

RELATIVE $\mathbb{A}^1$ -CONTRACTIBILITY OF KORAS-RUSSELL PROTOTYPES AND EXOTIC MOTIVIC SPHERES

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ABSTRACT. The Koras-Russell threefolds are a certain family of smooth, affine contractible threefolds exhibiting "exotic" behavior in the algebro-geometric context. Our goal in this note is to extend its $\mathbb{A}^1$ -contractibility from a field to a general base scheme. As a consequence, we also give a general strategy to extend the $\mathbb{A}^1$ -contractibility of Koras-Russell prototypes in higher dimensions over a general base scheme. As a major consequence, we establish the existence of "exotic" motivic spheres in all dimensions at least 4 over infinite perfect fields.

1. INTRODUCTION

One of the emerging problems in algebraic geometry is to uniquely characterize the affine n -space $\mathbb{A}_k^n$ among smooth affine $\mathbb{A}^1$ -contractible n -schemes over a field k . The uniqueness of $\mathbb{A}_k^n$ holds up to dimensions $n \leq 2$ over k (cf. [AD07; CR24a]) and fails from dimensions $n \geq 3$. When $\text{char } k > 0$, the failure is witnessed by the *non-cancellation* property of $\mathbb{A}_k^n$. Indeed, recall that an m -dimensional smooth affine k -variety X is said to be *cancellative* if $X \times_k \mathbb{A}_k^1 \cong_k \mathbb{A}_k^{m+1}$ implies that $X \cong_k \mathbb{A}_k^m$. This is the algebro-geometric version of the famously known *Zariski Cancellation Problem* (ZCP), which began its formulation in the setting of commutative algebra (cf. [Gup22, §2]). The ZCP is proven to be true for all fields in dimension 1 (cf. [AHE72; EH06]) and 2 (cf. [MS80; Rus81]). It is proven in negative for all dimensions ≥ 3 over fields of positive characteristic due to the presence of so-called *Asanuma-Gupta varieties* (see Remark 5.10) which was constructed in [Gup14a; Gup14b] building on a key results of [Asa87], thereby completely solving ZCP over fields of positive characteristics (see [Gup22] for comprehensive survey). Over a field of characteristic zero, it is now known that the uniqueness of $\mathbb{A}_k^3$ does not hold owing to the presence of *Koras-Russell threefolds*, a family of smooth affine threefolds defined by the polynomial equation

$$(1.1) \quad \mathcal{K} := \{x^m z = x + y^r + t^s\} \subset \mathbb{A}_k^4$$

where $m, r, s \geq 2$ are integers with r and s coprime. The counter-intuitive properties of $\mathcal{K}$ are that it is topologically contractible as an analytic manifold [KR97], $\mathbb{A}^1$ -contractible in the stable $\mathbb{A}^1$ -homotopy category $\text{SH}(\mathbb{C})$ [HKØ16], and even unstably $\mathbb{A}^1$ -contractible [DF18] in the Morel-Voevodsky's $\mathbb{A}^1$ -homotopy theory $\mathcal{H}(k)$ [MV99]; nevertheless, it is not isomorphic to $\mathbb{A}_k^3$ as witnessed by the eponymous Makar-Limanov [Mak96] and Dersken invariants [Der97, Chapter II, §3]. In a vast context, such smooth affine n -dimensional varieties that are $\mathbb{A}^1$ -contractible but not isomorphic to $\mathbb{A}^n$ will be called *exotic* (cf. Section 2). It is crucial to note that the ZCP remains open in characteristic zero. In this direction, [Dub09] proved that the cylinders over $\mathcal{K}$ have the same Makar-Limanov invariant as that of $\mathbb{A}_k^4$, but it is not known if $\mathcal{K} \times \mathbb{A}^1 \cong_k \mathbb{A}_k^4$.

Key words and phrases. Motivic Homotopy Theory, Koras-Russell Prototypes, Exotic Motivic Spheres.

We shall now briefly elaborate on the goals of this report. We begin by extending the $\mathbb{A}^1$ -contractibility of $\mathcal{K}$ from a field of characteristic zero to arbitrary fields.

Proposition. (Proposition 4.15) *For any perfect field k , the canonical morphism $f : \mathcal{K} \rightarrow \operatorname{Spec} k$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k .*

As a consequence, we obtain the analogous results over the integers $\operatorname{Spec} \mathbb{Z}$ and Noetherian base schemes S .

Theorem. (Theorem 4.16) *The smooth affine scheme $\mathcal{K} \rightarrow \operatorname{Spec} \mathbb{Z}$ is $\mathbb{A}^1$ -contractible in $\mathcal{H}(\mathbb{Z})$. In particular, $\mathcal{K}$ is an $\mathbb{A}^1$ -contractible scheme that is not isomorphic to $\mathbb{A}_{\mathbb{Z}}^3$.*

Corollary. (Corollary 4.17) *Let S be any Noetherian scheme with perfect residue fields. Then the canonical morphism $\mathcal{K} \rightarrow S$ is an $\mathbb{A}^1$ -weak equivalence in Spc_S of relative dimension 3.*

We remark that these results are essentially concise applications of the gluing theorem of Morel-Voevodsky ([MV99, Theorem 2.21]). A related consequence is that all these extended families of $\mathbb{A}^1$ -contractible varieties add to the list of "potential" counter-examples to the ZCP in positive characteristic (see Remark 5.10) and thereby, producing exotic varieties in mixed characteristics as by-products. Following these results, we extend the $\mathbb{A}^1$ -contractibility of the *generalized Koras-Russell varieties* constructed and studied by the authors in [DG25] from a field of characteristic zero to arbitrary (perfect) fields. For simplicity, let us now look at one of its simplest prototypes.

