

A gauge invariant Hamiltonian evolution across the black hole horizon in asymptotically AdS spacetimes

Anurag Kaushal,^{*} Naveen S. Prabhakar,[†] and Spenta R. Wadia[‡]
*International Centre for Theoretical Sciences-
Tata Institute of Fundamental Research,
Shivakote, Bengaluru 560089, India.*

We study the quantum dynamics of a probe scalar field in the background of a black hole in AAdS spacetime in the Hamiltonian formulation of general relativity in the maximal slicing gauge. The black hole solution in this gauge is expressed in terms of ‘wormhole coordinates’, a smooth coordinate system with constant time slices that cut across the horizon, and asymptote to the Killing time slices at the boundaries.

The quantum scalar field is expanded in terms of normalized solutions of the Klein-Gordon equation, that are valid at all points in spacetime. The operators that appear in the expansion are in the product space of the CFTs on the two spacetime boundaries, which are by definition gauge invariant under small bulk diffeomorphisms. The entangled Hartle-Hawking (HH) state arises naturally from this construction.

One of our main results is a well defined formula for the time dependent Hermitian Hamiltonian of the probe scalar in the product space of the two CFTs, which describes the time development of operators/states along the maximal slices. This Hamiltonian acting on the HH state creates a state of finite norm. Consequently there is a unitary description of horizon crossing scalar field excitations on top of the HH state. We also present a bulk reconstruction formula that evaluates an order parameter that signals horizon crossing in the boundary theory.

We calculate various bulk Wightman two-point functions on the two-sided BTZ black hole. We recover Hawking’s thermodynamic results in the exterior region when expressed in terms of BTZ coordinates that are related to the wormhole coordinates by a transformation which is singular at the horizon. We compute the two-point function with one insertion in the future/past interior and the other in the exterior. Both are related by a time reflection symmetry and asymptote to a non-zero constant as the coordinate time between the two points becomes large.

CONTENTS

I. Introduction and Summary	2	IV. The scalar field Hamiltonian for evolution along maximal slices	11
II. A review of the maximally sliced two-sided BTZ black hole	4	A. The expression for the Hamiltonian	11
A. The fully extended two-sided BTZ black hole	4	B. Hamiltonian in terms of the CFT modes $a_{\omega q}$, $\tilde{a}_{\omega q}$	12
B. The maximally sliced two-sided BTZ black hole in the wormhole coordinate system	5	C. The time evolution operator	13
C. The diffeomorphism to the Kruskal extension of the BTZ black hole	5	D. The Hamiltonian $\hat{H}(\bar{t})$ is a well-defined operator	13
D. Important features of the maximally sliced two-sided BTZ black hole solution	6	E. Diagonalizing the Hamiltonian	14
E. Large diffeomorphisms	7	F. Action of large diffeomorphisms on the scalar field	14
III. A probe scalar field in the maximally sliced BTZ black hole	8	V. An order parameter that signals horizon crossing	15
A. Klein-Gordon Mode expansions in Regions I and II	9	VI. Scalar field correlation functions in the maximally sliced two-sided BTZ black hole	15
B. The Hartle-Hawking mode functions	10	A. An expression for the scalar field two-point function	15
C. Smooth mode functions	10	B. Two-point function in Region I in BTZ coordinates	16
D. A bulk reconstruction formula in wormhole coordinates	11	C. The two-point Wightman functions for specific insertion points	16
		1. Timelike separated insertions	16
		2. Spacelike separated insertions	17
		D. The boundary limit of the correlators	17
		Acknowledgments	19

^{*} anuragkaushal314@gmail.com

[†] naveen.s.prabhakar@gmail.com

[‡] spenta.wadia@icts.res.in

A. Derivation of the maximally sliced two-sided BTZ black hole solution	19
1. The maximal slicing solution	19
2. Diffeomorphism to the two-sided BTZ black hole	20
3. Diffeomorphism to Kruskal coordinates	22
4. The wormhole coordinate	23
5. A second family of maximal slices	24
B. Probe scalar field is gauge invariant	24
C. Solutions of the Klein-Gordon equation in the BTZ spacetime	25
1. Separation of variables	25
2. Solution in Region I	26
3. Solution in region II	27
4. Solution in Region F	27
5. Solution in Region P	28
6. Analytic continuation to regions F and P	28
7. The Hartle-Hawking modes and Bogoliubov coefficients	30
8. Klein-Gordon normalization of the exterior mode functions	31
9. Klein-Gordon inner product of the Hartle-Hawking modes and the smeared modes	33
D. Bulk reconstruction of local scalar field operator in wormhole coordinates	34
References	35

I. INTRODUCTION AND SUMMARY

The quantum dynamics of matter in the presence of a black hole is an old subject beginning with Hawking's foundational work [1], where he calculated the temperature of the Schwarzschild black hole as a function of its mass, $T_{\text{BH}} = \frac{\hbar}{8\pi M G_N}$. A simpler exposition of the same result was given many years later by Haag and Fredenhagen [2] (also see the review by Witten [3]). This formula plus the first law of thermodynamics leads to the Bekenstein-Hawking formula for the entropy of the black hole, $S = \frac{A_{\text{hor}}}{4G_N \hbar}$. One of the key inputs in these calculations, which involves matter propagating in the black hole background, is the fact that the spacetime is restricted to be outside the horizon of the black hole and described by coordinates (e.g. Schwarzschild), that do not extend beyond the horizon.

One of the important milestones of string theory has been the explanation of black hole thermodynamics and Hawking radiation (for a class of supersymmetric black holes) in terms of the statistical mechanics of microscopic degrees of freedom that are unitary quantum mechanical systems [4, 5]. The AdS/CFT correspondence [6–8] made this explanation precise.

The Ryu-Takayanagi formula and its generalisations [9–13] for the von Neumann entropy enabled a new diagnostic of black hole evaporation in terms of the Page curve for an evaporating black hole in an AdS spacetime that is coupled to an external bath (a quantum field theory) at its boundary [14–19]. These are semi-classical bulk calculations in effective field theory without any reference to the dual degrees of freedom that reside on the boundary of AdS spacetime.

One wonders about the possibility, within the effective field theory description of the bulk, of a *unitary* Hamiltonian description of the crossing of the horizon by a wavepacket built out of a finite number of diffeomorphism invariant matter field excitations on top of the Hartle-Hawking state. In this paper we answer this question in the affirmative. The present work was partly motivated by the work of Leutheusser and Liu [20, 21], where a CFT operator which describes the time evolution of infalling observers in the black hole background was obtained using algebraic QFT methods. Ideas and questions related to infalling observers have been discussed previously in [22–24].

We study the quantum dynamics of a probe scalar field in the background of an eternal two-sided black hole in a spacetime that is asymptotically AdS (AAdS) paying special attention to gauge invariance. Though we present explicit formulas for the case of 2+1 dimensions, our main results are valid for $d + 1$ dimensions with $d > 2$.

We work in the framework of Hamiltonian General Relativity. An important result in mathematical relativity is that AAdS spacetimes can always be foliated by spatial slices that have zero or constant mean curvature [25–30]. In this paper we will work the $K = 0$ gauge, which is also known to maximize the spatial volume in the Lorentzian signature of spacetime; hence the name maximal slicing gauge.

It is possible to transform the two-sided $d + 1$ dimensional AdS-Schwarzschild black hole solution to the maximal slicing gauge, in which the d dimensional spatial slices are cylinders ($\mathbf{R} \times \mathbf{S}^{d-1}$) or *wormholes*. We specialize to $d = 2$ in this paper in which case we work with the two-sided BTZ black hole. We obtain a smooth spacetime metric in this gauge in what we call *wormhole coordinates* $(\bar{t}, x, \varphi)$. The time coordinate $\bar{t}$ labels the maximal slices and has the property that it is linearly related to the asymptotic AdS times at the two AAdS boundaries.[31] The spatial coordinate x is globally defined on the entire spatial slice. The two AAdS boundaries are at $x \rightarrow \pm\infty$, with x behaving as an areal radial coordinate in the asymptotic regions. The coordinate φ is an angular coordinate with period 2π . In higher dimensions, it is replaced by the usual set of angular coordinates for a $d - 1$ dimensional sphere $\mathbf{S}^{d-1}$.

These maximal slices have the following properties:

1. They smoothly cut across the horizon and the lapse function is non-zero, smooth and of one sign on the entire slice, including the bifurcate point. This is to be contrasted with the static BTZ or AdS-

Schwarzschild solution.

2. They asymptote to the usual Schwarzschild slices near the boundaries of the AAdS spacetime.

Section II reviews the maximal slicing gauge and its properties including various coordinate systems: Schwarzschild, Kruskal-Szekeres, areal and wormhole coordinates in detail.

In Section III, we quantize a probe free scalar field of mass m in this background. Using the extrapolate dictionary near the two boundaries, the scalar field can be expressed in terms of generalised free fields in the two dual CFTs associated to the two AAdS boundaries with conformal dimension $\Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + \ell^2 m^2}$. These generalized free field operators in the CFT are, by definition, invariant under small diffeomorphisms in the bulk. It is important to note that this description of the bulk scalar field in terms of gauge invariant generalized free fields on the two boundary CFTs is possible since we are considering a two sided description of the extended eternal black hole solution. If we work with a one sided description with one AAdS boundary, then the gauge invariance of the scalar modes behind the horizon needs to be discussed using gravitational ‘Wilson lines’ extending to the boundary.

We review Unruh’s procedure [32] of obtaining the scalar field solutions in the interior regions of the black hole by analytically continuing the exterior scalar field solutions in the lower-half Kruskal U and V planes. This procedure naturally gives a mode expansion of the scalar field in terms of the Hartle-Hawking modes, and picks out the Hartle-Hawking state as the state annihilated by these modes.

The Hartle-Hawking mode functions $h_{\omega q}$ are labelled by a continuous real parameter ω , which can take both positive and negative values, and a quantized angular momentum mode q . Though the parameter ω is allowed to take arbitrary large values, since we are working in the probe limit of the quantum scalar field, one should (as we will see later) work with ω of the order of η , the surface gravity of the black hole, and even then, only a finite number of such excitations about the Hartle-Hawking state to avoid backreaction effects.

If one looks closely at the intermediate steps of solving the Klein-Gordon equation, it can be seen that the ω -continuum is due to the presence of the null horizon in the black hole solution. Thus, there is a continuous spectrum of scalar field excitations about the Hartle-Hawking state. By the AdS/CFT extrapolate dictionary, this implies that, in the dual CFT, there is a continuum of energies in the neighbourhood the large energy state that is dual to the black hole in the bulk. This continuum is a result of working at leading order in large N in the dual CFT, where it is well-known that the spacing between energy levels in a neighbourhood of a state with energy N^2 goes as e^{-cN^2} , which is zero in the strict large N limit.

It turns out that the Hartle-Hawking modes $h_{\omega q}$, when

expressed in Kruskal-Szekeres coordinates, have derivatives that are singular at the horizon. Hence, as defined, the Hartle-Hawking modes are not smooth solutions of the Klein-Gordon equation in the neighbourhood of the horizons. We remedy this by smearing these modes in the ω -space with Hermite functions and show that the new mode functions are smooth at the horizons. Thus, the final expression for the scalar field that satisfies the Klein-Gordon equation at all spacetime points including at the horizons is expressed as a linear combination of the smeared Hartle-Hawking modes. These are in turn related to the operators of the two boundary CFTs by a Bogoliubov transformation. We also write down a bulk reconstruction formula in terms of two boundary to bulk kernels (corresponding to each boundary) in the wormhole coordinates.

In Section IV we discuss the scalar field Hamiltonian in this background, expressed in terms of the frequency-smeared Hartle-Hawking modes. The Hamiltonian is a time-dependent Hermitian operator that describes the unitary time evolution of operators/states as they traverse the horizon. We argue that it is a well defined operator as it’s action on the Hartle-Hawking state creates a state of finite norm.

The above facts would imply that the Wightman two-point correlators of the scalar field in wormhole coordinates retain information as in a unitary theory analogous to a quantum field theory in Minkowski spacetime. Indeed, we explicitly show that correlation functions in the interior remain finite and non-zero, even at late times.

In Section V we propose an order parameter that signals horizon crossing from the boundary perspective. This order parameter uses a bulk reconstruction formula presented in IIID and the explicit form of the time evolution operator. Such an order parameter has been discussed in [20] in the framework of algebraic QFT. It would be interesting to know the relation between these two constructions.

Section VI discusses the two-point function of the scalar field in the wormhole coordinates for various insertion points in spacetime, including the case in which the points are located on the boundaries of the spacetime.

Hawking’s result of thermal correlation functions in the bulk region outside the horizon is recovered only after we make a coordinate transformation from the wormhole coordinates to Schwarzschild coordinates. The formula for the black hole temperature as a function of its mass is recovered. It is important to note that this coordinate transformation is singular at the horizon.

In the situation where the two point function is for one scalar field insertion on each asymptotic boundary, we recover the result obtained in [33] which has the stark feature of decaying to zero beyond times of the order of the inverse temperature β of the black hole. As explained in [33], this is an artefact of working in the strict large N limit, in which case the dual CFT has a continuous spectrum of energies even though it is defined on a compact space ($\mathbf{S}^{d-1}$). In particular, at finite but large

N , it has been conjectured in [33] that additional saddle points other than the black hole may contribute to the scalar field two point function such that the decay to zero is modified to a small non-decaying answer which is consistent with unitarity of the dual CFT. However, we cannot see these effects in the current work since we are working in the infinite N limit.

It is worth noting that at large but finite N , our description of the scalar field dynamics in the BTZ black hole background would be valid only for times $\bar{t}$ much smaller than the scrambling time t_* . This is because, at times of order t_* , we expect that the given black hole background may not be the dominant saddle point due to backreaction effects becoming important [34]. We can also infer this from the observation that the dual CFT four-point out-of-time-ordered correlator ceases to factorize into the product of two two-point functions at times of order $t_* \sim \frac{\beta}{2\pi} \log N$ [35]. This indicates that the large N (corresponding to small G_N in the bulk) saddle point analysis in the dual CFT and in the bulk is no longer valid at times of order t_* . Since our analysis is in the large N limit, the scrambling time t_* is sent to infinity and the black hole remains the dominant saddle point for all times.

In a future work, it would be nice to study the black hole evaporation in our setup upon coupling the unitary scalar field theory in the bulk to a bath. This can be achieved by connecting the AAdS space to flat space regions similar to the studies in JT gravity [14–19] and the dual SYK model [36–39].

II. A REVIEW OF THE MAXIMALLY SLICED TWO-SIDED BTZ BLACK HOLE

The (AdS)-Schwarzschild black hole can be put in the in the maximal slicing gauge in any dimension $d \geq 2$. This is obtained by starting with a spherically symmetric ansatz for the metric and then solving the Einstein's equations together with the maximal slicing gauge condition, along the lines of [40]. The maximal slicing condition is a local condition on the extrinsic curvature K_{ij} of a given spatial slice:

$$K \equiv g^{ij} K_{ij} = 0, \quad (1)$$

where g^{ij} is the inverse of the spatial metric on the given spatial slice. The maximal slicing condition implies that the local spatial volume is a maximum under arbitrary small deformations of the slice.

In this paper, we mainly work with the maximally sliced $2+1$ dimensional BTZ black hole for concreteness. This solution was obtained in the companion paper [41] by systematically solving Einstein's equations in the Hamiltonian formulation. We also present an independent, self-contained derivation of the maximally sliced BTZ black hole solution in Appendix A. We review the necessary aspects of this solution in this section and refer the reader to Appendix A and the companion paper

[41] for more details. First, we quickly present some well-known facts about the BTZ black hole and its Kruskal extension.

A. The fully extended two-sided BTZ black hole

The BTZ black hole metric is [42, 43]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \quad f(r) = \frac{r^2}{\ell^2} - \frac{M}{2\pi}. \quad (2)$$

The spacetime has an asymptotic AdS boundary as $r \rightarrow \infty$. This metric has a coordinate singularity at the horizon

$$r = R_h = \ell \sqrt{\frac{M}{2\pi}}, \quad (3)$$

at which the time translation Killing vector becomes null. The above metric is valid for the range

$$\text{Region I: } R_h < r < \infty, \quad -\infty < t < \infty, \quad 0 \leq \varphi < 2\pi. \quad (4)$$

To go past the null horizon, one first defines the tortoise coordinate r_* :

$$r_* = \int_{-\infty}^r \frac{d\rho}{f(\rho)} = \frac{1}{2\eta} \log \frac{r - R_h}{r + R_h}, \quad (5)$$

where $\eta = R_h/\ell^2$ is the surface gravity of the black hole. The Kruskal-Szekeres coordinates are defined in terms of the Region I coordinates as

$$U = -\eta^{-1} e^{-\eta(t-r_*)}, \quad V = \eta^{-1} e^{\eta(t+r_*)}. \quad (6)$$

In Region I, we have $U < 0$ and $V > 0$. The past horizon $t \rightarrow -\infty, r \rightarrow R_h$ is described by the surface $V = 0$ whereas the future horizon $t \rightarrow \infty, r \rightarrow R_h$ is described by the surface $U = 0$. The BTZ black hole metric in terms of the Kruskal-Szekeres coordinates is

$$ds^2 = -4 \frac{R_h^2}{\ell^2} \frac{dU dV}{(1 + \eta^2 UV)^2} + R_h^2 \left(\frac{1 - \eta^2 UV}{1 + \eta^2 UV} \right)^2 d\varphi^2. \quad (7)$$

This metric has no singularities at the horizons $UV = 0$, and is valid in the range $|\eta^2 UV| < 1$, with $U, V \in \mathbf{R}$. This is the Kruskal extension of the BTZ black hole which we refer to as the two-sided BTZ black hole. See the left panel of Figure 1 for the Kruskal diagram.

There are four regions corresponding to the different signs of U and V where we can find BTZ-like coordinates in which the metric takes the form (2):

$$\text{I: } (-, +), \quad \text{F: } (+, +), \quad \text{II: } (+, -), \quad \text{P: } (-, -). \quad (8)$$

These four regions are shown in Figure 1. It is also useful to define the Penrose coordinates (p, q) :

$$p = \arctan(\eta V), \quad q = \arctan(\eta U). \quad (9)$$

The Penrose diagram of the fully extended BTZ black hole is displayed in the second panel in Figure 1.

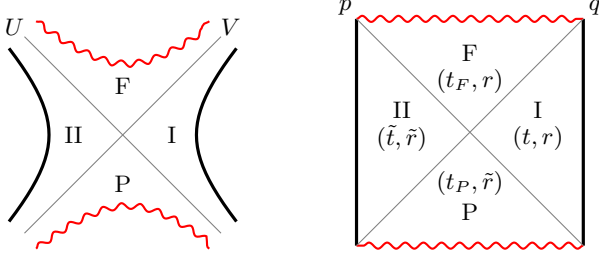


FIG. 1. The Kruskal (left) and Penrose (right) diagrams for the two-sided BTZ black hole solution. The two solid black lines are the asymptotic AdS boundaries, the diagonal gray lines are the event horizons and the wiggly red lines are the singularities. The horizons divide the two-sided black hole into four regions labelled I, II, F and P, each of which have a static Schwarzschild-like coordinate system which is displayed in the Penrose diagram.

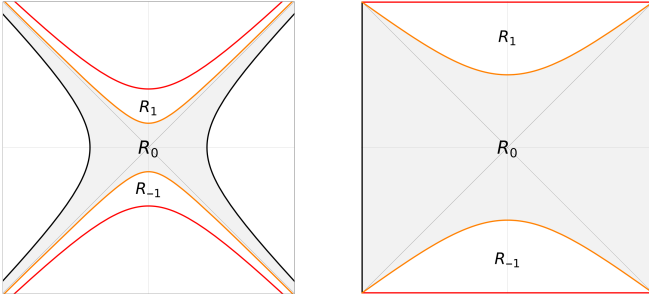


FIG. 2. The region R_0 (shaded gray) covered by the maximal slicing solution is demarcated by the orange lines in the BTZ Kruskal diagram (left) and Penrose diagram (right) of the fully extended BTZ black hole. The three regions R_0 , R_1 and R_{-1} cover the full Penrose diagram. The regions R_1 and R_{-1} (unshaded) are covered by a different family of maximal slices as is explained in the main text.

B. The maximally sliced two-sided BTZ black hole in the wormhole coordinate system

As mentioned in the introduction to this section, we derive the maximally sliced two-sided BTZ black hole solution in Appendix A. The spatial slices of this solution are *wormholes* or cylinders $\mathbf{R} \times \mathbf{S}^1$ connecting two AAdS boundaries.

This solution is presented in the *wormhole* coordinate system $(\bar{t}, x, \varphi)$ with ranges

$$-\infty < \bar{t} < \infty, \quad -\infty < x < \infty, \quad 0 \leq \varphi \leq 2\pi, \quad (10)$$

where φ is an angular coordinate with period 2π . The coordinate $\bar{t}$ is a time coordinate which labels spatial maximal slices, and x is a spatial coordinate which we call the

wormhole coordinate. The two AAdS boundaries are situated at $x = \pm\infty$ and the centre of the wormhole throat is at $x = 0$. The wormhole coordinate x has the virtue that it behaves as an areal radial coordinate in the region near the boundaries $x \sim \pm\infty$.

The spacetime metric for the maximally sliced two-sided BTZ black hole is

$$ds^2 = -N(x, \bar{t})^2 d\bar{t}^2 + \frac{\ell^2 (dx + N^x(x, \bar{t}) d\bar{t})^2}{x^2 + R_+(\bar{t})^2 - R_-(\bar{t})^2} + (x^2 + R_+(\bar{t})^2) d\varphi^2, \quad (11)$$

with

$$N^x(x, \bar{t}) = \frac{1}{x} \left(N(x, \bar{t}) T(\bar{t}) + R_+ \dot{R}_+ \right), \quad (12)$$

$$N(x, \bar{t}) = \frac{x \sqrt{x^2 + R_+^2 - R_-^2}}{\ell \sqrt{x^2 + R_+^2}} \times \left(1 + \dot{T} \ell^3 \int_x^\infty \frac{dy}{y^2} \frac{\sqrt{y^2 + R_+^2}}{(y^2 + R_+^2 - R_-^2)^{3/2}} \right), \quad (13)$$

$$R_\pm(T(\bar{t})) = \ell \sqrt{\frac{M}{4\pi}} \left[1 \pm \sqrt{1 - \left(\frac{4\pi T(\bar{t})}{M\ell} \right)^2} \right]^{1/2}, \quad (14)$$

and the function $T(\bar{t})$ is determined by the equation

$$\bar{t} = -T(\bar{t}) \ell^3 \int_0^\infty \frac{dy}{(y^2 - R_-^2) \sqrt{(y^2 + R_+^2)(y^2 + R_+^2 - R_-^2)}}. \quad (15)$$

The above metric is time dependent as is expected in absence of a global timelike Killing vector. Further it is completely smooth in the coordinate range (10).

C. The diffeomorphism to the Kruskal extension of the BTZ black hole

The smooth spacetime metric (11) is diffeomorphic to a region R_0 the two-sided BTZ black hole metric (7) written in Kruskal-Szekeres coordinates. The region R_0 is shown in Figure 2. It is bounded by the two branches of the hyperbola

$$UV = \frac{1}{\eta^2} \frac{\sqrt{2} - 1}{\sqrt{2} + 1}, \quad (16)$$

one branch each in Region F and Region P. The diffeomorphism between wormhole coordinates and Kruskal-Szekeres in the region R_0 is given by

$$\begin{aligned}
U(\bar{t}, x) &= \pm \eta^{-1} \exp \left(-\eta \bar{t} - R_h \int_x^\infty \frac{dy}{(y^2 - R_-^2) \sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+(\bar{t})^2 - R_-(\bar{t})^2}} + y \right) \right), \\
V(\bar{t}, x) &= \pm \eta^{-1} \exp \left(\eta \bar{t} + R_h \int_x^\infty \frac{dy}{(y^2 - R_-^2) \sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+(\bar{t})^2 - R_-(\bar{t})^2}} - y \right) \right).
\end{aligned} \tag{17}$$

The signs in front of the expressions on the right hand side correspond to the four regions in the two-sided black hole, see Figure 1 and equation (8) for the choice of signs for each region. The $\int$ symbol indicates that one has to use the Cauchy principal value to evaluate the integral when the integration range includes the pole at $y = \pm R_-(\bar{t})$. This happens when $x < R_-$ or $x < -R_-$. The diffeomorphism expressions are derived in Appendix A.

The coordinate transformation between the wormhole $(\bar{t}, x, \varphi)$ coordinates and the Kruskal-Szekeres (U, V, φ) coordinates is smooth and well-defined within the limit slices $\bar{t} = \pm\infty$ which are given by the equation (16) in Kruskal-Szekeres coordinates. It can be checked that the matrix of first partials, i.e., the Jacobian matrix is completely non-singular in the domain R_0 of the Kruskal diagram:

$$\det \begin{pmatrix} \frac{\partial U}{\partial \bar{t}} & \frac{\partial U}{\partial x} & 0 \\ \frac{\partial V}{\partial \bar{t}} & \frac{\partial V}{\partial x} & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq 0. \tag{18}$$

We compute this Jacobian near the horizon where it is most suspect to vanish or diverge. In particular, near the right future horizon ($U = 0, V > 0$), we can expand (17) around $x = R_-$ (location of the $U = 0$ horizon in the wormhole coordinates) to obtain

$$U \approx -e^{-\eta \bar{t}}(x - R_-), \quad V \approx \eta^{-1} e^{\left(\eta \bar{t} + \frac{R_h^2 x}{2R_+^2 R_-} \right)}. \tag{19}$$

From these we can calculate the Jacobian to be

$$\begin{aligned}
J &= \left(\frac{\partial U}{\partial \bar{t}} \frac{\partial V}{\partial x} - \frac{\partial U}{\partial x} \frac{\partial V}{\partial \bar{t}} \right) \\
&\approx e^{\frac{R_h^2 x}{2R_+^2 R_-}} \left[\left((x - R_-) + \eta^{-1} \dot{R}_- \right) \frac{R_h^2}{2R_+^2 R_-} \right. \\
&\quad \left. + \left(1 - \frac{R_h^2 x (2\dot{R}_+ R_- + R_+ \dot{R}_-)}{2\eta R_+^3 R_-^2} \right) \right].
\end{aligned} \tag{20}$$

This clearly remains finite and non-zero even on the horizon at $x = R_-$ at all times,

$$J(x = R_-) = e^{\frac{R_h^2}{2R_+^2}} \left(1 - \frac{R_h^2 \dot{R}_+}{\eta R_+^3} \right), \tag{21}$$

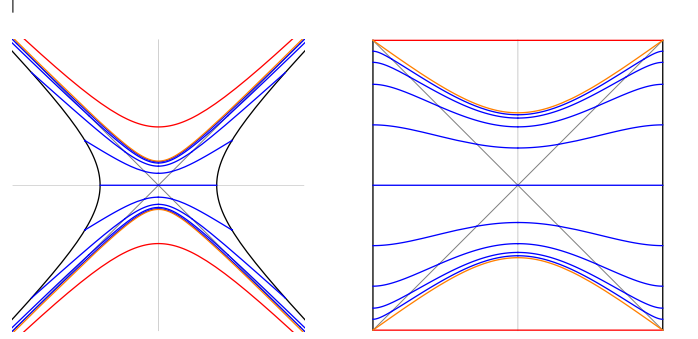


FIG. 3. Plots of constant $\bar{t} = 0, \pm 0.5, \pm 1, \pm 1.5, \pm 2$ slices, i.e., maximal slices (solid blue lines) in the BTZ Kruskal diagram (left) and Penrose diagram (right) of the fully extended BTZ black hole. As $\bar{t} \rightarrow \infty$, the final slice approaches $R_\infty = \ell \sqrt{M/4\pi}$ (orange curve) from below.

including at the bifurcate point at $\bar{t} = 0$,

$$J(x = 0, \bar{t} = 0) = \sqrt{e}. \tag{22}$$

We have used the fact that $R_+(0) = R_h$ and $\dot{R}_+(0) = 0$.

