

PREDICTING THE SIZE RANKING OF MINIMAL PRIMES IN THE GENERALISED GOLDBACH PARTITIONS

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To my Grandmothers

ABSTRACT. A scarcely known generalization of Goldbach’s conjecture introduced by Hardy and Littlewood states that for every pair of (relatively prime) positive integers m_1 and m_2 , every sufficiently large integer n satisfying certain simple congruence criteria can be (m_1, m_2) -partitioned as $n = m_1p + m_2q$ for some primes p and q . While the size of the minimal prime in the Goldbach partitions of even numbers has received prior attention, we extend this investigation to the general case of (m_1, m_2) -partitions. This question has a direct implication on the running times of verification algorithms of the generalised Goldbach conjecture. We study the rankings of the pairs (m_1, m_2) according to the sizes of the averages and maxima, respectively, of the minimal p in the (m_1, m_2) -partitions of numbers up to large thresholds, and propose a rank-order predicting function depending only on m_2 and the prime factors of m_1 . We computed both the average and the maximum of the minimal prime p in all (m_1, m_2) -partitions of integers up to 10^9 , for every pair of relatively prime coefficients $1 \leq m_1 \neq m_2 \leq 40$. Our function shows very high rank-order correlations with both the empirical averages and maxima of the minimal primes p (Spearman’s $\rho = 0.9949$ and 0.9958 , respectively). It also correctly predicts trends in the experimental data, for example, that for all relatively prime $1 \leq m_1 < m_2 \leq 40$, the average minimal p in the (m_1, m_2) -partitions of numbers up to 10^9 exceeds the analogous average for the (m_2, m_1) -partitions. We present numerical data, including the average and the maximum of the minimal p in the (m_1, m_2) -partitions of numbers up to 10^9 for each pair $1 \leq m_1 \neq m_2 \leq 20$ relatively prime, and the resulting size rankings.

1. INTRODUCTION

The even (or strong) Goldbach conjecture poses one of the longest-standing open problems in number theory, which has been intriguing wide audiences for nearly three centuries. Goldbach formulated his famous claim in 1742 in a letter to Euler, proposing – rephrased in modern form – that every even number greater than 2 can be expressed as the sum of two primes [4].

Although a general proof has been elusive, major relevant results have been achieved. In 1973 Chen proved that every sufficiently large even number is the sum of a prime and a semiprime (the product of at most two primes) [1]. In 2010 Lu showed that the number of even integers up to x which do not have Goldbach partitions is $O(x^{0.879})$ [17]. In 2013 Helfgott proved the odd (weak or ternary) Goldbach conjecture – a related but weaker statement than the even Goldbach conjecture – asserting that every odd number greater than 5 is the sum of three primes [7, 8].

Several efforts have been made to verify the even Goldbach conjecture empirically up to increasing limits [14, 15, 16, 5]. The current record is $4 \cdot 10^{18}$ set by Oliveira e Silva *et al.* in 2014 [13].

In 1894 Lemoine [10] suggested a stronger version of the weak Goldbach conjecture, stating that every odd number $n > 5$ can be expressed as $n = p + 2q$ for some primes p and q [2, p. 424].

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Although still unproven, Lemoine's conjecture (LC) has drawn more limited research interest than those of Goldbach. The highest limit of its empirical verification was 10^{10} [11] before 2024, when it was lifted to 10^{13} by the authors [9].

In 1923 Hardy and Littlewood [6] introduced the following generalization (GGC) of the even Goldbach conjecture, also generalizing LC: for every (relatively prime) positive integer m_1 and m_2 , every sufficiently large integer n satisfying certain simple congruence conditions can be expressed as $n = m_1p + m_2q$ for some primes p and q . Interestingly, this generalization appears to be hardly known and not to have been studied until 2017 [3]. The authors recently tested GGC empirically up to 10^{12} [and up to 10^{13} for selected pairs] for all relatively prime $1 \leq m_1, m_2 \leq 40$ [3, 9], presenting the largest values of n found satisfying the conditions of GGC without (m_1, m_2) -partitions. The small sizes of these largest counterexamples support the plausibility of GGC. Lemoine's conjecture was also confirmed up to a new record of 10^{13} .

The smallest value of p in all Goldbach partitions of an even number $n \geq 4$ is denoted by $p(n)$. The rate of growth of $p(n)$ has been the subject of prior research [5] and directly influences the time complexity of algorithms verifying Goldbach's conjecture by searching for the minimal partitions [5, 13]. Granville *et al.* [5] conjectured $p(n) = O(\log^2 n \log \log n)$. Granville also suggested two more precise, incompatible conjectures of the form $p(n) \leq (C + o(1)) \log^2 n \log \log n$, where C is 'sharp' in the sense that C is the smallest constant with this property: one with $C = C_2^{-1} \approx 1.51478$ and the other one with $C = 2e^{-\gamma} C_2^{-1} \approx 1.70098$, where $C_2 \approx 0.66016$ is the twin prime constant and $\gamma \approx 0.57722$ is the Euler's constant [13]. Empirical comparison of the plausibility of these conjectures was inconclusive due to the need for data up to even higher limits [13].

In the general setting of GGC, for any integers m_1, m_2 and n satisfying the conditions of GGC, we call an expression $n = m_1p + m_2q$ where p and q are primes, an (m_1, m_2) -partition of n . Let $p_{m_1, m_2}^*(n)$ denote the smallest value of p in all (m_1, m_2) -partitions of n if such partition exists [9]. The magnitude of $p_{m_1, m_2}^*(n)$ affects the runtime of corresponding verification algorithms [9]. To the authors' knowledge the only published results regarding $p_{m_1, m_2}^*(n)$ in general, beyond the classical case $p_{1, 1}^*(n) = p^*(n)$, are from 2024, when the average and maximum of the minimal $p_{m_1, m_2}^*(n)$ up to 10^9 were presented for every $1 \leq m_1 \neq m_2 \leq 20$ relatively prime [9].

In this paper we investigate how the relative sizes of the (average) values of the function $p_{m_1, m_2}^*(n)$ (where n is up to a sufficiently large threshold) depend on m_1 and m_2 . We study how the ranking of pairs (m_1, m_2) according to the size of the average values of $p_{m_1, m_2}^*(n)$ can be predicted based on m_1 and m_2 only. Guided by a heuristic argument, we introduce a function R_{m_1, m_2} depending only on m_2 and the prime factors of m_1 . This function exhibits excellent rank-order correlations with both the average and the maximum of $p_{m_1, m_2}^*(n)$ up to 10^9 over the pairs of relatively prime $1 \leq m_1 \neq m_2 \leq 40$. The corresponding Spearman's rank-order coefficients are 0.9949 (4 d.p.) for the averages and 0.9958 (4 d.p.) for the maxima. The function R_{m_1, m_2} also forecasts two notable observations emerging from our data: (1) For every pair $1 \leq m_1 < m_2 \leq 40$ relatively prime, the average value of $p_{m_1, m_2}^*(n)$ where $n \leq 10^9$ is larger than that of $p_{m_2, m_1}^*(n)$; and (2) The average value of $p_{m_1, m_2}^*(n)$ where $n \leq 10^9$, apart from m_2 , appears to be affected only by the distinct prime factors of m_1 , but hardly at all by the exact value of m_1 .