Definition. *Consider the family of smooth affine varieties defined by the equation*

$$\mathcal{X}_m := \{\underline{x}^n z = y^r + t^s + x_0\} \subset \mathbb{A}_k^{m+4} \cong \operatorname{Spec} k[x_0, \dots, x_m, y, z, t]$$

The canonical morphism $\mathcal{X}_m \rightarrow \operatorname{Spec} k$ makes it into a smooth affine scheme of dimension $(m+3)$, for all $m \geq 0$ and $n, r, s \geq 2$ with $\gcd(r, s) = 1$ and $\underline{x} := \prod_{i=0}^m x_i$.

Using Corollary 4.17, we essentially obtain the following results.

Theorem. (Theorem 5.4) *Let k be any (perfect) field. Then for all $m \geq 0$, the canonical morphism $\mathcal{X}_m \rightarrow \operatorname{Spec} k$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k .*

Corollary. (Corollary 5.9) *Let S be any Noetherian scheme with perfect residue fields. Then the canonical morphism $\mathcal{X}_m \rightarrow S$ is an $\mathbb{A}^1$ -weak equivalence in Spc_S .*

In the sequel, we study the existence of exotic motivic spheres: smooth n -dimensional schemes that are $\mathbb{A}^1$ -homotopic but not isomorphic to $\mathbb{A}_k^n \setminus \{0\}$ (see Definition 6.2). We adopt this terminology in the light of the identification of the $\mathbb{A}^1$ -homotopy type of the strictly¹ quasi affine smooth schemes $\mathbb{A}_k^n \setminus \{0\}$ with that of the motivic spheres $\mathbb{S}^{2n-1, n}$ ([MV99, Example 2.20]). The inception of this study is the following question.

Question. *Over a reasonably base scheme S , if two smooth schemes $X \simeq_{\mathbb{A}^1} \mathbb{A}_S^n \setminus \{0\}$ are $\mathbb{A}^1$ -homotopic, then are they isomorphic as S -schemes?*

Put differently, the question demands the *existence of exotic motivic spheres*. In Section 6.1, we shall provide an overview of our knowledge of this question. Our main goal here is to exhibit that the generalized Koras-Russell varieties pave the way for producing counterexamples to this question in higher dimensions, thereby closing this problem once and for all. The following result puts this perspective into words.

¹quasi affine but not affine

Theorem. (Theorem 6.8) *Let k be any infinite perfect field. Then for any $m \geq 0$, the strictly quasi-affine variety $\mathcal{X}_m \setminus \{\bullet\}$ is an exotic motivic sphere.*

Examples of motivic spaces that are $\mathbb{A}^1$ -homotopic to $\mathbb{A}^n \setminus \{0\}$ without being isomorphic to $\mathbb{A}^n \setminus \{0\}$ are not hard to construct and have appeared in the literature. However, the novelty here is that the ones produced in this article are, to the author's knowledge, the first family of examples belonging to the same dimension.

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2. HISTORICAL OUTSET

The characterization of the affine n -space $\mathbb{A}^n$ is a classical problem that traces back to the 1950s, aimed to classify the real Euclidean spaces $\mathbb{R}^n$ up to topological contractibility. In dimensions $n \leq 2$, $\mathbb{R}^n$ is the unique smooth manifold that is contractible; the story threw a curve ball in dimension 3 already. In an attempt to solve the Poincaré Conjecture in dimension 3, Whitehead put forth a proof [Whi34] where, in essence, he argued that an open contractible real 3-manifold that is homotopic to $\mathbb{R}^3$ is necessarily homeomorphic to $\mathbb{R}^3$. However, this argument collapsed owing to his own counter-example - now called the *Whitehead manifold* [Whi35]. Soon after this discovery, several higher-dimensional prototypes of Whitehead manifolds emerged (see [AØ21, §1.1] for a beautiful overview and chronological contributions), which apparently did not facilitate $\mathbb{R}^n$ to be characterized by its contractibility alone. Nevertheless, it was later observed that $\mathbb{R}^3$ stayed simply connected in the complement of any larger compact subset, while the Whitehead manifold did not, revealing that $\mathbb{R}^3$ is *simply connected at infinity*. Thus, invariants coming from topology at infinity found a home in the heart of such a characterization. Building on several mathematical predecessors, the final breakthrough happened in [Sie68], proving that $\mathbb{R}^n$ is the unique open contractible n -manifold that is simply connected at infinity for all $n \geq 3$. The aforementioned homotopical characterization of $\mathbb{R}^n$ is sometimes called the *Open Poincaré Conjecture*.

Recall from topology that a smooth complex n -manifold is said to be *exotic* if it is topologically contractible but not isomorphic to $\mathbb{C}^n$. In analogy, we say that a smooth affine n -dimensional variety over a field k is *exotic* if it is $\mathbb{A}^1$ -contractible but isomorphic to $\mathbb{A}_k^n$. As we will witness in this note, the algebro-geometric analog of the Whitehead manifold will be played by a family of smooth affine threefolds called the *Koras-Russell threefolds* $\mathcal{K}$ (Definition 4.1) and it is one of the overriding dreams in motivic homotopy theory to construct "motivic topology at infinity" characterizing $\mathbb{A}^n$ among spaces similar to $\mathcal{K}$ (see [AD07, §6.4], [DDØ22]).

We briefly introduce the following notions from affine algebraic geometry.