D. Important features of the maximally sliced two-sided BTZ black hole solution

Let us note some important properties of the maximally sliced two-sided BTZ spacetime metric in wormhole coordinates.

1. The constant $\bar{t}$ slices are maximal slices in the two-sided BTZ black hole geometry and go from the left asymptotic boundary to the right asymptotic boundary while smoothly cutting across the two horizons located at $x = \pm R_-(\bar{t})$, see Figure 3. The maximal slicing time $\bar{t}$ matches the boundary time $t(-\bar{t})$ on the right (left) AAdS boundary.
2. *Cosmic censorship.* The regions R_1 and R_{-1} in Figure 2 are not covered by the spacetime solution (11). Thus these maximal slices do not cover a portion of the black hole spacetime near the singularities. A type of ‘cosmic censorship’ is automatic in the maximal slicing gauge. This implies that the dynamics in the dual conformal field theories that is dictated by the maximal slicing solution in Region R_0 cannot access the singularity. In numerical

relativity, the exclusion of the region near the singularity is often a desired feature (see, for instance, [44, 45]).

We also note that a second family of maximal surfaces cover the regions R_1 and R_{-1} in Figure 2. These maximal slices lie completely inside the horizons and do not meet the AAdS boundaries. See Appendix A 5 for a description of this second family.

3. *Collapse of the lapse.* An important feature of maximal slicing gauge is that the while the lapse N is finite and positive (including at $x = 0$ at any finite $\bar{t}$), it vanishes as $\bar{t} \rightarrow \infty$. This has been noted before in the literature [40] and is referred to as ‘collapse of the lapse’. At late times, the lapse decays exponentially,

$$N(\bar{t} \gg \eta^{-1}, x) \approx N_0(x) e^{-\sqrt{2}\eta\bar{t}} \quad (23)$$

where the function $N_0(x)$ decreases for increasing values of $|x|$ and can be calculated numerically.

E. Large diffeomorphisms

Recall the maximal slicing gauge $K = g_{ij}K^{ij} = 0$ where g_{ij} is the metric on the spatial slice and K_{ij} is the extrinsic curvature. Demanding that the maximal slicing gauge is preserved under time evolution, i.e., $\partial_{\bar{t}}K = 0$, gives the following differential equation for the lapse N :

$$(D^2 - K_{ij}K^{ij} + 2\Lambda)N = 0. \quad (24)$$

The lapse $N(\bar{t}, x)$ given in (12) is the unique solution to the above differential equation that satisfies the AAdS boundary conditions

$$N(\bar{t}, x) \sim \frac{|x|}{\ell}, \quad \text{as } x \rightarrow \pm\infty. \quad (25)$$

The above boundary conditions are not the most general ones consistent with spherical symmetry on the spatial slice, and can be generalized to include two independent constants c and $\tilde{c}$:

$$N(\bar{t}, x) \sim \begin{cases} c \frac{x}{\ell}, & x \rightarrow +\infty, \\ -\tilde{c} \frac{x}{\ell}, & x \rightarrow -\infty. \end{cases} \quad (26)$$

We reserve the notation $N(\bar{t}, x)$ for the lapse corresponding to $c = \tilde{c} = 1$ which gives the symmetric maximal slicing foliation of the two-sided BTZ black hole discussed previously. The solution with general c and $\tilde{c}$ then correspond to large diffeomorphisms which preserve the maximal slicing gauge, which translate the time $\bar{t}$ on the left boundary by the amount $\tilde{c}$ and the time t on the right boundary by an amount c . We denote this solution by $\zeta_{\perp}$. There is a corresponding shift vector $(\zeta^x, \zeta^{\varphi})$ that can be obtained by solving the equations for the shift as in Appendix A. The large diffeomorphism $(\zeta_{\perp}(\bar{t}, x), \zeta^i(\bar{t}, x))$ on a given time slice labelled by $\bar{t}$ is given by

$$\begin{aligned} \zeta_{\perp}(\bar{t}, x) &= \begin{cases} \frac{x\sqrt{x^2 + R_+^2 - R_-^2}}{\ell\sqrt{x^2 + R_+^2}} \left[c + \ell^3 \frac{c + \tilde{c}}{2} \dot{T} \int_x^{\infty} \frac{dy}{y^2} \frac{\sqrt{y^2 + R_+^2}}{(y^2 + R_+^2 - R_-^2)^{3/2}} \right] & x > 0 \\ -\frac{x\sqrt{x^2 + R_+^2 - R_-^2}}{\ell\sqrt{x^2 + R_+^2}} \left[\tilde{c} + \ell^3 \frac{c + \tilde{c}}{2} \dot{T} \int_{-x}^{\infty} \frac{dy}{y^2} \frac{\sqrt{y^2 + R_+^2}}{(y^2 + R_+^2 - R_-^2)^{3/2}} \right] & x < 0 \end{cases}, \\ \zeta^x(\bar{t}, x) &= \frac{1}{x} (\zeta_{\perp}(\bar{t}, x)T(\bar{t}) + \frac{c+\tilde{c}}{2} R_+ \partial_{\bar{t}} R_+), \\ \zeta^{\varphi}(\bar{t}, x) &= 0. \end{aligned} \quad (27)$$

The diffeomorphism in (27) is given in terms of normal ($\zeta_{\perp}$) and tangential deformations (ζ^i) of the spatial surface Σ , which is well-suited for the Hamiltonian formulation. The spacetime vector field ζ^{μ} , $\mu = \bar{t}, x, \varphi$ which implements the large diffeomorphism $(\zeta_{\perp}, \zeta^i)$ in the max-

imally sliced spacetime (11) is given by

$$\begin{aligned} \zeta^{\bar{t}}(\bar{t}, x) &= N(\bar{t}, x)^{-1} \zeta_{\perp}(\bar{t}, x), \\ \zeta^i(\bar{t}, x) &= \zeta^i(\bar{t}, x) - N(\bar{t}, x)^{-1} N^i(\bar{t}, x) \zeta_{\perp}(\bar{t}, x), \end{aligned} \quad (28)$$

where N and N^i are the lapse and shift of the maximally sliced spacetime (11). The vector field above can be used to see the action of the large diffeomorphism correspond-

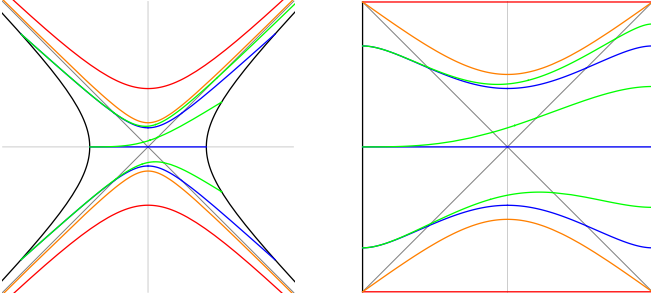


FIG. 4. The action of the large diffeomorphism corresponding $c = 1$, $\tilde{c} = 0$, which translations points on the right boundary but fixes points on the left boundary. The blue curves are the slices of the symmetric foliation and the green curves attached to the blue curves on the left boundary are their images under the large diffeomorphism.

ing to given values of c , $\tilde{c}$ on the maximal slices of the spacetime (11). We display the action of the large diffeomorphism corresponding to $c = 1$, $\tilde{c} = 0$ in Figure 4. One can generalize the boundary conditions (26) to include angular dependence through a factor of $\cos n\varphi$ or $\sin n\varphi$ and obtain an infinite family of large diffeomorphisms with the above angular dependence. These are expected to generate the Virasoro algebra which corresponds to the algebra of large diffeomorphisms of AAdS₃ spacetimes.

III. A PROBE SCALAR FIELD IN THE MAXIMALLY SLICED BTZ BLACK HOLE

A probe scalar field in a fixed pure gravity background is *gauge invariant* under small diffeomorphisms at quadratic order in perturbation theory in $\kappa^2 = 16\pi G_N$. [46] This is because (1) the metric and scalar degrees of freedom are decoupled at quadratic order in perturbation theory, and (2) after the background diffeomorphisms are completely fixed, the diffeomorphisms at $\mathcal{O}(\kappa)$ act purely in the metric fluctuation sector without involving the scalar field. See Appendix B for a more elaborate discussion.

The Hamiltonian of a massive scalar field in the background of an AdS-Schwarzschild black hole is

$$H = \int_{\Sigma} d^d x \left(N \left(\frac{1}{2\sqrt{g}} \pi_{\phi}^2 + \frac{1}{2} \sqrt{g} g^{ij} (\partial_i \phi \partial_j \phi + m^2 \phi^2) \right) + N^i \pi_{\phi} \partial_i \phi \right), \quad (29)$$

with the appropriate lapse and shift corresponding to maximal slicing of the black hole. The Hamiltonian equations of motion are

$$\begin{aligned} \partial_t \phi &= \frac{N}{\sqrt{g}} \pi_{\phi} + N^i \partial_i \phi, \\ \partial_t \pi_{\phi} &= \partial_i (N \sqrt{g} g^{ij} \partial_j \phi) - N \sqrt{g} m^2 \phi + \partial_i (N^i \pi_{\phi}). \end{aligned} \quad (30)$$

Note: As is usual in field theory, we work in the Heisenberg picture. Thus, the operators $\phi(x, t)$, $\pi_{\phi}(x, t)$ as well as the Hamiltonian H are all in the Heisenberg picture and the states are defined for all times. After promoting the fields ϕ, π_{ϕ} to operators, the equations above coincide with the Heisenberg equations of motion.

Plugging in the expression for π_{ϕ} in terms of $\dot{\phi}$, we get the Klein-Gordon equation

$$\begin{aligned} \partial_t (-N^{-1} \sqrt{g} (\partial_t \phi - N^i \partial_i \phi)) + \partial_i (N^{-1} \sqrt{g} N^i \partial_t \phi) \\ + \partial_i (N \sqrt{g} (g^{ij} - N^{-2} N^i N^j) \partial_j \phi) - N \sqrt{g} m^2 \phi = 0. \end{aligned} \quad (31)$$

It is expected that the initial value problem is well-posed for the above linear hyperbolic differential equation when suitable normalizable boundary conditions are imposed on the scalar field [47–58]. We describe these boundary conditions briefly. In the asymptotic region, say, near the right boundary ($r \rightarrow \infty$), the spacetime metric approaches that of AdS_d. The Klein-Gordon equation (31) has two linearly independent solutions $\phi_{\pm}$ in this region which behave in the $r \rightarrow \infty$ limit as

$$\phi_{\pm} \sim r^{-\Delta_{\pm}}, \quad \text{with} \quad \Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + \ell^2 m^2}. \quad (32)$$

When $m^2 > 0$, it can be shown that the solution ϕ_{+} is normalizable with respect to the Klein-Gordon norm whereas the solution ϕ_{-} is not (see below and Appendix C). The behaviour near the left boundary ($\tilde{r} \rightarrow \infty$) is analogous to the above.

Solving the Klein-Gordon equation directly in worm-hole coordinates, analytically, turns out to be hard due to the time dependence of the coefficients in the differential equation. However we note that given Cauchy initial data for the scalar field, there is no problem in solving the differential equation numerically since everything is smooth and well-defined. To solve the Klein-Gordon equation analytically, we adopt a different strategy as described below.

As is well-known, there is typically no preferred complete set of solutions for the Klein-Gordon equation in a general curved spacetime due a lack of time translation symmetry. Depending on the context, we can choose to expand the scalar field in an appropriate complete set of mode functions. Here, we are interested in the AdS/CFT interpretation of scalar field propagation in the maximally sliced BTZ black hole. Hence, it is useful to have an expansion of the scalar field in terms of modes which are relevant for AdS/CFT.

These mode functions are obtained by quantizing the scalar field separately in the two exterior regions of the AdS-Schwarzschild black hole in the static coordinate system which possess a timelike Killing symmetry. We then analytically continue them to the interior regions [32] (see also [22] for a more recent discussion) to obtain the Hartle-Hawking mode functions which are defined not only in the exterior regions but also in the interior regions F and P (refer to figure 1). These mode

functions are most naturally expressed in terms of the Kruskal-Szekeres coordinate system of the fully extended black hole. Then, to get a solution of the Klein-Gordon equation in maximal slicing coordinates, we do a diffeomorphism from the Kruskal-Szekeres coordinates to the maximal slicing coordinates.

One important point is that the mode functions obtained in the Killing time slicing oscillate infinitely rapidly with constant magnitude as one approaches the horizon. This implies that their derivatives with respect to the Kruskal-Szekeres coordinates are singular at the horizon. Thus we see that the expansion of the scalar field in terms of the Hartle-Hawking modes does not actually solve the Klein-Gordon equation on the horizons. This will present a problem when we consider derivatives of the scalar fields on a given maximal slice – in computing the Hamiltonian for instance – since the slice cuts across the horizon. However, we show that smearing the Hartle-Hawking modes with Hermite functions $\psi_n(\omega/\eta)$ in the frequency domain smoothens out these singularities. The smeared modes are labelled by the discrete index n of the Hermite functions. This discreteness also is natural from the point of view of quantization of scalar field in AAdS spacetimes since the scalar field is effectively in a gravitational box.

We then express the quadratic Hamiltonian in terms of the discrete scalar field expansion that is smooth everywhere and obtain a well-defined Hamiltonian operator which evolves the scalar field along the maximal slices. This operator describes the evolution of wavepackets which smoothly cross the horizon and hence describes infalling observers.

Finally we also calculate the two-point Wightman functions in the Hartle-Hawking state in the wormhole coordinates. For concreteness, from now on we specialize to $d = 2$, i.e., the BTZ black hole, where we can do explicit computations.

Before solving the Klein-Gordon equation, we note that for the BTZ black hole, there is another degree of freedom from the pure gravity sector at the same order in perturbation theory as the scalar field. This corresponds to the fluctuations in reduced phase space variables (m, T) or equivalently $(Q = 4\pi^2 \frac{T}{m^2}, P = m)$, see the companion paper [41]. [59] We can expand

$$Q = Q_0 + \kappa \delta Q, \quad P = P_0 + \kappa \delta P \quad (33)$$

where $\kappa = \sqrt{16\pi G_N}$. Here (Q_0, P_0) form the on-shell black hole solution while δQ and δP are fluctuations on the black hole background. Since we have already solved the constraints and obtained the gauge invariant reduced phase space, there are no further constraints on these fluctuations. Quantization of $(\delta Q, \delta P)$ is expected to lead to $L^2(\mathbf{R})$, the space of square-integrable functions on the real line. However note that the scalar field does not mix with this degree of freedom at quadratic order in the semi-classical approximation.

A. Klein-Gordon Mode expansions in Regions I and II

To obtain the most general solution of the Klein-Gordon equation (31), we start with the solutions in Regions I and II solved in terms of the original BTZ coordinates in these regions:

$$\phi(t, r, \varphi) = \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(a_{\omega q} F_{\omega q}(t, r, \varphi) + \text{c.c.} \right), \quad (34)$$

where $F_{\omega q}(t, r, \varphi)$ are solutions of the Klein-Gordon equation in Region I which satisfy the normalizability condition $F_{\omega q}(r) \sim r^{-\Delta_+}$ in the limit $r \rightarrow \infty$. (See equations (C3) to (C14) in Appendix C for the detailed expressions.) The $F_{\omega q}$ have the following Klein-Gordon inner products:

$$(F_{\omega q}, F_{\omega' q'})_{\text{KG}} = 4\pi\omega\delta(\omega - \omega'), \quad (F_{\omega q}^*, F_{\omega' q'})_{\text{KG}} = 0. \quad (35)$$

where the Klein-Gordon inner product is defined on the constant t surfaces with $R_h < r < \infty$ in Region I. There is a completely analogous story in Region II with the scalar field mode expansion

$$\phi(\tilde{t}, \tilde{r}, \varphi) = \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(\tilde{a}_{\omega q} F_{\omega q}^*(\tilde{t}, \tilde{r}, \varphi) + \text{c.c.} \right), \quad (36)$$

where we have an $F_{\omega q}^*$ in contrast to Region I since the time $\tilde{t}$ in Region II runs in the opposite sense to t in Region I.

The ladder operators $a_{\omega q}, a_{\omega q}^\dagger$ satisfy the commutation relations

$$[a_{\omega q}, a_{\omega' q'}^\dagger] = [\tilde{a}_{\omega q}, \tilde{a}_{\omega' q'}^\dagger] = \delta(\omega - \omega')\delta_{qq'}, \quad (37)$$

with all other commutators being zero. Throughout this paper we have put $\hbar = 1$. The boundary operator $\mathcal{O}$ in the dual CFT associated to the right boundary is obtained by the extrapolate limit:

$$\begin{aligned} \mathcal{O}(t, \varphi) &= \lim_{r \rightarrow \infty} r^{\Delta_+} \phi(t, r, \varphi), \\ &= \frac{R_h^{\Delta_+}}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \frac{1}{N_{\omega q}} (a_{\omega q} e^{-i\omega t - iq\varphi} + \text{c.c.}). \end{aligned} \quad (38)$$

Similarly, the operator in the CFT associated to the left boundary $\tilde{r} \rightarrow \infty$ is

$$\begin{aligned} \tilde{\mathcal{O}}(\tilde{t}, \varphi) &= \lim_{\tilde{r} \rightarrow \infty} \tilde{r}^{\Delta_+} \phi(\tilde{t}, \tilde{r}, \varphi), \\ &= \frac{R_h^{\Delta_+}}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \frac{1}{N_{\omega q}} (\tilde{a}_{\omega q} e^{i\omega \tilde{t} + iq\varphi} + \text{c.c.}). \end{aligned} \quad (39)$$

B. The Hartle-Hawking mode functions

The solutions in Region I and II above can be analytically continued into the Regions F and P using analyticity in the lower-half U and V planes following Unruh [32]. This process is described in detail in the Appendix C 6. The resulting mode functions are the Hartle-Hawking mode functions $h_{\omega q}(U, V, \varphi)$, in terms of which the scalar field mode expansion is

$$\phi(U, V, \varphi) = \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_{-\infty}^{\infty} d\omega \left(c_{\omega q} h_{\omega q}(U, V, \varphi) + \text{c.c.} \right), \quad (40)$$

see equations (C63), (C26), (C39) in Appendix C 6 for detailed expressions for the mode function $h_{\omega q}$. In contrast to the $F_{\omega q}$, the Hartle-Hawking mode functions are supported everywhere in the fully extended BTZ black hole. The Klein-Gordon inner products are

$$(h_{\omega q}, h_{\omega' q'})_{\text{KG}} = 2\pi R_h \delta(\omega - \omega'), \quad (h_{\omega q}^*, h_{\omega' q'})_{\text{KG}} = 0. \quad (41)$$

Here, the spatial slice on which the Klein-Gordon inner product is defined is the union of the $t = 0$ and $\bar{t} = 0$ slices in Regions I and II respectively. This coincides with the $U + V = 0$ slice in the Kruskal coordinates and the $\bar{t} = 0$ in maximal slicing coordinates.

The operators $c_{\omega q}$ for $\omega < 0$ are sometimes written as $\tilde{c}_{\omega q}$:

$$\tilde{c}_{\omega q} = c_{-\omega, -q}, \quad \text{for } \omega > 0. \quad (42)$$

The $c_{\omega q}$ and $\tilde{c}_{\omega q}$ are related to the Region I and II operators by the Bogoliubov transformations

$$\begin{aligned} c_{\omega q} &= \frac{a_{\omega q} e^{\pi\omega/2\eta} - \tilde{a}_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \\ \tilde{c}_{\omega q} &= \frac{\tilde{a}_{\omega q} e^{\pi\omega/2\eta} - a_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \end{aligned} \quad (43)$$

and their conjugates. The state annihilated by the $c_{\omega q}$, for $\omega > 0$, is the Hartle-Hawking vacuum:

$$c_{\omega q} |\text{HH}\rangle = \tilde{c}_{\omega q} |\text{HH}\rangle = 0, \quad \text{for } \omega > 0, q \in \mathbb{Z}. \quad (44)$$

As noted earlier, the Hartle-Hawking state exists for all times as we are working in the Heisenberg picture. The ladder operators $c_{\omega q}$, $\tilde{c}_{\omega q}$ satisfy

$$[c_{\omega q}, c_{\omega' q'}^\dagger] = [\tilde{c}_{\omega q}, \tilde{c}_{\omega' q'}^\dagger] = \delta(\omega - \omega') \delta_{qq'}, \quad (45)$$

with all other commutators being zero.

C. Smooth mode functions

The Hartle-Hawking mode expansion (40) is defined everywhere in the fully extended BTZ black hole geometry. However, the mode functions $h_{\omega q}(U, V, \varphi)$ are not

differentiable at the horizons (see [32] for instance) and, strictly speaking, do not satisfy the Klein-Gordon equation in a neighbourhood of the horizons. The singularities at the horizons can be seen as follows. In a neighbourhood of the future horizon $U = 0$, $V > 0$, $h_{\omega q}(U, V, \varphi)$ takes the form

$$\begin{aligned} h_{\omega q} &\sim \frac{e^{-iq\varphi}}{\sqrt{4\pi\omega} \sqrt{2 \sinh(\pi\omega/\eta)}} \times \\ &\times \begin{cases} e^{i\delta_{\omega q}} e^{\frac{\pi\omega}{2\eta}} (2\eta V)^{-\frac{i\omega}{\eta}} + e^{-i\delta_{\omega q}} e^{\frac{\pi\omega}{2\eta}} (-2\eta U)^{\frac{i\omega}{\eta}} & U < 0 \\ e^{i\delta_{\omega q}} e^{\frac{\pi\omega}{2\eta}} (2\eta V)^{-\frac{i\omega}{\eta}} + e^{-i\delta_{\omega q}} e^{-\frac{\pi\omega}{2\eta}} (+2\eta U)^{\frac{i\omega}{\eta}} & U > 0. \end{cases} \end{aligned} \quad (46)$$

Here, $\delta_{\omega q}$ is a phase factor and depends on ω , the surface gravity η , the scalar field mass m^2 , and the angular momentum quantum number q . In the large mass limit $m^2 \gg \ell^{-2}$, this phase simply becomes the phase of $\Gamma(i\omega/\eta)$, the Euler Γ -function. It is independent of the angular momentum q and depends only on the surface gravity of the black hole. In fact, in the large mass limit, one recovers the expression for the Hartle-Hawking mode functions for a scalar field in Rindler space with acceleration parameter η .

Clearly, taking derivatives of $h_{\omega q}$ with respect to U brings down powers of U^{-1} which blow up at the horizon. One way to resolve this issue is to work with the Fourier transformed mode functions [32]:

$$\hat{h}_{\vartheta q}(U, V, \varphi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi\eta} e^{i\omega\vartheta/\eta} h_{\omega q}(U, V, \varphi). \quad (47)$$

In the neighbourhood of the horizon and in the large mass limit of the scalar field, $\hat{h}_{\vartheta q}$ takes the form

$$\hat{h}_{\vartheta q}(U, V, \varphi) \sim \frac{1}{2\pi\sqrt{2\eta}} e^{-iq\varphi} \left(e^{2i\eta V e^{-\vartheta}} + e^{-2i\eta U e^{\vartheta}} \right). \quad (48)$$

It is easy to see that the above expressions are differentiable at $U = 0$. We expect the property of smoothness to hold for all masses even though we do not have an explicit expression like in the above large mass limit.

Since the scalar field is in an asymptotic AdS background which naturally acts as a gravitational box, it may be useful to work with modes which are labelled by a discrete index. To this end, we can smear the Hartle-Hawking mode functions with Hermite functions [60] $\psi_n(\omega/\eta)$ in ω -space and define the mode functions g_{nq} :

$$\begin{aligned} g_{nq}(U, V, \varphi) &:= \int_{-\infty}^{\infty} \frac{d\omega}{\eta} h_{\omega q}(U, V, \varphi) \psi_n(\omega/\eta) \\ &= (-i)^n \int_{-\infty}^{\infty} d\vartheta \hat{h}_{\vartheta q}(U, V, \varphi) \psi_n(\vartheta) \\ &\sim \frac{(-i)^n e^{-iq\varphi}}{2\pi\sqrt{2\eta}} \int_{-\infty}^{\infty} d\vartheta \left(e^{2i\eta V e^{-\vartheta}} + e^{-2i\eta U e^{\vartheta}} \right) \psi_n(\vartheta). \end{aligned} \quad (49)$$

where, in going to the second line, we have used the fact that the Fourier transform of a Hermite function $\psi_n(\omega/\eta)$ is the same Hermite function $(-i)^n \psi_n(\vartheta)$. The g_{nq} mode functions also have the important property that they are differentiable at the horizons $UV = 0$.

We can now expand the scalar field in terms of the g_{nq} mode functions:

$$\phi(U, V, \varphi) = \frac{1}{\sqrt{2\pi\ell}} \sum_{q \in \mathbb{Z}} \sum_{n=0}^{\infty} (e_{nq} g_{nq}(U, V, \varphi) + \text{c.c.}), \quad (50)$$

where the operators e_{nq} are defined in terms of $c_{\omega q}$ as

$$e_{nq} = \frac{1}{\sqrt{\eta}} \int_{-\infty}^{\infty} d\omega c_{\omega q} \psi_n(\omega/\eta), \quad (51)$$

and satisfy the canonical commutation relations

$$[e_{mq}, e_{nq'}^\dagger] = \delta_{mn} \delta_{qq'}, \quad [e_{mq}, e_{nq'}] = [e_{mq}^\dagger, e_{nq'}^\dagger] = 0. \quad (52)$$

Since e_{nq} only involve the $c_{\omega q}$ and not the $c_{\omega q}^\dagger$, they also annihilate the Hartle-Hawking state $|\text{HH}\rangle$:

$$e_{nq} |\text{HH}\rangle = 0, \quad n = 0, 1, \dots \quad (53)$$

The mode expansion (50) is smooth across the horizons and globally defined in the fully extended BTZ spacetime.

Finally, to obtain the general solution of the Klein-Gordon equation in the maximally sliced BTZ spacetime (31) which is smooth everywhere, we use the diffeomorphism between maximal slicing coordinates and Kruskal-Szekeres coordinates given in Appendix A 3 in the mode expansion (50). Define

$$\hat{g}_{nq}(\bar{t}, x, \varphi) = g_{nq}(U(\bar{t}, x), V(\bar{t}, x), \varphi). \quad (54)$$

We then finally have the general solution of the Klein-Gordon equation

$$\phi(\bar{t}, x, \varphi) = \frac{1}{\sqrt{2\pi\ell}} \sum_{q \in \mathbb{Z}} \sum_{n=0}^{\infty} (e_{nq} \hat{g}_{nq}(\bar{t}, x, \varphi) + \text{c.c.}), \quad (55)$$

which satisfy the normalizable boundary conditions:

$$\phi(\bar{t}, x, \varphi) \rightarrow \begin{cases} x^{-\Delta_+} \mathcal{O}(t, \varphi) & x \rightarrow +\infty, \\ (-x)^{-\Delta_+} \tilde{\mathcal{O}}(\tilde{t}, \varphi) & x \rightarrow -\infty, \end{cases} \quad (56)$$

where, recall that, $\mathcal{O}(t, \varphi)$ and $\tilde{\mathcal{O}}(\tilde{t}, \varphi)$ are generalized free field scalar operators dual to the bulk scalar field in the right and left CFTs.

