
DYNAMIC PHASE TRANSITIONS IN MEAN-FIELD GINZBURG–LANDAU MODELS: CONJUGATE FIELDS AND FOURIER-MODE SCALING

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ABSTRACT

Dynamic phase transitions of periodically forced mean-field ferromagnets are often described by a single order parameter and a scalar conjugate field. Building from previous work, we show that, at the critical period P_c of the mean-field Ginzburg–Landau (MFGL) dynamics with energy $F(m) = am^2 + bm^4 - hm$, the correct conjugate field is the entire even-Fourier component part of the applied field. The correct order parameter is $z_k = \sqrt{|m_k|^2 - |m_{k,c}|^2}$, where m_k is the k^{th} Fourier component of the magnetization $m(t)$, and $m_{k,c}$ is the k^{th} Fourier component at the critical period. Using high-accuracy limit-cycle integration and Fourier analysis, we first confirm that, for periodic fields that contain only odd components, the symmetry-broken branch below P_c exhibits $z_k \sim \varepsilon^{1/2}$ (computationally tested for modes $k \leq 30$), where $\varepsilon = (P_c - P)/P_c$. This provides strong enough evidence that the $1/2$ scaling holds for all Fourier modes. We then find three robust facts: (1) Exactly at P_c , adding a small perturbation composed of even Fourier components with an overall field multiplier h_{mult} yields $z_k \sim h_{\text{mult}}^{1/3}$ across many k . (2) Mode-resolved deviations obey a parity rule: $|\delta m_{2n}| \propto h_{\text{mult}}^{1/3}$ and $|\delta m_{2n+1}| \propto h_{\text{mult}}^{2/3}$. (3) The same findings persist in MFGL models where an m^6 replaces the m^4 term and come with simple one-period integral criteria to locate P_c .

1 Introduction

Melting ice, popping popcorn, and hot iron losing its magnetization are all examples of phase transitions—a physical phenomenon in which the macroscopic state of a system changes due to external conditions. Dynamic phase transitions (DPTs) are a special type of phase transition which occur when external forces change rapidly, rather than gradually. Unlike equilibrium phase transitions, during which the system has sufficient time to settle into equilibrium, DPTs involve time-dependent external forces, such as an oscillating magnetic field, which prevent the system from fully going through relaxation dynamics.

Dynamic order parameter Q is used to quantify DPTs that occur due to an oscillating magnetic field. Prior to 2014, Q was known as the time-averaged magnetization over one period of the oscillating field. If $Q \approx 0$, the average magnetization over period P is zero, and the system follows the oscillating field closely. If $Q \neq 0$, the system breaks the symmetry of the oscillating field. Robb et al. showed in 2014 that there are components to the dynamic order parameter beyond the period-averaged magnetization Q [1]. Their analysis was performed close to the critical period P_c —defined such that if the period of oscillations is over P_c , the system has a symmetric magnetization response with $Q = 0$. If the period is below P_c , the system has an asymmetric magnetization response with $Q \neq 0$.

Since the Robb–Ostrander work in 2014 [1], DPTs in magnetic systems have remained an active research topic, with continued theoretical, computational, and experimental developments [2, 3, 4, 5, 6, 7, 8]. This research topic still clearly involves open questions, which the results in this paper make a contribution toward understanding.

2 Background

The goal of this article is to extend Robb et al.'s approach to investigate the dynamic order parameter and the conjugate field using the mean-field Ginzburg–Landau (MFGL) model. In the MFGL model, each magnetic spin is assumed to experience a magnetic field equal to the average field of the system neglecting direct spin-spin interactions. This MFGL model has the following Ginzburg–Landau free-energy:

$$F(m) = am^2 + bm^4 - hm,$$

where a and b are material-specific constants, $m(t)$ is the magnetization, and $h(t)$ is the strength of the applied magnetic field [1]. Figure 1 shows what $F(m)$ looks like for different a .

