following 106 values in the following table. Here g is the genus of the curve $X_0^*(N)$.*

N	g
160, 173, 199, 200, 224, 225, 228, 247, 251, 259, 261, 264, 267, 273, 275, 280, 300, 306, 308, 311, 321, 322, 334, 335, 341, 342, 350, 354, 355, 356, 370, 374, 385, 399, 426, 483, 546, 570	4
157, 192, 208, 212, 216, 218, 226, 235, 237, 250, 263, 278, 216, 339, 364, 371, 376, 377, 382, 391, 393, 396, 402, 406, 407, 410, 413, 414, 418, 435, 438, 440, 442, 444, 465, 494, 495, 551, 555, 572, 574, 595, 630, 645, 663, 714, 770, 798, 910	5
163, 197, 211, 223, 244, 265, 269, 272, 274, 297, 301, 332, 336, 340, 359, 369, 375, 447, 470, 564, 598, 620, 627, 670, 780	6
193, 232, 241, 257, 268, 281, 288, 296, 320, 326, 360, 368, 372, 423, 456, 460, 504, 558, 658, 660	7
292, 304, 344, 412, 520, 532, 585, 990	8
328, 528, 560	9

Furthermore, the curve $X_0^*(N)$ has $\mathbb{C}$-gonality equal to 4 and $\mathbb{Q}$-gonality greater than 4 if and only if N is one of the following 16 values in the following table.

N	g
271, 291, 314, 325, 327, 338, 398, 451, 506, 561, 590, 609, 615, 690, 858	6
243	7

The genus 5 curve $X_0^*(378)$ also has $\mathbb{C}$-gonality equal to 4.

We use a variety of methods to prove these results. Each section roughly corresponds to one method. The paper is organized as follows:

- In Section 2 we eliminate all but finitely many levels N for which the curve $X_0^*(N)$ is not $\mathbb{Q}$-tetragonal.
- In Section 3 we determine the $\mathbb{Q}$-gonality of genus 4 curves and some genus 6 curves. We deal with the remaining genus 6 curves in Section 6.

- In Section 4 we give lower bounds on the $\mathbb{Q}$-gonality via $\mathbb{F}_p$-gonality, either by counting points over $\mathbb{F}_{p^n}$ or computing the Riemann-Roch dimensions of degree 4 effective $\mathbb{F}_p$-rational divisors.
- In Section 5 we determine the $\mathbb{C}$-gonality by computing Betti numbers $\beta_{i,j}$. We use them both to prove and disprove the existence of degree 4 morphisms over $\mathbb{C}$ (and $\mathbb{Q}$) to $\mathbb{P}^1$.
- In Section 6 we first use `Magma` to construct degree 4 rational morphisms to $\mathbb{P}^1$ by finding a degree 4 rational divisor $D \geq 0$ with Riemann-Roch dimension 2. In the second part of the section we construct quotient maps from $X_0^*(N)$. We use them to construct degree 4 rational maps to $\mathbb{P}^1$, but also to give lower bound on the $\mathbb{Q}$-gonality for some levels N.
- In Section 7 we combine all these results to obtain proofs of Theorems 1.1 and 1.2. After that we give a short discussion about the only remaining level for the $\mathbb{Q}$-gonality, $N = 378$.

A lot of the results in this paper rely on `Magma` computations. The version of `Magma` used in the computations is V2.28-3. This is important because, for example, the function X0Nstar() was extensively used for obtaining models of curves $X_0^*(N)$. This function is not present in the version V2.27-1, for example. The codes that verify all computations in this paper can be found on

`https://github.com/orlic1/gonality_X0_star`.

All computations were performed on the Euler server at the Department of Mathematics, University of Zagreb with a Intel Xeon W-2133 CPU running at 3.60GHz and with 64 GB of RAM.

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`https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves`.

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2. Preliminaries

We first give a very important result that we will use throughout the paper. It was briefly mentioned in the Introduction.

Proposition 2.1 ([25, Proposition A.1.]). *Let X be a curve of genus g over a field k.*

(i) If L is a field extension of k, then $\mathrm{gon}_L(X) \leq \mathrm{gon}_k(X)$.
(ii) If k is algebraically closed and L is a field extension of k, then $\mathrm{gon}_L(X) = \mathrm{gon}_k(X)$.
(iii) If $g \geq 2$, then $\mathrm{gon}_k(X) \leq 2g - 2$.
(iv) If $g \geq 2$ and $X(k) \neq \emptyset$, then $\mathrm{gon}_k(X) \leq g$.
(v) If k is algebraically closed, then $\mathrm{gon}_k(X) \leq \frac{g+3}{2}$.
(vi) If $\pi : X \to Y$ is a dominant k-rational map, then $\mathrm{gon}_k(X) \leq \deg \pi \cdot \mathrm{gon}_k(Y)$.
(vii) If $\pi : X \to Y$ is a dominant k-rational map, then $\mathrm{gon}_k(X) \geq \mathrm{gon}_k(Y)$.

In order to determine all $\mathbb{Q}$ and $\mathbb{C}$-tetragonal curves $X_0^*(N)$, we need to provide an upper bound for possible levels N. We use Abramovich's lower bound on the $\mathbb{C}$-gonality of modular curves [1, Theorem 0.1] with Kim and Sarnak's constant $\lambda = \frac{2^{15}}{325}$ [19, Appendix 2]. This result was more clearly stated in [17].

Theorem 2.2. [17, Theorem 1.2.] *Let X_Γ be the algebraic curve corresponding to a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ of index*

$$D_\Gamma = [\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma].$$

If X_Γ is d-gonal, then $D_\Gamma \leq \frac{2^{15}}{325} d$.

Corollary 2.3. *The curve $X_0^*(N)$ is not $\mathbb{C}$-tetragonal for $N \geq 12906$.*

Proof. If the curve $X_0^*(N)$ is tetragonal. then we have a composition map of degree $4 \cdot 2^{\omega(N)}$ ($\omega(N)$ is the number of different prime divisors of N)

$$X_0(N) \xrightarrow{2^{\omega_N}} X_0^*(N) \xrightarrow{4} \mathbb{P}^1.$$

Since the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is equal to $\psi(N) = N \prod_{q|N}(1 + \frac{1}{p}) \geq N$, we have the following inequality:

$$N \leq \psi(N) \leq \frac{2^{15}}{325} \cdot 4 \cdot 2^{\omega(N)}.$$

We now have 2 cases.