Definition 2.1. An $\mathbb{A}^n$ -*fiber space* over a scheme S is a smooth morphism of finite presentation $f : X \rightarrow S$ whose fibers $X_s := X \times_S \operatorname{Spec} \kappa(s)$ over all points $s \in S$ are isomorphic to the affine space $\mathbb{A}^n$ over the corresponding residue field $\kappa(s)$.

Definition 2.2. An $\mathbb{A}^n$ -fiber space $f : X \rightarrow S$ as above is called a *locally trivial $\mathbb{A}^n$ -bundle* in the Zariski (resp. Nisnevich, resp. étale) topology if every $s \in S$ admits a Zariski (resp. Nisnevich, resp. étale) neighborhood U in S such that $U \times_S X \cong U \times \mathbb{A}^n$ as schemes over U .

We adopt the terminology of " $\mathbb{A}^n$ -fiber space" as an alternative to the classical notion of " $\mathbb{A}^n$ -fibration" or "affine fibration", which is traditionally used by the authors from the affine algebraic geometry realm. This is to avoid any possible confusion, either with that of fibrations from the model category or that of topology.

CONVENTION

Unless specified otherwise, all schemes considered are smooth and are of finite type over a base scheme. Throughout, k will denote the base field and S will denote a Noetherian base scheme of finite Krull dimension. The category of smooth S -schemes of finite type is denoted by Sm_S . We denote the category of presheaves (resp. sheaves) by $\mathrm{Psh}(\mathrm{Sm}_S)$ (resp. $\mathrm{Shv}(\mathrm{Sm}_S)$) and the category of motivic S -spaces which are simplicial presheaves by $\mathrm{Spc}_S := \mathrm{sPsh}(\mathrm{Sm}_S)$. The unstable (resp. stable) $\mathbb{A}^1$ -homotopy category over a scheme S will be denoted by $\mathcal{H}(S)$ (resp. $\mathrm{SH}(S)$). For motivic spaces $X, Y \in \mathrm{Spc}_S$, its $\mathbb{A}^1$ -weak equivalence is denoted by $X \simeq Y$ (lives in $\mathcal{H}(S)$), while its isomorphism is denoted by $X \cong Y$ (lives in Sm_S).

3. MOTIVIC HOMOTOPY THEORY

3.1. Preliminaries. Motivic homotopy (also known as the $\mathbb{A}^1$ -homotopy) theory is a nascent domain in algebraic geometry, established by Morel and Voevodsky [MV99]. The conceptual aim was to provide a homotopical framework for algebraic varieties, tackling the then-open Milnor Conjecture (cf. [OVV07]) and the Bloch-Kato Conjecture (cf. [Voe03]). Soon after its formulation, it became clearer that the $\mathbb{A}^1$ -homotopy category provides a natural ground to several of Grothendieck's visions towards the universal theory of motives. The underpinning of the $\mathbb{A}^1$ -homotopy category is facilitated by the homotopy theory of the simplicial presheaves, equivalently, the category of *motivic S -spaces* $\mathrm{Spc}_S := \mathrm{sPsh}(\mathrm{Sm}_S)$ equipped with the discrete simplicial structure. The category Spc_S is then subjected to two formal Bousfield localizations, thereby producing the desired $\mathbb{A}^1$ -homotopy category $\mathcal{H}(S)$. Roughly speaking, the first localization inverts the affine line $\mathbb{A}^1$ by a process of $\mathbb{A}^1$ -localization, thereby producing motivic spaces that are $\mathbb{A}^1$ -invariant; in other words, a space $X \rightarrow S$ is $\mathbb{A}^1$ -invariant if the induced morphism

$$\mathrm{pr}_1^* : \mathrm{Hom}_{\mathrm{Spc}_S}(U, X) \rightarrow \mathrm{Hom}_{\mathrm{Spc}_S}(\mathbb{A}_U^1, X)$$

is a bijection for the canonical projection $\mathrm{pr}_1 : \mathbb{A}_U^1 \rightarrow U$, for every $U \in \mathrm{Sm}_S$. The $\mathbb{A}^1$ -localization functor $\mathbb{L}_{\mathbb{A}^1} : \mathrm{Psh}(\mathrm{Sm}_S) \rightarrow \mathrm{Psh}_{\mathbb{A}^1}(\mathrm{Sm}_S)$ is the left adjoint to the inclusion functor $\iota : \mathrm{Psh}_{\mathbb{A}^1}(\mathrm{Sm}_S) \hookrightarrow \mathrm{Psh}(\mathrm{Sm}_S)$ of presheaves that are $\mathbb{A}^1$ -invariant. The second localization inverts the descent maps into weak equivalences; that is, a simplicial presheaf $X \in \mathrm{Spc}_S$ satisfies *Nisnevich descent* if for every Čech covers $\tilde{U}_\bullet \rightarrow X$, we have that the $\mathrm{hocolim}_n U_n \xrightarrow{\sim} X$ is a weak equivalence of simplicial presheaves. The Nisnevich localization inverts these maps in the $\mathbb{A}^1$ -homotopy category via the functor $\mathbb{L}_{\mathrm{Nis}} : \mathrm{Psh}(\mathrm{Sm}_S) \rightarrow \mathbb{L}_{\mathrm{Nis}} \mathrm{Psh}(\mathrm{Sm}_S)$ which is the left derived functor of the identity functor. Although in a general context, one replaces the Čech covers by hypercovers and studies the Nisnevich hyperdescent (see [AE17, §3.4]). The $\mathbb{A}^1$ -homotopy category is then defined as the subcategory of $\mathrm{sPsh}(\mathrm{Sm}_S)$ consisting of simplicial presheaves those that are both $\mathbb{A}^1$ -invariant $\mathrm{sPsh}(\mathrm{Sm}_S)_{\mathbb{A}^1}$ and satisfying the Nisnevich descent $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$,

$$\mathcal{H}(S) := \mathrm{sPsh}(\mathrm{Sm}_S)_{\mathbb{A}^1} \cap \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \subset \mathrm{sPsh}(\mathrm{Sm}_S) =: \mathrm{Spc}_S.$$