By following the chain of definitions, it is easy to compute the Klein-Gordon inner products for the $\hat{g}_{nq}$ modes:

$$\begin{aligned} (\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} &= 2\pi\ell^2 \delta_{nn'} \delta_{qq'}, \quad (\hat{g}_{nq}^*, \hat{g}_{n'q'})_{\text{KG}} = 0, \\ (\hat{g}_{nq}^*, \hat{g}_{n'q'}^*)_{\text{KG}} &= -2\pi\ell^2 \delta_{nn'} \delta_{qq'}. \end{aligned} \quad (57)$$

D. A bulk reconstruction formula in wormhole coordinates

We can view the local bulk operator $\phi(\bar{t}, x, \varphi)$ as extended operator in the product space of the two boundary CFTs, with x as a label. More precisely, we can write down a bulk reconstruction formula for the local bulk operator $\phi(\bar{t}, x, \varphi)$ in the entire region R_0 (see figure 2), covered by the range of the wormhole coordinates $-\infty < \bar{t} < \infty$, $-\infty < x < \infty$, $0 \leq \varphi \leq 2\pi$, as

$$\begin{aligned} \phi(\bar{t}, x, \varphi) &= \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dt' \mathcal{K}(\bar{t}, x, \varphi; t', \varphi') \mathcal{O}(t', \varphi') \\ &\quad + \int_0^{2\pi} d\tilde{\varphi}' \int_{-\infty}^{\infty} d\tilde{t}' \tilde{\mathcal{K}}(\bar{t}, x, \varphi; \tilde{t}', \tilde{\varphi}') \tilde{\mathcal{O}}(\tilde{t}', \tilde{\varphi}'), \end{aligned} \quad (58)$$

where $\mathcal{O}(t', \varphi')$ and $\tilde{\mathcal{O}}(\tilde{t}', \tilde{\varphi}')$ are dual generalized free field operators in the right and left CFTs respectively. The exact form of $\mathcal{K}$ and $\tilde{\mathcal{K}}$ is provided in Appendix D, equation (D9). This formula is derived from the discrete mode expansion of the scalar field (55) in terms of the e_{nq} oscillators (51), and it naturally involves support from both the boundary CFTs, in contrast to the HKLL formula [61].

The boundary-to-bulk kernels $\mathcal{K}$ and $\tilde{\mathcal{K}}$ have the following important property. When the bulk spacetime point is located entirely in Region I, the kernel $\tilde{\mathcal{K}}$ vanishes. Similarly for a spacetime point in Region II, the kernel $\mathcal{K}$ vanishes. However in the interior regions F and P, both $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are necessarily non-zero. Later in Section V, this property will be crucial for defining an order parameter that detects crossing of the black hole horizon.

IV. THE SCALAR FIELD HAMILTONIAN FOR EVOLUTION ALONG MAXIMAL SLICES

A. The expression for the Hamiltonian

We now evaluate the scalar field Hamiltonian (29) on the maximally sliced BTZ black hole solution (11) after plugging in the field expansion (55) in terms of the smooth mode functions $\hat{g}_{nq}$. The Hamiltonian is written in condensed form as

$$H = \sum_{\substack{n, n' \\ q, q'}} \begin{pmatrix} e_{n'q'}^\dagger & e_{n'q'} \end{pmatrix} \begin{pmatrix} A_{nn'qq'}(\bar{t}) & B_{nn'qq'}^*(\bar{t}) \\ B_{nn'qq'}(\bar{t}) & A_{nn'qq'}^*(\bar{t}) \end{pmatrix} \begin{pmatrix} e_{nq} \\ e_{nq}^\dagger \end{pmatrix}, \quad (59)$$

where the kernels A and B are defined as

$$A_{nn'qq'}(\bar{t}) = \frac{1}{4\pi\ell^2} \int dx d\varphi \sqrt{g} \left[N \left(\mathcal{D}_{\bar{t}} \hat{g}_{n'q'}^* \mathcal{D}_{\bar{t}} \hat{g}_{nq} + g^{xx} \hat{g}_{n'q'}^* \hat{g}'_{nq} \right. \right. \\ \left. \left. + (m^2 + qq' g^{\varphi\varphi}) \hat{g}_{n'q'}^* \hat{g}_{nq} \right) + 2N^x \mathcal{D}_{\bar{t}} \hat{g}_{n'q'}^* \hat{g}'_{nq} \right], \quad (60)$$

and

$$B_{nn'qq'}(\bar{t}) = \frac{1}{4\pi\ell^2} \int dx d\varphi \sqrt{g} \left[N \left(\mathcal{D}_{\bar{t}} \hat{g}_{n'q'} \mathcal{D}_{\bar{t}} \hat{g}_{nq} + g^{xx} \hat{g}'_{n'q'} \hat{g}'_{nq} \right. \right. \\ \left. \left. + (m^2 + qq' g^{\varphi\varphi}) \hat{g}_{n'q'} \hat{g}_{nq} \right) + 2N^x \mathcal{D}_{\bar{t}} \hat{g}_{n'q'} \hat{g}'_{nq} \right], \quad (61)$$

where $\mathcal{D}_{\bar{t}} = n^\mu \partial_\mu = N^{-1}(\partial_{\bar{t}} - N^x \partial_x)$ is the normal derivative and $' = \partial_x$. The kernel $A_{nn'qq'}(\bar{t})$ satisfies

$$A_{nn'qq'}^* = A_{n'nq'q}. \quad (62)$$

The integrals in the above kernels A and B are finite because the mode functions $\hat{g}_{nq}$ are smooth everywhere in the bulk, and fall-off at the boundary as $|x|^{-\Delta_+}$, with $\Delta_+ > 1$. [62] The Hamiltonian (59) is Hermitian by construction and is written entirely in terms of the smooth scalar field modes e_{nq} .

The expectation value of H (59) in the Hartle-Hawking state $|\text{HH}\rangle$ is

$$\langle \text{HH} | H(\bar{t}) | \text{HH} \rangle = \sum_{n,q} A_{nnqq}(\bar{t}). \quad (63)$$

We can now define a normal-ordered Hamiltonian operator

$$\hat{H}(\bar{t}) = H(\bar{t}) - \langle \text{HH} | H(\bar{t}) | \text{HH} \rangle, \quad (64)$$

which generates time translations along maximal slices that probe the black hole interior.

B. Hamiltonian in terms of the CFT modes $a_{\omega q}$, $\tilde{a}_{\omega q}$

Recall that the e_{nq} operators are related to the Hartle-Hawking operators $c_{\omega q}$, $\tilde{c}_{\omega q}$, $\omega > 0$, by the formula

$$e_{nq} = \frac{1}{\sqrt{\eta}} \int_0^\infty d\omega c_{\omega q} \psi_n(\omega/\eta) + \frac{1}{\sqrt{\eta}} \int_0^\infty d\omega \tilde{c}_{\omega q} \psi_n(-\omega/\eta). \quad (65)$$

The Hartle-Hawking operators are in turn related to the operators $a_{\omega q}$, $\tilde{a}_{\omega q}$, $a_{\omega q}^\dagger$, $\tilde{a}_{\omega q}^\dagger$ in the two CFTs by the

Bogoliubov transformations (43):

$$c_{\omega q} = \frac{a_{\omega q} e^{\pi\omega/2\eta} - \tilde{a}_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \\ \tilde{c}_{\omega q} = \frac{\tilde{a}_{\omega q} e^{\pi\omega/2\eta} - a_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}. \quad (66)$$

Hence the Hamiltonian (59) serves as the time evolution operator in the dual product CFT of the two boundaries which entangles the degrees of freedom in the two CFTs. The time-dependent operators (72) demonstrate this entanglement explicitly since it is no longer possible to separate out the left and right CFT operators $a_{\omega q}$ and $\tilde{a}_{\omega q}$ under $\bar{t}$ evolution.

One can also obtain an ill-defined expression (see below) for the Hamiltonian (59) in terms of the Hartle-Hawking operators $c_{\omega q}$, $\omega \in \mathbf{R}$, by plugging in the expressions (65) for e_{nq} into (59). We get

$$H = \sum_{q,q'} \int_{-\infty}^\infty d\omega \int_{-\infty}^\infty d\omega' \\ \begin{pmatrix} c_{\omega'q'}^\dagger & c_{\omega'q'} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\omega\omega'qq'}(\bar{t}) & \mathcal{B}_{\omega\omega'qq'}^*(\bar{t}) \\ \mathcal{B}_{\omega\omega'qq'}(\bar{t}) & \mathcal{A}_{\omega\omega'qq'}^*(\bar{t}) \end{pmatrix} \begin{pmatrix} c_{\omega q} \\ c_{\omega q}^\dagger \end{pmatrix}, \quad (67)$$

with

$$\mathcal{A}_{\omega\omega'qq'}(\bar{t}) = \frac{1}{4\pi\ell^2} \int dx d\varphi \sqrt{g} \left[N \left(\mathcal{D}_{\bar{t}} h_{\omega'q'}^* \mathcal{D}_{\bar{t}} h_{\omega q} + g^{xx} h_{\omega'q'}^* h'_{\omega q} \right. \right. \\ \left. \left. + (m^2 + qq' g^{\varphi\varphi}) h_{\omega'q'}^* h_{\omega q} \right) + 2N^x \mathcal{D}_{\bar{t}} h_{\omega'q'}^* h'_{\omega q} \right], \quad (68)$$

and

$$\mathcal{B}_{\omega\omega'qq'}(\bar{t}) = \frac{1}{4\pi\ell^2} \int dx d\varphi \sqrt{g} \left[N \left(\mathcal{D}_{\bar{t}} h_{\omega'q'} \mathcal{D}_{\bar{t}} h_{\omega q} + g^{xx} h'_{\omega'q'} h'_{\omega q} \right. \right. \\ \left. \left. + (m^2 + qq' g^{\varphi\varphi}) h_{\omega'q'} h_{\omega q} \right) + 2N^x \mathcal{D}_{\bar{t}} h_{\omega'q'} h'_{\omega q} \right]. \quad (69)$$

The kernels $\mathcal{A}_{\omega\omega'qq'}$ and $\mathcal{B}_{\omega\omega'qq'}$ are ill-defined since the integration over x includes the horizons and the derivatives of the Hartle-Hawking mode functions that appear in the kernels are singular at the horizon as explained in Section III C.

Now, the Hamiltonian (70) can be further written in terms of the $a_{\omega q}$, $\tilde{a}_{\omega q}$, $a_{\omega q}^\dagger$, $\tilde{a}_{\omega q}^\dagger$ by using the Bogoliubov transformations (66) in (70). The kernel $\mathcal{A}_{\omega\omega'qq'}$ (similarly for $\mathcal{B}_{\omega\omega'qq'}$) has to be separated into four different parts corresponding to positive and negative ω , ω' , and

the Hamiltonian will be of the form

$$H = \sum_{q,q'} \int_0^\infty d\omega \int_0^\infty d\omega' \begin{pmatrix} a_{\omega'q'}^\dagger & a_{\omega'q'} & \tilde{a}_{\omega'q'}^\dagger & \tilde{a}_{\omega'q'} \end{pmatrix} \begin{pmatrix} 4 \times 4 \end{pmatrix}_{\omega\omega'qq'} \begin{pmatrix} a_{\omega q} \\ a_{\omega q}^\dagger \\ \tilde{a}_{\omega q} \\ \tilde{a}_{\omega q}^\dagger \end{pmatrix}, \quad (70)$$

which has a 4×4 matrix of kernels whose entries are all ill-defined due to the same divergences present in the kernels $\mathcal{A}_{\omega\omega'qq'}$, $\mathcal{B}_{\omega\omega'qq'}$ (68) and (69).

Thus, though one can write the Hamiltonian for maximal slicing evolution in terms of the CFT operators, the expression has divergences arising from the blue-shift singularities of the mode functions at the horizons and hence is an ill-defined operator when acting individually on the modes of the left or the right CFT.

C. The time evolution operator

Since this Hamiltonian is time-dependent, the time-evolution operator has to be constructed as the anti time-ordered exponential [63]

$$U(\bar{t}) = \bar{\mathcal{T}} \exp \left(-i \int_0^{\bar{t}} d\bar{u} \hat{H}(\bar{u}) \right). \quad (71)$$

This is a *unitary* operator which describes the evolution of the quantum scalar field along maximal slices. In particular, one can define the time-dependent creation operators which satisfy the Heisenberg equations of motion:

$$e_{nq}^\dagger(\bar{t}) = U(\bar{t})^\dagger e_{nq}^\dagger(0) U(\bar{t}). \quad (72)$$

We must note that the scalar field Hamiltonian (59) goes to zero for $\bar{t} \gg \eta^{-1}$ due to the phenomenon of the *collapse of the lapse* discussed in Section IID. This reflects the freezing of proper time in the maximally sliced BTZ geometry as one approaches the limit slices $\bar{t} \rightarrow \pm\infty$. To avoid spurious effects that may arise from the lapse becoming zero, we must restrict the range of $\bar{t}$ to be of the order of η^{-1} .

An immediate consequence of the unitarity of the time evolution operator in the maximal slicing gauge in the Schroedinger picture is that the time evolved Hartle-Hawking state,

$$|\text{HH}; \bar{t}\rangle = U(\bar{t}) |\text{HH}\rangle, \quad (73)$$

remains pure for all times. This means that

$$\rho(\bar{t})^2 = \rho(\bar{t}), \quad \rho(\bar{t}) = |\text{HH}; \bar{t}\rangle \langle \text{HH}; \bar{t}|, \quad (74)$$

so its von Neumann entropy vanishes at all times. The Hartle-Hawking state is supported on smooth spatial

slices which cut across the horizon and hence the crossing of the horizon by a wave-packet built out of a finite number of $e_n^\dagger$ excitations on top of the Hartle-Hawking state can be smoothly described by this Hamiltonian.

D. The Hamiltonian $\hat{H}(\bar{t})$ is a well-defined operator

We observe that our formalism is also capable describing the Killing time evolution in the exterior regions I and II. We simply impose a different boundary condition on the lapse, namely

$$N \sim \begin{cases} \frac{r}{\ell}, & r \rightarrow \infty \quad (\text{Region I}), \\ -\frac{\tilde{r}}{\ell}, & \tilde{r} \rightarrow \infty \quad (\text{Region II}). \end{cases} \quad (75)$$

With these boundary conditions, one recovers the static BTZ metric in the two exterior regions, which can be combined together by introducing a wormhole-like coordinate. The slices of this foliations are also smooth and satisfy the maximal slicing gauge but all of them pass through the bifurcate horizon at $r = \tilde{r} = R_h$ where the lapse vanishes. Following the steps of Section III, we land up with the Hamiltonian H_{Killing} , which turns out to be the same as ' $h_r - h_l$ ' [64] and generates time evolution along the Killing slices. Smoothness and the fact that the Cauchy slice is not split into parts, [65] ensures that H_{Killing} is a well-defined Hamiltonian. In fact, it annihilates the Hartle-Hawking vacuum, $H_{\text{Killing}} |\text{HH}\rangle = 0$.

The Hamiltonian $\hat{H}(\bar{t})$ in (59) corresponds to the lapse with positive boundary condition on both the asymptotic boundaries (25) and generates time evolution along maximal slices that go into the black hole interior. [66] The Hamiltonian $\hat{H}(\bar{t})$ has finite expectation value and matrix elements in the Hilbert space of small excitations obtained by acting with a finite number of creation operators $e_{nq}^\dagger$ above the Hartle-Hawking state, which we denote by $\mathcal{H}_{\text{HH}}$. Since these maximal slices are smooth and not split into regions, we expect $\|\hat{H}(\bar{t})|\text{HH}\rangle\|^2 = \langle \text{HH} | \hat{H}(\bar{t})^2 | \text{HH} \rangle < \infty$, i.e., the state $\hat{H}(\bar{t})|\text{HH}\rangle$ is normalizable. It is easy to see that

$$\langle \text{HH} | \hat{H}(\bar{t})^2 | \text{HH} \rangle = \sum_{\substack{n,n' \\ q,q'}} 2B_{nn'qq'}^*(\bar{t}) B_{n'nq'q}(\bar{t}) = 2\text{Tr}(B^*B). \quad (76)$$

Below we outline the reason for $\text{Tr}(BB^*)$ to be finite.

The range of n and q is bounded by the following considerations. Recall that Hermite functions are eigenfunctions of the quantum harmonic oscillator in one dimension. Given this it is easy to see that the spatial spread of the n 'th eigenstate, quantified by $\langle n | \hat{x}^2 | n \rangle$, is proportional to n . Hence a function with support on an interval of say size L , can be well approximated by summing n from zero to n_{max} , which is of the order of L^2 . In the present problem, there is already a natural scale in

the frequency space, namely the surface gravity η and the above estimate then gives $n_{max} \sim (\eta\ell)^2 \sim M$. [67]

The bound on the range of q comes from the perspective of scattering off a black hole where the horizon radius, acts as an impact parameter. The larger the angular momentum, the smaller is the amplitude of a particle to fall into the black hole. In-falling modes with an ‘impact parameter’ larger than the horizon radius are naturally cutoff. The relevant length scale here is the black hole horizon radius which in units of the AdS length ℓ , is proportional to the square-root of the mass parameter M , thus effectively restricting $|q| \lesssim q_{max} \sim \sqrt{M}$. Once the range of n and q is bounded, $\text{Tr}(B^*B)$ is finite and the state $\hat{H}(\bar{t})|\text{HH}\rangle$ has a finite norm and is in the Hilbert space $\mathcal{H}_{\text{HH}}$.

Another reason to expect $\text{Tr}(B^*B)$ to be finite is from the lens of algebraic quantum field theory. The operator algebra $\mathcal{A}_\phi$ of the quantum scalar field associated to a complete Cauchy slice (which is the case for maximal slices considered here) is Type I because the state on the full Cauchy slice, namely the Hartle-Hawking state, is pure. Since any automorphism of a Type I factor is always an inner automorphism [68], the evolution operator $U(\bar{t})$ defined in equation (71) belongs to the algebra $\mathcal{A}_\phi$. For a very short time interval ϵ , since $U(\bar{t} = \epsilon) \approx \mathbb{1} - i\epsilon\hat{H}(0)$, $\hat{H}(0)$ should also belong to $\mathcal{A}_\phi$. Conjugation by the time evolution operator $U(\bar{t})$ would then imply that $\hat{H}(\bar{t}) \in \mathcal{A}_\phi$ for all times.

E. Diagonalizing the Hamiltonian

Since the Hamiltonian $\hat{H}(\bar{t})$ is time dependent, the number operator $\mathcal{N}_{nq}(\bar{t}) = e_{nq}^\dagger(\bar{t})e_{nq}(\bar{t})$ is also time dependent in the Heisenberg picture. Equivalently the expectation value of the number operator $\mathcal{N}_{nq}(\bar{t} = 0)$ in the state $|\text{HH}; \bar{t}\rangle$ changes with $\bar{t}$. We will now remedy this by introducing a new basis of operators $(b_{nq}, b_{nq}^\dagger)$ related to the regularised Hartle-Hawking modes $(e_{nq}, e_{nq}^\dagger)$ by a Bogoliubov transformation.

The Hamiltonian $H(\bar{t})$ is Hermitian and hence can be diagonalized. Rewriting (59) more compactly as

$$H(\bar{t}) = \begin{pmatrix} e^{\dagger T} & e^T \end{pmatrix} \begin{pmatrix} A(\bar{t}) & B^*(\bar{t}) \\ B(\bar{t}) & A^*(\bar{t}) \end{pmatrix} \begin{pmatrix} e \\ e^\dagger \end{pmatrix}, \quad (77)$$

where we have arranged all the operators in a column,

$$e = \begin{pmatrix} e_{00} \\ e_{10} \\ \vdots \end{pmatrix}, \quad e^\dagger = \begin{pmatrix} e_{00}^\dagger \\ e_{10}^\dagger \\ \vdots \end{pmatrix}. \quad (78)$$

After diagonalization, the Hamiltonian will take the form

$$H(\bar{t}) = \begin{pmatrix} b^{\dagger T} & b^T \end{pmatrix} \begin{pmatrix} \mu(\bar{t}) & 0 \\ 0 & \mu(\bar{t}) \end{pmatrix} \begin{pmatrix} b \\ b^\dagger \end{pmatrix}, \quad (79)$$

where μ is a diagonal matrix. The new operators b are related to the old ones by a Bogoliubov transformation

which can be written in matrix form as

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = M \begin{pmatrix} e \\ e^\dagger \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} e \\ e^\dagger \end{pmatrix}. \quad (80)$$

Pushing this relation in the Hamiltonian (77), we find

$$\begin{pmatrix} A(\bar{t}) & B^*(\bar{t}) \\ B(\bar{t}) & A^*(\bar{t}) \end{pmatrix} = M^\dagger \begin{pmatrix} \mu(\bar{t}) & 0 \\ 0 & \mu(\bar{t}) \end{pmatrix} M \quad (81)$$

The Bogoliubov transformation matrix M is determined from the eigenvectors of the matrix $\begin{pmatrix} A(\bar{t}) & B^*(\bar{t}) \\ B(\bar{t}) & A^*(\bar{t}) \end{pmatrix}$. Since $A(\bar{t})$, $B(\bar{t})$ are time dependent, the Bogoliubov coefficients α, β are also time dependent.

The b operators define a new state $|\Omega; \bar{t}\rangle$ such that

$$b_I |\Omega; \bar{t}\rangle = 0 \quad \forall I, \quad (82)$$

where I is a label for the new modes. Using the Bogoliubov transformation

$$b_I = \sum_{n,q} (\alpha_{I,nq} e_{nq} + \beta_{I,nq} e_{nq}^\dagger), \quad (83)$$

we find that $|\Omega; \bar{t}\rangle$ is a squeezed state built on top of the Hartle-Hawking vacuum,

$$|\Omega; \bar{t}\rangle = \mathcal{N} \exp \left(-\frac{1}{2} \sum_{\substack{n,n' \\ q,q'}} e_{nq}^\dagger (\alpha^{-1} \beta)_{nn'qq'} e_{n'q'}^\dagger \right) |\text{HH}\rangle, \quad (84)$$

where $\mathcal{N}$ is the overall normalization. Note that the state $|\Omega; \bar{t}\rangle$ is also time dependent because the Bogoliubov coefficients are time dependent.

The state $|\Omega; \bar{t}\rangle$ can be thought as a co-moving state in which the occupation number $b_I^\dagger b_I$ vanishes at all times,

$$\langle \Omega; \bar{t} | b_I^\dagger b_I | \Omega; \bar{t} \rangle = 0. \quad (85)$$

The occupation number of the operator $e_{nq}^\dagger e_{nq}$ in the state $|\Omega; \bar{t}\rangle$ is given by

$$\langle \Omega; \bar{t} | e_{nq}^\dagger e_{nq} | \Omega; \bar{t} \rangle = (\beta^* \beta)_{nnqq}. \quad (86)$$

It would be interesting understand this state further and its implications for black hole physics.

F. Action of large diffeomorphisms on the scalar field

As reviewed in Section II E, there is a family of large diffeomorphisms $(\zeta_\perp, \zeta^i)$ (27) which preserve the maximal slicing gauge and act on the maximally sliced two-sided BTZ black hole. These are parametrized by the constants c and $\tilde{c}$ which correspond to the amount by which the large diffeomorphism translates the left and right boundary times $t, \tilde{t}$ respectively. The action of the

large diffeomorphism $(\zeta_\perp, \zeta^i)$ on the scalar field $\phi(\bar{t}, x, \varphi)$ is obtained by substituting $\zeta_\perp$ and ζ^i in place of the lapse N and shift N^i in the scalar field Hamiltonian

$$\begin{aligned} H[\zeta_\perp, \zeta^i] \\ = \int_\Sigma d^d x \left(\zeta_\perp \left(\frac{1}{2\sqrt{g}} \pi_\phi^2 + \frac{1}{2} \sqrt{g} g^{ij} (\partial_i \phi \partial_j \phi + m^2 \phi^2) \right) \right. \\ \left. + \zeta^i \pi_\phi \partial_i \phi \right). \end{aligned} \quad (87)$$

The action of the large diffeomorphism on ϕ and π_ϕ is obtained by taking commutators with the Hamiltonian generator (87). In particular, the original Hamiltonian (29) corresponds to the large diffeomorphism labelled by $c = \tilde{c} = 1$. The above formula generalizes the action to arbitrary values of c and $\tilde{c}$. One can also obtain expressions similar to (59) in terms of the smeared modes e_{nq} , $e_{nq}^\dagger$ by plugging in the mode expansion (55) for the scalar field in the above formula (87).

V. AN ORDER PARAMETER THAT SIGNALS HORIZON CROSSING

Recall that $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are the operators dual to the scalar field ϕ , by the extrapolate dictionary, in the right and left CFTs, respectively. Consider an operator A_L in the left CFT such that $[A_L, \tilde{\mathcal{O}}] \neq 0$. By definition $[A_L, \mathcal{O}] = 0$. Now consider the localized operator at $x_0 > 0$, given by

$$\phi_f(\bar{t}, x_0) = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dx \phi(\bar{t}, x, \varphi) f(x - x_0), \quad (88)$$

where $f(x - x_0)$ is a smooth function with compact support centred around $x = x_0$. [69] Using the bulk reconstruction formula (58) and recalling that $\tilde{\mathcal{K}}$ vanishes in the Region I, we see that $[A_L, \phi_f(\bar{t}, x_0)] = 0$ in Region I. However it becomes non-zero,

$$[A_L, \phi_f(\bar{t}, x_0)] = [A_L, U(\bar{t})^\dagger \phi_f(0, x_0) U(\bar{t})] \neq 0, \quad (89)$$

as soon as the localised operator has support in the interior region F, where $\tilde{\mathcal{K}}$ is non-zero. Here $U(\bar{t})$ is the unitary time evolution operator we constructed before in (71). If we define an ‘order parameter’,

$$\mathcal{H}(\bar{t}) = \langle \text{HH} | [A_L, \phi_f(\bar{t}, x_0)] | \text{HH} \rangle, \quad (90)$$

then it has the property

$$\mathcal{H}(\bar{t}) = \begin{cases} = 0 & (\bar{t}, x_0) \in \text{Region I} \\ \neq 0 & (\bar{t}, x_0) \in \text{Region F} \end{cases}. \quad (91)$$

This means that for a localised operator evolving in $\bar{t}$ along a constant x_0 trajectory (that is sufficiently close to the horizon) the order parameter $\mathcal{H}(\bar{t})$ becomes non-zero when the operator crosses the horizon at some $\bar{t}_0$. It

is important to note that using the bulk reconstruction formula (58), the order parameter (90) can be expressed entirely in terms of CFT operators on the two boundaries and hence the transition from zero to non-zero value signals horizon crossing in the boundary theory.

VI. SCALAR FIELD CORRELATION FUNCTIONS IN THE MAXIMALLY SLICED TWO-SIDED BTZ BLACK HOLE

Recall the observation that under $\bar{t}$ time evolution, the state of the scalar field in the two-sided BTZ black hole remains pure at all times:

$$\rho(\bar{t})^2 = \rho(\bar{t}), \quad \rho(\bar{t}) = |\text{HH}; \bar{t}\rangle \langle \text{HH}; \bar{t}|. \quad (92)$$

This implies that the correlation functions of the probe scalar field in the maximally sliced background in the Hartle-Hawking state must be that of a unitary theory, similar to that of quantum field theory in Minkowski spacetime, and in particular exhibit no thermality.

In this section, we demonstrate this unitary nature of our time evolution by explicit calculation of two-point Wightman functions of the probe scalar field.

A. An expression for the scalar field two-point function

Recall the mode expansion of the scalar field (50) in terms of smooth mode functions $g_{nq}(U, V, \varphi)$:

$$\phi(U, V, \varphi) = \frac{1}{\sqrt{2\pi\ell}} \sum_{q \in \mathbb{Z}} \sum_{n=0}^{\infty} (e_{nq} g_{nq}(U, V, \varphi) + \text{c.c.}), \quad (93)$$

where the operators e_{nq} annihilate the Hartle-Hawking state, $e_{nq} |\text{HH}\rangle = 0$. From the above expression for the two-point Wightman function of the scalar field in the maximal slicing coordinate system can be written as

$$\begin{aligned} G(\bar{t}, x, \varphi; \bar{t}', x', \varphi') &\equiv \langle \text{HH} | \phi(\bar{t}, x, \varphi) \phi(\bar{t}', x', \varphi') | \text{HH} \rangle \\ &= \frac{1}{2\pi\ell^2} \sum_{n,q} g_{nq}(U, V, \varphi) g_{nq}^*(U', V', \varphi'), \\ &= \frac{1}{2\pi\ell^2} \sum_{n,q} \hat{g}_{nq}(\bar{t}, x, \varphi) \hat{g}_{nq}^*(\bar{t}', x', \varphi'), \end{aligned} \quad (94)$$

where $U = U(\bar{t}, x)$, $V = V(\bar{t}, x)$, $U' = U(\bar{t}', x')$, $V' = V(\bar{t}', x')$ and the second equality is obtained by plugging in the the diffeomorphism from U, V to $\bar{t}, x$.