- $\omega(N) \geq 6$: In this case we have

$$\psi(N) \geq \prod_{q|N}(q+1) \geq 3 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdot 14 \cdot 18^{\omega(N)-6} > \frac{2^{15}}{325} \cdot 4 \cdot 2^{\omega(N)}.$$

  The last inequality can be easily checked by hand. Thus in this case we have a contradiction.
- $\omega(N) \leq 5$: In this case we have

$$N \leq \psi(N) \leq \frac{2^{15}}{325} \cdot 4 \cdot 2^5 < 12906.$$

□

We are now left with only finitely many levels N to consider. We can further reduce the number of possible levels such that $X_0^*(N)$ is $\mathbb{Q}$-tetragonal with Ogg's inequality [22, Theorem 3.1], stated in simpler form in [12, Lemma 3.1].

Lemma 2.4 (Ogg)**.** *Let p be a prime not dividing N. Then the number of $\mathbb{F}_{p^2}$ points on $X_0(N)$ is at least*

$$L_p(N) := \frac{p-1}{12}\psi(N) + 2^{\omega(N)}.$$

Here $\psi(N) = N\prod_{q|N}(1+\frac{1}{q})$ is the index of the congruence subgroup $\Gamma_0(N)$, and $\omega(N)$ is the number of different prime divisors of N.

Lemma 2.5. [21, Lemma 3.5] *Let $C/\mathbb{Q}$ be a curve, p a prime of good reduction for C, and q a power of p. Suppose $\#C(\mathbb{F}_q) > d(q+1)$. Then $\mathrm{gon}_{\mathbb{Q}}(C) > d$.*

If N is a prime power, then we have $X_0^+(N) = X_0^*(N)$. Since all tetragonal curves $X_0^+(N)$ were determined in [24], we can skip the proof for these levels. From now on we will therefore suppose that N is not a prime power.

We may also eliminate all levels N such that $g(X_0^*(N)) \leq 3$ because such curves have $\mathbb{Q}$-gonality at most 3. The curve $X_0^*(N)$ is of genus 4 if and only if

$$\begin{aligned} N \in \{&148, 160, 172, 200, 201, 202, 214, 219, 224, 225, 228, 242, 247, 253, 254, 259, 260, 261,\\ &262, 264, 267, 273, 275, 280, 300, 302, 305, 306, 308, 319, 321, 322, 323, 334, 335, 341,\\ &342, 345, 350, 351, 354, 355, 366, 370, 374, 385, 395, 399, 426, 434, 483, 546, 555, 570\}. \end{aligned}$$

After eliminating all hyperelliptic and trigonal levels N with $g \geq 5$ (these are $N \in \{253, 327, 302, 323, 351, 555\}$, other levels are prime powers - [11, 14]), as well as levels N that do not satisfy Lemmas 2.4 and 2.5 (with the least $p \nmid N$), there are 553 possible curves $X_0^*(N)$ of $\mathbb{Q}$-gonality 4 with genus $g \geq 5$. The largest levels out of them are

$$6930, 6510, 6090, 5460, 4830, 4620, 4290, 3990, 3570, 3360, 3150, 2940, 2730, 2560, \ldots$$

Also, we can eliminate levels $N \in \{2520, 2940, 3150, 3360, 4620, 4830, 5460, 6090, 6510, 6930\}$ with Abramovich's bound (Theorem 2.2). We are left with 543 levels N we need to check. The largest of them are

$$3990, 3570, 2730, 2580, \ldots$$

We will solve all of them (except $N = 378$) in the following sections. A complete list of these levels can be found on Github.

3. Curves of genus 4 and 6

In this section we determine the $\mathbb{Q}$ and $\mathbb{C}$-gonality of all curves $X_0^*(N)$ of genus 4 and some curves of genus 6. This is done with `Magma` functions Genus4GonalMap() and Genus6GonalMap(). We deal with the rest of genus 6 curves in Section 6 where we give an explicit map to $\mathbb{P}^1$ instead of just proving its existence.

Proposition 3.1. *The curve $X_0^*(N)$ has $\mathbb{Q}$-gonality equal to 3 for*

$$N \in \{148, 172, 201, 202, 214, 219, 242, 253, 254, 260, 262, 302, 305, 319, 323, 345, 351, 395, 434, 555\}.$$

The curve $X_0^*(N)$ has $\mathbb{C}$-gonality equal to 3 and $\mathbb{Q}$-gonality equal to 4 for

$$\begin{aligned} N \in \{&160, 200, 224, 225, 228, 247, 259, 261, 264, 267, 273, 275, 280, 300, 306, 308, 321,\\ &322, 334, 335, 341, 342, 350, 354, 355, 356, 370, 374, 385, 399, 426, 483, 546, 570\}. \end{aligned}$$

Proof. All these curves are of genus 4. We use the `Magma` function Genus4GonalMap() to obtain a degree 3 morphism to $\mathbb{P}^1$ (it exists due to Proposition 2.1 (v)). If the curve is $\mathbb{Q}$-trigonal, this morphism will be defined over $\mathbb{Q}$. Otherwise, the function gives a morphism over a quadratic or a biquadratic field [5] (even though the trigonal map can always be defined over a number field of degree ≤ 2). □

Proposition 3.2. *The curve $X_0^*(N)$ has $\mathbb{Q}$-gonality equal to 4 for the following 12 levels*

$$N \in \{265, 274, 297, 301, 369, 375, 447, 470, 564, 598, 627, 670\}.$$

The curve $X_0^*(N)$ has $\mathbb{C}$-gonality equal to 4 and $\mathbb{Q}$-gonality greater than 4 for the following 14 levels

$$N \in \{291, 314, 325, 327, 338, 398, 451, 506, 561, 590, 609, 615, 690, 858\}.$$

Proof. All these curves are of genus 6. We use the `Magma` function Genus6GonalMap() to obtain a degree 4 morphism to $\mathbb{P}^1$ (it exists due to Proposition 2.1 (v)). This morphism is defined over $\mathbb{Q}$ for all levels N in the first set.

For all levels N in the second set the function Genus6GonalMap() tells us that $X_0^*(N)$ is neither bielliptic nor isomorphic to a smooth plane quintic. Hence it has only finitely many gonal maps up to equivalence [10] and the function gives us the map defined over a minimal degree number field [5]. For all these levels N that degree 4 map is not defined over $\mathbb{Q}$, hence these curves are not $\mathbb{Q}$-tetragonal. However, they are $\mathbb{C}$-tetragonal due to Proposition 2.1 (v) (we know that they are not trigonal from [14]). □

We will construct degree 4 rational morphisms from genus 5 curves in a different way, by finding degree 4 effective divisors with Riemann-Roch dimension 2. This will be discussed further in Section 6.

4. $\mathbb{F}_p$-gonality

If $C/\mathbb{Q}$ is a curve and p a prime of good reduction for C, then we have an inequality

$$\mathrm{gon}_{\mathbb{F}_p}(C) \leq \mathrm{gon}_{\mathbb{Q}}(C).$$

$\mathbb{F}_p$ gonality is a powerful tool for giving a lower bound on the $\mathbb{Q}$-gonality. The reason for that is that there are only finitely many $\mathbb{F}_{p^n}$ points on $\mathbb{C}$ for any $n \in \mathbb{N}$. Hence we can compute Riemann-Roch spaces of all degree d $\mathbb{F}_p$-rational divisors $D \geq 0$.