After preliminaries, the main results are presented in Section 3. Section 4 includes remarks regarding the connection between the magnitude of $p_{m_1, m_2}^*(n)$ and the running times of verification algorithms for GGC. Section 5 summarizes some of the main conclusions. Section 6 contains the tables of relevant numerical data generated, including the following: the average and maximum values of $p_{m_1, m_2}^*(n)$ where $n \leq 10^9$, for every $1 \leq m_1 \neq m_2 \leq 20$ relatively prime, and the rankings

of all pairs $1 \leq m_1 \neq m_2 \leq 20$ relatively prime according to the size of the average $p_{m_1, m_2}^*(n)$ and that of the maximal $p_{m_1, m_2}^*(n)$, respectively, where $n \leq 10^9$, and according to the value R_{m_1, m_2} .

2. PRELIMINARIES

For every integer a and b , let $\gcd(a, b)$ denote the greatest common divisor of a and b . Hardy and Littlewood [6] introduced the following conjecture:

Generalised Goldbach conjecture (GGC). *Let m_1 and m_2 be positive integers such that $\gcd(m_1, m_2) = 1$. Then for every sufficiently large integer n satisfying the conditions*

- (1) $\gcd(n, m_1) = \gcd(n, m_2) = 1$ and
- (2) $n \equiv m_1 + m_2 \pmod{2}$,

there exist primes p and q such that

$$(2.1) \quad n = m_1 p + m_2 q.$$

Furthermore, they also conjectured the following estimate for the number of ways $N(n)$ in which an integer n satisfying the conditions of GGC can be expressed in the form 2.1:

$$N(n) \sim \frac{2C_2}{m_1 m_2} \frac{n}{(\log n)^2} \prod \left(\frac{p-1}{p-2} \right),$$

where C_2 is the twin prime constant, and the product is taken over all odd primes p which divide m_1, m_2 or n .

In the sequel n, m_1 , and m_2 denote positive integers such that m_1 and m_2 are relatively prime. The definitions and notation below were introduced earlier [9]:

Definition 2.1. An expression of the form 2.1 where p and q are primes is called an (m_1, m_2) -Goldbach partition (or (m_1, m_2) -partition) of n . We say that n can be (m_1, m_2) -partitioned if it possesses at least one (m_1, m_2) -partition.

Definition 2.2. If n can be (m_1, m_2) -partitioned then the smallest and the largest values of p [q] in all (m_1, m_2) -partitions of n are denoted by $p_{m_1, m_2}^*(n)$ [$q_{m_1, m_2}^*(n)$] and $p_{m_1, m_2}^{**}(n)$ [$q_{m_1, m_2}^{**}(n)$], respectively. We call $n = m_1 p_{m_1, m_2}^*(n) + m_2 q_{m_1, m_2}^{**}(n)$ the p -minimal (or q -maximal) and $n = m_1 p_{m_1, m_2}^{**}(n) + m_2 q_{m_1, m_2}^*(n)$ the p -maximal (or q -minimal) (m_1, m_2) -partition of n .

We let GGC_{m_1, m_2} denote the claim of GGC for given coefficients m_1, m_2 . Then $\text{GGC}_{1,1}$ and $\text{GGC}_{1,2}$ are Goldbach's and Lemoine's conjectures, respectively. Clearly, for every m_1, m_2 the conditions of GGC_{m_1, m_2} and GGC_{m_2, m_1} on n are equivalent, and every (m_1, m_2) -partition of n is also an (m_2, m_1) -partition if the order of terms is disregarded. Hence, a number n can be (m_1, m_2) -partitioned if and only if it can be (m_2, m_1) -partitioned, and in this case $p_{m_1, m_2}^*(n) = q_{m_2, m_1}^*(n)$ and $p_{m_1, m_2}^{**}(n) = q_{m_2, m_1}^{**}(n)$. Conjectures GGC_{m_1, m_2} and GGC_{m_2, m_1} are equivalent.

For every m_1, m_2 , the number $n = m_1 + m_2$ satisfies the conditions of GGC_{m_1, m_2} and cannot be (m_1, m_2) -partitioned. Hence, if GGC_{m_1, m_2} is true then there exists a largest positive integer satisfying the conditions of GGC_{m_1, m_2} that cannot be (m_1, m_2) -partitioned, which we denote by k_{m_1, m_2} . For every $1 \leq m_1, m_2 \leq 40$ relatively prime, the authors computed the largest integer less than or equal to 10^{12} satisfying the conditions of GGC_{m_1, m_2} that cannot be (m_1, m_2) -partitioned, and denoted it by $\hat{k}_{m_1, m_2}$ and hypothesised that $\hat{k}_{m_1, m_2} = k_{m_1, m_2}$. [9]. Clearly, we have $\hat{k}_{m_1, m_2} = \hat{k}_{m_2, m_1}$, and if GGC_{m_1, m_2} (and so GGC_{m_2, m_1}) holds, then $k_{m_1, m_2} = k_{m_2, m_1}$.

For every pair m_1, m_2 and every $L > \hat{k}_{m_1, m_2}$ for which there is at least one n satisfying the conditions of GGC_{m_1, m_2} such that $\hat{k}_{m_1, m_2} < n \leq L$, we refer to the average [maximum] value

of $p_{m_1, m_2}^*(n)$ over all $\hat{k}_{m_1, m_2} < n \leq L$ satisfying the conditions of GGC_{m_1, m_2} more succinctly as the *average [maximum] of $p_{m_1, m_2}^*(n)$ up to L* . For brevity, the average $p_{m_1, m_2}^*(n)$ up to L is also written as $\overline{p}_{m_1, m_2}^*$ *up to L* . Table 2 in Section 6 shows the average and maximum $p_{m_1, m_2}^*(n)$ up to 10^9 for each pair of relatively prime coefficients $1 \leq m_1 \neq m_2 \leq 20$.

3. FUNCTION R_{m_1, m_2} : COMPARING THE AVERAGE AND THE MAXIMUM $p_{m_1, m_2}^*(n)$ VALUES, RESPECTIVELY, FOR DIFFERENT PAIRS m_1, m_2

In this section we introduce the function R_{m_1, m_2} of m_1 and m_2 . For every positive integer n , the value $\varphi(n)$ of Euler's totient function at n is the number of positive integers less than or equal to n that are relatively prime to n . For every integer a and $m \neq 0$, the modulo m residue of a is denoted by $a \bmod m$, the modulo m residue class of a is denoted by $[a]_m$. The ring of modulo m residue classes and the multiplicative group of the reduced modulo m residue classes are denoted by $\mathbb{Z}/m\mathbb{Z}$ and by $\mathbb{Z}'_m$, respectively. For every $[a]_m, [b]_m \in \mathbb{Z}/m\mathbb{Z}$, $[a]_m + [b]_m$ and $[a]_m [b]_m$ respectively, stand for the sum and the product of $[a]_m$ and $[b]_m$ in $\mathbb{Z}/m\mathbb{Z}$. If $(a, m) = 1$ then $[a]_m^{-1}$ denotes the inverse of $[a]_m \in \mathbb{Z}'_m$ in $\mathbb{Z}'_m$. The set of all primes is denoted by P .