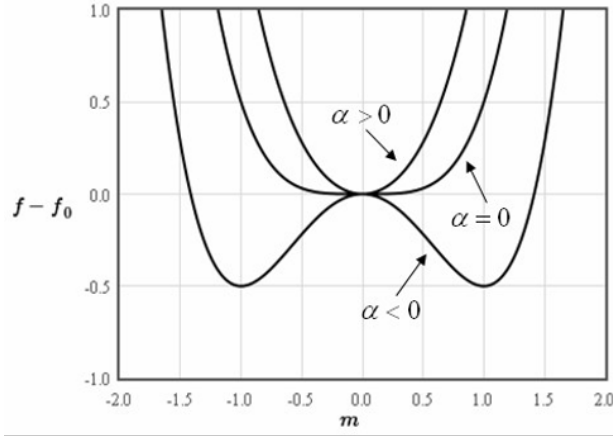


Figure 1: Free energy $F(m)$ vs magnetization $m(t)$, for varying values of the material-specific constant a .

Over time, every closed physical system evolves in such a way that its free energy gradually minimizes. The free energy in our model must be getting smaller with changes in magnetization. This principle allows us to determine how magnetization changes over time:

$$\frac{dm}{dt} = -\frac{dF}{dm} = -2am - 4bm^3 + h,$$

where the negative sign serves as a specification that the free energy minimizes as time progresses. Referring to Figure 1, when $a \geq 0$, the behavior of the system is predictable—it will change in such a way that $F(m)$ goes to its single minimum. However, when $a < 0$, there are two local minima, and this is the case we will investigate.

Robb et al. set $a = -\frac{3\sqrt{3}}{4}$ and $b = \frac{3\sqrt{3}}{8}$, which, for $h = 0$, yield free-energy minima at $m = \pm 1$ [1]. Figure 2 uses these parameters to show that as the strength of the applied field $h(t)$ gets closer to -1 , the left well of the graph gets larger than the right well, indicating a preference for negative magnetization $m(t)$. The equivalent happens as $h(t)$ approaches $+1$, which leads to positive magnetization. When $h(t)$ is zero, the system can exist in either state of magnetization.

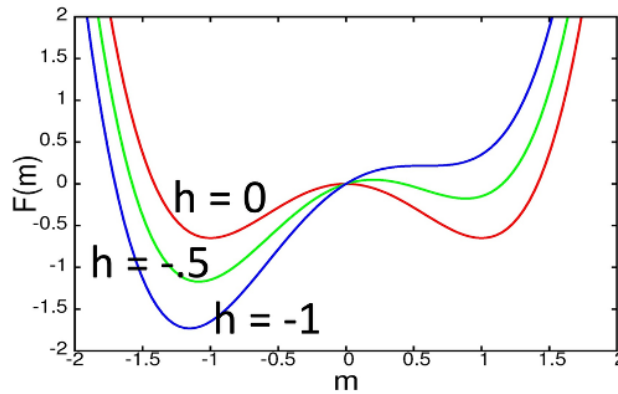


Figure 2: Free energy $F(m)$ vs magnetization $m(t)$, for varying values of h , where $a < 0$.

When the oscillation period P is large enough and $h(t)$ changes slowly, the system will gradually fall from one well to the other back and forth. This is represented in Figure 3A, where the system exhibits a symmetric hysteresis loop. However, if $h(t)$ changes rapidly, the system will not have enough time to transition between the two wells and will get stuck in one state of magnetization exhibiting a nonzero dynamic order parameter. Such behavior of the system is displayed in Figure 3B as two asymmetric hysteresis loops, where the initial $m(0)$ determines which loop the system gets stuck in. This separation in two loops is called bifurcation.

Both $m(t)$ and $h(t)$ are periodic functions with period P and can be represented as complex Fourier series:

$$m_k = \frac{1}{P} \int_0^P m(t) e^{-ikt} dt, \quad h_k = \frac{1}{P} \int_0^P h(t) e^{-ikt} dt.$$

Hence

$$m_0 = \frac{1}{P} \int_0^P m(t) dt, \quad h_0 = \frac{1}{P} \int_0^P h(t) dt,$$

which are the period-averaged magnetization and period-averaged conjugate field. m_0 turns out to be the previously identified dynamic order parameter Q . Robb et al.'s 2014 study demonstrated that the dynamic order parameter consists of multiple Fourier components of $m(t)$ beyond just m_0 .