This section solves the majority of levels listed in Section 2, 288 levels N.

Proposition 4.1. *The $\mathbb{F}_p$-gonality of $X_0^*(N)$ is at least 5 for the following 153 values of N:*

N	p	N	p	N	p	N	p	N	p	N	p	N	p	N	p
388	3	394	3	400	3	417	5	422	3	424	3	432	5	436	3
452	3	453	2	458	3	464	3	466	3	482	3	485	3	486	5
488	3	489	2	493	2	500	3	507	2	513	2	515	2	517	2
533	2	535	2	536	3	543	2	553	2	575	2	579	2	583	2
589	2	600	7	612	5	616	3	621	2	624	5	629	2	666	5
672	5	678	5	680	3	684	5	696	5	697	2	702	5	708	5
720	7	726	5	728	5	730	3	732	5	738	5	740	3	741	2
742	3	744	5	748	3	750	7	754	3	756	5	760	3	762	5

765	2	774	5	786	5	792	5	795	2	804	5	806	3	810	7
812	3	816	5	819	5	822	5	826	5	828	5	834	5	836	3
840	11	846	5	852	3	854	3	874	3	876	5	882	5	884	3
888	5	890	3	894	5	903	2	906	5	912	5	918	5	938	3
942	5	954	5	957	2	960	7	962	5	969	2	978	5	984	5
986	3	1038	5	1050	11	1062	5	1080	7	1092	5	1116	5	1170	7
1218	5	1230	7	1254	5	1260	11	1290	7	1302	5	1320	7	1326	5
1330	11	1386	5	1410	7	1428	5	1430	7	1470	11	1482	5	1518	5
1530	7	1554	5	1560	7	1590	7	1596	5	1638	5	1650	7	1680	11
1710	7	1770	7	1806	5	1830	7	1860	7	1890	11	1914	5	1938	5
1950	7	1980	7	2010	7	2070	7	2130	7	2310	13	2730	11	3570	11
3990	11														

Proof. Using `Magma`, we compute that there are no functions of degree ≤ 4 in $\mathbb{F}_p(X_0^*(N))$. We are able to reduce the number of degree 4 divisors that need to be checked by noting the following: If there exists a function f over $\mathbb{F}_p$ of a certain degree and if $\#X_0^*(N)(\mathbb{F}_p) > d(p+1)$, then there exists $c \in \mathbb{F}_p$ such that the function $g(x) := \frac{1}{f(x)-c}$ has the same degree and its polar divisor contains at least d $\mathbb{F}_p$-rational points, see [21, Lemma 3.1].

Therefore, if $\#X_0^*(N)(\mathbb{F}_p) > p+1$, it is enough to check divisors of the form $1+1+1+1$ and $1+1+2$, and if $\#X_0^*(N)(\mathbb{F}_p) > 2(p+1)$, it is enough to just check divisors of the form $1+1+1+1$. □

The computations for some higher levels N took several days to finish. For example, it took 2.5 days for $N = 3990$ and 7 days for $N = 3570$. The reason for that is that these curves are of high genus ($g(X_0^*(3990)) = 23, g(X_0^*(3570)) = 19$) and there are many $\mathbb{F}_{11}$-rational divisors of degree 4 that must be checked.

It also takes some time to find the model of $X_0^*(N)$ for higher levels N. It took 4 hours to find the model of $X_0^*(3990)$ with the function X0Nstar(). Other methods were unable to even find a model in a reasonable time.

We can also get a lower bound on the $\mathbb{Q}$-gonality with Lemma 2.5 by counting $\mathbb{F}_{p^n}$ points on the curve without actually computing the $\mathbb{F}_p$-gonality.

Proposition 4.2. *The $\mathbb{F}_p$-gonality of $X_0^*(N)$ is at least 5 for the following 135 values of N:*

N	q	$\#X_0^*(N)(\mathbb{F}_q)$	N	q	$\#X_0^*(N)(\mathbb{F}_q)$	N	q	$\#X_0^*(N)(\mathbb{F}_q)$
448	9	42	472	9	42	477	4	21
484	9	43	496	9	44	508	9	46
514	9	47	538	9	41	544	9	46
548	9	42	549	4	21	554	3	18
556	9	51	562	9	47	566	9	49
567	4	21	568	9	44	576	25	120
578	9	45	584	9	42	586	9	47
592	9	44	597	4	23	603	4	22
604	9	54	605	4	22	614	9	50

633	4	25	635	4	22	637	4	21
639	4	22	649	4	21	657	4	23
667	9	43	669	4	22	679	9	48
681	4	28	685	4	21	700	9	44
703	9	43	707	4	23	713	4	22
721	4	21	725	4	21	731	4	21
737	4	22	745	4	26	749	4	26
755	4	24	763	4	24	767	4	21
779	4	23	781	4	25	791	4	23
793	4	25	799	4	21	803	4	25
817	4	22	820	9	42	825	4	23
830	9	45	850	9	41	851	4	24
868	9	42	880	9	45	885	4	21
902	9	42	915	4	22	920	9	47
936	25	106	940	9	47	945	4	25
946	9	43	950	9	47	952	9	42
970	9	42	975	4	23	988	9	44
994	9	44	1002	25	108	1005	4	21
1008	25	108	1010	9	48	1014	25	114
1015	9	42	1022	9	48	1023	4	21
1026	25	110	1030	9	47	1034	9	43
1035	4	22	1054	9	42	1056	25	112
1065	4	22	1066	9	42	1071	4	22
1074	25	105	1085	4	21	1086	25	112
1095	4	23	1098	25	120	1102	9	41
1104	25	112	1105	4	22	1113	4	24
1118	9	51	1128	25	122	1131	4	23
1146	25	112	1158	25	111	1173	4	24
1182	25	119	1194	25	124	1200	49	213
1206	25	116	1209	4	23	1221	4	23
1235	4	21	1265	4	22	1295	4	23
1309	4	25	1540	9	43	1610	9	45
1716	25	110	1722	25	107	1785	4	21
1794	25	105	1974	25	115	2040	49	230
2046	25	111	2190	49	204	2280	49	236
2370	49	206	2490	49	222	4290	49	221

Proof. Using `Magma`, we calculate the number of $\mathbb{F}_q$ points on $X_0^*(N)$. It is now easy to check that $\#X_0^*(N)(\mathbb{F}_q) > 4(q+1)$ in all these cases.

For the majority of these levels we used the function FpnpointsforQuotientcurveX0NWN(), available on

https://github.com/FrancescBars/Magma-functions-on-Quotient-Modular-Curves/blob/main/funcions.m.