Our heuristics is based on the following proposition:

Proposition 3.1. *Let m_1, m_2 and n be pairwise relatively prime positive integers satisfying the conditions of GGC. If $m_2 \neq 1$ then denote the canonical decomposition of m_2 by $m_2 = r_1^{\alpha_1} \cdots r_k^{\alpha_k}$. For every prime $p < \frac{n}{m_1}$ and q the following two statements are equivalent:*

(1) *We have $n = m_1 p + m_2 q$.*

(2) *One of the following two cases holds:*

Case 1: The following conditions are all satisfied:

- (a) $q \nmid m_2$ and $q \nmid m_1$;
- (b) if $m_2 \neq 1$ then for every $1 \leq i \leq k$: p belongs to one of the following $r_i - 1$ different (reduced) modulo $r_i^{\alpha_i+1}$ residue classes: $[n]_{r_i^{\alpha_i+1}} [m_1]_{r_i^{\alpha_i+1}}^{-1} + j_i \cdot [r_i^{\alpha_i}]_{r_i^{\alpha_i+1}}$, where $1 \leq j_i \leq r_i - 1$;
- (c) p belongs to one of the following $q - 1$ different modulo q^2 residue classes: $[n]_{q^2} \cdot [m_1]_{q^2}^{-1} + \ell \cdot [q]_{q^2}$, where $1 \leq \ell \leq q - 1$ and
- (d) for every prime $s < K$ – where K can be chosen as any number $K \geq \frac{n}{m_2}$ – such that $s \neq q$, $s \nmid m_1$ and $s \nmid m_2$: p belongs to one of the following $s - 1$ different modulo s residue classes: $[n]_s \cdot [m_1]_s^{-1} + [j]_s$, where $1 \leq j \leq s - 1$.

Case 2: The following conditions are all satisfied:

- (e) $q | m_2$ (i.e. $q = r_\ell$ for some $1 \leq \ell \leq k$) and $q \nmid m_1$;
- (f) for every $1 \leq i \leq k$, $i \neq \ell$: p belongs to one of the following $r_i - 1$ different (reduced) modulo $r_i^{\alpha_i+1}$ residue classes: $[n]_{r_i^{\alpha_i+1}} \cdot [m_1]_{r_i^{\alpha_i+1}}^{-1} + j_i \cdot [r_i^{\alpha_i}]_{r_i^{\alpha_i+1}}$, where $1 \leq j_i \leq r_i - 1$;
- (g) p belongs to one of the following $r_\ell - 1$ different (reduced) modulo $r_\ell^{\alpha_\ell+2}$ residue classes: $[n]_{r_\ell^{\alpha_\ell+2}} \cdot [m_1]_{r_\ell^{\alpha_\ell+2}}^{-1} + j_\ell \cdot [r_\ell^{\alpha_\ell+1}]$, where $1 \leq j_\ell \leq r_\ell - 1$ and
- (h) for every prime $s < K$ – where K can be chosen as any number $K \geq \frac{n}{m_2}$ – such that $s \nmid m_1$ and $s \nmid m_2$: p belongs to one of the following $s - 1$ different modulo s residue classes: $[n]_s \cdot [m_1]_s^{-1} + [j]_s$, where $1 \leq j \leq s - 1$.

Proof. If $n = m_1p + m_2q$ then $q \nmid m_1$, since otherwise $q|m_1p$ and $q|m_2q$ would imply $q|n$, and so $q|\gcd(n, m_1) = 1$, a contradiction. Therefore we have the following two possible cases: either ($q \nmid m_1$ and $q \nmid m_2$) or ($q \nmid m_1$ and $q|m_2$). Clearly, $n = m_1p + m_2q \Leftrightarrow n - m_1p = m_2q$.

Case I: Suppose $q \nmid m_1$ and $q \nmid m_2$. Then if $m_2 = 1$ then $m_2q = q$. If $m_2 \neq 1$ then the canonical decomposition of m_2q is $m_2q = qr_1^{\alpha_1} \cdot \dots \cdot r_k^{\alpha_k}$. Since $0 < n - m_1p < n$, we have $n - m_1p = m_2q$ if and only if the following conditions are satisfied:

- (i) if $m_2 \neq 1$ then for every $1 \leq i \leq k$: $r_i^{\alpha_i} | (n - m_1p)$ and $r_i^{\alpha_i+1} \nmid (n - m_1p)$,
- (ii) $q | (n - m_1p)$ and $q^2 \nmid (n - m_1p)$ and
- (iii) for every prime $s < \frac{n}{m_2}$ such that $s \neq q$ and $s \nmid m_2$: $s \nmid (n - m_1p)$.

We show that Conditions (i), (ii) and (iii) above are equivalent to Conditions (b), (c) and (d), respectively, of the Proposition.

(i) For every $1 \leq i \leq k$:

- $r_i^{\alpha_i} | (n - m_1p) \Leftrightarrow [n]_{r_i^{\alpha_i}} = [m_1]_{r_i^{\alpha_i}} [p]_{r_i^{\alpha_i}} \Leftrightarrow [p]_{r_i^{\alpha_i}} = [n]_{r_i^{\alpha_i}} [m_1]_{r_i^{\alpha_i}}^{-1}$.
- $r_i^{\alpha_i+1} \nmid (n - m_1p) \Leftrightarrow [n]_{r_i^{\alpha_i+1}} \neq [m_1]_{r_i^{\alpha_i+1}} [p]_{r_i^{\alpha_i+1}} \Leftrightarrow [p]_{r_i^{\alpha_i+1}} \neq [n]_{r_i^{\alpha_i+1}} [m_1]_{r_i^{\alpha_i+1}}^{-1}$.

As $([p]_{r_i^{\alpha_i}} = [n]_{r_i^{\alpha_i}} [m_1]_{r_i^{\alpha_i}}^{-1} \text{ and } [p]_{r_i^{\alpha_i+1}} \neq [n]_{r_i^{\alpha_i+1}} [m_1]_{r_i^{\alpha_i+1}}^{-1}) \Leftrightarrow [p]_{r_i^{\alpha_i+1}} = [n]_{r_i^{\alpha_i+1}} [m_1]_{r_i^{\alpha_i+1}}^{-1} + j_i \cdot [r_i^{\alpha_i}]_{r_i^{\alpha_i+1}}$ for some $1 \leq j_i \leq r_i - 1$, Conditions (i) and (b) are equivalent.

(ii) We have

- $q | (n - m_1p) \Leftrightarrow [m_1]_q [p]_q = [n]_q \Leftrightarrow [p]_q = [n]_q [m_1]_q^{-1}$.
- $q^2 \nmid (n - m_1p) \Leftrightarrow [m_1]_{q^2} [p]_{q^2} \neq [n]_{q^2} \Leftrightarrow [p]_{q^2} \neq [n]_{q^2} [m_1]_{q^2}^{-1}$.

Since

$([p]_q = [n]_q [m_1]_q^{-1} \text{ and } [p]_{q^2} \neq [n]_{q^2} [m_1]_{q^2}^{-1}) \Leftrightarrow [p]_{q^2} = [n]_{q^2} [m_1]_{q^2}^{-1} + \ell \cdot [q]_{q^2}$ for some $1 \leq \ell \leq q-1$, Conditions (ii) and (c) are equivalent.

(iii) If $s|m_1$ then – since $\gcd(n, m_1) = 1 - s \nmid n$, and so $s \nmid (n - m_1p)$. Therefore Condition (iii) is equivalent to the following: for every prime s such that $s \neq q$, $s \nmid m_1$ and $s \nmid m_2$: $s \nmid (n - m_1p)$. For every prime s such that $s \neq q$, $s \nmid m_1$ and $s \nmid m_2$:

$s \nmid (n - m_1p) \Leftrightarrow [n]_s \neq [m_1]_s [p]_s \Leftrightarrow [p]_s \neq [n]_s [m_1]_s^{-1} \Leftrightarrow [p]_s = [n]_s [m_1]_s^{-1} + [j]_s$ for some $1 \leq j \leq s-1$. Therefore Conditions (iii) and (d) are equivalent. Hence in Case I, $n = m_1p + m_2q$ is equivalent to Conditions (a), (b), (c) and (d) of the Proposition being all satisfied.