The above expression for the two point function in terms of the Kruskal-Szekeres coordinates, and equivalently, maximal slicing coordinates, is well-defined everywhere in the maximal slicing coordinate system since the mode functions $g_{nq}(U, V, \varphi)$ are differentiable solutions of the Klein-Gordon equation, and consequently, so is the infinite sum in (94). [70] Recall that there are explicit expressions for the mode functions $\hat{g}_{nq}(\bar{t}, x, \varphi)$ which are

obtained by starting with the hypergeometric functions arising in Regions I and II and performing the appropriate chain of manipulations described in Section III.

A very similar formula can be written down for the two-point function of the scalar field in $d > 2$ maximally sliced AdS-Schwarzschild black hole backgrounds, and hence the observations we make below also hold in higher dimensions ($d > 2$).

For the two-sided BTZ black hole in $2 + 1$ dimensions, an alternate expression for the two-point function can be obtained by leveraging the fact that the two-sided BTZ geometry is a quotient of AdS_3 by a discrete group of translations and using the method of images [54, 55, 57]. The expression is in terms of Kruskal-Szekeres coordinates and is given by

$$G(\bar{t}, x, \varphi; \bar{t}', x', \varphi') = \langle \text{HH} | \phi(\bar{t}, x, \varphi) \phi(\bar{t}', x', \varphi') | \text{HH} \rangle \\ = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{z_n^2 - 1}} \left(z_n + \sqrt{z_n^2 - 1} \right)^{1-\Delta_+}, \quad (95)$$

where

$$z_n = \frac{1}{(1 + \eta^2 UV)(1 + \eta^2 U'V')} \times \\ \times \left((1 - \eta^2 UV)(1 - \eta^2 U'V') \cosh(\ell\eta(\varphi - \varphi' + 2n\pi)) \right. \\ \left. + 2\eta^2(VU' + UV') \right). \quad (96)$$

While the two expressions (94) and (95) look very different, they are equivalent upon doing the smooth coordinate transformation between the wormhole and Kruskal-Szekeres coordinates.

B. Two-point function in Region I in BTZ coordinates

The two-point function (94) (equivalently, (95)) with the insertion points in Region I can be rewritten in terms of the BTZ coordinates (t, r, φ) by using the coordinate transformation from wormhole coordinates $(\bar{t}, x, \varphi)$ to (t, r, φ) . Attention must be paid to the fact that this coordinate transformation is singular at the horizons since the BTZ coordinates are valid only in Region I. The correlation function turns out to be

$$G(t, r, \varphi; t', r', \varphi') = \\ = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{z_n^2 - 1}} \left(z_n + \sqrt{z_n^2 - 1} \right)^{1-\Delta_+}, \quad (97)$$

with

$$z_n(t, r, \varphi; t', r', \varphi') \\ = \frac{1}{R_h^2} \left(rr' \cosh(\ell\eta(\varphi - \varphi' + 2n\pi)) \right. \\ \left. - \sqrt{(r^2 - R_h^2)(r'^2 - R_h^2)} \cosh(\eta(t - t')) \right). \quad (98)$$

The above correlation function satisfies the KMS property

$$G(t, r, \varphi; t', r', \varphi') = G(t' - i\beta, r', \varphi'; t, r, \varphi), \quad (99)$$

where $\beta = 2\pi/\eta$. Thus, the correlators in Region I in the singular BTZ coordinates are that of a thermal system in Region I with temperature $\eta/2\pi$. The thermal density matrix of the system is obtained by tracing out the degrees of freedom in the Hartle-Hawking state that arise from Region II. Curiously, this tracing out is automatically performed by the singular coordinate transformation from wormhole coordinates to BTZ coordinates in Region I.

C. The two-point Wightman functions for specific insertion points

We now present the values of the two-point Wightman functions for various insertion points of the scalar field operators. Since the expression (95) is valid everywhere in the maximally extended BTZ black hole geometry, we will use it to study correlation functions with insertions throughout the extended spacetime, as a function of the wormhole coordinates $\bar{t}$ and x . We separately consider timelike and spacelike separated insertions below.

1. Timelike separated insertions

We study the two-point correlator with one insertion in the exterior region I and another in the black hole interior in region F,

$$G(\bar{t}, 0, 0; 0, x', 0) = \\ \langle \text{HH} | \phi(\bar{t}, x = 0, \varphi = 0) \phi(\bar{t}' = 0, x' > 0, \varphi' = 0) | \text{HH} \rangle. \quad (100)$$

For small $\bar{t}$, the two points can be spacelike separated, however for times large enough the separation is always timelike. This setup then provides a proxy for a correlator measured by an observer falling into the horizon. Keeping $x' > 0$ fixed, we plot this correlator as a function of the time $\bar{t}$ in Figure 5. Clearly, this correlator does not decay to zero; instead (the real part) saturates to a non-zero value while the imaginary part decays to zero. This seems to be the consequence of bulk unitarity.

We also note that for an insertion in region P ($\bar{t} < 0$) on the $x = 0, \varphi = 0$ line, the two-point function in (100) will saturate to the same finite non-zero value for large negative times as well, as shown in figure 6. This is a consequence of the fact that the background classical solution is invariant under $\bar{t} \rightarrow -\bar{t}$, and the scalar field transforms as $\phi(\bar{t}, x, \varphi) \rightarrow \phi(-\bar{t}, x, \varphi)$. We would not expect this to happen if our time evolution was dissipative.

As a special case, we put $x' = 0$ together with $\bar{t}' = 0$, which implies that $U' = V' = 0$, i.e., this insertion is exactly at the bifurcate horizon. In this case the correlator $G(\bar{t}, 0, 0; 0, 0, 0)$ also saturates to a finite non-zero value.

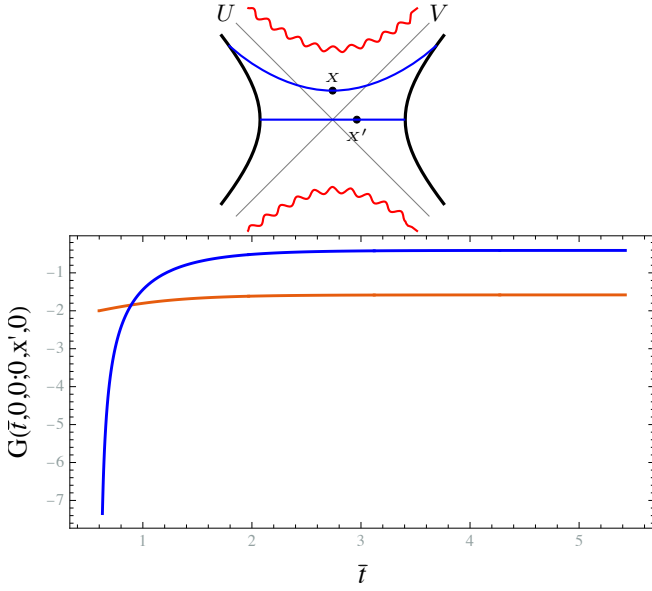


FIG. 5. The function (100) for timelike separated insertions is plotted as a function of time $\bar{t}$ with $x' = 0.5R_h$ held fixed. The imaginary part (blue) has the UV divergence at small times but decays at large times. On the other hand the real part (orange) saturates to a finite non-zero value at late times, showing that information is not lost even as $\bar{t} \rightarrow \infty$. The two insertions are also shown schematically in the Kruskal-Szekeres diagram in the top figure.

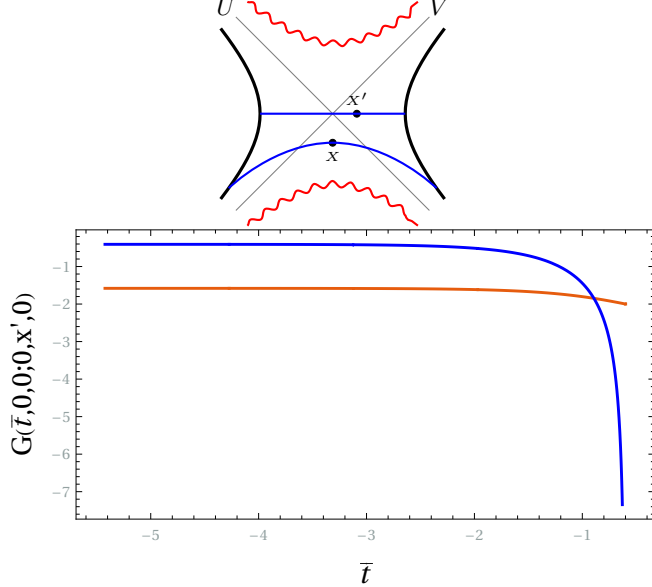


FIG. 6. The function (100) for timelike separated insertions is plotted for $\bar{t} < 0$ with $x' = 0.5R_h$ held fixed. The imaginary part (blue) has the UV divergence at small times but decays at large times. On the other hand the real part (orange) saturates to a finite non-zero value at large negative times, which follows from the time reflection property. The two insertions are also shown schematically in the Kruskal-Szekeres in the top figure.

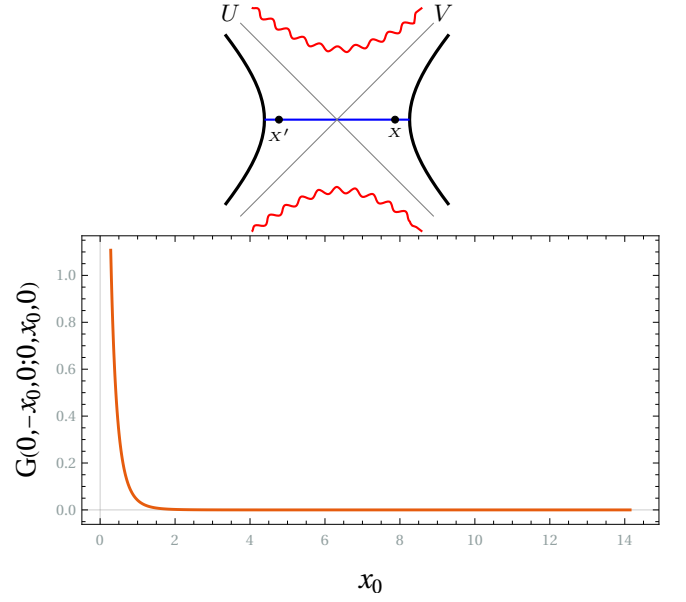


FIG. 7. The Wightman correlator (101) plotted as a function of the spatial separation x_0 on the $\bar{t} = 0$ slice. The real part clearly decays. The imaginary part simply vanishes and is therefore not shown. For this plot, we have taken $\Delta_+ = 3$. The insertions are also shown schematically in the Kruskal-Szekeres diagram in the top figure.

2. Spacelike separated insertions

When the two insertions in regions I and II each, the separation is always spacelike. First we consider the two insertions on the $\bar{t} = 0$ slice and study

$$G(0, x_0, 0; 0, -x_0, 0) = \langle \text{HH} | \phi(\bar{t} = 0, x = x_0, \varphi = 0) \phi(\bar{t}' = 0, x' = -x_0, \varphi' = 0) | \text{HH} \rangle \quad (101)$$

as a function of the separation x_0 . This correlation function decays with increasing x_0 as can be seen in Figure 7. This is analogous to the decay of the massive scalar field two-point function on a constant time slice in Minkowski spacetime as the separation between the points increases.

We then also consider insertions near the two asymptotic boundaries in regions I and II, but at different time slices,

$$G(\bar{t}, x_0, 0; 0, -x_0, 0) = \langle \text{HH} | \phi(\bar{t}, x = x_0, \varphi = 0) \phi(\bar{t}' = 0, x' = -x_0, \varphi' = 0) | \text{HH} \rangle. \quad (102)$$

With x_0 held fixed, we find that this correlation function decays as $\bar{t}$ increases, as is evident from Figure 8.

D. The boundary limit of the correlators

In this subsection, we evaluate the two point function with one scalar field insertion near the right boundary

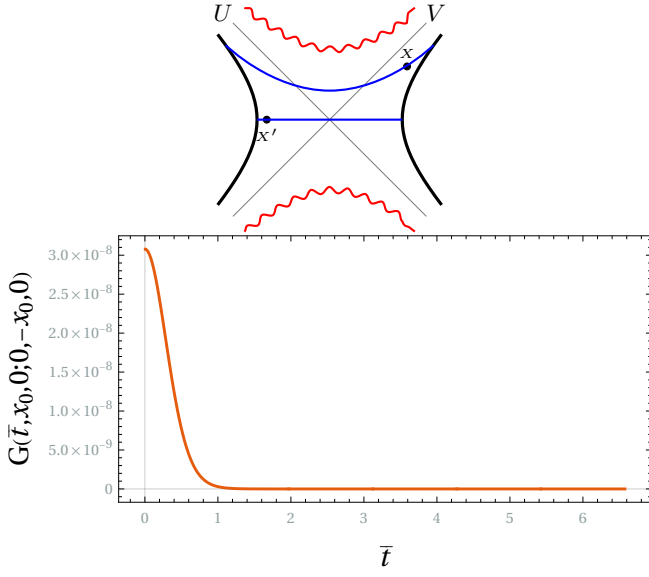


FIG. 8. The Wightman correlator (102) plotted as a function of the time $\bar{t}$, keeping $x_0 = 10R_h$ fixed. The real part clearly decays. The imaginary part simply vanishes and is therefore not shown. For this plot, we have taken $\Delta_+ = 3$. The two insertions are also shown schematically in the Kruskal-Szekeres diagram in the top figure.

and one near the left boundary by considering $x = \varepsilon^{-1}$ and $x' = -\varepsilon^{-1}$, in the limit that the cutoff ε is taken to zero.

Since the dependence of ε comes through the dependence of U and V in the correlation function (94) or (95), we have to look at the $\varepsilon \rightarrow 0$ limit of the expansions of U and V in region I,

$$U = -\eta^{-1}e^{-\eta\bar{t}}\left(1 - R_h\varepsilon + \frac{1}{2}(R_h\varepsilon)^2 - \frac{-3 - 2\frac{T(\bar{t})}{T_\infty} + 3\sqrt{1 - \frac{T(\bar{t})^2}{T_\infty^2}}}{12}(R_h\varepsilon)^3 + \mathcal{O}(\varepsilon^4)\right), \quad (103a)$$

$$V = \eta^{-1}e^{\eta\bar{t}}\left(1 - R_h\varepsilon + \frac{1}{2}(R_h\varepsilon)^2 + \frac{-3 + 2\frac{T(\bar{t})}{T_\infty} + 3\sqrt{1 - \frac{T(\bar{t})^2}{T_\infty^2}}}{12}(R_h\varepsilon)^3 + \mathcal{O}(\varepsilon^4)\right). \quad (103b)$$

It is easy to see that in the limit $\varepsilon \rightarrow 0$, the Kruskal-

Szekeres coordinates become

$$\lim_{\varepsilon \rightarrow 0} U(\bar{t}, \varepsilon^{-1}) = -\eta^{-1}e^{-\eta\bar{t}}, \quad \lim_{\varepsilon \rightarrow 0} V(\bar{t}, \varepsilon^{-1}) = \eta^{-1}e^{\eta\bar{t}}. \quad (104)$$

In the BTZ black hole background, we can substitute the above values for U , V , and the appropriate values for U' , V' (which are in region II), in the expression (95) to get

$$\lim_{\varepsilon \rightarrow 0} \langle \text{HH} | \phi(\bar{t}, \varepsilon^{-1}, \varphi) \phi(\bar{t}', -\varepsilon^{-1}, \varphi') | \text{HH} \rangle = (\varepsilon^{\Delta_+})^2 \times \sum_{n \in \mathbf{Z}} \left(\cosh(\eta(\bar{t} - \bar{t}')) + \cosh(\ell\eta(\varphi - \varphi' + 2n\pi)) \right)^{-\Delta_+}, \quad (105)$$

where recall that $\Delta_+ = 1 + \sqrt{1 + \ell^2 m^2}$ is the conformal dimension of the generalized free field operator $\mathcal{O}$ dual to the bulk scalar field ϕ .

The two-point function of the generalized free field $\mathcal{O}$ in the dual CFT is obtained by using the usual extrapolate dictionary (106):

$$\phi(\bar{t}, x, \varphi) \rightarrow \begin{cases} x^{-\Delta_+} \mathcal{O}(t, \varphi) & x \rightarrow +\infty, \\ (-x)^{-\Delta_+} \tilde{\mathcal{O}}(\bar{t}, \varphi) & x \rightarrow -\infty, \end{cases} \quad (106)$$

The Hartle-Hawking state is dual to the thermofield double state in the product of the two CFTs. Then we get, in the product of the dual CFTs on the left and right boundaries,

$$\langle \text{TFD} | \mathcal{O}(t, \varphi) \tilde{\mathcal{O}}(t', \varphi') | \text{TFD} \rangle = \sum_{n \in \mathbf{Z}} \left(\cosh(\eta(t + t')) + \cosh(\ell\eta(\varphi - \varphi' + 2n\pi)) \right)^{-\Delta_+}, \quad (107)$$

which agrees with the expression computed in [33].[71] This expression clearly decays at large times. The correlation function (107) above can also be obtained from the extrapolate limit of the bulk correlator (102) with non-zero t' . Then the decay is not surprising from the bulk point of view, since we already expected it to decay to zero (see figure 8).

Using the bulk reconstruction formula (58), the two-point Wightman correlator in (100), with one insertion in Region I and another in Region F, can be written as[72]

$$\begin{aligned} & \langle \text{HH} | \phi(\bar{t}, 0, 0) \phi(0, x' > 0, 0) | \text{HH} \rangle \\ &= \int_0^{2\pi} d\varphi' d\varphi'' \int_{-\infty}^{\infty} dt' dt'' \left(\mathcal{K}(\bar{t}, 0, 0; t', \varphi') \mathcal{K}(0, x', 0; \bar{t}', \varphi'') \langle \text{HH} | \mathcal{O}(t', \varphi') \mathcal{O}(\bar{t}', \varphi'') | \text{HH} \rangle \right. \\ & \quad \left. + \tilde{\mathcal{K}}(\bar{t}, 0, 0; t', \varphi') \mathcal{K}(0, x', 0; \bar{t}', \varphi'') \langle \text{HH} | \tilde{\mathcal{O}}(t', \varphi') \mathcal{O}(\bar{t}', \varphi'') | \text{HH} \rangle \right). \end{aligned} \quad (108)$$

For the insertion in Region I, the second term involving $\tilde{O}$ vanishes. The bulk correlation function is thus an integral transform of boundary correlators. While the extrapolate limit of bulk correlation functions decay to zero, the correlation function deep in the bulk need not decay to zero due to the complicated form of the $\mathcal{K}$ and $\tilde{\mathcal{K}}$ kernels in (108). This is precisely what we have observed in the sample calculations in this section.

ACKNOWLEDGMENTS

We would like to thank Rajesh Gopakumar, Hong Liu, Suresh Govindarajan, Alok Laddha, Gautam Mandal, Suvrat Raju, Ashoke Sen, Ronak Soni, Sandip Trivedi, Shiraz Minwalla, Edward Witten, and especially Raghu Mahajan and Kyriakos Papadodimas, for discussions and comments. We also acknowledge the organizers and participants of the ‘Workshop on Canonical Gravity’, held at Chennai Mathematical Institute, for discussions. S. R. W. would like to thank the CERN Theory Division for hospitality, and the Infosys Foundation Homi Bhabha Chair at ICTS-TIFR for its support.

Appendix A: Derivation of the maximally sliced two-sided BTZ black hole solution

In this appendix, we outline the solution of Einstein’s equations in the maximal slicing gauge in $2 + 1$ dimensions which gives the maximally sliced two-sided BTZ black hole. A very similar procedure gives the maximally sliced two sided AdS-Schwarzschild black hole in $d + 1$ dimensions for $d > 2$. We closely follow [40] in which the maximal slicing of the flat Schwarzschild black hole in $3 + 1$ dimensions was derived [73].

1. The maximal slicing solution

The general ADM ansatz for a non-rotating metric is

$$ds^2 = -\alpha(r, \bar{t})^2 d\bar{t}^2 + A(r, \bar{t}) \left(dr + \frac{\beta(r, \bar{t})}{A(r, \bar{t})} d\bar{t} \right)^2 + r^2 d\varphi^2. \quad (\text{A1})$$

The spatial metric is

$$ds^2 = g_{ij} dx^i dx^j = A(\bar{t}, r) dr^2 + r^2 d\varphi^2. \quad (\text{A2})$$

The Ricci scalar of g_{ij} is

$$R = \frac{\partial_r A}{r A^2}. \quad (\text{A3})$$

We use the notation $\dot{}$ to denote a derivative with respect to the time $\bar{t}$. The extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N} (\dot{g}_{ij} - 2D_{(i} N_{j)}), \quad (\text{A4})$$

where N is the lapse and N^i is the shift. Here, $N = \alpha$, $N^r = \beta/A$, $N^\varphi = 0$. We then have $N_r = g_{rr} N^r = \beta$. The non-zero components of the extrinsic curvature for the ansatz are [74]

$$K_{rr} = \frac{1}{2\alpha} \left(\partial_{\bar{t}} A + \beta \frac{\partial_r A}{A} - 2\partial_r \beta \right), \quad K_{\varphi\varphi} = -\frac{r\beta}{A\alpha}. \quad (\text{A5})$$

The maximal slicing gauge condition is

$$K = g^{ij} K_{ij} = 0, \quad (\text{A6})$$

which, with the above ansatz plugged in, becomes

$$\partial_{\bar{t}} A - \beta \partial_r \log \left(\frac{r^2 \beta^2}{A} \right) = 0. \quad (\text{A7})$$

Let us look at the momentum constraints $\mathcal{H}^j = -2D_i K^{ij} = 0$. The momentum constraint $\mathcal{H}^\varphi$ is automatically satisfied due to the angular symmetry. The radial momentum constraint $\mathcal{H}^r$ becomes the simple equation

$$\partial_r K_{\varphi\varphi} = 0, \quad \text{with solution} \quad K_{\varphi\varphi} = -T(\bar{t}), \quad (\text{A8})$$

where T is independent of the spatial coordinates but can be a function of time $\bar{t}$. Using the expression for $K_{\varphi\varphi}$ from (A5), we then get

$$\frac{r\beta}{A\alpha} = T(\bar{t}). \quad (\text{A9})$$

We next solve the Hamiltonian constraint $\mathcal{H}_\perp = K^{ij} K_{ij} - K^2 - R + 2\Lambda = 0$ which becomes the following first order equation for A :

$$2 \frac{T(\bar{t})^2}{r^4} - \frac{\partial_r A}{r A^2} + 2\Lambda = 0, \quad (\text{A10})$$

The solution is

$$A(r) = \left(\frac{r^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{r^2} \right)^{-1}, \quad (\text{A11})$$

where M is the integration constant. It turns out that M will be ADM mass of the solution, and it can be shown that it is constant under time evolution using the $\dot{K}_{rr}$ Einstein equation.

The maximal slicing condition (A7) can now be solved for the lapse α to get

$$\alpha(r, \bar{t}) = A(r, \bar{t})^{-1/2} \left(c - \dot{T} \int_{\infty}^r \frac{d\rho}{\rho} A(\rho, \bar{t})^{3/2} \right). \quad (\text{A12})$$

The integration constant is c , and is imposed as a boundary condition on α as $r \rightarrow \infty$:

$$\lim_{r \rightarrow \infty} \alpha = c \frac{r}{\ell}, \quad (\text{A13})$$

which is the familiar asymptotic AdS boundary condition for the lapse. The usual AAdS boundary conditions set

$c = 1$ so that the time $\bar{t}$ agrees with the time t of the asymptotic AdS metric near the boundary.

$$\text{AAdS boundary condition : } c = 1. \quad (\text{A14})$$

Note that T as a function of $\bar{t}$ is still undetermined.

Thus, the solution to the Einstein's equations in maximal slicing gauge is

$$\begin{aligned} ds^2 &= -\alpha(r, \bar{t})^2 d\bar{t}^2 + A(r, \bar{t}) \left(dr + \frac{\alpha(r, \bar{t}) T(\bar{t})}{r} d\bar{t} \right)^2 \\ &\quad + r^2 d\varphi^2, \\ A(r, \bar{t}) &= \left(\frac{r^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{r^2} \right)^{-1}, \\ \alpha(r, \bar{t}) &= A(r, \bar{t})^{-1/2} \left(1 - \dot{T}(\bar{t}) \int_{\infty}^r \frac{d\rho}{\rho} A(\rho, \bar{t})^{3/2} \right). \end{aligned} \quad (\text{A15})$$

The above solution has coordinate singularities since the component g_{rr} blows up at the roots of the equation $r^4 - \frac{M}{2\pi} \ell^2 r^2 + \ell^2 T^2 = 0$ which appears in the denominator of $A(r, \bar{t})$, see (A15). The roots of the equation are the following two repeated roots

$$R_{\pm}(T) = \ell \sqrt{\frac{M}{4\pi}} \left[1 \pm \sqrt{1 - \left(\frac{4\pi T(\bar{t})}{M\ell} \right)^2} \right]^{1/2}, \quad (\text{A16})$$

Thus, there are coordinate singularities at $r = R_{\pm}(T)$. From the expressions above, it is clear that the range of the function $T(\bar{t})$ is restricted to

$$-T_{\infty} \leq T \leq T_{\infty}, \quad \text{with} \quad T_{\infty} = \frac{\ell M}{4\pi}. \quad (\text{A17})$$

It is also clear that $R_{\pm}(T)$ satisfy

$$R_{-}(T) \leq R_{+}(T), \quad (\text{A18})$$

with the equality satisfied only when $T = T_{\infty} = \ell M/4\pi$:

$$R_{+}(T) = R_{-}(T) = \ell \sqrt{\frac{M}{4\pi}} \quad \text{when} \quad T = \pm T_{\infty}. \quad (\text{A19})$$

Below, we consider the diffeomorphism of the above solution to the two-sided BTZ black hole. We shall see that the above solution covers only a part of the two-sided BTZ black hole which includes the right boundary. Coming in from the boundary $r = \infty$, the above metric stops being valid at the first coordinate singularity $r = R_{+}(T)$. We call this the *right areal chart*.

Beyond this, it turns out that we need another chart to cover a similar region that starts from the left boundary of the two-sided BTZ black hole and comes up to the radius $R_{+}(T)$. To describe this solution, we use a different areal radial coordinate $\tilde{r}$ (which coincides with the areal radial coordinate of the BTZ solution in Region II, see

Figure 1). The solution in the maximal slicing gauge is the same as earlier, with obvious changes in notation:

$$\begin{aligned} ds^2 &= -\tilde{\alpha}(\tilde{r}, \bar{t})^2 d\bar{t}^2 + \tilde{A}(\tilde{r}, \bar{t}) \left(d\tilde{r} + \frac{\tilde{\alpha}(\tilde{r}, \bar{t}) \tilde{T}(\bar{t})}{\tilde{r}} d\bar{t} \right)^2 \\ &\quad + \tilde{r}^2 d\varphi^2, \\ \tilde{A}(\tilde{r}, \bar{t}) &= \left(\frac{\tilde{r}^2}{\ell^2} - \frac{M}{2\pi} + \frac{\tilde{T}(\bar{t})^2}{\tilde{r}^2} \right)^{-1}, \\ \tilde{\alpha}(\tilde{r}, \bar{t}) &= \tilde{A}(\tilde{r}, \bar{t})^{-1/2} \left(1 - \dot{\tilde{T}}(\bar{t}) \int_{\infty}^{\tilde{r}} \frac{d\rho}{\rho} \tilde{A}(\rho, \bar{t})^{3/2} \right). \end{aligned} \quad (\text{A20})$$

We call this solution the *left areal chart*.