It computes the newforms f such that the corresponding abelian variety A_f is (op to isogeny) in the decomposition of the Jacobian $J_0^*(N)$ and, after that, the characteristic polynomial

$$P(x) = \prod_f (x^2 - a_p(f)x + p) = \prod_1^{2g} (x - \alpha_i).$$

Here g is the genus of $X_0^*(N)$. The number of $\mathbb{F}_{p^n}$-points on $X_0^*(N)$ is then equal to $p^n + 1 - \sum_1^{2g} \alpha_i^n$. For some non-squarefree levels N this function fails to give the correct decomposition and returns a higher number of points than the correct one. In these cases we found the number of $\mathbb{F}_q$-points via the model of the curve. More details regarding the code are on Github.

Most levels were computed very fast, with some higher levels taking $5 - 10$ minutes and $N = 4290$ taking 30 minutes. □

5. Betti numbers

Graded Betti numbers $\beta_{i,j}$ are helpful when determining the $\mathbb{C}$-gonality of a curve. We will use the indexation of Betti numbers as in [18, Section 1.]. The results we mention can be found there and in [21, Section 3.1.]. In this section we solve the majority of remaining levels, 135 levels N.

Definition 5.1. For a curve X and divisor D of degree d, g_d^r is a subspace V of the Riemann-Roch space $L(D)$ such that $\dim V = r + 1$.

Therefore, we want to determine whether $X_0^*(N)$ has a g_4^1. Green's conjecture relates graded Betti numbers $\beta_{i,j}$ with the existence of g_d^r.

Conjecture 5.2 (Green, [9])**.** *Let X be a curve of genus g. Then $\beta_{p,2} \neq 0$ if and only if there exists a divisor D on X of degree d such that a subspace g_d^r of $L(D)$ satisfies $d \leq g-1$, $r = \ell(D) - 1 \geq 1$, and $d - 2r \leq p$.*

The "if" part of this conjecture has been proven in the same paper.

Theorem 5.3 (Green and Lazarsfeld, Appendix to [9])**.** *Let X be a curve of genus g. If $\beta_{p,2} = 0$, then there does not exist a divisor D on X of degree d such that a subspace g_d^r of $L(D)$ satisfies $d \leq g - 1$, $r \geq 1$, and $d - 2r \leq p$.*

Corollary 5.4 ([24, Corollary 3.20])**.** *Let X be a curve of genus $g \geq 5$ with $\beta_{2,2} = 0$. Suppose that X is neither hyperelliptic nor trigonal. Then $\mathrm{gon}_{\mathbb{C}}(X) \geq 5$.*

We can say something more about the curve from the value of the Betti number $\beta_{2,2}$.

Theorem 5.5 ([28, Theoremw 4.1, 4.4])**.** *Let $C \subseteq \mathbb{P}^{g-1}$ be a reduced irreducible canonical curve over $\mathbb{C}$ of genus $g \geq 7$. Then $\beta_{2,2}$ has one of the values in the following table.*

$\beta_{2,2}$	$(g-4)(g-2)$	$\binom{g-2}{2} - 1$	$g - 4$	0
linear series	$\exists g_3^1$	$\exists g_6^2$ or g_8^3	$\exists$ a single g_4^1	no g_4^1

Notice that the indexation of Betti numbers is different here and in [28], where they are indexed as $\beta_{2,4}$.

The existence of a single g^1_4 is useful as the following result tells us that in that case we have a rational morphism of degree ≤ 4 to $\mathbb{P}^1$.

Theorem 5.6 ([26, Theorem 5])**.** *Let C/k be a smooth projective curve. Suppose that for a fixed r and d there is only one g^r_d, giving a morphism $f : C_{k^{\text{sep}}} \to \mathbb{P}^r_{k^{\text{sep}}}$. Then there exists a Brauer-Severi variety $\mathcal{P}$ defined over k together with a k-morphism $g : C \to \mathcal{P}$ such that $g \oplus_k k^{\text{sep}} : C_{k^{\text{sep}}} \to \mathcal{P}_{k^{\text{sep}}} \cong \mathbb{P}^r_{k^{\text{sep}}}$ is equal to f.*

Corollary 5.7. *Let $C/\mathbb{Q}$ be smooth projective curve such that there is only one g^1_d. Then there exists a conic $\mathcal{P}$ defined over $\mathbb{Q}$ and a degree d rational map $g : C \to \mathcal{P}$. If additionally C has at least one rational point, then $\mathcal{P}$ is $\mathbb{Q}$-isomorphic to $\mathbb{P}^1$.*

Proof. In dimension 1 Brauer-Severi varieties are conics. The degree of f (and of g) cannot be $\leq d-1$ since g^1_d is unique. Therefore, it is of degree d. If $C(\mathbb{Q}) \neq \emptyset$, then $\mathcal{P}(\mathbb{Q}) \neq \emptyset$ and it is isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$. □

Proposition 5.8. *The $\mathbb{C}$-gonality of $X^*_0(N)$ is at least 5 for the following 127 values of N. Here g is the genus of $X^*_0(N)$.*

N	g
298, 309, 324, 358, 363, 365, 411, 425, 437, 445, 446, 450, 474, 478, 490, 492, 498, 501, 518, 519, 525, 527, 530, 550, 582, 623, 636, 638, 646, 671, 710, 861, 870, 897, 924	7
333, 346, 356, 362, 381, 403, 405, 408, 415, 427, 468, 480, 505, 511, 534, 540, 545, 552, 559, 573, 580, 581, 606, 651, 654, 665, 682, 715, 759, 782, 805, 814, 930, 966, 1001, 1020, 1155, 1190	8
352, 384, 386, 387, 392, 404, 428, 441, 454, 459, 469, 471, 473, 475, 481, 497, 502, 516, 522, 524, 526, 531, 537, 539, 542, 588, 591, 594, 602, 610, 611, 618, 622, 642, 644, 650, 655, 689, 693, 695, 705, 735, 777, 790, 855, 860, 935, 987, 1045, 1110, 1122, 1140, 1365	9
416	10

Proof. For all these levels we compute that $\beta_{2,2} = 0$. We know from [14] that these curves have $\mathbb{C}$-gonality at least 4. Thus by Corollary 5.4 they are not tetragonal over $\mathbb{C}$.

The computations for $g = 7$ levels took only several seconds, for $g = 8$ up to several minutes, and for $g = 9$ they could take around 1 hour. The computation for the genus 10 level $N = 416$ took around 4 hours.

In the code Betti numbers are computed with the `Magma` function BettiNumber(_,2,4). The indexation of $\beta_{i,j}$ in that function is different from that in this paper. It is the same indexation as in [27] and [28], and the reader can consult [27, Table 1] to check this result.