However, note that for a simplicial presheaf $\mathcal{F}$, $\mathbb{L}_{\mathbb{A}^1}(\mathbb{L}_{\text{Nis}}(\mathcal{F}))$ need not be a Nisnevich sheaf and $\mathbb{L}_{\text{Nis}}(\mathbb{L}_{\mathbb{A}^1}(\mathcal{F}))$ need not be $\mathbb{A}^1$ -invariant (cf. [MV99, §3, Example 2.7]). This issue is circumvented by considering the *motivic localization functor* $\mathbb{L}_{\text{mot}} : \text{Psh}(\text{Sm}_S) \rightarrow \mathcal{H}(S)$ which is the left adjoint of the inclusion $\iota : \mathcal{H}(S) \hookrightarrow \text{Psh}(\text{Sm}_S)$ defined as

$$\mathbb{L}_{\text{mot}} := \text{colim}(\mathbb{L}_{\text{Nis}} \rightarrow \mathbb{L}_{\mathbb{A}^1} \mathbb{L}_{\text{Nis}} \rightarrow \mathbb{L}_{\mathbb{A}^1} \mathbb{L}_{\text{Nis}} \mathbb{L}_{\mathbb{A}^1} \rightarrow \dots) = (\mathbb{L}_{\mathbb{A}^1} \circ \mathbb{L}_{\text{Nis}})^{\circ \mathbb{N}}.$$

Being a left adjoint, it preserves colimits. Furthermore, it is locally cartesian and preserves finite products ([Hoy17, Proposition 3.15 & 4.1]). For a detailed background on the construction of the motivic homotopy category, we redirect the reader to [Hoy17; AE17; AØ21]. Let us now recollect some basic jargon in the $\mathbb{A}^1$ -homotopy category.

Definition 3.1. The $\mathbb{A}^1$ -connected component sheaf of a scheme $X \rightarrow S$ is the Nisnevich sheaf associated to the presheaf

$$U \in \text{Sm}_S \mapsto \text{Hom}_{\mathcal{H}(S)}(U, X)$$

for all $U \in \text{Sm}_S$. The $\mathbb{A}^1$ -connected component sheaf of X is denoted by $\pi_0^{\mathbb{A}^1}(X)$ and is a Nisnevich sheaf of sets on Sm_S . We say that a scheme X is $\mathbb{A}^1$ -connected if $\pi_0^{\mathbb{A}^1}(X)$ is isomorphic to the trivial sheaf S .

Example 3.2. Every $\mathbb{A}^1$ -connected smooth scheme X over a locally Henselian scheme S admits an S -point ([MV99], Remark 2.5). In other words, any variety that does not possess a rational point cannot be $\mathbb{A}^1$ -connected as a variety.

Definition 3.3. A k -scheme X is $\mathbb{A}^1$ -rigid if the induced morphism

$$\text{pr}_1^* : \text{Hom}_{\text{Sm}_S}(U, X) \rightarrow \text{Hom}_{\text{Sm}_S}(U \times \mathbb{A}^1, X)$$

is a bijection, for the canonical projection morphism $\text{pr}_1 : \mathbb{A}_U^1 \rightarrow U$, for every $U \in \text{Sm}_S$.

Example 3.4. Any 0-dimensional smooth scheme over a field is $\mathbb{A}^1$ -rigid ([AØ21, Example 2.1.10]). The affine variety $\mathbb{G}_m$ is $\mathbb{A}^1$ -rigid ([AØ21, Example 4.1.10]). Any smooth projective curve of positive genus and any Abelian variety are $\mathbb{A}^1$ -rigid varieties. In the view of $\mathbb{A}^1$ -connectedness, an $\mathbb{A}^1$ -rigid variety X is $\mathbb{A}^1$ -connected if and only if it is trivial $X \cong \text{Spec } k$.

For more examples, one can refer to [CR24a, §2], [AØ21, §4.1]. In parallel to topology, one defines the analogous notion of path-connectedness in algebraic geometry via images of the affine line $\mathbb{A}^1$.

Definition 3.5. A k -scheme X is $\mathbb{A}^1$ -chain connected if for any finitely generated separable field extension L/k and for all rational points x and $y \in X(L)$, there exists rational points $x = x_0, \dots, x_{n-1}$, $x_n = y \in X(L)$ and an elementary $\mathbb{A}^1$ -weak equivalence $f_i : \mathbb{A}_L^1 \rightarrow X$ between x_{i-1} and x_i , that is, $f_i(0) = x_{i-1}$ and $f_i(1) = x_i$ for $1 \leq i \leq n$.

Example 3.6. Affine n -spaces or more generally, any n -dimensional smooth k -variety covered by affine spaces in the sense of [AM11, Definition 2.2.10] are $\mathbb{A}^1$ -chain connected. Any $\mathbb{A}^1$ -chain connected variety is $\mathbb{A}^1$ -connected ([AM11, Proposition 2.2.7]) but the converse is not true ([BHS15, §4.1]). The smooth exotic surfaces such as Ramanujam surface and tom Dieck-Petrie surfaces cannot be $\mathbb{A}^1$ -chain connected as they are not even $\mathbb{A}^1$ -connected (this is a consequence of the characterization of $\mathbb{A}^2$ ([CR24a, Theorem 1.1]) over a field of characteristic zero).