Case II: Suppose $q \nmid m_1$ and $q|m_2$. Then $q = r_\ell$ for some $1 \leq \ell \leq k$ and the canonical decomposition of m_2q is $m_2q = r_1^{\alpha_1} \cdot \dots \cdot r_{\ell-1}^{\alpha_{\ell-1}} \cdot q^{\alpha_\ell+1} \cdot r_{\ell+1}^{\alpha_{\ell+1}} \cdot \dots \cdot r_k^{\alpha_k}$.

Similarly to Case I it can be shown that in Case II, $n = m_1p + m_2q$ is equivalent to Conditions (iv), (v) and (vi) below being all satisfied, where Conditions (iv), (v) and (vi) are equivalent to Conditions (f), (g) and (h), respectively, of the Proposition:

- (iv) for every $1 \leq i \leq k$, $i \neq \ell$: $r_i^{\alpha_i} | (n - m_1p)$ and $r_i^{\alpha_i+1} \nmid (n - m_1p)$,
- (v) $q^{\alpha_\ell+1} | (n - m_1p)$ and $q^{\alpha_\ell+2} \nmid (n - m_1p)$ and
- (vi) for every prime $s < \frac{n}{m_2}$ such that $s \nmid m_2$: $s \nmid (n - m_1p)$.

Hence in Case II, $n = m_1p + m_2q$ is equivalent to Conditions (e), (f), (g) and (h) of the Proposition being all satisfied.

□

Next we describe a heuristic argument to deduce the function R_{m_1, m_2} of ordered pairs (m_1, m_2) of relatively prime coefficients. In the following let m_1, m_2 and n be given positive integers satisfying the conditions of GGC.

Condition (b) in Case 1 of Proposition 3.1 means that if $m_2 \neq 1$ then $\forall 1 \leq i \leq k$ the prime p belongs to one of certain $r_i - 1$ reduced modulo $r_i^{\alpha_i+1}$ classes. Hence, using the Chinese Remainder Theorem and the Prime Number Theorem for Arithmetic Progressions (PNTAP), for a given prime q , the ratio of primes satisfying this condition among all primes in a 'large' interval can be estimated as

$$\prod_{1 \leq i \leq k} \frac{r_i - 1}{\varphi(r_i^{\alpha_i+1})} = \prod_{1 \leq i \leq k} \frac{r_i - 1}{r_i^{\alpha_i+1} - r_i^{\alpha_i}} = \prod_{1 \leq i \leq k} \frac{1}{r_i^{\alpha_i}} = \frac{1}{m_2}.$$

If $q|n$ then Condition (c) means that $q|(n - m_1p)$ and $q^2 \nmid n - m_1p$. Then $q|m_1p$, and since n and m_1 are relatively prime, $q|p$ and so $q = p$. Therefore, in this case there is at most one p , namely $p = q$ which can satisfy Condition (c). Hence, the ratio of those primes p in a 'large' interval satisfying the condition tends to 0, as the length of the interval approaches infinity. If $q \nmid n$ then Condition (c) means that the prime p belongs to one of certain $q - 1$ reduced modulo q^2 residue classes, so in this case, by PNTAP the ratio of primes p satisfying Condition (c) among all primes in a 'large' interval can be estimated by $\frac{q-1}{\varphi(q^2)} = \frac{q-1}{q^2-q} = \frac{1}{q}$.

Let q be a given prime. Condition (d) means that for every prime s such that $s \neq q$, $s \nmid m_1$ and $s \nmid m_2$ we have the following: If $s|n$ then $p \nmid s$, that is $p \neq s$, and if $s \nmid n$ then either $p = s$ or p belongs to one of certain $s - 2$ reduced modulo s residue classes. The ratio of primes p satisfying Condition (d) among all primes in a 'large' interval can be estimated by

$$\prod_{\substack{s \in P, s \neq q, s < \frac{n}{m_2} \\ s \nmid nm_1m_2}} \frac{s-2}{\varphi(s)} = \prod_{\substack{s \in P, s \neq q, s < \frac{n}{m_2} \\ s \nmid nm_1m_2}} \frac{s-2}{s-1},$$

(Note that since $n \equiv m_1 + m_2 \pmod{2}$, exactly one of n, m_1 and m_2 is even, and so $s \nmid nm_1m_2$ implies that in the factors of the above product $s \neq 2$.)

If for some primes p and q we have $q|n$ and $n = m_1p + m_2q$ then $q|m_1p$, which – since $\gcd(n, m_1) = 1$ – implies $q|p$, hence $p = q$, and so $n = (m_1 + m_2)q$. Therefore, if $q|n$, then ignoring the case of the single number $n = (m_1 + m_2)q$, the number of primes satisfying $n = m_1p + m_2q$ is 0. Furthermore, since for given m_1, m_2, n and K no prime p can satisfy Conditions (c) and (d) simultaneously for more than one prime $q \leq K$, the ratio of those primes p for which there exists a prime $q \leq \frac{n}{m_2}$, $q \nmid m_1m_2$ such that Conditions (b) (c) and (d) are simultaneously satisfied, in a 'large' interval can be estimated as

$$\begin{aligned} (3.1) \quad \sum_{\substack{q \in P, q < \frac{n}{m_2} \\ q \nmid nm_1m_2}} \left(\frac{1}{m_2q} \cdot \prod_{\substack{s \in P, s \neq q \\ s \nmid nm_1m_2 \\ s < \frac{n}{m_2}}} \frac{s-2}{s-1} \right) &= \frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s < \frac{n}{m_2} \\ s \nmid nm_1m_2}} \frac{s-2}{s-1} \right) \cdot \sum_{\substack{q \in P, q < \frac{n}{m_2} \\ q \nmid nm_1m_2}} \frac{q-1}{q(q-2)} \\ &\approx \frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s < \frac{n}{m_2} \\ s \nmid nm_1m_2}} \frac{s-2}{s-1} \right) \cdot \sum_{\substack{q \in P, q < \frac{n}{m_2} \\ q \nmid nm_1m_2}} \frac{1}{q-1}. \end{aligned}$$

Case 2: Estimated ratio of primes $p < \frac{n}{m_1}$ satisfying the conditions of Case 2 for some prime $q|m_2$, among all primes less than $\frac{n}{m_1}$:

$$(3.2) \quad \sum_{q \in P, q|m_2} \left(\frac{1}{m_2 q} \cdot \prod_{\substack{s \in P, s \leq \frac{n}{m_2} \\ s \nmid nm_1 m_2}} \frac{s-2}{s-1} \right) = \frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s \leq \frac{n}{m_2} \\ s \nmid nm_1 m_2}} \frac{s-2}{s-1} \right) \cdot \sum_{q \in P, q|m_2} \frac{1}{q}$$

$$\approx \frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s \leq \frac{n}{m_2} \\ s \nmid nm_1 m_2}} \frac{s-2}{s-1} \right) \cdot \sum_{q \in P, q|m_2} \frac{1}{q-1}.$$

The sum of the above estimates yields an estimate for the ratio of primes $p < \frac{n}{m_1}$ satisfying the conditions of Case 1 or Case 2 for some prime q among all primes less than $\frac{n}{m_1}$:

$$(3.3) \quad \frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s \leq \frac{n}{m_2} \\ s \nmid nm_1 m_2}} \frac{s-2}{s-1} \right) \cdot \sum_{\substack{q \in P, q \leq \frac{n}{m_2} \\ q \nmid nm_1}} \frac{1}{q-1}$$

For ease of calculation define

$$(3.4) \quad R'_{m_1, m_2}(n) := \frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s \leq n \\ s \nmid nm_1 m_2}} \frac{s-2}{s-1} \right) \cdot \sum_{\substack{q \in P, q \leq n \\ q \nmid nm_1}} \frac{1}{q-1}.$$