2. Diffeomorphism to the two-sided BTZ black hole

In this subsection, we obtain the diffeomorphism from the maximal slicing solution obtained previously to the two-sided BTZ black hole. Our strategy is to look at the diffeomorphism in each of the four Regions of the two-sided BTZ black hole separately and then put them together. The right areal chart solution (A15) will be compared with the solution in Region I and it can be continued into Regions F and P till the radius $r = R_{+}(T)$. Similarly, Region II and remaining parts of Regions F and P will be covered by the left areal chart solution (A20).

Recall the standard BTZ metric in Region I:

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-1} dr^2 + r^2 d\varphi^2, \quad f(r) = \frac{r^2}{\ell^2} - \frac{M}{2\pi}, \quad (\text{A21})$$

with the Killing event horizon located at $r = R_h$ with

$$R_h = \ell \sqrt{\frac{M}{2\pi}}. \quad (\text{A22})$$

See Figure 9 for a pictorial description of the various regions of the two-sided BTZ black hole.

The simplest diffeomorphism is to map the radial coordinate r and angle φ in the above metric to r and φ in the the maximally sliced solution (A15). The only non-trivial function to compute then is $t(\bar{t}, r)$, i.e., the BTZ coordinate time t as a function of $\bar{t}$ and r . Plugging this into (A21) and comparing with (A1), we get

$$\partial_{\bar{t}} t = \epsilon \alpha \sqrt{A}, \quad \partial_r t = -\epsilon \frac{T \sqrt{A}}{r f(r)}, \quad (\text{A23})$$

where ϵ is a sign. In Region I for $t > 0$, since t increases as r decreases along a maximal slice, we must choose $\epsilon = +1$. Similarly, $\epsilon = -1$ for the $t < 0$ part of Region I. These signs can be inferred visually by tracking the change in r and t along a maximal slice in Figure 3.

Suppose we consider a maximal slice that starts at the boundary in the $t > 0$ part of Region I. Then, the first

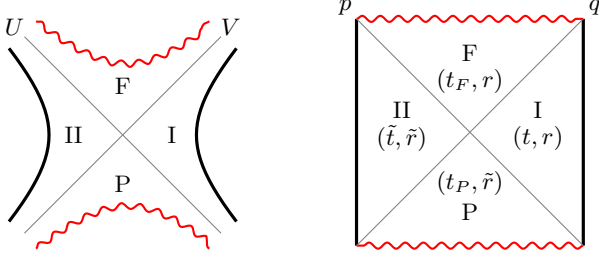


FIG. 9. The Kruskal (left) and Penrose (right) diagrams for the two-sided BTZ black hole solution. The two solid black lines are the asymptotic AdS boundaries, the diagonal gray lines are the event horizons and the wiggly red lines are the singularities. The horizons divide the two-sided black hole into four regions labelled I, II, F and P, each of which have a static Schwarzschild-like coordinate system which is displayed in the Penrose diagram.

equation in (A23) gives

$$\lim_{r \rightarrow \infty} t = \bar{t} + t_0, \quad (\text{A24})$$

where t_0 is an arbitrary constant which sets the origin of time on the boundary in Region I. We take this constant t_0 to be zero since we would like to identify the origin of t and $\bar{t}$. Integrating the second equation gives the diffeomorphism $t(r, \bar{t})$ in Region I:

$$t(r, \bar{t}) = \bar{t} + T(\bar{t}) \int_r^\infty d\rho \frac{A(\rho, \bar{t})^{1/2}}{\rho f(\rho)}. \quad (\text{A25})$$

Next, we study the diffeomorphism in Region F. The BTZ metric in Region F is

$$ds^2 = -f(r)^2 dt_F^2 + f(r)^{-1} dr^2 + r^2 d\varphi^2, \quad r < \ell \sqrt{\frac{M}{2\pi}}, \quad (\text{A26})$$

with $r < R_h$. The diffeomorphism is obtained by solving the equations

$$\partial_{\bar{t}} t_F = \epsilon \alpha \sqrt{A}, \quad \partial_r t_F = -\epsilon \frac{T \sqrt{A}}{r f(r)}, \quad (\text{A27})$$

where the sign ϵ must be chosen to be $+1$ for the $t_F > 0$ part of Region F and -1 in the $t_F < 0$ part of Region F.

Note that the derivative $\partial t_F / \partial r$ blows up at the roots $R_\pm(T)$ (A16). It is easy to see that these roots satisfy

$$R_-(T) \leq R_+(T) \leq R_h, \quad (\text{A28})$$

where recall that R_h is the horizon radius (A22). Thus, both $R_\pm$ are located in Region F with $r < R_h$. As one comes in from the horizon $r = R_h$ on a constant $\bar{t}$ slice, the divergence of $\partial t_F / \partial r$ at $r = R_+$ is first attained. The divergence of $\partial t_F / \partial r$ indicates that the constant $\bar{t}$ slice becomes tangent to the constant r hyperbola at $r = R_+$. Beyond this the areal radial coordinate ceases to be a good coordinate since the tangent condition implies that

continuing beyond this point increases r again to values greater than R_+ .

Integrating the second equation in (A27) for points on the maximal slice that are in the $t_F > 0$ part of Region F, we get

$$t_F(r, \bar{t}) - t_F(R_+(T), \bar{t}) = -T(\bar{t}) \int_{R_+(T)}^r d\rho \frac{A(\rho, \bar{t})^{1/2}}{\rho f(\rho)}, \quad (\text{A29})$$

where we have chosen the lower limit of integration to be $R_+(T)$ where the solution has a coordinate singularity. The above diffeomorphism in Region F can be extended to Region I as well by taking $r > \ell \sqrt{\frac{M}{2\pi}}$ in (A29); the integration range will now have a simple pole at the horizon which has to be dealt with the usual Cauchy principal-value prescription. After this, it is easy to see that it satisfies the second equation in (A23). So, the diffeomorphism in Region I is

$$t(r, \bar{t}) - t_F(R_+(T), \bar{t}) = -T(\bar{t}) \oint_{R_+(T)}^r d\rho \frac{A(\rho, \bar{t})^{1/2}}{\rho f(\rho)}. \quad (\text{A30})$$

Note that the reference time $\hat{t}_F(R_+, \bar{t})$ is not yet fixed. We fix this later after we obtain maximal slices in Region II as well below.

The function $T(\bar{t})$ is implicitly determined by using the boundary condition (A24) on t as $r \rightarrow \infty$:

$$\bar{t} = t_F(R_+(T), \bar{t}) - T(\bar{t}) \oint_{R_+(T)}^\infty d\rho \frac{A(\rho, \bar{t})^{1/2}}{\rho f(\rho)}. \quad (\text{A31})$$

We now describe the maximal slicing solution that is diffeomorphic to a region of the fully extended BTZ black hole that includes only the boundary in Region II, which we call the left boundary. Recall the BTZ metric in Region II:

$$ds^2 = -f(\tilde{r})^2 d\tilde{t}^2 + f(\tilde{r})^{-1} d\tilde{r}^2 + \tilde{r}^2 d\varphi^2, \quad \tilde{r} > \ell \sqrt{\frac{M}{2\pi}}. \quad (\text{A32})$$

Note that $\tilde{t}$ is chosen to increase in the opposite sense to the that of the time t in Region I. Thus, the $\bar{t} > 0$ maximal slice actually starts from the boundary in the $t > 0$ part of Region I, enters the $t_F > 0$ part of Region F at the future horizon $U = 0, V > 0$, goes to the $t_F < 0$ part of Region F and exits Region F at the past horizon $U > 0, V = 0$ and reaches the boundary in Region II in the $\tilde{t} < 0$ part.

To obtain maximal slices in the $\tilde{t} < 0$ region, we use the left areal chart maximal slicing solution (A20). The diffeomorphism to the BTZ metric in Region II is obtained by solving the equations

$$\partial_{\tilde{t}} \tilde{t} = \epsilon \tilde{\alpha} \sqrt{A(\tilde{r}, \tilde{t})}, \quad \partial_{\tilde{r}} \tilde{t} = -\epsilon \frac{\tilde{T} \sqrt{A(\tilde{r}, \tilde{t})}}{\tilde{r} f(\tilde{r})}, \quad (\text{A33})$$

where the sign ϵ must be chosen to be -1 for the $\tilde{t} < 0$ part of Region II and $+1$ in the $\tilde{t} > 0$ part of Region

II. The first equation with $\epsilon = -1$ gives the boundary condition

$$\lim_{\tilde{r} \rightarrow \infty} \tilde{t} = -\tilde{t} - \tilde{t}_0. \quad (\text{A34})$$

Following the same procedure as earlier, the coordinate transformation to BTZ coordinates in the $\tilde{t} < 0$ part of Region II is

$$\tilde{t}(\tilde{r}, \tilde{t}) - t_F(R_+(\tilde{T}), \tilde{t}) = \tilde{T} \int_{R_+(\tilde{T})}^{\tilde{r}} d\rho \frac{A(\rho, \tilde{t})^{1/2}}{\rho f(\rho)}, \quad (\text{A35})$$

where $t_F(R_+(\tilde{T}))$ is the t_F coordinate of the point $\tilde{r} = R_+(\tilde{T})$ in Region F where the metric (A20) has a coordinate singularity. The $r \rightarrow \infty$ limit of (A35) gives the following implicit expression for $\tilde{T}(\tilde{t})$:

$$-\tilde{t} = t_F(R_+(\tilde{T}), \tilde{t}) + \tilde{T} \int_{R_+(\tilde{T})}^{\infty} d\rho \frac{A(\rho, \tilde{t})^{1/2}}{\rho f(\rho)}. \quad (\text{A36})$$

We match the maximal slices from the left and right charts. Continuity in the radial coordinate forces the reference points $t_F(R_+(T))$ and $t_F(R_+(\tilde{T}))$ to be the same. This in turn requires the functions $\tilde{T}(\tilde{t})$ and $T(\tilde{t})$ to be the same, i.e., $\tilde{T}(\tilde{t}) = T(\tilde{t})$. Since these slices are tangent to the same constant r hyperbola, they are guaranteed to be smooth at $r = R_+(T(\tilde{t}))$. Adding (A31) and (A36) we get

$$t_F(R_+(T), \tilde{t}) = 0. \quad (\text{A37})$$

Taking the difference of (A31) and (A36) gives

$$\tilde{t} = -T \int_{R_+(T)}^{\infty} d\rho \frac{A(\rho, \tilde{t})^{1/2}}{\rho f(\rho)}. \quad (\text{A38})$$

This equation gives an implicit equation for T in terms of $\tilde{t}$.

Thus, the two areal radial coordinate charts together cover a region R_0 of the two-sided BTZ black hole which includes both AAdS boundaries. The region R_0 is depicted in Figure 2. It is demarcated by two constant r hyperbolas, one in Region F and one in Region P. These correspond to the $\tilde{t} \rightarrow \pm\infty$ spatial slices respectively.

One way to obtain the location of these *limit slices* is as follows. Recall that the maximum value of $|T|$ that is allowed by the expression for $R_{\pm}$ in (A16) is

$$-T_{\infty} \leq T \leq T_{\infty}, \quad T_{\infty} = \frac{\ell M}{4\pi}. \quad (\text{A39})$$

At this value of T , the two roots R_+ and R_- coincide and become

$$R_+(T = T_{\infty}) = R_-(T = T_{\infty}) = R_{\infty} \equiv \ell \sqrt{\frac{M}{4\pi}}. \quad (\text{A40})$$

From the implicit equation for T (A38), it is clear that when $T = \pm T_{\infty}$, there is a pole in the integrand at

$\rho = R_{\infty}$ due to the $A(\rho, \tilde{t})^{1/2}$ factor. This pole in the integrand is non-removable since it occurs at the end of the integration range. This leads to $\tilde{t} \rightarrow \pm\infty$ when $T = \pm T_{\infty}$. Thus, the limit slices $\tilde{t} \rightarrow \pm\infty$ are located at

$$r = R_{\infty} = \ell \sqrt{\frac{M}{4\pi}}. \quad (\text{A41})$$

Since $R_{\infty} < R_h$, these limit slices are hyperbolas which completely lie in Regions F and P. In Kruskal-Szekeres coordinates, these limit slices are the two branches of the hyperbola

$$UV = \eta^2 \frac{R_h - R_{\infty}}{R_h + R_{\infty}} = \eta^2 \frac{\sqrt{2} - 1}{\sqrt{2} + 1}. \quad (\text{A42})$$

These are displayed in Figure 2 as the orange curves in Regions F and P.

3. Diffeomorphism to Kruskal coordinates

Recall from Section II that the Kruskal coordinates were defined in terms of the Region I coordinates as $U = -\eta^{-1} e^{-\eta(t-r_*)}$ and $V = \eta e^{\eta(t+r_*)}$, where r_* is the tortoise coordinate

$$r_* = \int_{\infty}^r \frac{\ell^2 d\rho}{\rho^2 - R_h^2} = \frac{1}{2\eta} \log \frac{r - R_h}{r + R_h}. \quad (\text{A43})$$

Using the expression for the diffeomorphism $t(r, \tilde{t})$ (A25) for the BTZ time coordinate t from the right chart $(\tilde{t}, r)$, we get the following diffeomorphism from the right areal chart $(\tilde{t}, r)$ in Region I to (U, V) :

$$V(\tilde{t}, r) = \eta^{-1} e^{\eta \tilde{t} - R_h \int_r^{\infty} \frac{\rho^2 d\rho}{(\ell T + \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}) \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}}}, \quad (\text{A44})$$

$$U(\tilde{t}, r) = -\eta^{-1} e^{\eta \tilde{t} - R_h \int_r^{\infty} \frac{\rho^2 d\rho}{(\ell T - \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}) \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}}}. \quad (\text{A45})$$

When $T > 0$, i.e., when $\tilde{t} > 0$, the integral in the expression for U has a pole at $\rho = R_h$ which we can see by using $R_+(T)^2 + R_-(T)^2 = R_h^2$ for all T and $R_+^2 R_-^2 = \ell^2 T^2$. When the lower limit r of the integral is set to $r = R_h$, the integral diverges, so that U is driven to zero at the future horizon. For $r < R_h$, the integral can be evaluated using the principal value prescription. The additional $i\pi/\eta$ which comes from the contour integral around the pole flips the sign of U when $r < R_h$. The expression for v is regular at $r = R_h$ for $T > 0$, and hence it can be continued into Region F without encountering divergences.

When $T < 0$, i.e., when $\tilde{t} < 0$, the integrand for v has a pole at $\rho = R_h$ and the integral for v diverges to $-\infty$

when $r = R_h$. This drives $V = \eta^{-1}e^{\eta v}$ to zero at the past horizon. To continue into Region P from Region I, the integral for $r < R_h$ has to be evaluated with the principal value prescription. The imaginary part $i\pi/\eta$ from the contour integral around the pole flips the sign of V as one crosses the past horizon.

Thus, the diffeomorphism for the Kruskal-Szekeres coordinates U, V from the right areal radial coordinate chart $(\bar{t}, r)$ which covers Region I fully including the right AAdS boundary, and half each of Region F and Region P is

$$V(\bar{t}, r) = \pm \eta^{-1} e^{\eta \bar{t} - R_h f_r^\infty} \frac{\rho^2 d\rho}{(\ell T + \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}) \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}}, \quad (\text{A46})$$

$$U(\bar{t}, r) = \pm \eta^{-1} e^{\eta \bar{t} - R_h f_r^\infty} \frac{\rho^2 d\rho}{(\ell T - \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}) \sqrt{(\rho^2 - R_+^2)(\rho^2 - R_-^2)}}, \quad (\text{A47})$$

where the signs for Region I are $(-, +)$, Region F are $(+, +)$, and for Region P are $(+, -)$. A similar diffeomorphism can be written down for the left areal radial coordinate chart $(\bar{t}, \tilde{r})$ with $\tilde{r} > R_+(T)$.

A few constant $\bar{t}$ slices are plotted in Figure 3.

4. The wormhole coordinate

While we have matched the left and right charts, the metric component $g_{rr} = A(r)$ still diverges at the locus of matching points. We can remedy this by defining a new coordinate by

$$x^2 = r^2 - R_+(T(\bar{t}))^2. \quad (\text{A48})$$

This coordinate transformation is valid everywhere except the final limit slice where $r = R_+(T_\infty) = R_\infty = \ell \sqrt{M/4\pi}$. In this case, the entire final slice lies at $x = 0$ and x fails to be a good spatial coordinate on this final slice.

In terms of these coordinates $(\bar{t}, x, \varphi)$, the metric is given by

$$ds^2 = -N^2 d\bar{t}^2 + g_{xx} (dx + N^x d\bar{t})^2 + (x^2 + R_+(\bar{t})^2) d\varphi^2, \quad (\text{A49})$$

with

$$N(\bar{t}, x) = \frac{x \sqrt{x^2 + R_+^2 - R_-^2}}{\ell \sqrt{x^2 + R_+^2}} \times \left(1 + \dot{T} \ell^3 \int_x^\infty \frac{dy}{y^2} \frac{\sqrt{y^2 + R_+^2}}{(y^2 + R_+^2 - R_-^2)^{3/2}} \right), \quad (\text{A50a})$$

$$N^x(\bar{t}, x) = \frac{1}{x} \left(N(\bar{t}, x) T + R_+ \dot{R}_+ \right), \quad (\text{A50b})$$

$$g_{xx}(\bar{t}, x) = \frac{\ell^2}{x^2 + R_+^2 - R_-^2}. \quad (\text{A50c})$$

All components of the above spacetime metric are smooth and regular everywhere except at the limit slices $\bar{t} \rightarrow \pm\infty$ where the entire slice lies at $x = 0$.

First, we can see that metric component g_{xx} is regular everywhere since the denominator is positive everywhere except the limit slice where $R_+ = R_- = R_\infty$ and $x = 0$.

Similarly, the lapse and shift are regular and smooth everywhere. For instance, the Taylor expansion near $x = 0$ is

$$\lim_{x \rightarrow 0^+} N = \frac{\dot{T} \ell^2}{R_+^2 - R_-^2} + \dot{T} \ell^2 \frac{R_+^2 + R_-^2}{R_+^2 (R_+^2 - R_-^2)} x^2 + \mathcal{O}(x^4). \quad (\text{A51})$$

Thus we have constructed a coordinate chart $(\bar{t}, x, \varphi)$ with metric (A49), whose constant time slices are maximal, go from the left asymptotic boundary to the right asymptotic boundary while smoothly cutting across the two horizons.

The Kruskal-Szekeres coordinates in Region I are related to wormhole coordinates by

$$U(\bar{t}, x) = -\eta^{-1} e^{-\eta \bar{t} - R_h f_x^\infty} \frac{dy}{(y^2 - R_-^2) \sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+ (\bar{t})^2 - R_- (\bar{t})^2}} + y \right), \quad (\text{A52})$$

$$V(\bar{t}, x) = \eta^{-1} e^{\eta \bar{t} + R_h f_x^\infty} \frac{dy}{(y^2 - R_-^2) \sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+ (\bar{t})^2 - R_- (\bar{t})^2}} - y \right). \quad (\text{A53})$$

In Region F they are related by

$$U(\bar{t}, x) = \eta^{-1} e^{-\eta \bar{t} - R_h f_x^\infty} \frac{dy}{(y^2 - R_-^2) \sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+ (\bar{t})^2 - R_- (\bar{t})^2}} + y \right), \quad (\text{A54})$$

$$V(\bar{t}, x) = \eta^{-1} e^{\eta \bar{t} + R_h f_x^\infty} \frac{dy}{(y^2 - R_-^2) \sqrt{y^2 + R_+^2}} \left(\frac{\ell T}{\sqrt{y^2 + R_+ (\bar{t})^2 - R_- (\bar{t})^2}} - y \right). \quad (\text{A55})$$

Similarly one can write down the diffeomorphisms in regions II and P.

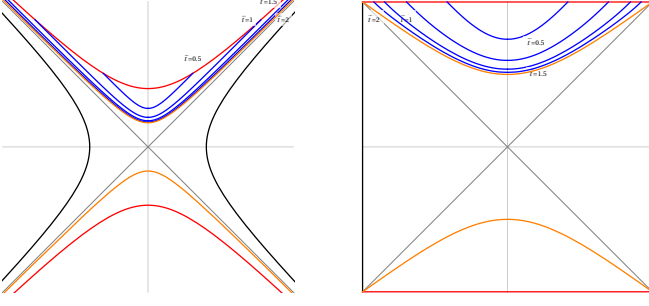


FIG. 10. Constant $\bar{t}$ slices of the second family are drawn in Kruskal-Szekeres diagram on the left and the Penrose diagram on the right. The red curves are the singularities at $r = 0$ while the black curves are the asymptotic boundaries. The maximal slices are drawn in blue. These slices start at the singularity, grow outwards and end at the final slice at $R_\infty = \ell\sqrt{\frac{M}{4\pi}}$ (orange curve) as $\bar{t} \rightarrow \infty$, thus cover region R_1 . Similarly one can cover the region R_{-1} by a similar set of slices of the second family.

5. A second family of maximal slices

There exist another family of maximal slices that lie completely inside the horizon. These spatial slices are sausage-like: they are cylindrical and cap off at the two ends at $r = 0$. We define the globally defined ‘sausage’ coordinate (analogous to the wormhole coordinate x for the first family)

$$z(r, \bar{t}) = \begin{cases} \sqrt{R_-(\bar{t})^2 - r^2} & \text{right chart} \\ -\sqrt{R_-(\bar{t})^2 - \tilde{r}^2} & \text{left chart} \end{cases}. \quad (\text{A56})$$

Since $z \in [-R_-(T), R_-(T)]$, the range of the sausage coordinate z is time dependent. The metric in these coordinates is given by

$$ds^2 = -N^2 d\bar{t}^2 + g_{zz} (dz + N^z d\bar{t})^2 + (R_-(\bar{t})^2 - z^2) d\varphi^2, \quad (\text{A57})$$

with

$$N(\bar{t}, z) = \frac{z\sqrt{z^2 + R_+^2 - R_-^2}}{\ell\sqrt{R_-^2 - z^2}} \left(1 + \dot{T}\ell^3 \times \int_{\sqrt{R_-^2 - z^2}}^{\infty} \frac{d\rho}{\rho} \left(\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} + \frac{T(\bar{t})^2}{\rho^2} \right)^{-3/2} \right), \quad (\text{A58a})$$

$$N^z(\bar{t}, z) = -\frac{1}{z} \left(N(\bar{t}, z)T + R_- \dot{R}_- \right), \quad (\text{A58b})$$

$$g_{zz}(\bar{t}, z) = \frac{\ell^2}{z^2 + R_+^2 - R_-^2}. \quad (\text{A58c})$$

The coordinate transformation to the BTZ coordinates (t_F, r, φ) in Region F is given by

$$r^2 = R_-(T(\bar{t}))^2 - z^2, \quad (\text{A59a})$$

$$t_F = -T(\bar{t}) \int_{R_-(T)}^{\sqrt{R_-^2 - z^2}} \frac{d\rho}{\rho \left(\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} \right) \sqrt{\frac{\rho^2}{\ell^2} - \frac{M}{2\pi} + \frac{T^2}{\rho^2}}}. \quad (\text{A59b})$$

These second family of slices are plotted in Figure 10. They start from $r = 0$ and cover the interior of the black hole up to a final limit slice at constant radial value $R_\infty = \ell\sqrt{M/4\pi}$, i.e., the region R_1 in Figure 2. The first family of maximal slices reached the same final slice but from the exterior (see Figure 3). Similarly one can cover the region R_{-1} by a similar set of slices of the second family. Together the two families foliate the entire black hole spacetime.

Appendix B: Probe scalar field is gauge invariant

Since we are interested in treating the scalar field as a perturbation around a pure gravity solution of the Einstein’s equations, we would like to establish its gauge invariance in this situation. This analysis is done in arbitrary spacetime dimension $d + 1$.

Recall that the gravitational Hamiltonian is given by (here $\kappa^2 = 16\pi G_N$ is the gravitational constant)

$$H[N, N_i] = \int_{\Sigma} d^d x (N\mathcal{H}_\perp + N_i \mathcal{H}^i) + H_\partial, \quad (\text{B1})$$

where $\mathcal{H}_\perp$ and $\mathcal{H}^i$ are respectively the Hamiltonian and momentum constraints

$$\begin{aligned} \mathcal{H}_\perp &= \frac{\kappa^2}{\sqrt{g}} \left(\pi^{ij} \pi_{ij} - \frac{1}{d-1} \pi^2 \right) - \frac{1}{\kappa^2} \sqrt{g} (R - 2\Lambda) + \rho, \\ \mathcal{H}^i &= -2D_j \pi^{ij} + S^i, \\ H_\partial &= -\frac{2}{\kappa^2} N \sqrt{\sigma} k + 2r_i N_j \pi^{ij} - \mathcal{B}_{\text{ct}}. \end{aligned} \quad (\text{B2})$$

Here, $\pi = g_{ij} \pi^{ij}$ is the trace of π^{ij} and the quantities ρ and S^i are the scalar field energy density and momentum density respectively,

$$\begin{aligned} \rho &= \frac{\pi_\phi^2}{2\sqrt{g}} + \sqrt{g} \left(\frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi + V(\phi) \right), \\ S^i &= g^{ij} \pi_\phi \partial_j \phi, \end{aligned} \quad (\text{B3})$$

with $V(\phi)$ being the scalar field potential.

Note that the scalar field appears in the Hamiltonian at $\mathcal{O}(\kappa^0)$ whereas the gravity part starts at $\mathcal{O}(\kappa^{-2})$. Since we want all terms corresponding to the pure gravity background to contribute at the same order of κ , we write down perturbations as

$$\begin{aligned} g_{ij} &= \hat{g}_{ij} + \kappa h_{ij}, & \pi^{ij} &= \frac{1}{\kappa^2} (\hat{P}^{ij} + \kappa p^{ij}), \\ N_i &= \hat{N}_i + \kappa \beta_i, & N &= \hat{N} (1 + \kappa \alpha), \end{aligned} \quad (\text{B4})$$

where the hatted quantities correspond to the pure gravity background solution and the perturbations are h_{ij} , p^{ij} , β_i and α . The scalar field is zero in the background and hence is purely a perturbation.