Definition 3.7. A motivic S -space $\mathcal{X}$ is said to be $\mathbb{A}^1$ -contractible if the canonical map $\mathcal{X} \rightarrow S$ is an $\mathbb{A}^1$ -weak equivalence in Spc_S or equivalently, an isomorphism in $\mathcal{H}(S)$.

Example 3.8. Here are some of the basic examples of $\mathbb{A}^1$ -contractible spaces in nature.

- » By construction, the affine line $\mathbb{A}^1$ is $\mathbb{A}^1$ -contractible. And so, by induction on the dimension $n \geq 1$, the affine n -spaces form the prototypical examples of $\mathbb{A}^1$ -contractibles. In fact, the map

$$\mathbb{A}^n \times \mathbb{A}^1 \rightarrow \mathbb{A}^n \quad (x_1, \dots, x_n, t) \mapsto (tx_1, \dots, tx_n)$$

corresponds to the usual radial rescaling morphism, which is an $\mathbb{A}^1$ -weak equivalence.

- » The singular cuspidal curves $C_{r,s}$ defined by the equation $\{y^r - t^s = 0\}$, for coprime integers r, s are $\mathbb{A}^1$ -contractible ([AD07, Example 2.1]). In fact, the normalization map

$$\mathbb{A}^1 \rightarrow C_{r,s} \quad w \mapsto (w^s, w^r)$$

is an $\mathbb{A}^1$ -weak equivalence of motivic spaces.

- » Any vector bundle morphism is an $\mathbb{A}^1$ -homotopy equivalence ([MV99, Example 2.2]). More generally, any Zariski (resp. Nisnevich) locally trivial bundles with $\mathbb{A}^1$ -contractible fibers are $\mathbb{A}^1$ -weak equivalences.
- » Any $\mathbb{A}^1$ -contractible variety X is necessarily $\mathbb{A}^1$ -connected. Hence, X is $\mathbb{A}^1$ -rigid if and only if X is trivial.

3.2. Functoriality in Motivic Homotopy Theory. We shall discuss the functoriality in the motivic homotopy theory with a special emphasis on the base change property that is crucial for our context. For a versatile background on functoriality, we refer to [MV99, §3], [Ayo07; CD19]. For other recent accounts, see also e.g., [Hoy17, §4], [ELSØ22, §2], and [Rön25, §3]. We shall now outsource several results from the articles mentioned above, wherein our contribution is the style of the presentation and the explicit illustration $\mathbb{A}^1$ -contractibility in the relative setting (§3.3).

The Base Change. Let $f : T \rightarrow S$ be a morphism of base schemes². Then the pullback along f defines a functor

$$\begin{aligned} f^\bullet : \mathrm{Sm}_S &\longrightarrow \mathrm{Sm}_T \\ U &\longmapsto U_T := U \times_S T. \end{aligned}$$

Precomposition with $f^\bullet$, we obtain another colimit preserving functor at the level of motivic spaces defined as

$$\begin{aligned} f_* : \mathrm{Spc}_T &\longrightarrow \mathrm{Spc}_S \\ \mathcal{F} &\longmapsto \mathcal{F} \circ f^\bullet := U \mapsto \mathcal{F}(U_T). \end{aligned}$$

By the left Kan extension, we see that f_* admits a left adjoint $f^* : \mathrm{Spc}_S \rightarrow \mathrm{Spc}_T$ on the level of motivic spaces, which is strict symmetric monoidal and preserves limits. Since every (pointed) motivic space is a colimit of representable motivic spaces, we can characterize f^* by the following formula

$$(3.9) \quad f^*(X_+) = (X \times_S T)_+ \quad \text{for all } X \in \mathrm{Sm}_S.$$

Smooth Base Change. If furthermore, $f : T \rightarrow S$ is a smooth morphism of base schemes, then composition with f defines a functor $f_\bullet : \mathrm{Sm}_T \rightarrow \mathrm{Sm}_S$. Precomposition with this functor defines another functor $f^\star : \mathrm{Spc}_S \rightarrow \mathrm{Spc}_T$ which itself admits a left adjoint $f_\sharp : \mathrm{Spc}_T \rightarrow \mathrm{Spc}_S$ via (enriched) Kan extension, that is $f_\sharp \dashv f^\star \dashv f_*$. Again, using the fact that any motivic space is a colimit of representable ones, $f_\sharp$ can be characterized by the formula

$$f_\sharp(X \xrightarrow{g} T)_+ = (X \xrightarrow{g} T \xrightarrow{f} S)_+$$

²By a base scheme, we mean a Noetherian separated scheme of finite Krull dimension

for every $X \in \mathrm{Spc}_T$. In other words, $f_\#$ is simply induced by the forgetful functor $\mathcal{U} : \mathrm{Sm}_S \rightarrow \mathrm{Sm}_T$. On the other hand, if $g : Z \rightarrow S$ is a morphism, then the canonical S -morphism $Z \times_S T$ defines a map $B(Z) \rightarrow f_* f^\star B(Z)$ which is natural in Z and $B \in \mathrm{Spc}_T$, and consequently, a natural transformation $\mathrm{Id}_{\mathrm{Spc}_S} \rightarrow f_* \circ f^\star$. Moreover, for a smooth morphism $\phi : X \rightarrow Y$, the adjoint $f^* \rightarrow f^\star$ of the natural transformation $\mathrm{Id}_{\mathrm{Spc}_S} \rightarrow f_* \circ f^\star$ is a natural transformation ([Rön25, Lemma 3.7]). Denoting by Lf^* (resp. RF_*) the left (resp. right) derived Quillen functor on the level of $\mathbb{A}^1$ -homotopy categories, the following provides us with the concise base change functoriality relating the category of motivic spaces and the associated $\mathbb{A}^1$ -homotopy category.