Then for all sufficiently large n satisfying the conditions of GGC_{m_1, m_2} :

$$(3.5) \quad \frac{R'_{1,1}(n)}{R'_{m_1, m_2}(n)} = \left(\prod_{\substack{s \in P, s \nmid n \\ 2 < s \leq n}} \frac{s-2}{s-1} \right) \cdot \left(\sum_{\substack{q \in P, s \nmid n \\ q \leq n}} \frac{1}{q-1} \right) / \left(\frac{1}{m_2} \cdot \left(\prod_{\substack{s \in P, s \leq n \\ s \nmid nm_1 m_2}} \frac{s-2}{s-1} \right) \cdot \left(\sum_{\substack{q \in P, q \leq n \\ q \nmid nm_1}} \frac{1}{q-1} \right) \right)$$

$$= m_2 \cdot \left(\prod_{\substack{s \in P, 2 < s \leq n \\ s|m_1 m_2}} \frac{s-2}{s-1} \right) \cdot \left(1 + \left(\sum_{\substack{q \in P, q \leq n \\ q|m_1}} \frac{1}{q-1} \right) / \left(\sum_{\substack{q \in P, q \leq n \\ q \nmid nm_1}} \frac{1}{q-1} \right) \right)$$

$$\text{Define } R_{m_1, m_2} := m_2 \cdot \prod_{\substack{s \in P, 2 < s \\ s|m_1 m_2}} \frac{s-2}{s-1}.$$

We hypothesize that there is a ‘strong’ rank-order correlation between the following three values across all pairs of relatively prime positive integers (m_1, m_2) : (1) $\overline{p}^*_{m_1, m_2}$ up to any sufficiently large threshold; (2) the maximum of $p^*_{m_1, m_2}$ up to any sufficiently large threshold and (3) R_{m_1, m_2} .

We determined the ascending rankings of all ordered pairs of relatively prime coefficients $1 \leq m_1 \neq m_2 \leq 40$ by $\overline{p}^*_{m_1, m_2}$ up to 10^9 (Ranking 1), by the maximum of $p^*_{m_1, m_2}(n)$ up to 10^9 (Ranking 2) and by R_{m_1, m_2} (Ranking 3). Subfigures 1a, 1b and 1c in Figure 1 display the scatter

plots of Ranking 3 (vertical axis) against Ranking 1 (horizontal axis), Ranking 2 (vertical axis) against Ranking 1 (horizontal axis), and Ranking 2 (vertical axis) against Ranking 1 (horizontal axis), respectively. Table 3 shows Rankings 1, 2 and 3 of all relatively prime coefficients $1 \leq m_1 \neq m_2 \leq 20$. The Spearman's rank-order correlation coefficient between R_{m_1, m_2} and $\overline{p}_{m_1, m_2}^*$ up to 10^9 is 0.9949 (4 d.p.) and between R_{m_1, m_2} and the maximum of $p_{m_1, m_2}^*(n)$ up to 10^9 is 0.9958 (4 d.p.). These data suggest very strong rank correlation between R_{m_1, m_2} and $\overline{p}_{m_1, m_2}^*$ up to 10^9 and also between R_{m_1, m_2} and the maximum of $p_{m_1, m_2}^*(n)$ up to 10^9 , supporting our hypothesis.

The function R_{m_1, m_2} also predicts two further observations emerging from our data. Note that the value of R_{m_1, m_2} depends only on m_2 and on the set of prime divisors of m_1 , but not on the actual value of m_1 . Based on this we can expect $\overline{p}_{m_1, m_2}^*$ and $\overline{p}_{m'_1, m_2}^*$ to be ‘very close’ to each other, if m_1 and m'_1 have the same set of prime divisors. This prediction is in remarkable correspondence with our experimental data, as it is illustrated by Table 1 in Section 6. The other observation is the following: For every pair $1 \leq m_1 < m_2 \leq 40$ relatively prime we found that $\overline{p}_{m_2, m_1}^* < \overline{p}_{m_1, m_2}^*$ up to 10^9 , which is predicted by the fact that we have $R_{m_2, m_1} < R_{m_1, m_2}$ when $m_1 < m_2$.

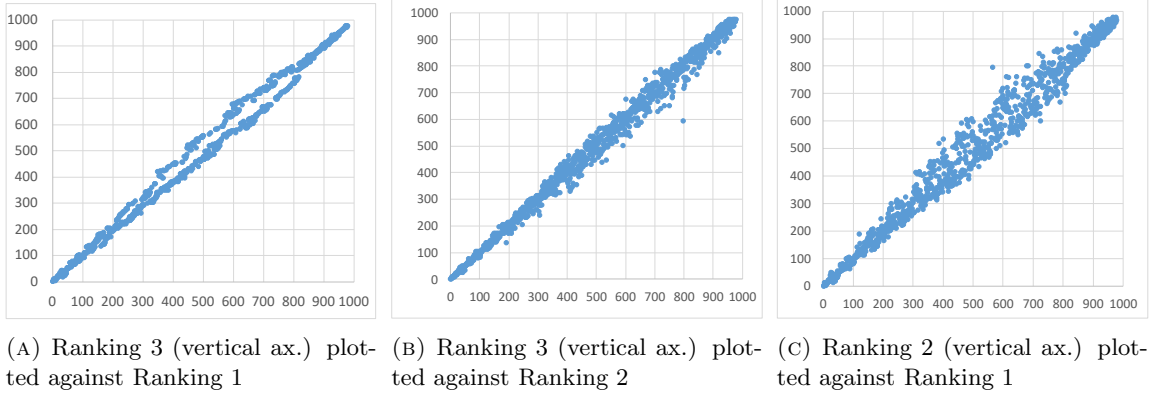


FIGURE 1. Subfigures 1a and 1b plot the ascending ranking of all pairs of relatively prime $1 \leq m_1 \neq m_2 \leq 40$ by R_{m_1, m_2} on the vertical axis against their ascending ranking by $\overline{p}_{m_1, m_2}^*$ up to 10^9 and against their ascending ranking by the maximum of p_{m_1, m_2}^* up to 10^9 , respectively. Subfigure 1c displays the ascending ranking of the same pairs m_1, m_2 by the maximum of p_{m_1, m_2}^* up to 10^9 against their ascending ranking by $\overline{p}_{m_1, m_2}^*$ up to 10^9 .

Figure 2 shows R_{m_1, m_2} plotted against $\overline{p}_{m_1, m_2}^*$ up to 10^9 , for all pairs of relatively prime coefficients $1 \leq m_1 \neq m_2 \leq 40$. The points are clearly clustered ‘very close’ to a smooth looking curve. The diagram suggests a relationship close to linear, but we hypothesize that the relationship can be approximated better by some sublinear function, because of the decreasing trend which can be observed in the quotients $R_{m_1, m_2} / \overline{p}_{m_1, m_2}^*$ as $\overline{p}_{m_1, m_2}^*$ increases. The approximation of this relationship is a topic for further research.