The Hamiltonian dynamics can now be analyzed order by order in κ . First, we describe the dynamics of the pure gravity background by solving the constraints and gauge conditions for the background and analyzing the Hamiltonian dynamics on the reduced phase space involving only the gravitational variables. This was discussed in detail in the companion paper [41]. Skipping over the details of the calculation, we find that the dynamics of the fluctuations are then governed by the quadratic Hamiltonian

$$H_2 = \int_{\Sigma} d^d x \left(\alpha \mathcal{C}_{\perp}(h_{ij}, p^{ij}) + \beta_i \mathcal{C}^i(h_{ij}, p^{ij}) + \mathcal{H}_{\text{grav}}(h_{ij}, p^{ij}) + \mathcal{H}_s(\phi, \pi_{\phi}) \right) + \text{boundary terms}, \quad (\text{B5})$$

where $\mathcal{C}_{\perp}$ and $\mathcal{C}^i$ are the linearized Hamiltonian and momentum constraints that depend (in addition to the pure gravity background) only on the gravity fluctuations,

$$\begin{aligned} \mathcal{C}_{\perp} &= \hat{N} (2\hat{P}^{ij} p_{ij} + 2h_{ik} \hat{P}^k_j \hat{P}^{ij} - h \hat{P}^{ij} \hat{P}_{ij} \\ &\quad - \hat{g} (\hat{D}^a \hat{D}^b h_{ab} - \hat{D}^2 h - h^{ac} \hat{R}_{ac})) , \\ \mathcal{C}^l &= 2\hat{D}_i p^{il} + \hat{P}^{ij} (\hat{D}_i h^l_j + \hat{D}_j h^l_i - \hat{D}^l h_{ij}) . \end{aligned} \quad (\text{B6})$$

$\mathcal{H}_{\text{grav}}(h_{ij}, p^{ij})$ and $\mathcal{H}_s(\phi, \pi_{\phi})$ are the dynamical, non-constraint, quadratic Hamiltonians for the metric and scalar degrees of freedom respectively,

$$\begin{aligned} &\mathcal{H}_{\text{grav}}(h_{ij}, p^{ij}) \\ &= \frac{\hat{N}}{\sqrt{\hat{g}}} \left\{ -\frac{1}{2} h (2\hat{P}^{ij} p_{ij} + 2h_{ik} \hat{P}^k_j \hat{P}^{ij}) \right. \\ &\quad + \frac{1}{4} (h^{ij} h_{ij} + \frac{1}{2} h^2) \hat{P}^{ij} \hat{P}_{ij} \\ &\quad + p^{ij} p_{ij} + 2p^{ij} h_{ik} \hat{P}^k_j + \hat{P}^{ij} \hat{P}^{kl} h_{ik} h_{jl} + 2\hat{P}^{ij} h_{ik} p^k_j \Big\} \\ &\quad - \hat{N} \sqrt{\hat{g}} \left\{ \frac{1}{4} (\frac{1}{2} h^2 - h_{ij} h^{ij}) (\hat{R} - 2\Lambda) \right. \\ &\quad + \frac{1}{2} h (\hat{D}^a \hat{D}^b h_{ab} - \hat{D}^2 h - h^{ac} \hat{R}_{ac}) \\ &\quad + \frac{3}{4} \hat{D}^a h^{bc} \hat{D}_a h_{bc} + h^{ab} \hat{D}^2 h_{ab} - \hat{D}_b h^{bc} \hat{D}^a h_{ac} - h^{bc} \hat{D}_b \hat{D}^a h_{ac} \\ &\quad + h^{ab} \hat{D}_a \hat{D}_b h + \hat{D}_a h^{ac} \hat{D}_c h - \frac{1}{2} \hat{D}_b h^{ac} \hat{D}_a h^b_c - \frac{1}{4} \hat{D}^a h \hat{D}_a h \\ &\quad \left. - h^{ac} \hat{D}_b \hat{D}_a h^b_c + h^{ae} h_e^c \hat{R}_{ac} \right\} \\ &\quad - (\hat{N}^l p^{ij} - \hat{N}_k h^{kl} \hat{P}^{ij}) C_{ijl} , \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \mathcal{H}_s(\phi, \pi_{\phi}) &= \hat{N} \left(\frac{1}{2\sqrt{\hat{g}}} \pi_{\phi}^2 + \sqrt{\hat{g}} (\frac{1}{2} \hat{g}^{ij} \partial_i \phi \partial_j \phi + V(\phi)) \right) \\ &\quad - \hat{N}_i \hat{g}^{ij} \pi_{\phi} \partial_j \phi . \end{aligned} \quad (\text{B8})$$

The important feature of the quadratic Hamiltonian H_2 (B5) is that the gravitational perturbations h_{ij}, p^{ij} and scalar fluctuations ϕ, π_{ϕ} do not mix. In $d = 2$,

since there are no local fluctuating gravitational degrees of freedom, the quadratic Hamiltonian H_2 (B5) involves only the scalar perturbations. Moreover, the linearized constraints do not contain the scalar field perturbations and hence do not act on the scalar field. Thus, if one has succeeded in completely fixing the diffeomorphism freedom on the background pure gravity solution, then *the scalar field is completely gauge invariant to this order in perturbation theory*.

Appendix C: Solutions of the Klein-Gordon equation in the BTZ spacetime

1. Separation of variables

The 2 + 1 components of the BTZ metric (2) are

$$N(r) = \sqrt{f(r)}, \quad N_i = 0, \quad g_{rr} = \frac{1}{f(r)}, \quad g_{\varphi\varphi} = r^2, \quad g_{r\varphi} = 0, \quad (\text{C1})$$

which gives $\sqrt{g} = r f(r)^{-1/2}$. The wave equation takes the form

$$-f^{-1} \ddot{\phi} + r^{-1} \partial_r (r f \partial_r \phi) + r^{-1} \partial_{\varphi} (r^{-1} \partial_{\varphi} \phi) - m^2 \phi = 0. \quad (\text{C2})$$

Since ∂_t and ∂_{φ} are Killing vectors, we can separate variables and write a particular mode of the scalar field as

$$F_{\omega q}(t, r, \varphi) \equiv e^{-i\omega t} e^{-iq\varphi} f_{\omega q}(r), \quad (\text{C3})$$

where ω, q are the frequency and angular momentum along the circle respectively. Changing variables to the dimensionless $\rho = r^2/R_h^2$ and defining

$$f_{\omega q}(\rho) = (\rho - 1)^{\alpha} \rho^{\gamma} \chi(\rho), \quad (\text{C4})$$

$$\text{with } \alpha = -i \frac{\omega}{2\eta}, \quad \gamma = i \frac{q\ell}{2R_h}, \quad (\text{C5})$$

we see that χ satisfies the hypergeometric equation:

$$\rho(1-\rho)\chi''(\rho) + (c - (a+b+1)\rho)\chi'(\rho) - ab\chi(\rho) = 0, \quad (\text{C6})$$

where we have defined

$$\begin{aligned} a &= \alpha + \gamma + \frac{1}{2} \Delta_+, \quad b = \alpha + \gamma + \frac{1}{2} \Delta_-, \\ c &= 2\gamma + 1, \quad \Delta_{\pm} = 1 \pm \sqrt{1 + \ell^2 m^2}. \end{aligned} \quad (\text{C7})$$

Note that Δ_+ is the conformal dimension of the operator dual to ϕ in the dual CFT. The hypergeometric equation has regular singular points at $\rho = 0$, $\rho = 1$ and $\rho = \infty$, corresponding to the origin (or inner horizon), the horizon (or the outer horizon), and infinity. There are solutions readily available (as a convergent series via the Frobenius method) in the neighbourhood of each of these points. This greatly facilitates further analysis. For instance, in the neighbourhood of $z = 0$,

$${}_2F_1(a_1, a_2; b_1; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(b_1)_n} \frac{z^n}{n!}, \quad (\text{C8})$$

with

$$(a)_n = a(a+1) \cdots (a+n-1), \quad (\text{C9})$$

so that ${}_2F_1(a_1, a_2; b_1; 0) = 1$. Below, we list the solutions $\chi(\rho)$ that are regular in the neighbourhood of the three singular points $\rho = \infty, 1, 0$. We use the notation of DLMF [75, §15.10]. The solutions regular at $\rho = 1$ are

$$\begin{aligned} w_3(\rho) &= {}_2F_1(a, b; a+b-c+1; 1-\rho), \\ w_4(\rho) &= (1-\rho)^{c-a-b} {}_2F_1(c-a, c-b; c-a-b+1; 1-\rho), \end{aligned} \quad (\text{C10})$$

the solutions regular at $\rho = \infty$ are

$$\begin{aligned} w_5(\rho) &= e^{i\pi a} \rho^{-a} {}_2F_1\left(a, a-c+1; a-b+1; \frac{1}{\rho}\right), \\ w_6(\rho) &= e^{i\pi b} \rho^{-b} {}_2F_1\left(b, b-c+1; b-a+1; \frac{1}{\rho}\right), \end{aligned} \quad (\text{C11})$$

and the solutions regular at $\rho = 0$ are

$$\begin{aligned} w_1(\rho) &= {}_2F_1(a, b; c; \rho), \\ w_2(\rho) &= \rho^{1-c} {}_2F_1(a-c+1, b-c+1; 2-c; \rho). \end{aligned} \quad (\text{C12})$$

2. Solution in Region I

Let us look at the solution in the exterior region I. Recalling that $f_{\omega q}(\rho) = (\rho-1)^\alpha \rho^\gamma \chi(\rho)$, the general solution in the neighbourhood of $\rho = \infty$ is a linear combination of $(\rho-1)^\alpha \rho^\gamma w_5$ and $(\rho-1)^\alpha \rho^\gamma w_6$ in (C11):

$$\begin{aligned} f_{\omega q}(\rho) &= C_{\omega q} e^{-i\pi a} (\rho-1)^\alpha \rho^\gamma w_5(\rho) + C'_{\omega q} e^{-i\pi b} (\rho-1)^\alpha \rho^\gamma w_6(\rho) \\ &= C_{\omega q} (\rho-1)^\alpha \rho^{\gamma-a} {}_2F_1\left(a, a-c+1; a-b+1; \frac{1}{\rho}\right) \\ &\quad + C'_{\omega q} (\rho-1)^\alpha \rho^{\gamma-b} {}_2F_1\left(b, b-c+1; b-a+1; \frac{1}{\rho}\right), \end{aligned} \quad (\text{C13})$$

for some constants $C_{\omega q}$ and $C'_{\omega q}$. We find that the normalizable component of $f_{\omega q}(\rho)$ w.r.t. the Klein-Gordon inner product is proportional to w_5 and is given by

$$f_{\omega q}(\rho) = C_{\omega q} (\rho-1)^\alpha \rho^{\gamma-a} {}_2F_1\left(a, a-c+1; a-b+1; \frac{1}{\rho}\right). \quad (\text{C14})$$

The constant $C_{\omega q}$ can be fixed by demanding that $F_{\omega q} = e^{-i\omega t} e^{-iq\varphi} f_{\omega q}$ satisfy the canonical Klein-Gordon inner product

$$(F_{\omega' q'}, F_{\omega q})_{\text{KG}} = 4\pi\omega\delta(\omega-\omega')\delta_{qq'}. \quad (\text{C15})$$

We do this in Appendix C 8 below and obtain

$$C_{\omega q} = \frac{1}{N_{\omega q} \sqrt{2\pi R_h}}, \quad N_{\omega q} = \left| \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} \right|. \quad (\text{C16})$$

The mode expansion for the scalar field in Region I reads

$$\phi(t, r, \varphi) = \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} a_{\omega q} F_{\omega q}(t, r, \varphi) + \text{c.c.} \quad (\text{C17})$$

We then have

$$a_{\omega q} = \frac{1}{\sqrt{4\pi\omega}} (F_{\omega q}, \phi)_{\text{KG}}. \quad (\text{C18})$$

Using that $F_{\omega q}$ satisfy the canonical Klein-Gordon inner product (C15), we get the commutation relations

$$[a_{\omega q}, a_{\omega' q'}^\dagger] = \delta(\omega-\omega')\delta_{qq'}, \quad [a_{\omega q}, a_{\omega' q'}] = [a_{\omega q}^\dagger, a_{\omega' q'}^\dagger] = 0. \quad (\text{C19})$$

a. Solutions regular at the horizon. The mode functions (C14) have a good Taylor series near $\rho = \infty$. To obtain an expression near the horizon for the mode function in Region I, we have to first use a connection formula to write $w_5(\rho)$ in terms of hypergeometric functions that have a good expansion near $\rho = 1$:

$$\begin{aligned} w_5(\rho) &= e^{i\pi a} \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} w_3(\rho) \\ &\quad + e^{i\pi(c-b)} \frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)} w_4(\rho). \end{aligned} \quad (\text{C20})$$

Plugging in the values of a, b, c (C7) in the coefficients, we see that, when ω and q are real,

$$\begin{aligned} \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} &= \left(\frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)} \right)^* \\ &= \frac{\Gamma(\Delta_+) \Gamma(i\omega/\eta)}{\Gamma\left(\frac{1}{2}(\Delta_+ + \frac{i\omega}{\eta} - \frac{iq\ell}{R_h})\right) \Gamma\left(\frac{1}{2}(\Delta_+ + \frac{i\omega}{\eta} + \frac{iq\ell}{R_h})\right)}. \end{aligned} \quad (\text{C21})$$

Let

$$\frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)} = N_{\omega q} e^{i\delta_{\omega q}}, \quad (\text{C22})$$

with $N_{\omega q}$ the magnitude and $\delta_{\omega q}$ the phase of the left-hand side. We have already encountered $N_{\omega q}$ in the normalization of the mode function (C14). We can then write

$$\begin{aligned} f_{\omega q} &= \frac{e^{-i\pi a}}{N_{\omega q} \sqrt{2\pi R_h}} (\rho-1)^\alpha \rho^\gamma w_5(\rho), \\ &= \frac{1}{\sqrt{2\pi R_h}} \left(e^{i\delta_{\omega q}} (\rho-1)^\alpha \rho^\gamma w_3(\rho) \right. \\ &\quad \left. + e^{i\pi(c-b-a)} e^{-i\delta_{\omega q}} (\rho-1)^\alpha \rho^\gamma w_4(\rho) \right), \end{aligned} \quad (\text{C23})$$

which can be written more explicitly using (C10) as [76]

$$\begin{aligned} f_{\omega q} &= \frac{\rho^\gamma}{\sqrt{2\pi R_h}} \left(e^{i\delta_{\omega q}} (\rho-1)^\alpha {}_2F_1(a, b; a+b-c+1; 1-\rho) \right. \\ &\quad \left. + e^{-i\delta_{\omega q}} (\rho-1)^{-\alpha} {}_2F_1(c-a, c-b; c-a-b+1; 1-\rho) \right). \end{aligned} \quad (\text{C24})$$

It will be useful to write the full mode function $F_{\omega q}$ which also includes the t and φ dependence succinctly as

$$F_{\omega q} = e^{-i\omega t - iq\varphi} f_{\omega q} = \frac{1}{\sqrt{2\pi R_h}} (F_{\omega q}^1 + F_{\omega q}^2), \quad (C25)$$

with

$$\begin{aligned} F_{\omega q}^1 &= e^{-i\omega t - iq\varphi} e^{i\delta_{\omega q}} (\rho - 1)^\alpha \rho^\gamma \times \\ &\quad \times {}_2F_1(a, b; a + b - c + 1; 1 - \rho), \\ F_{\omega q}^2 &= e^{-i\omega t - iq\varphi} e^{-i\delta_{\omega q}} (\rho - 1)^{-\alpha} \rho^\gamma \times \\ &\quad \times {}_2F_1(c - a, c - b; c - a - b + 1; 1 - \rho), \end{aligned} \quad (C26)$$

Near the horizon $\rho = 1$ in Region I, we have $\rho - 1 \sim 4e^{2\eta r_*}$ (with r_* the tortoise coordinate in Region I) so that the above functions behave as

$$\begin{aligned} F_{\omega q}^1 &\sim e^{-iq\varphi} e^{i\delta_{\omega q}} (2\eta V)^{-i\omega/\eta}, \\ F_{\omega q}^2 &\sim e^{-iq\varphi} e^{-i\delta_{\omega q}} (-2\eta U)^{i\omega/\eta}. \end{aligned} \quad (C27)$$

Thus, the scalar field behaves as

$$\begin{aligned} \phi(U, V, \varphi) &\approx \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} a_{\omega q} e^{-iq\varphi} \times \\ &\quad (e^{i\delta_{\omega q}} (2\eta V)^{-i\omega/\eta} + e^{-i\delta_{\omega q}} (-2\eta U)^{i\omega/\eta}) + \text{c.c.} \end{aligned} \quad (C28)$$

3. Solution in region II

We write the ansatz for a mode of the scalar field in Region II as

$$\tilde{F}_{\omega q}(\tilde{t}, \tilde{r}, \varphi) = e^{i\omega \tilde{t}} e^{iq\varphi} \tilde{f}_{\omega q}(\tilde{r}). \quad (C29)$$

The solution for $\tilde{f}_{\omega q}$ that is normalizable has the same functional form as (C14) above, but with the choices

$$\begin{aligned} \tilde{\alpha} &= \alpha^* = \frac{i\omega}{2\eta}, \quad \tilde{\gamma} = \gamma^* = -\frac{iq\ell}{2R_h}, \quad \Delta_\pm = 1 \pm \sqrt{1 + \ell^2 m^2}, \\ \tilde{c} &= 2\tilde{\gamma} + 1, \quad \tilde{a} = \tilde{\alpha} + \tilde{\gamma} + \frac{1}{2}\Delta_+, \quad \tilde{b} = \tilde{\alpha} + \tilde{\gamma} + \frac{1}{2}\Delta_-. \end{aligned} \quad (C30)$$

The mode function in region II is then

$$\tilde{f}_{\omega q}(\tilde{\rho}) = C_{\omega q} (\tilde{\rho} - 1)^{\tilde{\alpha}} \tilde{\rho}^{\tilde{\gamma} - \tilde{a}} {}_2F_1\left(\tilde{a}, \tilde{a} - \tilde{c} + 1; \tilde{a} - \tilde{b} + 1; \frac{1}{\tilde{\rho}}\right). \quad (C31)$$

$\tilde{\alpha}$ and $\tilde{\gamma}$ are chosen as above so that $\tilde{f}_{\omega q}^*$ is exactly the same function of ρ as $f_{\omega q}$ in Region I (C14). This choice is useful when we define the Hartle-Hawking modes.

The scalar field mode expansion in Region II is then

$$\phi(\tilde{t}, \tilde{r}, \varphi) = \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \tilde{a}_{\omega q} \tilde{F}_{\omega q}(\tilde{t}, \tilde{r}, \varphi) + \text{c.c.}, \quad (C32)$$

with the canonical commutation relations

$$[\tilde{a}_{\omega q}, \tilde{a}_{\omega' q'}^\dagger] = \delta(\omega - \omega'), \quad [\tilde{a}_{\omega q}, \tilde{a}_{\omega' q'}] = [\tilde{a}_{\omega q}^\dagger, \tilde{a}_{\omega' q'}^\dagger] = 0. \quad (C33)$$

We note the following property of $\tilde{F}_{\omega q}$ under complex conjugation:

$$\tilde{F}_{-\omega, -q}^* = \tilde{F}_{\omega q} = F_{-\omega, -q}. \quad (C34)$$

We also record mode functions that are regular near the horizon $\tilde{\rho} = 1$. From the relations (C34), it follows that

$$\tilde{F}_{\omega q} = \frac{1}{\sqrt{2\pi R_h}} (F_{\omega q}^{1*} + F_{\omega q}^{2*}). \quad (C35)$$

The near-horizon $\tilde{\rho} \rightarrow 1$ behaviour of the scalar field is then

$$\begin{aligned} \phi(U, V, \varphi) &\approx \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \tilde{a}_{\omega q} e^{iq\varphi} \times \\ &\quad (e^{-i\delta_{\omega q}} (-2\eta V)^{i\omega/\eta} + e^{i\delta_{\omega q}} (2\eta U)^{-i\omega/\eta}) + \text{c.c.} \end{aligned} \quad (C36)$$

4. Solution in Region F

The ansatz in Region F is

$$e^{-i\omega t_F} e^{-iq\varphi} f_{\omega q}^F(r). \quad (C37)$$

As usual, we write $f^F(r) = (1 - \rho)^\alpha \rho^\gamma \chi(\rho)$ so that χ satisfies the hypergeometric equation (C6). There is no normalizability condition imposed on the solution in Region F. Hence, both linearly independent solutions to the Klein-Gordon equation are admitted. We write them in terms of hypergeometric functions w_3, w_4 (C10) which are defined in a neighbourhood of the horizon $\rho = 1$. The mode expansion of the scalar field is then

$$\begin{aligned} \phi(t_F, r, \varphi) &= \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(C_{\omega q}^F G^1(t_F, r, \varphi) \right. \\ &\quad \left. + C_{\omega q}^{F'} G^2(t_F, r, \varphi) + \text{c.c.} \right), \end{aligned} \quad (C38)$$

where $C_{\omega q}^F$ and $C_{\omega q}^{F'}$ are some constants, and $G_{\omega q}^1$ and $G_{\omega q}^2$ are defined as

$$\begin{aligned} G_{\omega q}^1 &= e^{i\delta_{\omega q}} e^{-i\omega t_F - iq\varphi} (1 - \rho)^\alpha \rho^\gamma \times \\ &\quad \times {}_2F_1(a, b; a + b - c + 1; 1 - \rho), \\ G_{\omega q}^2 &= e^{-i\delta_{\omega q}} e^{-i\omega t_F - iq\varphi} (1 - \rho)^{-\alpha} \rho^\gamma \times \\ &\quad \times {}_2F_1(c - a, c - b; c - a - b + 1; 1 - \rho). \end{aligned} \quad (C39)$$

We fix the constants $C_{\omega q}^F$ and $C_{\omega q}^{F'}$ by matching with the mode expansions in Regions I and II at the horizons.

The near horizon behaviour of the G^i can be obtained by writing $1 - \rho \sim 4e^{2\eta r_F}$. Then,

$$\begin{aligned} G_{\omega q}^1 &\sim e^{-iq\varphi} e^{i\delta_{\omega q}} (2\eta V)^{-i\omega/\eta}, \\ G_{\omega q}^2 &\sim e^{-iq\varphi} e^{-i\delta_{\omega q}} (2\eta U)^{i\omega/\eta}, \end{aligned} \quad (C40)$$

which implies

$$\begin{aligned} \phi(U, V, \varphi) &\sim \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \times \\ &\times \left[e^{-iq\varphi} (C_{\omega q}^F e^{i\delta_{\omega q}} (2\eta V)^{-i\omega/\eta} + C_{\omega q}^{F'} e^{-i\delta_{\omega q}} (2\eta U)^{i\omega/\eta}) \right. \\ &\left. + e^{iq\varphi} (C_{\omega q}^{F*} e^{-i\delta_{\omega q}} (2\eta V)^{i\omega/\eta} + C_{\omega q}^{F'*} e^{i\delta_{\omega q}} (2\eta U)^{-i\omega/\eta}) \right]. \end{aligned} \quad (C41)$$

This expression should match the expression from Region I at the future horizon $U = 0$ and that from Region II at the past horizon $V = 0$ so that the scalar field is continuous across the horizons [22]. As $U \rightarrow 0$, the U dependent part oscillates progressively faster and can be set to zero in all expressions (similarly for the $V \rightarrow 0$ case). The near-horizon expression for the scalar field in Region I is

$$\begin{aligned} \phi(t, r, \varphi) &\approx \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \times \\ &\times \left[a_{\omega q} e^{-iq\varphi} (e^{i\delta_{\omega q}} (2\eta V)^{-i\omega/\eta} + e^{-i\delta_{\omega q}} (-2\eta U)^{i\omega/\eta}) \right. \\ &\left. + a_{\omega q}^\dagger e^{iq\varphi} (e^{-i\delta_{\omega q}} (2\eta V)^{i\omega/\eta} + e^{i\delta_{\omega q}} (-2\eta U)^{-i\omega/\eta}) \right]. \end{aligned} \quad (C42)$$

Similarly, the near-horizon expression for the scalar field in Region II is

$$\begin{aligned} \phi(\tilde{t}, \tilde{r}, \varphi) &\approx \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \times \\ &\times \left[\tilde{a}_{\omega q} e^{iq\varphi} (e^{-i\delta_{\omega q}} (-2\eta V)^{i\omega/\eta} + e^{i\delta_{\omega q}} (2\eta U)^{-i\omega/\eta}) \right. \\ &\left. + \tilde{a}_{\omega q}^\dagger e^{-iq\varphi} (e^{i\delta_{\omega q}} (-2\eta V)^{-i\omega/\eta} + e^{-i\delta_{\omega q}} (2\eta U)^{i\omega/\eta}) \right]. \end{aligned} \quad (C43)$$

Matching (C41) with (C42) and (C43) at $U = 0$ and $V = 0$ respectively, we get

$$C_{\omega q}^F = a_{\omega q}, \quad C_{\omega q}^{F'} = \tilde{a}_{\omega q}^\dagger. \quad (C44)$$

Thus, the mode expansion in Region F is

$$\begin{aligned} \phi(t_F, r, \varphi) &= \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \times \\ &\times \left(a_{\omega q} G_{\omega q}^1 + \tilde{a}_{\omega q}^\dagger G_{\omega q}^2 + \text{c.c.} \right). \end{aligned} \quad (C45)$$

5. Solution in Region P

The ansatz in Region P for the scalar field is

$$e^{i\omega t_P} e^{iq\varphi} f_{\omega q}^P(\tilde{r}), \quad (C46)$$

with the solutions

$$\begin{aligned} f_{\omega q}^P(\tilde{r}) &= C_{\omega q}^P (1 - \tilde{\rho})^{\tilde{\alpha}} \tilde{\rho}^{\tilde{\gamma}} w_3(\tilde{\rho}) + C_{\omega q}^{P'} (1 - \tilde{\rho})^{\tilde{\alpha}} \tilde{\rho}^{\tilde{\gamma}} w_4(\tilde{\rho}), \\ &= C_{\omega q}^P (1 - \tilde{\rho})^{\tilde{\alpha}} \tilde{\rho}^{\tilde{\gamma}} {}_2F_1\left(\tilde{a}, \tilde{b}; \tilde{a} + \tilde{b} - \tilde{c} + 1; 1 - \tilde{\rho}\right) \\ &\quad + C_{\omega q}^{P'} (1 - \tilde{\rho})^{-\tilde{\alpha}} \tilde{\rho}^{\tilde{\gamma}} {}_2F_1\left(\tilde{c} - \tilde{a}, \tilde{c} - \tilde{b}; \tilde{c} - \tilde{a} - \tilde{b} + 1; 1 - \tilde{\rho}\right), \end{aligned} \quad (C47)$$

for some constants $C_{\omega q}^P$ and $C_{\omega q}^{P'}$. Note that the functions above are simply G^{1*} and G^{2*} respectively so that the scalar field in Region P has the mode expansion

$$\begin{aligned} \phi(t_P, \tilde{r}, \varphi) &= \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(C_{\omega q}^P G_{\omega q}^{1*}(t_P, \tilde{r}, \varphi) \right. \\ &\quad \left. + C_{\omega q}^{P'} G_{\omega q}^{2*}(t_P, \tilde{r}, \varphi) + \text{c.c.} \right). \end{aligned} \quad (C48)$$

The near horizon expression is

$$\begin{aligned} \phi(t_P, \tilde{r}, \varphi) &\sim \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \times \\ &\times \left[e^{iq\varphi} (C_{\omega q}^P e^{-i\delta_{\omega q}} (-2\eta V)^{i\omega/\eta} + C_{\omega q}^{P'} e^{i\delta_{\omega q}} (-2\eta U)^{-i\omega/\eta}) \right. \\ &\left. + e^{-iq\varphi} (C_{\omega q}^{P*} e^{i\delta_{\omega q}} (-2\eta V)^{-i\omega/\eta} + C_{\omega q}^{P'*} e^{-i\delta_{\omega q}} (-2\eta U)^{i\omega/\eta}) \right]. \end{aligned} \quad (C49)$$

At the future horizon $U = 0$, (C49) has to be matched with the expression (C43) in Region II, whereas at the past horizon $V = 0$, it has to be matched with the expression (C42) in Region I. We get

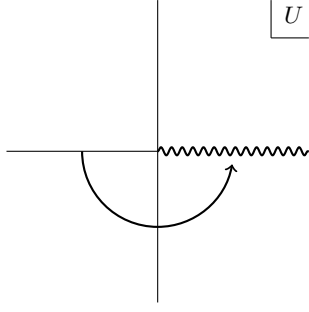
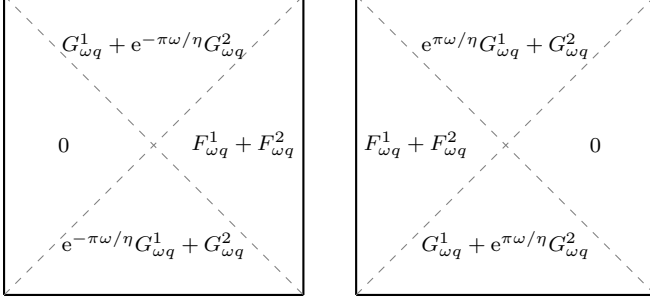
$$C_{\omega q}^P = \tilde{a}_{\omega q}, \quad C_{\omega q}^{P'} = a_{\omega q}^\dagger, \quad (C50)$$

so that the mode expansion in Region P is

$$\begin{aligned} \phi(t_P, \tilde{r}, \varphi) &= \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \times \\ &\times \left(\tilde{a}_{\omega q} G_{\omega q}^{1*} + a_{\omega q}^\dagger G_{\omega q}^{2*} + \text{c.c.} \right). \end{aligned} \quad (C51)$$

6. Analytic continuation to regions F and P

In the previous subsections, we solved the Klein-Gordon equation separately in each Region and then matched the expressions across the horizons. These mode functions were analytic in the lower-half t plane (where t has to be replaced by its counterpart in each Region). One can also consider functions which are analytic in the lower-half U or V planes. This consideration is motivated by the simple observation, due to Unruh [32], that translations in the V coordinate are a symmetry of the metric on the future horizon $U = 0$ (and similarly, translations in U on the future horizon $V = 0$). One can then naturally write down mode functions $e^{-i\omega V}$ which

FIG. 11. Analytic continuation in the lower-half U -plane.FIG. 12. The above are Penrose diagrams for the fully extended BTZ black hole. **Left:** The analytic continuation of Region I mode $F_{\omega q} = F_{\omega q}^1 + F_{\omega q}^2$ to rest of the fully extended BTZ spacetime. **Right:** The analytic continuation of Region II conjugate modes $F_{\omega q}^* = F_{\omega q}^1 + F_{\omega q}^2$.

are analytic in the lower-half V plane. Following Unruh, let us try to obtain appropriate analytic continuations of the mode functions $F_{\omega q}^i$ such that they are analytic in the lower-half U and V planes.