One of the requirements of Theorem 5.5 was that the curve must be canonical. The function X0Nstar() gives a canonical model, although not Petri's model. The provided model consists of cubics instead of quadrics, as mentioned in Section 3. □

Proposition 5.9. *The curve $X^*_0(N)$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to 4 for the following 8 values of N:*

N	g	$\beta_{2,2}$
$320, 326, 368, 658$	7	3
$304, 585$	8	4
$528, 560$	9	5

Proof. For these levels we computed that $\beta_{2,2} = g - 4$. Hence there exists a unique g^1_4 by Theorem 5.5. Since curves $X_0^*(N)$ have a rational cusp, Corollary 5.7 now tells us that we have a rational map to $\mathbb{P}^1$ of degree 4. Hence all these curves have $\mathbb{Q}$ and $\mathbb{C}$-gonality 4 as they are not trigonal by [14]. □

The reason why we computed Betti numbers only for curves of genus $g \leq 9$ (with the exception of $N = 416$) is the Tower theorem.

Theorem 5.10 (The Tower Theorem)**.** *Let C be a curve defined over a perfect field k such that $C(k) \neq 0$ and let $f : C \to \mathbb{P}^1$ be a non-constant morphism over $\overline{k}$ of degree d. Then there exists a curve C' defined over k and a non-constant morphism $C \to C'$ defined over k of degree d' dividing d such that the genus of C' is $\leq (\frac{d}{d'} - 1)^2$.*

Corollary 5.11. [21, Corollary 4.6. (ii)] *Let C be a curve defined over $\mathbb{Q}$ with $\mathrm{gon}_{\mathbb{C}}(C) = 4$ and $g(C) \geq 10$ and such that $C(\mathbb{Q}) \neq \emptyset$. Then $\mathrm{gon}_{\mathbb{Q}}(C) = 4$.*

Therefore, for curves $X_0^*(N)$ of genus $g \geq 10$ it is enough to prove that $\mathbb{Q}$-gonality is greater than 4 and the $\mathbb{C}$-gonality will also be greater than 4.

6. Morphisms $X_0^*(N) \to Y$

In this section we construct rational morphisms from curves $X_0^*(N)$ and use them to obtain gonality bounds for the remaining 93 levels N. We first present results that prove the existence of degree 4 rational morphisms to $\mathbb{P}^1$.

Proposition 6.1. *The curve $X_0^*(N)$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to 4 for the following 24 levels*

$$N \in \{218, 226, 235, 237, 250, 278, 339, 382, 391, 393, 402, 406,$$
$$407, 413, 418, 435, 438, 465, 494, 551, 555, 574, 595, 645\}.$$

Proof. All these curves have genus 5. Using `Magma` we found a rational divisor that is a sum of 4 rational points and has Riemann-Roch dimension equal to 2. We searched for rational points using the `Magma` function PointSearch(). □

Proposition 6.2. *The curve $X_0^*(N)$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to 4 for the following 11 levels*

$$N \in \{288, 371, 377, 410, 423, 442, 663, 714, 770, 798, 910\}.$$

Proof. The curve $X_0^*(423)$ has genus 7 (and $\beta_{2,2} = 9$), other curves in the list have genus 5. For these levels there is no rational divisor of the form $1 + 1 + 1 + 1$ and Riemann-Roch dimension ≥ 2. Therefore, we had to search for higher degree points.

We searched for quadratic points $(x_0, \ldots, x_k)$ by intersecting the curve $X_0^*(N)$ with hyperplanes of the form

$$b_0x_0 + b_1x_1 + b_2x_2 = 0,$$

where $b_0, \ldots, b_k \in \mathbb{Z}$ are coprime and chosen up to a certain bound, a similar idea as in [6, Section 3.2]. The main idea is that the first 3 coordinates of a quadratic points must be linearly dependent over $\mathbb{Z}$.

We constructed a scheme from the intersection of $X_0^*(N)$ with hyperplanes $b_0x_0 + b_1x_1 + b_2x_2 = 0$ and searched for points using the function PointsOverSplittingField(). After that, we input these points and construct degree 4 divisors out of them, as in Proposition 6.1. More details are in the code.

In all of these cases we found a function of degree 4 lying in the Riemann-Roch space of a divisor of the form $1 + 1 + 2$, that is, $P_1 + P_2 + Q + \sigma(Q)$, where $P_1, P_2 \in X_0^*(N)(\mathbb{Q})$, and Q is one of the quadratic points we found. □

We now move on to the morphisms from $X_0^*(N)$ induced by involutions.

Proposition 6.3 ([11, Proposition 1], [4, Proposition 4.15])**.** *Let N be an integer divisible by 8 and let $2^\nu \mid\mid N$. Put $S_\mu = \begin{bmatrix} 1 & \frac{1}{\mu} \\ 0 & 1 \end{bmatrix}$. Then $V_2 = S_2 w_{2^\nu} S_2$ defines an involution on $X_0(N)$ defined over $\mathbb{Q}$ that commutes with all Atkin-Lehner involutions w_r.*

Hence, if $8 \mid N$, then V_2 is an involution of $X_0^*(N)$ defined over $\mathbb{Q}$. Since we can construct degree 2 morphisms from involutions, we have a degree 2 rational map $X_0^*(N) \to X_0^*(N)/\langle V_2 \rangle$.

Proposition 6.4 ([11, Proposition 1], [4, Proposition 4.15])**.** *Let N be an integer such that $9 \mid\mid N$. Put $S_\mu = \begin{bmatrix} 1 & \frac{1}{\mu} \\ 0 & 1 \end{bmatrix}$. Then $V_3 = S_3 w_9 S_3^2$ defines an involution on $X_0(N)$ and $X_0^*(N)$ defined over $\mathbb{Q}(\sqrt{-3})$. Morover. if V_3 is an involution of $X_0(N)/W$, then it is defined over $\mathbb{Q}$ if and only if $w_9 \in W$.*

Hence, if $9 \mid\mid N$, then V_3 is an involution of $X_0^*(N)$ defined over $\mathbb{Q}$ and have a degree 2 rational map $X_0^*(N) \to X_0^*(N)/\langle V_3 \rangle$. A noticeable difference from V_2 is that V_3 does not necessarily commute with all Atkin-Lehner involutions w_r. Thus it does not need to be an involution on all quotients of $X_0(N)$.

We can compute the genera of these quotients by counting the newforms f with A_f in the decomposition of the Jacobian $J_0^*(N)$. We use the following results. Lemma 6.5 describes how to lift newforms to a higher level with desired Atkin-Lehner action and Propositions 6.6, 6.7 tell us how to calculate the genera of quotients $X_0^*(N)/\langle V_2 \rangle, X_0^*(N)/\langle V_3 \rangle$.