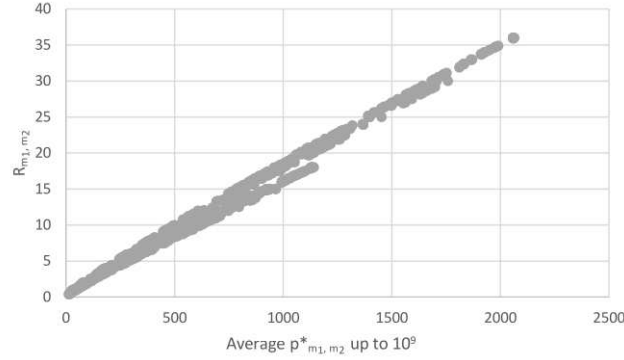


FIGURE 2. R_{m_1, m_2} (vertical axis) plotted against $\overline{p^*}_{m_1, m_2}$ up to 10^9 (horizontal axis) for every pair of relatively prime coefficients $1 \leq m_1, m_2 \leq 40$.

4. CONNECTIONS BETWEEN THE MAGNITUDE OF $p^*_{m_1, m_2}(n)$ AND THE RUNNING TIMES OF VERIFYING ALGORITHMS OF GGC

We applied four (two in case $m_1 = m_2 = 1$) different natural algorithms to test GGC for all pairs of relatively prime coefficients $1 \leq m_1, m_2 \leq 40$ up to 10^{12} [9]. These algorithms search for $p^*_{m_1, m_2}(n)$ or $p^*_{m_2, m_1}(n)$ in ascending order among the ‘small’ primes or for $q^{**}_{m_1, m_2}(n)$ or $q^{**}_{m_2, m_1}(n)$ in descending order among ‘large’ primes. According to expectations, for each pair m_1, m_2 , algorithms searching for the ‘large’ primes in descending order were faster than those looking for the ‘small’ primes in ascending order.

Based on our data we hypothesize that the significant part of the computation is searching for the (m_1, m_2) -partitions of numbers after prime generation. Therefore, an estimate for the magnitude of the functions $p^*_{m_1, m_2}(n)$ is required to approximate the time complexities of these algorithms.

When looking for $p^*_{m_1, m_2}(n)$ [$p^*_{m_2, m_1}(n)$], only primes p satisfying

$$(4.1) \quad m_1 p \equiv n \pmod{m_2} \quad [m_2 p \equiv n \pmod{m_1}]$$

were checked as candidates. An analogous idea was also used when searching for $q^{**}_{m_1, m_2}(n)$ [$q^{**}_{m_2, m_1}(n)$]. We introduced the function

$$g_L(m_1, m_2) := \frac{\overline{p^*}_L(m_1, m_2)}{\varphi(m_2) \ln \overline{p^*}_L(m_1, m_2)},$$

where $\overline{p^*}_L(m_1, m_2)$ is the average of $p^*_{m_1, m_2}(n)$ up to L . For 476 out of all 489 pairs $m_1 < m_2$ tested, we observed that the algorithm searching for $p^*_{m_1, m_2}(n)$ was faster than the one looking for $p^*_{m_2, m_1}(n)$ if and only if $g_{10^9}(m_1, m_2) < g_{10^9}(m_2, m_1)$ [9].

In less optimized versions of the algorithms where all primes are tested in order as candidates regardless of the congruence conditions 4, we predict that if $m_1 < m_2$ then searching for $p^*_{m_2, m_1}(n)$ is faster than for $p^*_{m_1, m_2}(n)$, since $\overline{p^*}_L(m_2, m_1) < \overline{p^*}_L(m_1, m_2)$ (where $L = 10^9$) in our data, which is also anticipated by $R_{m_2, m_1} < R_{m_1, m_2}$.

5. CONCLUSION AND CLOSING REMARKS

The function

$$R_{m_1, m_2} := m_2 \cdot \prod_{\substack{s \in P, 2 < s \\ s | m_1 m_2}} \frac{s-2}{s-1}$$

exhibits very strong rank-order correlation with the average and the maximum of $p_{m_1, m_2}^*(n)$ up to 10^9 on the pairs $1 \leq m_1 \neq m_2 \leq 40$ of relatively prime coefficients, with Spearman's rank-order coefficients $\rho = 0.9949$ and 0.9958 , respectively. By our heuristics we expect this strong rank-order correlation to hold also in general, in cases of all larger thresholds $N \geq 10^9$ and relatively prime positive coefficients m_1, m_2 . The function R_{m_1, m_2} helps explain observed trends in our empirical data: (1) For every pair $m_1 < m_2$ we found that the average of $p_{m_2, m_1}^*(n)$ up to 10^9 is less than that of $p_{m_1, m_2}^*(n)$, in line with $R_{m_2, m_1} < R_{m_1, m_2}$. (2) For every pair m_1, m_2 and m'_1, m_2 such that m_1 and m'_1 have the same set of prime divisors, the averages of $p_{m_1, m_2}^*(n)$ and $p_{m'_1, m_2}^*(n)$, respectively, up to 10^9 are strikingly close to equal (illustrated by Table 1) as predicted by $R_{m_1, m_2} = R_{m'_1, m_2}$.

For the size of the minimal primes in the classical Goldbach-partitions of even numbers Granville *et al.* conjectured $p(n) = O(\log^2 n \log \log n)$. Granville also proposed two more precise, incompatible conjectures of the form $p(n) \leq (C + o(1)) \log^2 n \log \log n$, where C is 'sharp' in the sense that C is the smallest constant with this property: one with $C = C_2^{-1} \approx 1.51478$ and the other one with $C = 2e^{-\gamma} C_2^{-1} \approx 1.70098$, where $C_2 \approx 0.66016$ is the twin prime constant and $\gamma \approx 0.57722$ is the Euler's constant [13]. In order to gain information about the running times of the verification algorithms for GGC, estimates of the functions $p_{m_1, m_2}^*(n)$ are required. This is a goal of future research.

6. TABLES OF DATA

TABLE 1. The average $\overline{p}_{m_1, m_2}^*$ up to 10^9 for some pairs $1 \leq m_1, m_2 \leq 40$ relatively prime, where $m_2 = 1, 2, \dots, 10$. The values of m_1 in each block of the table have the same set of prime factors.

m_1	m_2	$\overline{p}_{m_1, m_2}^*$ up to 10^9	m_1	m_2	$\overline{p}_{m_1, m_2}^*$ up to 10^9	m_1	m_2	$\overline{p}_{m_1, m_2}^*$ up to 10^9	m_1	m_2	$\overline{p}_{m_1, m_2}^*$ up to 10^9
2	1	32.80032	2	3	69.3523	6	5	93.49197	10	7	207.4333
4	1	32.80632	4	3	69.36289	12	5	93.51101	20	7	207.4348
8	1	32.80586	8	3	69.36958	18	5	93.51134	40	7	207.4686
16	1	32.80342	16	3	69.36135	24	5	93.49586	3	8	211.8724
32	1	32.80091	32	3	69.35155	36	5	93.48327	9	8	211.9202
3	1	20.07168	5	3	55.3382	14	5	141.3145	27	8	211.8799
9	1	20.07204	25	3	55.32683	28	5	141.2603	5	8	295.392
27	1	20.07323	10	3	51.95065	5	6	118.7446	25	8	295.2399
5	1	26.76726	20	3	51.9557	25	6	118.7147	2	9	241.7804
25	1	26.76609	40	3	51.95532	2	7	277.1068	4	9	241.822
6	1	17.27898	14	3	57.04462	4	7	277.1688	8	9	241.7959
12	1	17.2793	28	3	57.04528	8	7	277.1186	16	9	241.8264
18	1	17.2799	3	4	97.75677	16	7	277.1205	32	9	241.8369
24	1	17.28291	9	4	97.77398	32	7	277.0654	5	9	184.588
36	1	17.27778	27	4	97.74765	3	7	154.0733	25	9	184.5113
10	1	23.95771	5	4	135.3879	9	7	154.1084	10	9	180.7615
20	1	23.95715	25	4	135.3506	27	7	154.0745	20	9	180.7731
40	1	23.95944	2	5	172.1371	5	7	211.9499	40	9	180.7754
14	1	26.55656	4	5	172.1871	25	7	211.871	14	9	198.5837
28	1	26.55535	8	5	172.142	6	7	149.3165	28	9	198.5533
3	2	43.62564	16	5	172.1673	12	7	149.3576	3	10	208.8865
9	2	43.62763	32	5	172.1022	18	7	149.359	9	10	208.9756
27	2	43.62843	3	5	97.29165	24	7	149.344	27	10	208.9667
5	2	60.48209	9	5	97.31536	36	7	149.3266			
25	2	60.46878	27	5	97.29562						