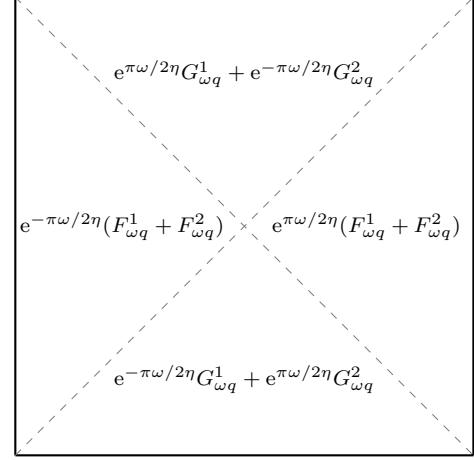
The mode functions $F_{\omega q}^i$ (defined in (C26)) in Region I are valid for $U < 0$, $V > 0$, and are certainly not analytic in the lower-half U and V planes. However, one can make them analytic in the lower-half U plane by appropriately defining their continuation to $U > 0$, across the future horizon $U = 0$ (and similarly, across the past horizon $V = 0$). Recall from (C27) that $F_{\omega q}^1 \sim V^{-i\omega/\eta}$. Hence, this continues uneventfully across $U = 0$. The function $F_{\omega q}^2 \sim U^{i\omega/\eta}$ requires the introduction of a branch cut in the complex U plane to be well-defined. Let us take the branch cut along the positive U axis and define

$$\text{For } U > 0, \quad \underbrace{U}_{\text{in F}} = e^{i\pi} \underbrace{(-U)}_{\text{in I}}. \quad (\text{C52})$$

This corresponds to a rotation by π in the counter-clockwise direction in the lower-half U plane, see Figure 11. This ensure that $U^{i\omega/\eta}$ is analytic in the lower-half

plane. How do we implement this for the full function $F_{\omega q}^2$ and not just its near-horizon expression? Recall the definition of the Kruskal U in Regions I and F:

$$\text{I: } U = -\eta^{-1} e^{-\eta(t-r_*)}, \quad \text{F: } U = \eta^{-1} e^{-\eta(t_F-r_{*F})}. \quad (\text{C53})$$

FIG. 13. The definition of the Hartle-Hawking modes $h_{\omega q}$ obtained by the Unruh procedure.

To get formulas in Region F, we replace $t - r_*$ by $t - r_* - i\pi/\eta$ in all formulas in Region I. But since the local coordinates in Region F are called t_F and r_F , we relabel t and r_* as t_F and r_F respectively, after the replacement. Effectively, we have

$$t - r_* \rightarrow t_F - r_{*F} - i\pi/\eta, \quad t + r_* \rightarrow t_F + r_{*F}, \quad (\text{C54})$$

that is, in continuing from Region I $\rightarrow$ F:

$$t \rightarrow t_F - i\pi/2\eta, \quad r_* \rightarrow r_{*F} + i\pi/2\eta \Rightarrow \rho - 1 \rightarrow e^{i\pi}(1 - \rho). \quad (\text{C55})$$

Next, the continuation across the past horizon $V = 0$ involves taking V to $-V$ through the lower half complex V plane while keeping U fixed:

$$\text{For } V < 0, \quad \underbrace{V}_{\text{in P}} = e^{-i\pi} \underbrace{(-V)}_{\text{in I}} = -e^{-i\pi} \eta^{-1} e^{\eta(t+r_*)}, \quad (\text{C56})$$

which continuing from Region I $\rightarrow$ P, corresponds to

$$t \rightarrow t_P - i\pi/2\eta, \quad r_* \rightarrow r_{*P} - i\pi/2\eta \Rightarrow \rho - 1 \rightarrow e^{-i\pi}(1 - \bar{\rho}). \quad (\text{C57})$$

Thus, we apply the above rules to $F_{\omega q}^i$ to obtain the appropriate analytic continuation to Regions F and P. We then get

$$\begin{aligned}
\text{Region I to F: } & F_{\omega q}^1(t, r, \varphi) \rightarrow G_{\omega q}^1(t_F, r, \varphi), \quad F_{\omega q}^2(t, r, \varphi) \rightarrow e^{-\pi\omega/\eta} G_{\omega q}^2(t_F, r, \varphi), \\
\text{Region I to P: } & F_{\omega q}^1(t, r, \varphi) \rightarrow e^{-\pi\omega/\eta} G_{\omega q}^1(t_P, \tilde{r}, \varphi), \quad F_{\omega q}^2(t, r, \varphi) \rightarrow G_{\omega q}^2(t_P, \tilde{r}, \varphi).
\end{aligned} \tag{C58}$$

The analytic continuation of the $F_{\omega q}^i$ are then

$$F_{\omega q}^1 = \begin{cases} F_{\omega q}^1 & \text{Region I} \\ G_{\omega q}^1 & \text{Region F} \\ e^{-\pi\omega/\eta} G_{\omega q}^1 & \text{Region P} \\ 0 & \text{Region II.} \end{cases} \quad F_{\omega q}^2 = \begin{cases} F_{\omega q}^2 & \text{Region I} \\ e^{-\pi\omega/\eta} G_{\omega q}^2 & \text{Region F} \\ G_{\omega q}^2 & \text{Region P} \\ 0 & \text{Region II.} \end{cases} \tag{C59}$$

This is summarized on the left of Figure 12.

Note: Most importantly, the function presented (C59) is analytic for negative values of ω as well, since the procedure above did not involve positivity of ω .

Similarly, we can work out the continuation rules for Region II to Regions F and P:

$$\begin{aligned}
\text{Region II} \rightarrow \text{Region F: } & \tilde{t} \rightarrow t_F + i\pi/2\eta, \quad \tilde{r}_* \rightarrow r_{*F} + i\pi/2\eta \Rightarrow \tilde{\rho} - 1 \rightarrow e^{i\pi}(1 - \rho), \\
\text{Region II} \rightarrow \text{Region P: } & \tilde{t} \rightarrow t_P + i\pi/2\eta, \quad \tilde{r}_* \rightarrow r_{*P} - i\pi/2\eta \Rightarrow \tilde{\rho} - 1 \rightarrow e^{-i\pi}(1 - \tilde{\rho}).
\end{aligned} \tag{C60}$$

The Region II modes (C35) involve F^{i*} . Under the rules (C60), the analytic continuation to Regions F and P are

$$\begin{aligned}
\text{Region II to F: } & F_{\omega q}^{1*}(\tilde{t}, \tilde{r}, \varphi) \rightarrow e^{-\pi\omega/\eta} G_{\omega q}^{1*}(t_F, r, \varphi), \quad F_{\omega q}^{2*}(\tilde{t}, \tilde{r}, \varphi) \rightarrow G_{\omega q}^{2*}(t_F, r, \varphi), \\
\text{Region II to P: } & F_{\omega q}^{1*}(\tilde{t}, \tilde{r}, \varphi) \rightarrow G_{\omega q}^{1*}(t_P, \tilde{r}, \varphi), \quad F_{\omega q}^{2*}(\tilde{t}, \tilde{r}, \varphi) \rightarrow e^{-\pi\omega/\eta} G_{\omega q}^{2*}(t_P, \tilde{r}, \varphi).
\end{aligned} \tag{C61}$$

We also need the analytic continuation of the conjugate mode functions in Region II. These simply involve the $F_{\omega q}^i$. Applying the rules (C60), we get

$$\begin{aligned}
\text{Region II to F: } & F_{\omega q}^1(\tilde{t}, \tilde{r}, \varphi) \rightarrow e^{\pi\omega/\eta} G_{\omega q}^1(t_F, r, \varphi), \quad F_{\omega q}^2(\tilde{t}, \tilde{r}, \varphi) \rightarrow G_{\omega q}^2(t_F, r, \varphi), \\
\text{Region II to P: } & F_{\omega q}^1(\tilde{t}, \tilde{r}, \varphi) \rightarrow G_{\omega q}^1(t_P, \tilde{r}, \varphi), \quad F_{\omega q}^2(\tilde{t}, \tilde{r}, \varphi) \rightarrow e^{\pi\omega/\eta} G_{\omega q}^2(t_P, \tilde{r}, \varphi).
\end{aligned} \tag{C62}$$

The analytic continuations of the Region II conjugate modes are summarized on the right in Figure 12. Observe that the analytic continuation of the rescaled Region I mode function $e^{\pi\omega/2\eta} F_{\omega q} = e^{\pi\omega/2\eta} (F^1 + F^2)$ and the rescaled Region II mode function $e^{-\pi\omega/2\eta} \tilde{F}_{\omega q}^* = e^{-\pi\omega/2\eta} (F^1 + F^2)$ agree on Region F and Region P. Using these analytic continuations, we can define a function on the fully extended BTZ black hole that is analytic in the lower half U, V planes as

$$h_{\omega}(U, V, \varphi) = \frac{1}{\sqrt{4\pi\omega} \sqrt{2 \sinh(\pi\omega/\eta)}} \times \begin{cases} e^{\pi\omega/2\eta} (F_{\omega q}^1 + F_{\omega q}^2) & \text{Region I} \\ e^{\pi\omega/2\eta} G_{\omega q}^1 + e^{-\pi\omega/2\eta} G_{\omega q}^2 & \text{Region F} \\ e^{-\pi\omega/2\eta} G_{\omega q}^1 + e^{\pi\omega/2\eta} G_{\omega q}^2 & \text{Region P} \\ e^{-\pi\omega/2\eta} (F_{\omega q}^1 + F_{\omega q}^2) & \text{Region II.} \end{cases} \tag{C63}$$

Note that these modes are analytic in the lower half U and V planes for both positive and negative ω . Thus, ω is a label running over all real numbers for these modes,

and only when we include ω of both signs do we get a complete set of modes.

7. The Hartle-Hawking modes and Bogoliubov coefficients

The scalar field can then be expanded in terms of these Hartle-Hawking modes:

$$\phi(U, V, \varphi) = \frac{1}{\sqrt{2\pi R_h}} \sum_{q \in \mathbb{Z}} \int_{-\infty}^{\infty} d\omega c_{\omega q} h_{\omega q} + \text{c.c.} \tag{C64}$$

For future use, we define $\tilde{c}_{\omega q} = c_{-\omega, -q}$ for $\omega > 0$. Restricting to Region I, we get

$$\begin{aligned}
\phi(U, V, \varphi)|_{\text{I}} &= \sum_{q \in \mathbb{Z}} \int_0^{\infty} \frac{d\omega}{\sqrt{4\pi\omega} \sqrt{2 \sinh \frac{\pi\omega}{\eta}}} \times \\
&\times \left((c_{\omega q} e^{\pi\omega/2\eta} + \tilde{c}_{\omega q}^\dagger e^{-\pi\omega/2\eta}) F_{\omega q} \right. \\
&\left. + (\tilde{c}_{\omega q} e^{-\pi\omega/2\eta} + c_{\omega q}^\dagger e^{\pi\omega/2\eta}) F_{\omega q}^* \right). \tag{C65}
\end{aligned}$$

Comparing with the mode expansion in Region I (C17), we get the following relations between the operators $a_{\omega q}$ and $c_{\omega q}$, $\tilde{c}_{\omega q}$:

$$\begin{aligned} a_{\omega q} &= \frac{c_{\omega q} e^{\pi\omega/2\eta} + \tilde{c}_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \\ a_{\omega q}^\dagger &= \frac{\tilde{c}_{\omega q}^\dagger e^{\pi\omega/2\eta} + c_{\omega q} e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}. \end{aligned} \quad (\text{C66})$$

Similarly, restricting the global mode expansion (C64) to Region II, and comparing with the mode expansion in Region II (C32), we get

$$\begin{aligned} \tilde{a}_{\omega q} &= \frac{\tilde{c}_{\omega q} e^{\pi\omega/2\eta} + c_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \\ \tilde{a}_{\omega q}^\dagger &= \frac{\tilde{c}_{\omega q}^\dagger e^{\pi\omega/2\eta} + c_{\omega q} e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}. \end{aligned} \quad (\text{C67})$$

We can invert the relations (C66) and (C67) together to get the Bogoliubov transformations

$$\begin{aligned} c_{\omega q} &= \frac{a_{\omega q} e^{\pi\omega/2\eta} - \tilde{a}_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \\ \tilde{c}_{\omega q} &= \frac{\tilde{a}_{\omega q} e^{\pi\omega/2\eta} - a_{\omega q}^\dagger e^{-\pi\omega/2\eta}}{\sqrt{2 \sinh(\pi\omega/\eta)}}, \end{aligned} \quad (\text{C68})$$

and their conjugates. The commutation relations for $c_{\omega q}$, $\tilde{c}_{\omega q}$ for $\omega > 0$ can be obtained from those of the $a_{\omega q}$, $\tilde{a}_{\omega q}$:

$$\begin{aligned} [c_{\omega q}, c_{\omega' q'}^\dagger] &= [\tilde{c}_{\omega q}, \tilde{c}_{\omega' q'}^\dagger] = \delta(\omega - \omega') \delta_{qq'}, \\ [c_{\omega q}, \tilde{c}_{\omega' q'}] &= [c_{\omega q}, c_{\omega' q'}] = [\tilde{c}_{\omega q}, \tilde{c}_{\omega' q'}] = 0. \end{aligned} \quad (\text{C69})$$

8. Klein-Gordon normalization of the exterior mode functions

The Klein-Gordon inner product between two solutions ϕ_1 and ϕ_2 of the Klein-Gordon equation is defined by

$$(\phi_1, \phi_2)_{\text{KG}} = i \int_{\Sigma} d^{d-1} x \sqrt{g} n^\mu (\phi_1^* D_\mu \phi_2 - D_\mu \phi_1^* \phi_2), \quad (\text{C70})$$

where Σ is a constant time slice and n^μ is the future directed unit vector normal to it. In a static spacetime with metric $-N^2 dt^2 + g_{ij} dx^i dx^j$, the normal to the constant time t slice in the coordinate basis $(\partial/\partial t, \partial/\partial x^i)$ is given by $n^\mu \partial_\mu = N^{-1} \partial_t$. Thus, the inner product becomes

$$(\phi_1, \phi_2)_{\text{KG}} = i \int_{\Sigma} d^{d-1} x \sqrt{g} N^{-1} (\phi_1^* \dot{\phi}_2 - \dot{\phi}_1^* \phi_2). \quad (\text{C71})$$

For Region I of the BTZ black hole, any constant t slice is described by the range of coordinates $R_h < r < \infty$, $0 \leq \varphi < 2\pi$. Recall that after writing $F_{\omega q}(t, r, \varphi) = e^{-i\omega t} e^{-iq\varphi} f_{\omega q}(r)$, the wave equation for $f_{\omega q}(r)$ in terms of the variable $\rho = r^2/R_h^2$ reads

$$\frac{d}{d\rho}(\rho(\rho-1)) \frac{df_{\omega q}}{d\rho} = \left(\frac{-1}{4(\rho-1)} \frac{\omega^2}{\eta^2} + \frac{q^2 \ell^2}{4\rho R_h^2} + \frac{\ell^2 m^2}{4} \right) f_{\omega q}. \quad (\text{C72})$$

The Klein-Gordon norm between the modes $F_{\omega q}$, $F_{\omega' q'}$ is

$$\begin{aligned} (F_{\omega' q'}, F_{\omega q})_{\text{KG}} &= 2\pi(\omega + \omega') e^{i(\omega' - \omega)t} \delta_{qq'} \int_{R_h}^{\infty} dr r f^{-1} f_{\omega' q}^* f_{\omega q}, \\ &= \pi \ell^2 (\omega + \omega') e^{i(\omega' - \omega)t} \delta_{qq'} \int_1^{\infty} \frac{d\rho}{\rho - 1} f_{\omega' q}^* f_{\omega q}, \\ &= -\pi \ell^2 \eta (\omega + \omega') e^{i(\omega' - \omega)t} \delta_{qq'} \int_{-\infty}^0 dr_* \coth \eta r_* f_{\omega' q}^* f_{\omega q}. \end{aligned} \quad (\text{C73})$$

Let us analyze the integrals near the end-points which are the horizon at $\rho = 1$ and the boundary at $\rho = \infty$.

a. Near the boundary. The two linearly independent solutions of the equation (C72) can be written as

$$\begin{aligned} f_{\omega q}^\pm(\rho) &= C_{\omega q}^\pm (\rho - 1)^\alpha \rho^{-\alpha - \frac{1}{2}\Delta_\pm} \times \\ &\quad \times {}_2F_1\left(\alpha + \gamma + \frac{1}{2}\Delta_\pm, \alpha - \gamma + \frac{1}{2}\Delta_\pm; \Delta_\pm; \frac{1}{\rho}\right), \end{aligned} \quad (\text{C74})$$

where $C_{\omega q}^\pm$ are undetermined normalization constants. We see that the integrand near $\rho = \infty$ behaves as $\rho^{-\Delta_\pm - 1}$ for the two solutions so that the integral behaves as $\rho^{-\Delta_\pm}$. From the expression $\Delta_\pm = 1 \pm \sqrt{1 + \ell^2 m^2}$, it is clear that, for $m^2 > 0$, the normalizable solution is $f_{\omega q}^+$. For the range $-\ell^{-2} < m^2 < 0$, both solutions are normalizable but one set of normalizable functions may be more appropriate, depending on the context.

b. Near the horizon. The integrand has a pole at the horizon $\rho = 1$ and the integral seems to be singular near the horizon. However, the expression for the inner product in terms of the tortoise coordinate r_* (the last line of (C73)) displays no singularity at the horizon as $r_* \rightarrow -\infty$. Near the horizon, the mode function $f_{\omega q}^+$ becomes a plane wave in terms of the tortoise coordinate:

$$f_{\omega q}^+ = 2 C_{\omega q}^+ N_{\omega q} \cos\left(\delta_{\omega q} - \omega r_* - \frac{\omega}{\eta} \log 2\right) + \mathcal{O}(e^{2\eta r_*}). \quad (\text{C75})$$

Since $\coth \eta r_* \rightarrow -1$ as $r_* \rightarrow -\infty$, the integral near the horizon gives a delta function $\delta(\omega - \omega')$. We see below that the Klein-Gordon inner product is precisely this delta function and hence the mode functions $F_{\omega q}$ are delta function normalized.

c. Evaluating the Klein-Gordon inner product We show that the integral over the radial coordinate in (C73) is in fact a sum of boundary terms on the spatial slice. From the equation (C72) satisfied by $f_{\omega q}$, we find that

$$f_{\omega'q}^* \frac{d}{d\rho} \left(\rho(\rho-1) \frac{df_{\omega q}}{d\rho} \right) - f_{\omega q} \frac{d}{d\rho} \left(\rho(\rho-1) \frac{df_{\omega'q}^*}{d\rho} \right) = -\frac{1}{4(\rho-1)} \frac{(\omega^2 - \omega'^2)}{\eta^2} f_{\omega'q}^* f_{\omega q}. \quad (C76)$$

Integrating both sides over ρ , we get

$$\int_1^\infty d\rho f_{\omega'q}^* \frac{d}{d\rho} \left(\rho(\rho-1) \frac{df_{\omega q}}{d\rho} \right) - \int_1^\infty d\rho f_{\omega q} \frac{d}{d\rho} \left(\rho(\rho-1) \frac{df_{\omega'q}^*}{d\rho} \right) = -\frac{(\omega^2 - \omega'^2)}{4\eta^2} \int_1^\infty \frac{d\rho}{\rho-1} f_{\omega'q}^* f_{\omega q}. \quad (C77)$$

Integrating by parts in both terms on the left-hand side, we see that the volume terms cancel each other, and there is only a surface contribution left:

$$\left[\rho(\rho-1) \left(f_{\omega'q}^* \frac{df_{\omega q}}{d\rho} - f_{\omega q} \frac{df_{\omega'q}^*}{d\rho} \right) \right]_1^\infty = -\frac{(\omega^2 - \omega'^2)}{4\eta^2} \int_1^\infty \frac{d\rho}{\rho-1} f_{\omega'q}^* f_{\omega q}. \quad (C78)$$

The right hand side is proportional to the Klein-Gordon inner product (C73) of $F_{\omega'q}$ and $F_{\omega q}$:

$$\left[\rho(\rho-1) \left(f_{\omega'q}^* \frac{df_{\omega q}}{d\rho} - f_{\omega q} \frac{df_{\omega'q}^*}{d\rho} \right) \right]_1^\infty = \frac{(\omega' - \omega)}{4\pi\eta^2\ell^2} e^{-i(\omega' - \omega)t} (F_{\omega'q}, F_{\omega q})_{\text{KG}}. \quad (C79)$$

Let us first study the boundary term at $\rho = \infty$. As $\rho \rightarrow \infty$, the solutions $f_{\omega q}^\pm(\rho)$ in (C74) behave as

$$f_{\omega q}^\pm(\rho) = C_{\omega q}^\pm \rho^{-\Delta_\pm/2} \left(1 + \frac{4(\alpha^2 - \gamma^2) + \Delta_\pm^2}{4\Delta_\pm} \frac{1}{\rho} + \mathcal{O}(\rho^{-2}) \right), \quad (C80)$$

$$\frac{df_{\omega q}^\pm}{d\rho} = C_{\omega q}^\pm \rho^{-\Delta_\pm/2-1} \left(-\frac{\Delta_\pm}{2} + \frac{(4(\alpha^2 - \gamma^2) + \Delta_\pm^2)(2 + \Delta_\pm)}{8\Delta_\pm} \frac{1}{\rho} + \mathcal{O}(\rho^{-2}) \right). \quad (C81)$$

Plugging this into the left hand side at $\rho = \infty$, the leading contribution cancels, while the subleading contribution is

$$\left[\rho(\rho-1) \left(f_{\omega'q}^{\pm*} \frac{df_{\omega q}^\pm}{d\rho} - f_{\omega q}^\pm \frac{df_{\omega'q}^{\pm*}}{d\rho} \right) \right]_{\rho \rightarrow \infty} = C_{\omega'q}^{\pm*} C_{\omega q}^\pm \frac{(2 + \Delta_\pm)(\omega^2 - \omega'^2)}{8\Delta_\pm\eta^2} \rho^{-\Delta_\pm} \Big|_{\rho \rightarrow \infty}. \quad (C82)$$

Thus, we indeed see that, when $m^2 > 0$, $f_{\omega q}^+$ is normalizable since $\Delta_+ > 0$ whereas $f_{\omega q}^-$ is not normalizable since $\Delta_- < 0$. The upper limit in (C79) simply vanishes for the normalizable solution.

Recall the behaviour (C75) of $f_{\omega q}^+$ near the horizon $\rho = 1$. This gives

$$f_{\omega'q}^{+*} \rho(\rho-1) \frac{df_{\omega q}^+}{d\rho} = C_{\omega'q}^{+*} N_{\omega'q} C_{\omega q}^+ N_{\omega q} \frac{\omega}{\eta} \left[\sin \left(\delta_{\omega q} + \delta_{\omega'q} - (\omega + \omega') r_* - \frac{\omega + \omega'}{\eta} \log 2 \right) \right. \\ \left. + \sin \left(\delta_{\omega q} - \delta_{\omega'q} - (\omega - \omega') r_* - \frac{\omega - \omega'}{\eta} \log 2 \right) \right] + \mathcal{O}(e^{2\eta r_*}). \quad (C83)$$

Using, $\lim_{r_* \rightarrow -\infty} \sin(-\omega r_*)/\omega = \pi\delta(\omega)$, we get

$$\frac{1}{(\omega - \omega')} \lim_{r_* \rightarrow -\infty} f_{\omega'q}^{+*} \rho(\rho-1) \frac{df_{\omega q}^+}{d\rho} = |C_{\omega'q}^+|^2 N_{\omega q}^2 \frac{\pi\omega}{\eta} (\delta(\omega - \omega') + \delta(\omega + \omega')). \quad (C84)$$

Since $\omega, \omega' \geq 0$, the second delta function never clicks. Therefore, we have

$$\frac{1}{(\omega - \omega')} \left[\rho(\rho-1) \left(f_{\omega'q}^{+*} \frac{df_{\omega q}^+}{d\rho} - f_{\omega q}^+ \frac{df_{\omega'q}^{+*}}{d\rho} \right) \right]_{\rho \rightarrow 1^+} = |C_{\omega'q}^+|^2 N_{\omega q}^2 \frac{\pi}{\eta} 2\omega\delta(\omega - \omega'), \quad (C85)$$

from which we find the Klein-Gordon inner product

$$(F_{\omega'q}, F_{\omega q})_{\text{KG}} = |C_{\omega'q}^+|^2 N_{\omega q}^2 4\pi^2 R_h 2\omega\delta(\omega - \omega'). \quad (C86)$$

Similarly, we find

$$(F_{\omega'q}^*, F_{\omega q}^*)_{\text{KG}} = -|C_{\omega'q}^+|^2 N_{\omega q}^2 4\pi^2 R_h 2\omega\delta(\omega - \omega'), \quad (F_{\omega'q}^*, F_{\omega q})_{\text{KG}} = 0. \quad (C87)$$

Setting

$$C_{\omega q}^+ = \frac{1}{N_{\omega q} \sqrt{2\pi R_h}}, \quad (\text{C88})$$

we have

$$f_{\omega q}^+(\rho) = \frac{1}{N_{\omega q} \sqrt{2\pi R_h}} (\rho - 1)^\alpha \rho^{-\alpha - \frac{1}{2}\Delta_+} {}_2F_1\left(\alpha + \gamma + \frac{1}{2}\Delta_+, \alpha - \gamma + \frac{1}{2}\Delta_+; \Delta_+; \frac{1}{\rho}\right), \quad (\text{C89})$$

so that the modes $F_{\omega q} = e^{-i\omega t} e^{-iq\varphi} f_{\omega q}$ satisfy the canonical Klein-Gordon inner products

$$(F_{\omega' q'}, F_{\omega q})_{\text{KG}} = 4\pi\omega\delta(\omega - \omega')\delta_{qq'}, \quad (F_{\omega' q'}^*, F_{\omega q}^*)_{\text{KG}} = -4\pi\omega\delta(\omega - \omega')\delta_{qq'}, \quad (F_{\omega' q'}, F_{\omega q})_{\text{KG}} = 0. \quad (\text{C90})$$

9. Klein-Gordon inner product of the Hartle-Hawking modes and the smeared modes

Recall the Klein-Gordon inner product

$$(\phi_1, \phi_2)_{\text{KG}} = i \int_{\Sigma} d^{d-1}x \sqrt{g} n^\mu (\phi_1^* \partial_\mu \phi_2 - \partial_\mu \phi_1^* \phi_2). \quad (\text{C91})$$

For a given $d - 1$ dimensional slice Σ in the spacetime geometry and given functions ϕ_1 and ϕ_2 are fixed, different coordinate systems give the same answer for the inner product. This is because $n^\mu \partial_\mu$ is covariant and so is $d^{d-1}x \sqrt{g}$. We demonstrate this in the case of the time-symmetric slice in the fully extended BTZ black hole.