Proposition 3.10. Let $f : T \rightarrow S$ be a morphism of schemes, let Spc_S and Spc_T be the category of motivic spaces endowed with their respective local projective model structures, and let $\mathcal{H}(S)$ and $\mathcal{H}(T)$ be their respective localizations with their $\mathbb{A}^1$ -local projective model structure. Then the following holds:

- (1) The adjoint pair (f^*, f_*) induces Quillen adjunctions

$$\mathrm{Spc}_S \xrightleftharpoons[f^*]{f_*} \mathrm{Spc}_T \quad \text{and} \quad \mathcal{H}(S) \xrightleftharpoons[Lf^*]{Rf_*} \mathcal{H}(T).$$

- (2) If $f : T \rightarrow S$ is smooth, then the adjoint pair $(f_\#, f^\star)$ induces Quillen adjunctions

$$\mathrm{Spc}_T \xrightleftharpoons[f_\#]{f^\star} \mathrm{Spc}_S \quad \text{and} \quad \mathcal{H}(T) \xrightleftharpoons[Rf_\#]{Lf^\star} \mathcal{H}(S).$$

Proof. The fact that (f^*, f_*) (resp. $(f_\#, f^\star)$ when f is smooth) induces a Quillen adjunction between the corresponding model categories is a consequence of using the projective model structure, and this follows from [Lur09, Proposition A.2.8.7]. Indeed, the first assertion follows from the observation that if $U_\bullet \rightarrow X$ is a Nisnevich hypercover of a smooth S -scheme X , then $(U_T)_\bullet \rightarrow X_T$ is a Nisnevich hypercover of X_T and that for every smooth S -scheme X , $(X \times_S \mathbb{A}_S^1) \cong X_T \times_T \mathbb{A}_T^1$ which implies by [Lur09, Proposition A.3.7.9] that the Quillen adjunction

$$\mathrm{Spc}_S \xrightleftharpoons[\mathrm{id} \circ f_*]{f_\# \circ \mathrm{id}} \mathrm{Spc}_T$$

induces a Quillen adjunction

$$\mathcal{H}(S) \xrightleftharpoons[\mathrm{id} \circ f_* \circ \mathrm{id}]{\mathrm{id} \circ f^\star \circ \mathrm{id}} \mathcal{H}(T)$$

The second assertion follows for the same reasons after observing that when $f : T \rightarrow S$ is smooth, a Nisnevich hypercover $U_\bullet \rightarrow X$ of a smooth T -scheme X is again a Nisnevich hypercover of X considered as a smooth S -scheme via the composition with u and that for every smooth T -scheme X , $X \times_T \mathbb{A}_T^1 \cong X \times_S \mathbb{A}_S^1$ as schemes over S . $\square$

Corollary 3.11. Let $f : T \rightarrow S$ be a morphism of schemes. Then the following holds:

- (1) For every morphism $u : Y \rightarrow X$ between smooth S -schemes which is an $\mathbb{A}^1$ -weak equivalence in Spc_S , the morphism $u_T : Y_T \rightarrow X_T$ is an $\mathbb{A}^1$ -weak equivalence in Spc_T .
- (2) If $f : T \rightarrow S$ is smooth, then every morphism $u : Y \rightarrow X$ between smooth T -schemes which is an $\mathbb{A}^1$ -weak equivalence in Spc_T is also an $\mathbb{A}^1$ -weak equivalence in Spc_S .

Proof. For the first assertion, we consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spc}_S & \xrightarrow{f^*} & \mathrm{Spc}_T \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ \mathcal{H}(S) & \xrightarrow{Lf^*} & \mathcal{H}(T) \end{array}$$

The functor $f^* : \mathrm{Spc}_S \rightarrow \mathrm{Spc}_T$ maps the presheaves associated to the smooth S -schemes X and Y to the presheaves associated to the smooth T -schemes X_T and Y_T , respectively, and maps the natural transformation associated to $u : Y \rightarrow X$ onto that associated to $u_T : Y_T \rightarrow X_T$. On the other hand, by construction of the projective model structure on Spc_S , the presheaves associated to X and Y are cofibrant objects of Spc_S , whence of $\mathcal{H}(S)$ by definition of left Bousfield localization. Thus, $u : Y \rightarrow X$ is an $\mathbb{A}^1$ -weak equivalences between cofibrant objects of Spc_S , and since f^* is a left Quillen functor (that is, using the adjunction $f^* \dashv f_*$), it follows from Brown's lemma (see Lemma A.2) that the image $u_T : Y_T \rightarrow X_T$ of $u : Y \rightarrow X$ is $\mathbb{A}^1$ -weak equivalence in Spc_T . The second assertion follows from the same reasoning using the commutative diagram

$$\begin{array}{ccc} \mathrm{Spc}_T & \xrightarrow{f_\#} & \mathrm{Spc}_S \\ \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ \mathcal{H}(T) & \xrightarrow{Lf_\#} & \mathcal{H}(S) \end{array}$$

and the fact that $f_\#$ maps the presheaves associated to the smooth T -schemes X and Y to those associated to X and Y considered as smooth S -schemes via the composition with $f : T \rightarrow S$. $\square$

Proposition 3.12. Let $u : X \rightarrow S$ be a smooth $\mathbb{A}^1$ -contractible S -scheme. Then for any arbitrary morphism of schemes $f : T \rightarrow S$, the induced map $u_T : X_T \rightarrow T$ is a smooth $\mathbb{A}^1$ -contractible T -scheme, where we have denoted the scheme-theoretic pullback by $X_T := X \times_S T$.