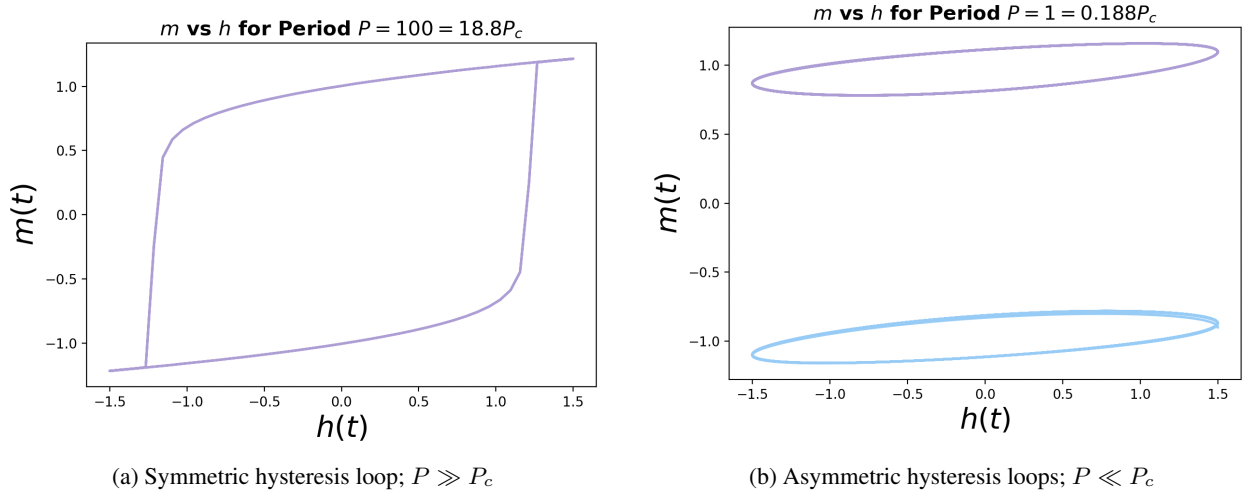


Figure 3: Hysteresis behavior in dynamic systems.

In equilibrium phase transitions, the relationship between magnetization and applied field at the critical temperature T_c is $m \propto h^{1/\delta}$, where $\delta = 3$ for the MFGL model [2]. In the DPT, an oscillating applied field $h(t)$ causes either a symmetric or an asymmetric magnetization response $m(t)$. Research before Robb et al.'s 2014 paper showed that the relationship $m_0 \approx (h_0)^{1/3}$ holds at $P = P_c$ for the DPT in the MFGL model. The exponent $1/3$ is the same as in the equilibrium transition.

Robb et al.'s 2014 study showed that odd Fourier components of $h(t)$ determine the location of P_c . Above and at P_c , the response to odd Fourier components of $h(t)$ consists solely of odd Fourier components of $m(t)$. Below P_c , the even components of $m(t)$ take on non-zero values as well, each undergoing a bifurcation. At P_c , introducing even Fourier components of $h(t)$ affects both the even and odd Fourier components of $m(t)$. Specifically, through simulations for $j, k \leq 30$, Robb et al. discovered that, at P_c :

$$\delta m_{k,e} \propto (\delta h_{j,e})^{1/3}, \quad \delta m_{k,e} \propto (\delta h_{j,o})^{1/3}, \quad \delta m_{k,o} \propto (\delta h_{j,e})^{1/3}, \quad \delta m_{k,o} \propto (\delta h_{j,o})^{1/3},$$

where e denotes even components, and o denotes odd.

Robb et al. additionally showed that, by starting from the TDGL equation and projecting onto Fourier modes, the perturbation equations can be grouped into a sum of three contributions that balance the Fourier component of an added applied field. For the $k = 0$ mode the equation becomes:

$$\delta h_0 = \underbrace{\left[-i0\omega \delta m_0 - 2a \delta m_0 - 12b \sum_{n_1, n_2} m_{n_1, c} m_{n_2, c} \delta m_{-n_1-n_2} \right]}_{T_1(\delta m)} - \underbrace{\left[12b \sum_{n_1, n_2} m_{n_1, c} \delta m_{n_2} \delta m_{-n_1-n_2} \right]}_{T_2(\delta m, \delta m)} - \underbrace{\left[4b \sum_{n_1, n_2} \delta m_{n_1} \delta m_{n_2} \delta m_{-n_1-n_2} \right]}_{T_3(\delta m, \delta m, \delta m)}.$$

Since $k = 0$, the $i0\omega \delta m_0$ term vanishes. At $P = P_c$ the linear coefficient of the even sector crosses through zero, so small even fields are opposed primarily by the cubic term, yielding the cube-root response $\delta m_e \sim (\delta h_e)^{1/3}$; by coupling, odd modes scale as $|\delta m_{2n+1}| \sim (\delta h_e)^{2/3}$. For $P > P_c$ a small but nonzero linear part produces a crossover from the $h^{1/3}$ law to a linear law as $h \rightarrow 0$, with the crossover shifting to smaller h as $P \rightarrow P_c^+$ (see Figure 4).