Lemma 6.5 ([11, Lemma 1])**.** *Let M be a positive integer. Let M_0 be a positive divisor of M and let d be a positive divisor of $\frac{M}{M_0}$. For a prime divisor p of M define integers α, β, γ by*

$$p^\alpha \mid\mid M,\ p^{\alpha-\beta} \mid\mid M',\ p^\gamma \mid\mid d.$$

If $f(\tau) \in S_2^0(M')$ is a newform on $\Gamma_0(M')$, then f lifts to an eigenform for all w_m^M with $m \mid\mid M$. In particular, if $f|w_{p^{\alpha-\beta}}^{M'} = \lambda_p' f$ $(= \pm f)$ and if $\beta > 2\gamma$ (resp. $\beta = 2\gamma$), then

$$f(d\tau) \pm p^{\beta-2\gamma}\lambda_p' f(p^{\beta-2\gamma}d\tau) \text{ (resp. } f(d\tau))$$

becomes an eigenform for $w^M_{p^{\alpha-\beta}}$ with eigenvalue ± 1 (resp. λ'_p).

Now we will consider maps $X_0^*(N) \to X_0^*(N)/\langle V_2\rangle, X_0^*(N)/\langle V_3\rangle$. For simplicity we will write $f^{(d)}(\tau) = f(d\tau)$ in the following results.

Proposition 6.6 ([11, Proposition 2])**.** *Let N be a positive integer such that $8 \mid N$. Let N' be a positive divisor of N and let d be a positive divisor of $\frac{N}{N'}$. Define integers α, β, γ by*

$$2^\alpha \mid\mid N,\ 2^{\alpha-\beta} \mid\mid N',\ 2^\gamma \mid\mid d$$

so that $N = 2^\alpha M$ and $N' = 2^{\alpha-\beta}M'$ for odd integers $M' \mid M$- Let f be a newform on $\Gamma_0(N')$ such that $f|w^{N'}_{2^{\alpha-\beta}} = \lambda f$, and put

$$g^{(d)} = f^{(d)} \pm 2^{\beta-2\gamma}\lambda f^{(2^{\beta-2\gamma}d)}.$$

(i) If $\alpha - \beta \geq 2$, then

$$\begin{cases} g^{(d)}|V_2 = -g^{(d)} & \text{if } \beta > \gamma = 0, \\ g^{(d)}|V_2 = +g^{(d)} & \text{if } \beta - \gamma > \gamma > 0, \\ f^{(d)}|V_2 = \lambda f^{(d)} & \text{if } \beta = 2\gamma. \end{cases}$$

(ii) If $\alpha - \beta = 1$, then

$$\begin{cases} (g^{(d)} + \lambda g^{(2d)})|V_2 = -(g^{(d)} + \lambda g^{(2d)}) & \text{if } \beta > \gamma = 0, \\ g^{(d)}|V_2 = +g^{(d)} & \text{if } \beta - \gamma > \gamma > 0, \\ f^{(d)}|V_2 = \lambda f^{(d)} & \text{if } \beta = 2\gamma. \end{cases}$$

Proposition 6.7 ([11, Proposition 3])**.** *Let $N = 9M$ be a positive integer such that $3 \nmid M$. Let M' be a positive divisor of M and let d be a positive divisor of $\frac{M}{M'}$.*

(i) Let f be a newform on $\Gamma_0(9M)$ such that $f|w_9^{9M'} = +f$. Then $f^{(d)}$ is an eigenform of V_3 with eigenvalue $+1$.

(ii) Let f be a newform on $\Gamma_0(3M)$ such that $f|w_3^{3M'} = \lambda f$. Then $f^{(d)} + 3\lambda f^{(3d)}$ is an eigenform of V_3 with eigenvalue -1.

Example 6.8 ([11, Example 2])**.** *Let $n = 360 = 8 \cdot 9 \cdot 5$. A basis of $S_2^*(360)$ is given by (the newforms are given with their LMFDB labels* [20]*)*

$$\begin{aligned} &g_1^{(1)},\ g_1^{(3)},\ \text{where } f_1 = 20.2.a.a, \\ &g_2^{(1)},\ g_2^{(2)},\ \text{where } f_2 = 30.2.a.a, \\ &g_3^{(1)},\ \text{where } f_3 = 36.2.a.a, \\ &g_4^{(1)},\ \text{where } f_4 = 90.2.a.b, \\ &g_5^{(1)},\ \text{where } f_5 = 120.2.a.a. \end{aligned}$$

We can check that these newforms have the correct Atkin-Lehner signs on

`https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/?level_type=divides&level=360&weight=2&char_order=1`

Thus $g(X_0^*(360)) = 7$. By Proposition 6.6 we can see that only $g_2^{(2)}$ and $g_5^{(1)}$ form a basis of $S_2^*(360)^{V2}$. Therefore the curve $X_0^*(360)/\langle V_2\rangle$ is of genus 2.

Example 6.9. *Let $n = 414 = 2 \cdot 9 \cdot 23$. A basis of $S_2^*(414)$ is given by (the newforms are given with their LMFDB labels* [20]*)*

$$g_1^{(1)}, \text{ where } f_1 = 69.2.a.a,$$
$$g_2^{(1)}, \text{ where } f_2 = 138.2.a.a,$$
$$g_3^{(1)}, \text{ where } f_3 = 138.2.a.b,$$
$$g_4^{(1)}, \text{ where } f_4 = 207.2.a.b \text{ (dim} = 2).$$

We can check that these newforms have the correct Atkin-Lehner signs on

`https://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/?level_type=divides&level=414&weight=2&char_order=1`

Thus $g(X_0^*(360)) = 5$. The newforms of level $3 \mid\mid M$ will never lift to eigenforms with eigenvalue $+1$ for V_3 because of Proposition 6.7. Proposition 6.7 also tells us that, due to action of w_9, the two eigenforms $f_4^{(1)}$ form a basis for $S_2^*(414)^{V2}$. Therefore the curve $X_0^*(414)/\langle V_3 \rangle$ is of genus 2*.*

Proposition 6.10. *The curve $X_0^*(N)$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to* 4 *for the following* 14 *values of N. Here* dim *denotes the dimensions of newforms in the table.*

N	*newforms for V_2*	dim	N	*newforms for V_2*	dim
192	*24.2.a.a, 192.2.a.a*	1, 1	208	*26.2.a.b, 208.2.a.b*	1, 1
216	*72.2.a.a, 216.2.a.a*	1, 1	232	*58.2.a.a, 232.2.a.a*	1, 1
272	*34.2.a.a, 272.2.a.a*	1, 1	296	*37.2.a.a, 296.2.a.a*	1, 1
328	*82.2.a.a, 328.2.a.a*	1, 1	336	*42.2.a.a, 112.2.a.a*	1, 1
344	*43.2.a.a, 344.2.a.b*	1, 1	360	*30.2.a.a, 120.2.a.a*	1, 1
376	*376.2.a.b*	2	440	*88.2.a.a, 440.2.a.a*	1, 1
456	*57.2.a.a, 152.2.a.a*	1, 1	520	*65.2.a.a, 520.2.a.a*	1, 1