Proof. This is a straightforward application of Corollary 3.11 (1). $\square$

Corollary 3.13. The Proposition 3.12 has several interesting implications.

- (1) Let X be a smooth $\mathbb{A}^1$ -contractible scheme over a field k . Then for every field extension $k \subset L$ (separable or not), $X_L := X \times_{\mathrm{Spec} k} \mathrm{Spec} L$ is an $\mathbb{A}^1$ -contractible L -scheme.
- (2) Let $f : X \rightarrow S$ be a smooth $\mathbb{A}^1$ -contractible S -scheme. Then for every point s of S (closed or not) with residue field $\kappa(s)$, the scheme theoretic fiber $X_s := X \times_S \mathrm{Spec} \kappa(s)$ of f over the point s is a smooth $\mathbb{A}^1$ -contractible $\kappa(s)$ -scheme.
- (3) Let $i : Z \hookrightarrow S$ be a closed immersion of schemes. If $f : X \rightarrow S$ is an $\mathbb{A}^1$ -contractible S -scheme, then $\mathrm{pr}_2 : X \times_S Z \rightarrow Z$ is an $\mathbb{A}^1$ -contractible Z -scheme.

3.3. Relative $\mathbb{A}^1$ -Contractibility of Smooth Schemes. A morphism of S -schemes $\phi : X \rightarrow Y$ is *relatively $\mathbb{A}^1$ -weak equivalence* if ϕ is an $\mathbb{A}^1$ -weak equivalence in Spc_Y . The following portrays that relative $\mathbb{A}^1$ -weak equivalences behave well under *smooth* base change.

Corollary 3.14. Let $f : Y \rightarrow X$ and $g : Z \rightarrow Y$ be smooth morphisms between smooth schemes over a scheme S . If $g : Z \rightarrow Y$ is an $\mathbb{A}^1$ -weak equivalence in Spc_Y and $f : Y \rightarrow X$ is an $\mathbb{A}^1$ -weak equivalence in Spc_X , then $h = f \circ g : Z \rightarrow X$ is an $\mathbb{A}^1$ -weak equivalence in Spc_X .

Proof. Since $f : Y \rightarrow X$ is a smooth morphism, Corollary 3.11 (2) implies that $g : Z \rightarrow Y$ is an $\mathbb{A}^1$ -weak equivalence in Spc_X and the assertion then follows from the "two out of three" axiom for weak equivalence in a model category. $\square$

In the setting of Corollary 3.14, the implication " f and g are $\mathbb{A}^1$ -weak equivalence in $\mathrm{Spc}_X \Rightarrow g$ is an $\mathbb{A}^1$ -weak equivalence in Spc_Y " does not hold in general (see Example 3.15). Put differently, being "relatively" $\mathbb{A}^1$ -contractible is a subtle notion, and the need for such a distinction in a general setting is demonstrated by the following example. Observe also that this example is a non-pathological and, in fact, a fundamental instance.

Example 3.15. Consider the smooth morphism of k -schemes

$$(3.16) \quad \begin{array}{ccc} g : Z := \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^1 =: Y & & \mathbb{A}_k^2 \xrightarrow{g} \mathbb{A}_k^1 \\ & (x, y) \mapsto x^2 y^2 + y & \searrow \quad \swarrow \\ & & \mathrm{Spec} k \end{array}$$

We have that both Y and Z is $\mathbb{A}^1$ -contractible over X . Therefore, g is an $\mathbb{A}^1$ -weak equivalence in Spc_k . But g cannot be an $\mathbb{A}^1$ -weak equivalence in $\mathrm{Spc}_{\mathbb{A}_k^1}$. Suppose it was so, then by Corollary 3.13, every fiber of f over a point of $\mathbb{A}_k^1$ would be an $\mathbb{A}^1$ -contractible scheme over the corresponding residue field. But observe that the fiber of f over 0 is the disjoint union of a copy of $\mathbb{A}_k^1$ and a copy of $\mathbb{G}_{m,k}$. The latter component is not even $\mathbb{A}^1$ -connected (as $\mathbb{G}_m$ is $\mathbb{A}^1$ -rigid) in Spc_k and so the fiber cannot be $\mathbb{A}^1$ -connected, whence cannot be $\mathbb{A}^1$ -contractible. Therefore, we conclude that g cannot be a relative $\mathbb{A}^1$ -weak equivalence.