TABLE 2. Average (avg) and maximum (max) values of $p_{m_1, m_2}^*(n)$ and $q_{m_1, m_2}^*(n)$ up to 10^9 for each pair of relatively prime coefficients $1 \leq m_1 < m_2 \leq 20$.

m_1	m_2	$P_{m_1, m_2}^*(n)$		$Q_{m_1, m_2}^*(n)$		m_1	m_2	$P_{m_1, m_2}^*(n)$		$Q_{m_1, m_2}^*(n)$		m_1	m_2	$P_{m_1, m_2}^*(n)$		$Q_{m_1, m_2}^*(n)$	
		avg	max	avg	max			avg	max	avg	max			avg	max	avg	max
1	1					4	9	241.822	7927	97.774	3001	9	13	333.584	10193	222.26	6761
1	2	80.839	3037	32.8	1609	4	11	494.758	19507	160.372	5939	9	14	331.513	10067	198.584	6337
1	3	72.911	2371	20.072	743	4	13	607.515	24919	163.502	6311	9	16	463.174	13627	241.826	7219
1	4	181.026	6971	32.806	1453	4	15	327.845	9257	73.338	2153	9	17	464.765	13007	227.655	6481
1	5	176.526	6833	26.767	1093	4	17	841.539	29669	167.531	6553	9	19	528.846	15649	229.533	6301
1	6	157.484	4969	17.279	643	4	19	960.026	32801	168.921	6947	9	20	449.818	13921	180.773	5519
1	7	281.84	9431	29.376	1129	5	6	118.745	3457	93.492	2801	10	11	370.28	13093	339.42	12241
1	8	393.604	15497	32.806	1493	5	7	211.95	8969	145.513	6871	10	13	454.483	15731	345.898	11117
1	9	245.866	8431	20.072	647	5	8	295.392	10369	172.142	6229	10	17	632.311	21647	354.305	14369
1	10	382.522	13009	23.958	1153	5	9	184.588	5333	97.315	2731	10	19	721.599	25057	357.248	13033
1	11	500.068	17093	31.678	1499	5	11	375.582	11839	156.587	5881	11	12	304.228	8821	267.184	8293
1	12	342.648	11261	17.279	673	5	12	257.838	7309	93.511	2969	11	13	542.8	17299	452.035	16829
1	13	612.063	23663	32.294	1297	5	13	459.273	16477	159.551	5521	11	14	542.423	20359	406.549	15227
1	14	611.042	20359	26.557	1129	5	14	459.822	15773	141.315	4651	11	15	295.49	8941	203.796	5527
1	15	332.373	9127	15.379	557	5	16	638.409	24677	172.167	6451	11	16	754.067	26839	494.633	17863
1	16	849.623	33997	32.803	1597	5	17	637.215	22751	163.446	5657	11	17	751.415	25621	463.029	17713
1	17	846.422	32779	33.084	1381	5	18	398.036	10499	93.511	2963	11	18	470.391	14251	267.222	7681
1	18	529.975	15313	17.28	701	5	19	726.834	25609	164.784	5711	11	19	856.789	27581	466.872	15467
1	19	964.977	33791	33.364	1321	6	7	149.317	4597	129.94	3923	11	20	734.055	26497	370.239	12853
1	20	825.834	29209	23.957	1069	6	11	267.139	8543	139.856	4813	12	13	328.85	9871	310.206	9479
2	3	69.352	2083	43.626	1399	6	13	328.759	10883	142.486	4957	12	17	459.61	13033	317.598	10657
2	5	172.137	6379	60.482	2459	6	17	459.546	14731	145.948	4201	12	19	523.692	14699	320.198	9437
2	7	277.107	12011	66.282	2663	6	19	523.593	16703	147.126	4423	13	14	552.943	19889	499.815	16843
2	9	241.78	7129	43.628	1549	7	8	323.92	12589	277.119	11197	13	15	301.049	8539	250.574	7151
2	11	494.633	21107	71.487	3061	7	9	202.501	5717	154.108	4271	13	16	768.659	28463	607.268	25127
2	13	607.339	21383	72.924	3049	7	10	315.347	9769	207.433	6841	13	17	765.986	25747	566.894	21851
2	15	327.714	9049	32.917	1031	7	11	411.838	15131	249.973	9439	13	18	479.435	15199	328.849	9277
2	17	841.438	30859	74.71	3121	7	12	282.761	9137	149.358	4663	13	19	873.509	33703	571.528	22079
2	19	959.87	34039	75.341	3001	7	13	504.267	18593	254.754	10099	13	20	748.115	27953	454.483	14851
3	4	97.757	2939	69.363	2411	7	15	274.865	7499	116.406	3583	14	15	270.331	7789	248.994	6689
3	5	97.292	2909	55.338	1709	7	16	700.594	22783	277.12	10357	14	17	693.436	25121	566.271	20717
3	7	154.073	4517	60.42	1789	7	17	698.137	24109	260.916	11069	14	19	791.453	27277	570.866	20873
3	8	211.872	6869	69.37	2383	7	18	436.631	13367	149.359	4481	15	16	347.11	9521	327.84	8893
3	10	208.887	6359	51.951	1471	7	19	796.433	27583	263.145	10289	15	17	349.899	9539	308.256	8179
3	11	271.626	8231	64.839	2113	7	20	682.396	23689	207.435	6841	15	19	398.729	10979	310.687	9109
3	13	333.472	10733	66.064	1999	8	9	241.796	7027	211.92	6961	16	17	841.42	30727	787.381	29531
3	14	331.38	10259	57.045	1867	8	11	494.594	18481	348.832	13499	16	19	959.865	35327	793.796	28631
3	16	463.076	13553	69.361	2239	8	13	607.287	23887	355.623	12107	17	18	491.044	14149	459.684	12953
3	17	464.638	12503	67.628	2269	8	15	327.799	9091	158.333	4817	17	19	894.547	33721	790.894	26927
3	19	528.697	15217	68.167	2063	8	17	841.438	31081	364.403	15749	17	20	765.917	28429	632.339	25237
3	20	449.579	12659	51.956	1579	8	19	959.894	42727	367.416	13999	18	19	523.747	14897	495.035	16943
4	5	172.187	7109	135.388	5521	9	10	208.976	6469	180.762	5501	19	20	772.014	28729	721.715	24071
4	7	277.169	11497	148.746	5939	9	11	271.709	8363	218.093	6827						

TABLE 3. Ascending rankings of all pairs of relatively prime coefficients $1 \leq m_1 \neq m_2 \leq 20$ by average $p_{m_1, m_2}^*(n)$ and by maximum $p_{m_1, m_2}^*(n)$ up to 10^9 and by R_{m_1, m_2} , respectively.