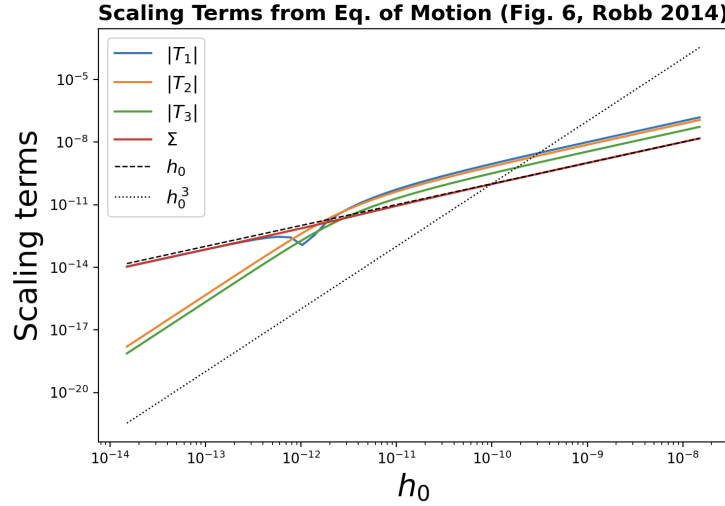


Figure 4: Scaling of the terms T_1, T_2, T_3 and their sum vs h_0 (re-created from Robb 2014 [1]).

3 What We Found

3.1 Computationally

1. We first verified from Robb et al. (2014) the behavior below the critical period using applied fields composed only of odd Fourier components [1]. Writing $z_k = \sqrt{||m_k|^2 - |m_{k,c}|^2|}$ for the magnitude of the deviation of the k -th Fourier component from the symmetric ($h_e = 0$) solution, we obtained, for ($k \leq 7$) and $P < P_c$:

$$z_k \propto \varepsilon^{1/2}, \quad \text{where } \varepsilon = \frac{P_c - P}{P_c},$$

while changes in the odd components of the applied field shifted the numerical value of P_c without altering the scaling exponent (see Figure 5).

2. At $P = P_c$, including a conjugate field consisting of any combination of even Fourier components multiplied by an overall small factor h_{mult} led to the scaling

$$z_k \propto h_{\text{mult}}^{1/3},$$

over several orders of magnitude and for many k , with or without the term h_0 (see Figure 6).

3. Mode-resolved deviations obey a parity rule (see Figure 7):

$$|\delta m_{2n}| \propto h_{\text{mult}}^{1/3}, \quad |\delta m_{2n+1}| \propto h_{\text{mult}}^{2/3}.$$

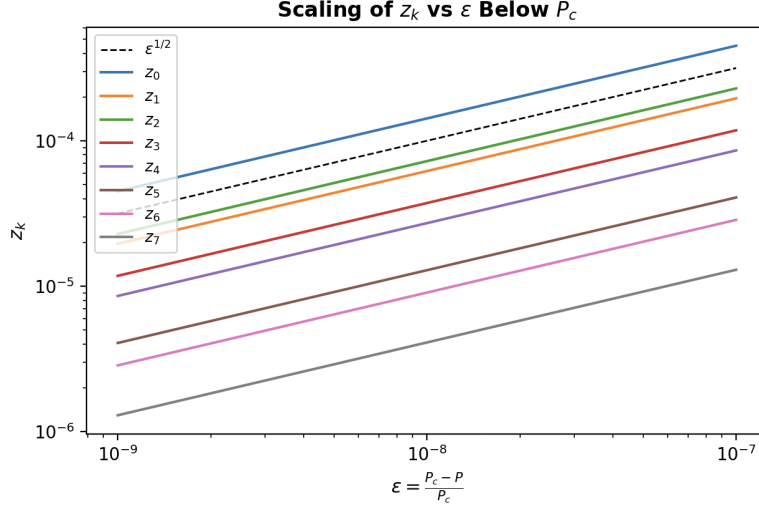


Figure 5: Scaling of z_k for $k = 0, \dots, 7$ vs $\varepsilon = \frac{P_c - P}{P_c}$ for $P < P_c$ (re-created from Robb 2014 [1]).