Proof. For these levels we compute that the genus of $X_0^*(N)/\langle V_2 \rangle$ is equal to 2 as in Example 6.8. The composition map $X_0^*(N) \to X_0^*(N)/\langle V_2 \rangle \to \mathbb{P}^1$ is a degree 4 rational morphism. □

Proposition 6.11. *The curve $X_0^*(N)$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to* 4 *for the following* 6 *values of N. Here* dim *denotes the dimensions of newforms in the table.*

N	*newforms for V_3*	dim	N	*newforms for V_3*	dim
414	*207.2.a.b*	2	495	*99.2.a.a*	1
504	*36.2.a.a, 504.2.a.a*	1, 1	558	*558.2.a.b*	1
630	*315.2.a.c*	2	990	*99.2.a.a, 990.2.a.a*	1, 1

Proof. For these levels we compute that the genus of $X_0^*(N)/\langle V_3\rangle$ is equal to 1 or 2 as in Example 6.9. The composition map $X_0^*(N) \to X_0^*(N)/\langle V_3\rangle \to \mathbb{P}^1$ is a degree 4 rational morphism. □

When $4 \mid\mid N$, the curve $X_0^*(N)$ has a modular involution.

Proposition 6.12 ([11, Proposition 7])**.** *Let $N = 4M$ with M being odd. We have the isomorphism*

$$X_0^*(N) \cong X_0(N)/\left\langle \{w_{p^r}\}_{p|M} \cup S_2 \right\rangle \cong X_0(2M)/\left\langle \{w_{p^r}\}_{p|M} \right\rangle.$$

The first isomorphism is obtained by conjugating $\Gamma_0(N)$ by $S_2 w_4 S_2 = V_2$ and the second by conjugating $\left\langle \{w_{p^r}\}_{p|M} \cup S_2 \cup \Gamma_0(N) \right\rangle$ by $\alpha_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Thus both isomorphisms are defined over $\mathbb{Q}$.

Proposition 6.13. *The curve $X_0^*(N)$ has $\mathbb{Q}$ and $\mathbb{C}$-gonality equal to 4 for the following 18 levels*

$$N \in \{212, 244, 268.292, 316, 332, 340, 364, 372, 396, 412, 444, 460, 532, 572, 620, 660, 780\}.$$

Proof. All these levels are such that $4 \mid\mid N$. Thus we have a degree 2 rational morphism

$$X_0^*(N) \cong X_0(2M)/\left\langle \{w_{p^r}\}_{p|M} \right\rangle \to X_0^*(2M).$$

All these curves $X_0^*(2M)$ are of genus 2 ([11, Remark 1]), so we have a degree 4 rational morphism to $\mathbb{P}^1$. □

Proposition 6.14. *The curve $X_0^*(N)$ has $\mathbb{Q}$-gonality at least 5 for the following 20 levels*

$$N \in \{596.900, 948, 996, 1012, 1032, 1036, 1044, 1068, 1164, 1212,$$
$$1380, 1740, 1848, 1932, 2100, 2220, 2340, 2460, 2580\}.$$

Proof. All these levels are such that $4 \mid\mid N$. Thus we have a degree 2 rational morphism

$$X_0^*(N) \cong X_0(2M)/\left\langle \{w_{p^r}\}_{p|M} \right\rangle \to X_0^*(2M).$$

Moreover, we have proven in Propositions 4.1, 3.2, and 5.8 that these curves $X_0^*(2M)$ are not $\mathbb{Q}$-tetragonal. Therefore, the curves $X_0^*(N)$ are also not $\mathbb{Q}$-tetragonal by Proposition 2.1 (vii). □

7. Proofs of the main theorems

The proof of Theorem 1.1 is given in Proposition 3.1 where we determine the fields of definition of trigonal maps. We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. If N is a prime power, then $X_0^*(N) = X_0^+(N)$. All tetragonal curves $X_0^+(N)$ were determined in [24]. Let us suppose now that N is not a prime power. Furthermore, all hyperelliptic and trigonal curves $X_0^*(N)$ were determined in [11] and [14]. Thus we may suppose that the $\mathbb{C}$-gonality of $X_0^*(N)$ is at least 4.

The gonality of genus 4 curves was determined in Proposition 3.1. We gave the list of all possible tetragonal levels N is Section 2. For levels N of genus $g \geq 5$ in the table of the theorem we prove the existence of a degree 4 rational morphism to $\mathbb{P}^1$ in Propositions 3.2, 5.9, 6.1, 6.2, 6.10, 6.11, and 6.13.

For the remaining levels N we disproved the existence of a degree 4 map as follows. For genus 6 curves we used Proposition 3.2 to prove that their $\mathbb{Q}$-gonality is larger than 4. However, the $\mathbb{C}$-gonality is equal to 4 due to Proposition 2.1 (v).

For curves of genus $g = 7, 8, 9$ and for $N = 416$ ($g = 10$) we used Proposition 5.8 to prove that the $\mathbb{C}$-gonality is larger than 4. For other curves of genus $g \geq 10$ we used Propositions 4.1, 2.5, and 6.14 to prove that their $\mathbb{Q}$-gonality is larger than 4. By Corollary 5.11, their $\mathbb{C}$-gonality is then also larger than 4.

Since the curve $X_0^*(378)$ is of genus 5, Proposition 2.1 (v) tells us that its $\mathbb{C}$-gonality is at most 4. Therefore, it is equal to 4 since it is neither hyperelliptic nor trigonal. □

7.1. $N = 378$**.** The curve $X_0^*(378)$ has degree 4 rational morphisms over $\mathbb{F}_p$ to $\mathbb{P}^1$ for all primes $p \leq 79$ of good reduction. However, we were unable to find a degree 4 rational map from the genus 5 curve $X_0^*(378)$ to $\mathbb{P}^1$ with Propositions 6.1 and 6.2. We found 2 rational points on it using the `Magma` function PointSearch() with the height bound up to 10000000. Hence these are most likely the only rational points on the curve.