m_1	m_2	Ranking of (m_1, m_2) by			m_1	m_2	Ranking of (m_1, m_2) by		
		$\max p_{m_1, m_2}^*$	$\text{avg } p_{m_1, m_2}^*$	R_{m_1, m_2}			$\max p_{m_1, m_2}^*$	$\text{avg } p_{m_1, m_2}^*$	R_{m_1, m_2}
15	1	1	1	1	13	15	115	128	118
6	1	2	2	4	11	12	117	129	123
12	1	4	3	4	17	15	110	130	120
18	1	5	4	4	13	12	131	131	125
3	1	6	5	4	19	15	123	132	121
9	1	3	6	4	7	10	134	133	150
20	1	8	7	8.5	17	12	144	134	129.5
10	1	12	8	8.5	19	12	129	135	133
14	1	11	9	11.5	7	8	158	136	151
5	1	9	10	8.5	2	15	121	137	129.5
7	1	10	11	11.5	8	15	122	138	129.5
11	1	20	12	13	16	15	118	139	129.5
13	1	13	13	14	4	15	126	140	129.5
2	1	24	14	19.5	6	13	146	141	143
16	1	23	15	19.5	18	13	127	142	143
8	1	19	16	19.5	12	13	135	143	143
4	1	17	17	19.5	3	14	139	144	137
15	2	7	18	8.5	9	14	136	145	137
17	1	15	19	15	1	15	124	146	129.5
19	1	14	20	16	3	13	145	147	143
3	2	16	21	19.5	9	13	138	148	143
9	2	21	22	19.5	11	10	156	149	153.5
10	3	18	23	24	1	12	151	150	148
20	3	22	24	24	13	10	149	151	155
5	3	25	25	24	15	16	132	152	148
14	3	27	26	26.5	11	8	168	153	158
7	3	26	27	26.5	15	17	133	154	146
5	2	38	28	35	17	10	175	155	156
11	3	31	29	28	13	8	155	156	159
13	3	28	30	29	19	10	165	157	157
7	2	39	31	39	17	8	189	158	165
17	3	34	32	30	19	8	172	159	168
19	3	29	33	31	20	11	160	160	161
2	3	30	34	35	10	11	166	161	161
16	3	33	35	35	5	11	153	162	161
4	3	37	36	35	1	10	163	163	165
8	3	36	37	35	1	8	186	164	175
11	2	50	38	40	5	18	143	165	153.5
1	3	35	39	35	15	19	147	166	152
13	2	49	40	41	14	11	183	167	179
15	4	32	41	35	7	11	180	168	179
17	2	51	42	44.5	7	18	167	169	165
19	2	47	43	48	3	20	159	170	165
1	2	48	44	50	9	20	171	171	165
6	5	41	45	44.5	13	11	193	172	193
12	5	45	46	44.5	10	13	188	173	185
18	5	44	47	44.5	20	13	178	174	185
3	5	42	48	44.5	5	13	191	175	185

m_1	m_2	Ranking of (m_1, m_2) by			m_1	m_2	Ranking of (m_1, m_2) by		
		$\max p_{m_1, m_2}^*$	$\text{avg } p_{m_1, m_2}^*$	R_{m_1, m_2}			$\max p_{m_1, m_2}^*$	$\text{avg } p_{m_1, m_2}^*$	R_{m_1, m_2}
9	5	40	49	44.5	6	17	177	176	171
3	4	43	50	50	12	17	164	177	171
9	4	46	51	50	18	17	161	178	171
15	7	53	52	52	5	14	190	179	183
5	6	52	53	53	17	11	198	180	194
7	6	54	54	54	3	16	169	181	175
5	4	70	55	65	9	16	170	182	175
11	6	63	56	55	3	17	157	183	171
14	5	61	57	67.5	9	17	162	184	171
13	6	65	58	56	19	11	185	185	195
7	5	96	59	67.5	11	18	174	186	177
17	6	55	60	57	13	18	181	187	179
19	6	57	61	58	17	18	173	188	181
7	4	77	62	69	8	11	200	189	198
6	7	60	63	61	2	11	208	190	198
12	7	62	64	61	16	11	199	191	198
18	7	58	65	61	4	11	202	192	198
3	7	59	66	61	19	18	195	193	182
9	7	56	67	61	14	13	194	194	201.5
11	5	76	68	71.5	1	11	196	195	198
1	6	66	69	65	7	13	201	196	201.5
15	8	64	70	65	6	19	192	197	189
13	5	71	71	74	12	19	176	198	189
11	4	78	72	77	18	19	179	199	189
17	5	73	73	75	3	19	182	200	189
13	4	81	74	78	9	19	187	201	189
19	5	74	75	76	1	18	184	202	192
17	4	88	76	84.5	11	14	205	203	203
19	4	97	77	90	11	13	197	204	205
2	5	84	78	84.5	13	14	203	205	204
8	5	79	79	84.5	17	14	206	206	206
16	5	85	80	84.5	17	13	211	207	208
4	5	101	81	84.5	19	14	207	208	207
1	5	92	82	84.5	19	13	212	209	209
10	9	68	83	71.5	16	13	224	210	213
20	9	69	84	71.5	8	13	217	211	213
1	4	99	85	92	2	13	209	212	213
5	9	67	86	71.5	4	13	221	213	213
14	9	82	87	84.5	1	14	204	214	210
7	9	75	88	84.5	1	13	215	215	213
15	11	72	89	79	10	17	210	216	217
10	7	93	90	99.5	20	17	225	217	217
20	7	94	91	99.5	5	17	213	218	217
3	10	83	92	84.5	5	16	220	219	219
9	10	86	93	84.5	7	20	216	220	220
3	8	95	94	92	14	17	223	221	221.5
9	8	98	95	92	7	17	219	222	221.5
5	7	120	96	99.5	7	16	214	223	223
11	9	91	97	94	10	19	222	224	225
13	9	90	98	95	20	19	218	225	225
17	9	87	99	96	5	19	226	226	225
19	9	80	100	97	11	20	229	227	227
2	9	102	101	105.5	13	20	235	228	228

m_1	m_2	Ranking of (m_1, m_2) by			m_1	m_2	Ranking of (m_1, m_2) by		
		$\max p_{m_1, m_2}^*$	$\text{avg } p_{m_1, m_2}^*$	R_{m_1, m_2}			$\max p_{m_1, m_2}^*$	$\text{avg } p_{m_1, m_2}^*$	R_{m_1, m_2}
8	9	100	102	105.5	11	17	227	229	231
4	9	109	103	105.5	11	16	230	230	232
16	9	104	104	105.5	17	20	236	231	229
1	9	114	105	105.5	13	17	228	232	233
15	14	89	106	99.5	13	16	237	233	234
11	7	130	107	119	19	20	239	234	230
15	13	103	108	102	17	16	241	235	237.5
13	7	137	109	122	19	17	231	236	239
5	12	105	110	105.5	14	19	232	237	235.5
17	7	148	111	124	19	16	238	238	240
19	7	140	112	126	7	19	234	239	235.5
6	11	116	113	113	1	20	240	240	237.5
12	11	112	114	113	16	17	243	241	243
18	11	107	115	113	2	17	244	242	243
14	15	108	116	109.5	8	17	245	243	243
3	11	111	117	113	4	17	242	244	243
9	11	113	118	113	1	17	246	245	243
7	15	106	119	109.5	1	16	251	246	246
2	7	154	120	137	11	19	233	247	247
8	7	150	121	137	13	19	248	248	248
16	7	141	122	137	17	19	249	249	249
4	7	152	123	137	16	19	253	250	252
1	7	128	124	137	2	19	252	251	252
7	12	125	125	116	8	19	254	252	252
5	8	142	126	148	4	19	247	253	252
11	15	119	127	117	1	19	250	254	252

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