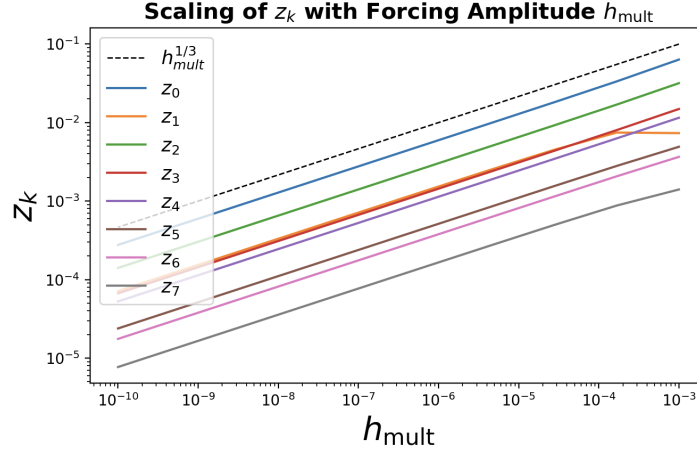


Figure 6: Scaling of z_k with an overall field multiplier h_{mult} .

3.2 Analytically

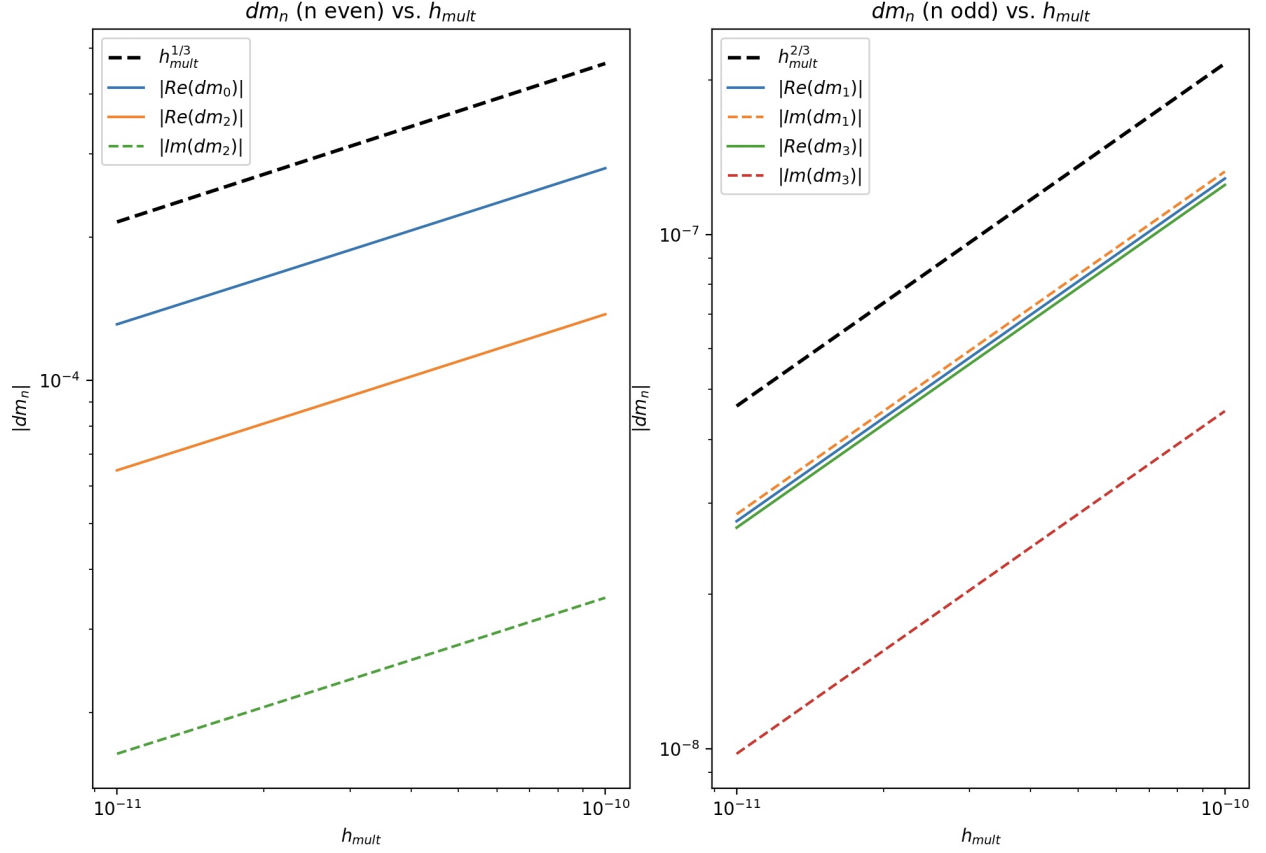
Fourier-mode equation at P_c . Starting from the TDGL equation