Since this genus 5 curve is not trigonal by [11], its Petri's model is the intersection of 3 quadrics

$$2x_1^2 - 210x_2^2 + 2045x_2x_3 - 5578x_3^2 + 138x_1x_4 - 3688x_2x_4 + 20248x_3x_4 - 15144x_4^2 + 352x_1x_5 +$$

$$+9046x_2x_5 - 67294x_3x_5 + 123188x_4x_5 - 259244x_5^2,$$

$$2x_1x_2 - 28x_2^2 + 224x_2x_3 - 587x_3^2 + 6x_1x_4 - 368x_2x_4 + 2189x_3x_4 - 1770x_4^2 + 96x_1x_5 + 718x_2x_5 -$$

$$-6945x_3x_5 + 14446x_4x_5 - 30598x_5^2,$$

$$-x_2^2 + x_1x_3 + 6x_2x_3 - 21x_3^2 - x_1x_4 - 15x_2x_4 + 117x_3x_4 - 114x_4^2 + 13x_1x_5 + 4x_2x_5 - 335x_3x_5 +$$

$$+938x_4x_5 - 2012x_5^2.$$

We can check this with `Magma` code

```
X:=X0Nstar(378);
CanonicalImage(X,CanonicalMap(X));
```

,

The function Genus5GonalMap() yields a degree 4 map to $\mathbb{P}^1$ defined over the quadratic field $\mathbb{Q}(\sqrt{-7})$, although this is not necessarily the smallest possible field. For a general curve of genus 5 ($\mathrm{gon}_{\mathbb{C}} = 4$) there are infinitely many equivalence classes of gonal maps which are parametrized by a plane quintic curve F [5]. This plane quintic is in the output of the function Genus5GonalMap().

We found 2 rational points on F and these 2 are likely the only rational points. If we could find all rational points on F, we could determine if there is a degree 4 rational map $X_0^*(378) \to \mathbb{P}^1$, see [7, Section 3.4]. However, this is generally a difficult problem.

The curve $X_0^*(378)$ could have $\mathbb{Q}$-gonality equal to 4 or 5, although most of the arguments presented here suggest that it is 5.

References

[1] D. Abramovich, *A linear lower bound on the gonality of modular curves*, Internat. Math. Res. Notices, (1996), pp. 1005–1011. 1, 2

[2] F. Bars, *Bielliptic modular curves*, J. Number Theory, 76 (1999), pp. 154–165. 1

[3] F. Bars, J. González, and M. Kamel, *Bielliptic quotient modular curves with N square-free*, J. Number Theory, 216 (2020), pp. 380–402. 1

[4] F. Bars, M. Kamel, and A. Schweizer, *Bielliptic quotient modular curves of $X_0(N)$*, Math. Comp., 92 (2022), pp. 895–929. 1, 6.3, 6.4

[5] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), pp. 235–265. Computational algebra and number theory (London, 1993). 3, 3, 7.1

[6] J. Box, *Quadratic points on modular curves with infinite Mordell-Weil group*, Math. Comp., 90 (2021), pp. 321–343. 6

[7] M. Derickx, B. Mazur, and S. Kamienny, *Rational families of 17-torsion points of elliptic curves over number fields*, in Number theory related to modular curves: Momose memorial volume. Proceedings of the Barcelona-Boston-Tokyo Number Theory Seminar in memory of Fumiyuki Momose, Barcelona, Spain, May 21–23, 2012, Providence, RI: American Mathematical Society (AMS), 2018, pp. 81–104. 7.1

[8] M. Furumoto and Y. Hasegawa, *Hyperelliptic Quotients of Modular Curves $X_0(N)$*, Tokyo J. Math., 22 (1999), pp. 105 – 125. 1

[9] M. L. Green, *Koszul cohomology and the geometry of projective varieties. Appendix: The nonvanishing of certain Koszul cohomology groups (by Mark Green and Robert Lazarsfeld)*, J. Differ. Geom., 19 (1984), pp. 125–167, 168–171. 5.2, 5.3

[10] M. Harrison, *Explicit solution by radicals, gonal maps and plane models of algebraic curves of genus 5 or 6*, J. Symbolic Comput., 51 (2013), pp. 3–21. 3

[11] Y. Hasegawa, *Hyperelliptic Modular Curves $X_0^*(N)$*, Acta Arith., 81 (1997), pp. 369 – 385. 2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8, 6.12, 6, 7, 7.1

[12] Y. Hasegawa and M. Shimura, *Trigonal modular curves*, Acta Arith., 88 (1999), pp. 129–140. 2

[13] ——, *Trigonal modular curves $X_0^{+d}(N)$*, Proc. Japan Acad., Ser. A, 75 (1999), pp. 172–175. 1

[14] ——, *Trigonal modular curves $X_0^*(N)$*, Proc. Japan Acad., Ser. A, 76 (2000), pp. 83–86. 1, 2, 3, 5, 7

[15] ——, *Trigonal quotients of modular curves $X_0(N)$.*, Proc. Japan Acad., Ser. A, 82 (2006), pp. 15–17. 1

[16] D. Jeon, *Bielliptic modular curves $X_0^+(N)$*, J. Number Theory, 185 (2018), pp. 319–338. 1

[17] D. Jeon, C. H. Kim, and E. Park, *On the torsion of elliptic curves over quartic number fields*, J. London Math. Soc. (2), 74 (2006), pp. 1–12. 2, 2.2

[18] D. Jeon and E. Park, *Tetragonal modular curves*, Acta Arith., 120 (2005), pp. 307–312. 5

[19] H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. Appendix 2: Refined estimates towards the Ramanujan and Selberg conjectures (by Henry Kim and Peter Sarnak)*, J. Amer. Math. Soc., 16 (2002), p. 139–183. 1, 2

[20] T. LMFDB Collaboration, *The L-functions and modular forms database*. `http://www.lmfdb.org`, 2021. [Online; accessed 5 February 2021]. 6.8, 6.9

[21] F. Najman and P. Orlić, *Gonality of the modular curve $X_0(N)$*, Math. Comp, (2023), pp. 863–886. 1, 2.5, 4, 5, 5.11

[22] A. P. Ogg, *Hyperelliptic modular curves*, Bull. Soc. Math. France, 102 (1974), pp. 449–462. 1, 2

[23] P. Orlić, *Tetragonal modular quotients $X_0^{+d}(N)$*, J. Number Theory, 276 (2025), pp. 98–114. 1

[24] ——, *Tetragonal modular quotients $X_0^+(N)$*, Acta Arith., 217 (2025), pp. 261–275. 1, 2, 5.4, 7

[25] B. Poonen, *Gonality of modular curves in characteristic p*, Math. Res. Lett., 14 (2007), pp. 691–701. 2.1

[26] J. Roé and X. Xarles, *Galois descent for the gonality of curves*, Math. Res. Lett, 25 (2018), pp. 1567–1589. 5.6

[27] F.-O. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann., 275 (1986), pp. 105–137. 5

[28] F.-O. Schreyer, *A standard basis approach to syzygies of canonical curves*, J. Reine Angew. Math., 421 (1991), pp. 83–123. 5.5, 5, 5

[29] P. G. Zograf, *Small eigenvalues of automorphic laplacians in spaces of parabolic forms*, J. Soviet Math., 36 (1987), pp. 106–114. 1

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