The time-symmetric slice is described in BTZ coordinates as the union of the $t = 0$ slice in Region I and $\tilde{t} = 0$ slice in Region II. In wormhole coordinates, it is the $\tilde{t} = 0$ slice and in Kruskal coordinates, it is the $\tau = \frac{\eta}{2}(U + V) = 0$ slice. We summarize the various quantities required for the Klein-Gordon norm in Table I. The quantities in different coordinate systems can be obtained from each other by the appropriate diffeomorphisms. Suppose we want to compute

	BTZ	Kruskal	Wormhole
Slice	$\{t = 0\} \cup \{\tilde{t} = 0\}$	$\tau = 0$	$\tilde{t} = 0$
Spatial range	$\{0 < r < \infty\} \cup \{0 < \tilde{r} < \infty\}$	$-1 < \chi < 1$	$-\infty < x < \infty$
$n^\mu \partial_\mu$	$f(r)^{-1/2} \partial_t - f(\tilde{r})^{-1/2} \partial_{\tilde{t}}$	$\frac{1 - \chi^2}{2\ell} \partial_\tau$	$N^{-1} \partial_{\tilde{t}}$
$d^2x \sqrt{g}$	$r f(r)^{-1/2} dr d\varphi, \tilde{r} f(\tilde{r})^{-1/2} d\tilde{r} d\varphi$	$2\ell R_h \frac{1 + \chi^2}{(1 - \chi^2)^2} d\chi d\varphi$	$\ell dx d\varphi$

TABLE I. The various quantities required for the Klein-Gordon inner product

the Klein-Gordon inner product of the smeared modes $\hat{g}_{nq}(\bar{t}, x, \varphi)$ and $\hat{g}_{n'q'}(\bar{t}, x, \varphi)$ on the $\bar{t} = 0$ slice. We have the formula

$$(\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} = \int_{\Sigma} dx d\varphi \ell N(\bar{t} = 0, x)^{-1} \left(\hat{g}_{nq}^* \frac{\partial \hat{g}_{n'q'}}{\partial \bar{t}} - \frac{\partial \hat{g}_{nq}^*}{\partial \bar{t}} \hat{g}_{n'q'} \right). \quad (\text{C92})$$

We can evaluate the above integral directly in wormhole coordinates, but that would be tedious. Instead, we first go back to their expression in terms of Hartle-Hawking mode functions:

$$\hat{g}(\bar{t}(U, V), x(U, V), \varphi) = g_{nq}(U, V, \varphi) = \int_{-\infty}^{\infty} \frac{d\omega}{\eta} h_{\omega q}(U, V, \varphi) \psi_n(\omega/\eta). \quad (\text{C93})$$

Plugging this into the Klein-Gordon inner product and also changing to Kruskal coordinates, we get

$$(\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} = \int_{-\infty}^{\infty} \frac{d\omega}{\eta} \psi_n(\omega/\eta) \int_{-\infty}^{\infty} \frac{d\omega'}{\eta} \psi_{n'}(\omega'/\eta) \int_{\Sigma} d\chi d\varphi R_h \frac{1 - \chi^2}{1 + \chi^2} \left(h_{\omega q}^* \frac{\partial h_{\omega'q'}}{\partial \tau} - \frac{\partial h_{\omega q}^*}{\partial \tau} h_{\omega'q'} \right). \quad (\text{C94})$$

Now, once again, we can map back to the original BTZ coordinates on each of the $t = 0$ and $\tilde{t} = 0$ slices and use the expression for the Hartle-Hawking mode functions in terms of the original mode functions $F_{\omega q}$ in Regions I and II.

This gives

$$\begin{aligned}
(\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} &= 2\pi R_h \int_{-\infty}^{\infty} \frac{d\omega}{\eta} \psi_n(\omega/\eta) \int_{-\infty}^{\infty} \frac{d\omega'}{\eta} \psi_{n'}(\omega'/\eta) \frac{1}{4\pi \sqrt{\omega\omega'} 2 \sinh(\pi\omega/\eta) 2 \sinh(\pi\omega'/\eta)} \times \\
&\quad \times \left[e^{\pi(\omega+\omega')/2\eta} \int_{\Sigma_R} dr d\varphi r f(r)^{-1} \left(F_{\omega q}^* \frac{\partial F_{\omega'q'}}{\partial t} - \frac{\partial F_{\omega q}^*}{\partial t} F_{\omega'q'} \right) \right. \\
&\quad \left. - e^{-\pi(\omega+\omega')/2\eta} \int_{\Sigma_L} d\tilde{r} d\tilde{\varphi} \tilde{r} f(\tilde{r})^{-1} \left(F_{\omega q}^* \frac{\partial F_{\omega'q'}}{\partial \tilde{t}} - \frac{\partial F_{\omega q}^*}{\partial \tilde{t}} F_{\omega'q'} \right) \right]. \quad (\text{C95})
\end{aligned}$$

The integrands have a pole at the horizon due to the $f(r)^{-1}$ factors. However, recall from our discussion in Appendix C8 after (C73) that, when we transform to tortoise coordinates the integral is well-defined and leads to a delta function normalization for the $F_{\omega q}$.

The integrals over Σ_R and Σ_L are the Klein-Gordon inner products of $F_{\omega q}$, $F_{\omega'q'}$ in their original regions of quantization and have been computed previously:

$$(F_{\omega'q'}, F_{\omega q})_{\text{KG}} = 4\pi\omega\delta(\omega - \omega')\delta_{qq'}, \quad (F_{\omega'q'}^*, F_{\omega q}^*)_{\text{KG}} = -4\pi\omega\delta(\omega - \omega')\delta_{qq'}, \quad (F_{\omega'q'}, F_{\omega q}^*)_{\text{KG}} = 0. \quad (\text{C96})$$

Plugging this into the previous expression, we get

$$\begin{aligned}
(\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} &= 2\pi R_h \int_{-\infty}^{\infty} \frac{d\omega}{\eta} \psi_n(\omega/\eta) \int_{-\infty}^{\infty} \frac{d\omega'}{\eta} \psi_{n'}(\omega'/\eta) \frac{1}{4\pi \sqrt{\omega\omega'} 2 \sinh(\pi\omega/\eta) 2 \sinh(\pi\omega'/\eta)} \times \\
&\quad \times \left[e^{\pi(\omega+\omega')/2\eta} 4\pi\omega\delta(\omega - \omega')\delta_{qq'} - e^{-\pi(\omega+\omega')/2\eta} 4\pi\omega\delta(\omega - \omega')\delta_{qq'} \right]. \quad (\text{C97})
\end{aligned}$$

Carrying out the ω, ω' integrals, we get

$$(\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} = \frac{2\pi R_h}{\eta} \int_{-\infty}^{\infty} \frac{d\omega}{\eta} \psi_n(\omega/\eta) \psi_{n'}(\omega/\eta) = 2\pi\ell^2 \delta_{nn'} \delta_{qq'}. \quad (\text{C98})$$

We can compute the other inner products analogously and obtain

$$(\hat{g}_{nq}, \hat{g}_{n'q'})_{\text{KG}} = 2\pi\ell^2 \delta_{nn'} \delta_{qq'}, \quad (\hat{g}_{nq}^*, \hat{g}_{n'q'})_{\text{KG}} = 0, \quad (\hat{g}_{nq}^*, \hat{g}_{n'q'}^*)_{\text{KG}} = -2\pi\ell^2 \delta_{nn'} \delta_{qq'}. \quad (\text{C99})$$

Finally, the Hartle-Hawking modes' inner products can be computed along the same lines:

$$(h_{\omega'q'}, h_{\omega q})_{\text{KG}} = 2\pi R_h \delta(\omega - \omega') \delta_{qq'}, \quad (h_{\omega'q'}^*, h_{\omega q}^*)_{\text{KG}} = -2\pi R_h \delta(\omega - \omega') \delta_{qq'}, \quad (h_{\omega'q'}, h_{\omega q}^*)_{\text{KG}} = 0. \quad (\text{C100})$$

Appendix D: Bulk reconstruction of local scalar field operator in wormhole coordinates

Recall the mode expansion in region I

$$\phi(t, r, \varphi) = \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(a_{\omega q} F_{\omega q}(t, r, \varphi) + \text{c.c.} \right), \quad (\text{D1})$$

with $F_{\omega q}(t, r, \varphi) = e^{-i\omega t} e^{-iq\varphi} f_\omega(r)$. This allows us to invert this equation, namely,

$$a_{\omega q} f_\omega(r) = \frac{1}{4\pi^2} \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dt' e^{i\omega t'} e^{iq\varphi'} \phi(t', r, \varphi'). \quad (\text{D2})$$

On taking the extrapolate limit, i.e., $\lim_{r \rightarrow \infty} r^{\Delta_+} \phi(t', r, \varphi') = \mathcal{O}(t', \varphi')$, we get an expression

for the right CFT oscillator

$$a_{\omega q} = \frac{N_{\omega q} \sqrt{2\pi R_h}}{4\pi^2 R_h^{\Delta_+}} \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dt' e^{i\omega t'} e^{iq\varphi'} \mathcal{O}(t', \varphi'). \quad (\text{D3})$$

Similar expression exists for the scalar field in region II. Recall

$$\phi(\tilde{t}, \tilde{r}, \tilde{\varphi}) = \sum_{q \in \mathbb{Z}} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left(\tilde{a}_{\omega q} \tilde{F}_{\omega q}(\tilde{t}, \tilde{r}, \tilde{\varphi}) + \text{c.c.} \right), \quad (\text{D4})$$

with $\tilde{F}_{\omega q}(\tilde{t}, \tilde{r}, \tilde{\varphi}) = e^{i\omega \tilde{t}} e^{iq\tilde{\varphi}} \tilde{f}_\omega(r)$. This gives

$$\tilde{a}_{\omega q} = \frac{N_{\omega q} \sqrt{2\pi R_h}}{4\pi^2 R_h^{\Delta_+}} \int_0^{2\pi} d\tilde{\varphi}' \int_{-\infty}^{\infty} d\tilde{t}' e^{-i\omega \tilde{t}'} e^{-iq\tilde{\varphi}'} \tilde{\mathcal{O}}(\tilde{t}', \tilde{\varphi}'). \quad (\text{D5})$$

Now let us go back to the discrete scalar field expansion in terms of the smooth mode functions,

$$\phi(\bar{t}, x, \varphi) = \frac{1}{\sqrt{2\pi\ell}} \sum_{q \in \mathbb{Z}} \sum_{n=0}^{\infty} (e_{nq} \hat{g}_{nq}(\bar{t}, x, \varphi) + \text{c.c.}), \quad (\text{D6})$$

and recall that the e_{nq} oscillators are combinations of $a_{\omega q}$ and $\tilde{a}_{\omega q}$:

$$\begin{aligned} e_{nq} &= \int_0^\infty \frac{d\omega}{\sqrt{\eta}} (c_{\omega q} \psi_n(\omega/\eta) + c_{-\omega q} \psi_n(-\omega/\eta)) = \int_0^\infty \frac{d\omega}{\sqrt{\eta}} (c_{\omega q} \psi_n(\omega/\eta) + \tilde{c}_{\omega, -q} \psi_n(-\omega/\eta)) \\ &= \int_0^\infty \frac{d\omega}{\sqrt{2\eta \sinh(\pi\omega/\eta)}} \left[(a_{\omega q} e^{\pi\omega/2\eta} - \tilde{a}_{\omega q}^\dagger e^{-\pi\omega/2\eta}) \psi_n(\omega/\eta) + (\tilde{a}_{\omega, -q} e^{\pi\omega/2\eta} - a_{\omega, -q}^\dagger e^{-\pi\omega/2\eta}) \psi_n(-\omega/\eta) \right] \\ &= \int_0^\infty \frac{d\omega}{\sqrt{2\eta \sinh(\pi\omega/\eta)}} \left[(a_{\omega q} e^{\pi\omega/2\eta} \psi_n(\omega/\eta) - a_{\omega, -q}^\dagger e^{-\pi\omega/2\eta} \psi_n(-\omega/\eta)) + (\tilde{a}_{\omega, -q} e^{\pi\omega/2\eta} \psi_n(-\omega/\eta) - \tilde{a}_{\omega q}^\dagger e^{-\pi\omega/2\eta} \psi_n(\omega/\eta)) \right]. \end{aligned} \quad (\text{D7})$$

Plugging in the individual oscillator expressions (D3) and (D5), we get a bulk expression for the scalar field everywhere in the wormhole coordinates (including the part of interior regions F and P contained in R_0), in terms of the left and right CFT operators

$$\phi(\bar{t}, x, \varphi) = \int_0^{2\pi} d\varphi' \int_{-\infty}^{\infty} dt' \mathcal{K}(\bar{t}, x, \varphi; t', \varphi') \mathcal{O}(t', \varphi') + \int_0^{2\pi} d\tilde{\varphi}' \int_{-\infty}^{\infty} d\tilde{t}' \tilde{\mathcal{K}}(\bar{t}, x, \varphi; \tilde{t}', \tilde{\varphi}') \tilde{\mathcal{O}}(\tilde{t}', \tilde{\varphi}'), \quad (\text{D8})$$

with

$$\begin{aligned} \mathcal{K}(\bar{t}, x, \varphi; t', \varphi') &= \frac{N_{\omega q} \sqrt{R_h}}{4\pi^2 \ell R_h^{\Delta_+}} \sum_{n,q} \hat{g}_{nq}(\bar{t}, x, \varphi) \times \\ &\times \int_0^\infty \frac{d\omega}{\sqrt{2\eta \sinh(\pi\omega/\eta)}} \left(e^{\pi\omega/2\eta} \psi_n(\omega/\eta) e^{i\omega t' + iq\varphi'} - e^{-\pi\omega/2\eta} \psi_n(-\omega/\eta) e^{-i\omega t' + iq\varphi'} \right) + \text{c.c.}, \end{aligned} \quad (\text{D9})$$

$$\begin{aligned} \tilde{\mathcal{K}}(\bar{t}, x, \varphi; \tilde{t}', \tilde{\varphi}') &= \frac{N_{\omega q} \sqrt{R_h}}{4\pi^2 \ell R_h^{\Delta_+}} \sum_{n,q} \hat{g}_{nq}(\bar{t}, x, \varphi) \times \\ &\times \int_0^\infty \frac{d\omega}{\sqrt{2\eta \sinh(\pi\omega/\eta)}} \left(e^{\pi\omega/2\eta} \psi_n(-\omega/\eta) e^{-i\omega \tilde{t}' + iq\tilde{\varphi}'} - e^{-\pi\omega/2\eta} \psi_n(\omega/\eta) e^{i\omega \tilde{t}' + iq\tilde{\varphi}'} \right) + \text{c.c.} \end{aligned} \quad (\text{D10})$$

Note that the bulk spacetime dependence is coming entirely from the the mode functions $\hat{g}_{nq}(\bar{t}, x, \varphi)$. Conse-

quently these boundary-to-bulk kernels satisfy the Klein-Gordon equation (31) in the wormhole coordinates.

-
- | | |
|---|---|
| <p>[1] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975), [Erratum: Commun.Math.Phys. 46, 206 (1976)].</p> <p>[2] K. Fredenhagen and R. Haag, Commun. Math. Phys. 127, 273 (1990).</p> <p>[3] E. Witten, Eur. Phys. J. Plus 140, 430 (2025), arXiv:2412.16795 [hep-th].</p> <p>[4] A. Strominger and C. Vafa, Phys. Lett. B 379, 99 (1996), arXiv:hep-th/9601029.</p> <p>[5] J. R. David, G. Mandal, and S. R. Wadia, Phys. Rept. 369, 549 (2002), arXiv:hep-th/0203048.</p> <p>[6] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998), arXiv:hep-th/9711200.</p> <p>[7] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998), arXiv:hep-th/9802150.</p> | <p>[8] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B 428, 105 (1998), arXiv:hep-th/9802109.</p> <p>[9] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006), arXiv:hep-th/0603001.</p> <p>[10] V. E. Hubeny, M. Rangamani, and T. Takayanagi, JHEP 07, 062, arXiv:0705.0016 [hep-th].</p> <p>[11] A. Lewkowycz and J. Maldacena, JHEP 08, 090, arXiv:1304.4926 [hep-th].</p> <p>[12] T. Faulkner, A. Lewkowycz, and J. Maldacena, JHEP 11, 074, arXiv:1307.2892 [hep-th].</p> <p>[13] N. Engelhardt and A. C. Wall, JHEP 01, 073, arXiv:1408.3203 [hep-th].</p> <p>[14] G. Penington, JHEP 09, 002, arXiv:1905.08255 [hep-th].</p> <p>[15] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield,</p> |
|---|---|

- JHEP **12**, 063, arXiv:1905.08762 [hep-th].
- [16] A. Almheiri, R. Mahajan, J. Maldacena, and Y. Zhao, JHEP **03**, 149, arXiv:1908.10996 [hep-th].
- [17] A. Almheiri, R. Mahajan, and J. Maldacena, Islands outside the horizon (2019), arXiv:1910.11077 [hep-th].
- [18] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, JHEP **03**, 205, arXiv:1911.11977 [hep-th].
- [19] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, Rev. Mod. Phys. **93**, 035002 (2021), arXiv:2006.06872 [hep-th].
- [20] S. Leutheusser and H. Liu, Phys. Rev. D **108**, 086019 (2023), arXiv:2110.05497 [hep-th].
- [21] S. A. W. Leutheusser and H. Liu, Phys. Rev. D **108**, 086020 (2023), arXiv:2112.12156 [hep-th].
- [22] K. Papadodimas and S. Raju, JHEP **10**, 212, arXiv:1211.6767 [hep-th].
- [23] K. Papadodimas and S. Raju, Phys. Rev. Lett. **112**, 051301 (2014), arXiv:1310.6334 [hep-th].
- [24] K. Papadodimas and S. Raju, Phys. Rev. D **89**, 086010 (2014), arXiv:1310.6335 [hep-th].
- [25] L. Andersson, P. Chrusciel, and H. Friedrich, Commun. Math. Phys. **149**, 587 (1992).
- [26] L. Andersson and P. T. Chrusciel, Diss. Math. **355**, 1 (1996).
- [27] A. Sakovich, Class. Quant. Grav. **27**, 245019 (2010), arXiv:0910.4178 [gr-qc].
- [28] P. T. Chrusciel and G. J. Galloway, Maximal hypersurfaces in asymptotically Anti-de Sitter spacetime (2022), arXiv:2208.09893 [gr-qc].
- [29] E. Witten, Adv. Theor. Math. Phys. **27**, 311 (2023), arXiv:2212.08270 [hep-th].
- [30] E. Witten, *Canonical Quantization in Anti de Sitter Space*, [LINK] (2017), Talk at Princeton Center for Theoretical Science.
- [31] The time $\bar{t}$ can be identified with the formula for time presented in [77], where it was shown that the notion of time derived from the Einstein-Hamilton-Jacobi equation in terms of the configuration space data precisely matches with the time in ADM decomposition of the spacetime metric.
- [32] W. G. Unruh, Phys. Rev. D **14**, 870 (1976).
- [33] J. M. Maldacena, JHEP **04**, 021, arXiv:hep-th/0106112.
- [34] S. H. Shenker and D. Stanford, JHEP **03**, 067, arXiv:1306.0622 [hep-th].
- [35] J. Maldacena, S. H. Shenker, and D. Stanford, JHEP **08**, 106, arXiv:1503.01409 [hep-th].
- [36] J. Maldacena and A. Milekhin, JHEP **04**, 258, arXiv:1912.03276 [hep-th].
- [37] A. Almheiri, A. Milekhin, and B. Swingle, JHEP **08**, 034, arXiv:1912.04912 [hep-th].
- [38] Y. Chen, X.-L. Qi, and P. Zhang, JHEP **06**, 121, arXiv:2003.13147 [hep-th].
- [39] A. Gaikwad, A. Kaushal, G. Mandal, and S. R. Wadia, JHEP **08**, 171, arXiv:2210.15579 [hep-th].
- [40] F. Estabrook, H. Wahlquist, S. Christensen, B. DeWitt, L. Smarr, and E. Tsiang, Phys. Rev. D **7**, 2814 (1973).
- [41] A. Kaushal, N. S. Prabhakar, and S. R. Wadia, 2+1 dimensional gravity in aads spacetimes with spatial wormhole slices: Reduced phase space dynamics and the btz black hole (2025).
- [42] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992), arXiv:hep-th/9204099.
- [43] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993), [Erratum: Phys.Rev.D 88, 069902 (2013)], arXiv:gr-qc/9302012.
- [44] E. Gourgoulhon, 3+1 formalism and bases of numerical relativity (2007), arXiv:gr-qc/0703035.
- [45] T. W. Baumgarte and S. L. Shapiro, *Numerical Relativity: Solving Einstein's Equations on the Computer* (Cambridge University Press, 2010).
- [46] This is consistent with the observations in [78, 79] regarding the existence of local gauge-invariant observables in perturbation theory about backgrounds which do not have isometries.
- [47] S. J. Avis, C. J. Isham, and D. Storey, Phys. Rev. D **18**, 3565 (1978).
- [48] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982).
- [49] P. Breitenlohner and D. Z. Freedman, Annals Phys. **144**, 249 (1982).
- [50] S. W. Hawking, Phys. Lett. B **126**, 175 (1983).
- [51] M. Henneaux and C. Teitelboim, Phys. Lett. B **142**, 355 (1984).
- [52] L. Mezincescu and P. K. Townsend, Annals Phys. **160**, 406 (1985).
- [53] M. Henneaux and C. Teitelboim, Commun. Math. Phys. **98**, 391 (1985).
- [54] G. Lifschytz and M. Ortiz, Phys. Rev. D **49**, 1929 (1994), arXiv:gr-qc/9310008.
- [55] I. Ichinose and Y. Satoh, Nucl. Phys. B **447**, 340 (1995), arXiv:hep-th/9412144.
- [56] V. Balasubramanian, P. Kraus, and A. E. Lawrence, Phys. Rev. D **59**, 046003 (1999), arXiv:hep-th/9805171.
- [57] E. Keski-Vakkuri, Phys. Rev. D **59**, 104001 (1999), arXiv:hep-th/9808037.
- [58] A. Ishibashi and R. M. Wald, Class. Quant. Grav. **21**, 2981 (2004), arXiv:hep-th/0402184.
- [59] The coordinates (Q, P) are related to by a canonical transformation to Kuchar's mass and time shift variables [80], for the BTZ black hole.
- [60] There may be other choices for the smearing functions which achieve similar results. For instance, we can take Hermite functions $\psi_n(\omega/\alpha)$ for any positive real parameter α . The Hermite functions $\psi_n(z)$ are defined in terms of the Hermite polynomials $H_n(z)$ as
- $$\psi_n(z) = \frac{1}{(\sqrt{\pi}2^n n!)^{1/2}} e^{-z^2/2} H_n(z),$$
- $$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.$$
- They satisfy the orthogonality and completeness relations
- $$\int_{-\infty}^{\infty} dz \psi_n(z) \psi_m(z) = \delta_{nm},$$
- $$\sum_{n=0}^{\infty} \psi_n(z) \psi_n(z') = \delta(z - z').$$
- [61] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, Phys. Rev. D **74**, 066009 (2006), arXiv:hep-th/0606141.
- [62] We can split the x integral as a bulk term and two terms near the boundary at $x = \pm\infty$,
- $$\int_{-\infty}^{\infty} dx = \int_{-\infty}^{-X} dx + \int_{-X}^X dx + \int_X^{\infty} dx$$
- for some large and positive value X . The bulk term is finite since all the quantities are smooth and finite. Near

the boundary, the mass term in $A_{nn'qq'}$ goes as,

$$m^2 \int_X^\infty dx \sqrt{g} N \hat{g}_{n'q'}^* \hat{g}_{nq} \propto \int_X^\infty dx x^{1-2\Delta_+} \sim x^{2(1-\Delta_+)} \Big|_X^\infty,$$

which is finite for $\Delta_+ > 1$. Similarly contribution from all the other terms is also finite.

- [63] The time evolution operator satisfies the differential equation

$$\partial_{\bar{t}} U(\bar{t}, 0) = -i U(\bar{t}, 0) H(\bar{t}),$$

where $H(\bar{t})$ is the Hamiltonian in the Heisenberg picture. Integrating this equation gives

$$U(\bar{t}, 0) = 1 - i \int_0^{\bar{t}} dt' U(\bar{t}', 0) H(\bar{t}').$$

We can iterate this further to get the Dyson series

$$\begin{aligned} U(\bar{t}, 0) = & 1 - i \int_0^{\bar{t}} d\bar{t}' H(\bar{t}') + (-i)^2 \int_0^{\bar{t}} d\bar{t}' \int_0^{\bar{t}'} d\bar{t}'' H(\bar{t}'') H(\bar{t}') \\ & + \cdots + (-i)^n \int_0^{\bar{t}} d\bar{t}' \int_0^{\bar{t}'} d\bar{t}'' \cdots \int_0^{\bar{t}^{(n-1)}} d\bar{t}^{(n)} \times \\ & \times H(\bar{t}^{(n)}) \cdots H(\bar{t}'') H(\bar{t}') + \cdots, \end{aligned}$$

which is written compactly as anti time-ordered exponential in (71).

- [64] h_r and h_l are the bulk Hamiltonians that evolve in the Schwarzschild time t and $\bar{t}$ in Region I and Region II respectively, see equation 2.7 of [68] for the precise definitions.
- [65] It is important that the Cauchy slice is not split, since there can be divergent fluctuations near the splitting point.
- [66] This is analogous to evolution with the Minkowski Hamiltonian as opposed to the Rindler time evolution. Minkowski time slices enter the Milne region while the Rindler slices do not.
- [67] Another justification for the bound on the range of n is by numerically studying the mode functions g_{nq} . For example, using the near horizon expressions, one can check that their magnitude decreases with increasing n .
- [68] E. Witten, JHEP **10**, 008, arXiv:2112.12828 [hep-th].

- [69] An example of a smooth function with compact support is given by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2 - a^2}\right) & |x| < a \\ 0 & |x| > a, \end{cases} \quad (\text{D11})$$

where a is positive number that controls the support of the function $f(x)$.

- [70] This is because the scalar field $\phi(\bar{t}, x, \varphi)$ in (55) itself satisfies the Klein-Gordon equation (31).
- [71] See also [81] for computation of the left-right boundary correlator using the bulk geodesic approximation in the large scalar field mass limit.
- [72] Note that any correlator with insertions in region R_0 can be expressed in terms of boundary operators. We consider this correlator as a specific example.
- [73] See also the textbook [45] for a detailed discussion.
- [74] It is convenient to use a Mathematica package such as diffgeo.m by Matthew Headrick.
- [75] DLMF, *NIST Digital Library of Mathematical Functions*, <https://dlmf.nist.gov/>, Release 1.2.3 of 2024-12-15, f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [76] To obtain this expression, we have absorbed the phase $e^{-i\pi 2\alpha}$ into $(1 - \rho)^{-2\alpha}$ to obtain $(\rho - 1)^{-2\alpha}$:
- $$e^{-i\pi 2\alpha} (1 - \rho)^{-2\alpha} = e^{-2\alpha \log((1-\rho)e^{-i\pi})} = (\rho - 1)^{-2\alpha}.$$
- [77] A. Kaushal, N. S. Prabhakar, and S. R. Wadia, Phys. Rev. D **111**, 106006 (2025), arXiv:2405.18486 [hep-th].
- [78] E. Bahiru, A. Belin, K. Papadodimas, G. Sarosi, and N. Vardian, Phys. Rev. D **108**, 086035 (2023), arXiv:2209.06845 [hep-th].
- [79] S. Antonini, C.-H. Chen, H. Maxfield, and G. Penington, An apologia for islands (2025), arXiv:2506.04311 [hep-th].
- [80] K. V. Kuchar, Phys. Rev. D **50**, 3961 (1994), arXiv:gr-qc/9403003.
- [81] G. Mandal, R. Sinha, and N. Sorokhaibam, JHEP **01**, 036, arXiv:1405.6695 [hep-th].