$$\frac{dm(t)}{dt} = -2a m(t) - 4b m^3(t) + h(t), \quad (1)$$

write $m(t) = \sum_{k \in \mathbb{Z}} m_k e^{ik\omega t}$ and $h(t) = \sum_{k \in \mathbb{Z}} h_k e^{ik\omega t}$ with $\omega = 2\pi/P$. Projecting (1) onto the k -th mode gives:

$$ik\omega m_k = -2a m_k - 4b \sum_{n_1+n_2+n_3=k} m_{n_1} m_{n_2} m_{n_3} + h_k. \quad (2)$$

Let $P = P_c$ and add a perturbation δh_k so that $m_k = m_{k,c} + \delta m_k$. Using the perturbed m_k in Eq. (2) yields linear, quadratic, and cubic contributions in δm that balance δh_k , which we denote T_1, T_2, T_3 (see text and Fig. 4). This structure underlies the exponents $1/3$ (even k) and $2/3$ (odd k) at $P = P_c$.

Figure 7: Scaling of δm_n for even and odd n with h_{mult} .

Even/odd decomposition and the P_c criterion. Decompose $m(t) = m_o(t) + m_e(t)$ and $h(t) = h_o(t) + h_e(t)$ into odd/even parts and insert into (1) to obtain the coupled equations:

$$\begin{aligned}\frac{dm_o}{dt} &= -2a m_o - 4b m_o^3 - 12b m_o m_e^2 + h_o, \\ \frac{dm_e}{dt} &= -2a m_e - 4b m_e^3 - 12b m_o^2 m_e + h_e.\end{aligned}$$

For $h_e = 0$, $m_e = 0$ satisfies the second equation identically and the first becomes

$$\frac{dm_o}{dt} = -2a m_o - 4b m_o^3 + h_o,$$

whose solutions graph as symmetric hysteresis loops. By the argument in Robb 2014 (p. 3), a perturbation δm_o of a symmetric solution m_o gives the stability condition

$$\int_0^{P_c} (2a + 12b m_o^2(t)) dt = 0,$$

indicating the role of the *odd* part of $h(t)$ in locating P_c .

Model variants and their P_c criteria. The same procedure applies to other polynomial free energies. For example:

$$F(m) = am^2 + bm^6 - hm \quad \Rightarrow \quad \frac{dm}{dt} = -2a m - 6b m^5 + h,$$

leading to

$$\int_0^{P_c} (2a + 30b m_o^4(t)) dt = 0,$$

and the same scalings $z_k \sim h_{\text{mult}}^{1/3}$ at P_c and $z_k \sim \varepsilon^{1/2}$ below P_c . For

$$F(m) = am^4 + bm^6 - hm \quad \Rightarrow \quad \frac{dm}{dt} = -4a m^3 - 6b m^5 + h,$$

the criterion becomes

$$\int_0^{P_c} (12a m_o^2(t) + 30b m_o^4(t)) dt = 0,$$

and the same parity behavior persists.

For a generic form $F(m) = am^p + bm^q - hm$ with equation of motion

$$\frac{dm}{dt} = -pa m^{p-1} - qb m^{q-1} + h,$$

one obtains the one-period criterion

$$\int_0^{P_c} [p(p-1)a m^{p-2} - q(q-1)b m^{q-2}] dt = 0,$$

which can be used to locate P_c .

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