Remark 3.17. If $u : X \rightarrow S$ is a smooth $\mathbb{A}^1$ -contractible scheme in Spc_S , then its base change with respect to an arbitrary morphism $f : T \rightarrow S$ is recovered as that of the (usual) scheme-theoretic pullback. As a consequence, this implies crucially that the following diagram

$$\begin{array}{ccc} \mathrm{Sm}_S & \xrightarrow{f^\bullet} & \mathrm{Sm}_T \\ \downarrow Y & & \downarrow Y \\ \mathrm{Spc}_S & \xrightarrow{f^*} & \mathrm{Spc}_T \\ \mathbb{L}_{mot,S} \downarrow & & \downarrow \mathbb{L}_{mot,T} \\ \mathcal{H}(S) & \xrightarrow{Lf^*} & \mathcal{H}(T) \end{array}$$

commutes. In other words, for a smooth $\mathbb{A}^1$ -contractible scheme $u : X \rightarrow S$, we have that

$$Lf^*(\mathbb{L}_{mot,S}(X)) \simeq \mathbb{L}_{mot,T}(f^*(X)).$$

Retreating to Corollary 3.11 (1), one is intrigued to ask the following question:

Question 3.18. For a smooth morphism of schemes $f : X \rightarrow S$, does the property that all fibers of f over points of s are $\mathbb{A}^1$ -contractible over the corresponding residue fields imply that X is an $\mathbb{A}^1$ -contractible S -scheme?

Such a desired property (Question 3.18) holds true in the stable $\mathbb{A}^1$ -homotopy category $\mathrm{SH}(S)$ owing to the fact that the family of functors $i_s^! : \mathrm{SH}(S) \rightarrow \mathrm{SH}(\kappa(s))$, $s \in S$, is conservative, see e.g. [DFJK21, B20]. However, as a consequence of the *gluing theorem* [MV99, Theorem 2.21], it also holds in the unstable $\mathbb{A}^1$ -homotopy category.

Proposition 3.19. For a scheme $f : X \rightarrow S$ smooth over a Noetherian scheme S of finite Krull dimension, the following properties are equivalent:

- (1) The morphism $f : X \rightarrow S$ is an $\mathbb{A}^1$ -weak-equivalence in Spc_S .

- (2) For every point s of S with residue field $\kappa(s)$, the scheme theoretic fiber of f over s is a smooth $\mathbb{A}^1$ -contractible $\kappa(s)$ -scheme.

Proof. See [ABEH25, Proposition 2.1]. $\square$

3.4. Milnor-Witt K -Theory. To prove the $\mathbb{A}^1$ -contractibility of Koras-Russell threefolds over an arbitrary field, we will need a description of the endomorphism ring of the motivic spheres $\mathbb{A}^n \setminus \{0\}$ that is intertwined with the Milnor-Witt K -theory. The Milnor Witt K -theory is essentially an extension of Milnor's K -theory, which cleverly mixes the generators and relations of Milnor's K -theory and Grothendieck Witt ring. Interested readers can refer to [Mor12; Fas20; Dég23; BCDFØ25] for an extensive background.

Definition 3.20. Let F be any arbitrary field and $F^\times$ be its units. The *Milnor-Witt K -theory* or *Milnor-Witt ring*, denoted as $K_*^{\text{MW}}(F)$, is defined as the $\mathbb{Z}$ -graded associative unital ring freely generated by the formal symbols $[a]$ of degree +1, for $a \in F^\times$ and a symbol η of degree -1 (called the *Hopf element*) subjected to the following relations:

- (1) $[a][1 - a] = 0$, for any $a \neq 0, 1$ (Steinberg relations)
- (2) $[ab] = [a] + [b] + \eta[a][b]$, for any $a, b \in F^\times$
- (3) $\eta[a] = [a]\eta$, for any $a \in F^\times$
- (4) $\eta(\eta[-1] + 2) = 0$

It is related to the Milnor K -theory [Mil70] by killing η , that is, $K_*^{\text{MW}}/(\eta) = K_*^{\text{M}}(F)$. It is related to the Grothendieck-Witt ring of symmetric bilinear forms by killing the hyperbolic form $h := \langle -1, 1 \rangle$, that is, $K_*^{\text{MW}}F/(h) = \text{GW}(F)$. There are various standard relations that one immediately derives by working with the generator and the relations. One such important element is the ϵ in degree 0 described as follows:

Lemma 3.21. Let $\epsilon := -\langle -1 \rangle \in K_0^{\text{MW}}(F)$. For any $n \in \mathbb{Z}$, let

$$n_\epsilon = \begin{cases} \sum_{i=1}^n \langle (-1)^{i-1} \rangle & \text{if } n > 0. \\ 0 & \text{if } n = 0. \\ \epsilon \sum_{i=1}^{-n} \langle (-1)^{i-1} \rangle & \text{if } n < 0 \end{cases}$$

Then the following holds.

- (1) For any $\alpha \in K_n^{\text{MW}}(F)$ and $\beta \in K_m^{\text{MW}}(F)$, we have $\alpha\beta = \epsilon^{mn}\beta\alpha$
- (2) For any $n \in \mathbb{Z}$ and $a \in F^\times$, we have $[a^n] = n_\epsilon[a]$.

Residue homomorphism: One of the fundamental tools to build a Gersten-type (cochain) complex with coefficients in the Milnor-Witt K -theory is the *residue homomorphism*. Let F be a field and $v : F \rightarrow \mathbb{Z} \cup \{-\infty\}$ be a discrete valuation with residue field $\kappa(v)$ and valuation ring $\mathcal{O}_v$ and choose a uniformizing parameter $\pi = \pi_v$ of v .

Theorem 3.22. There is a unique homomorphism of graded Abelian groups

$$\partial_v^\pi : K_*^{\text{MW}}(F) \rightarrow K_{*-1}^{\text{MW}}(\kappa(v))$$

of degree -1 such that ∂_v^π commutes with the multiplication by η and

- (1) $\partial_v^\pi([\pi, u_1, \dots, u_n]) = [\bar{u}_1, \dots, \bar{u}_n]$
- (2) $\partial_v^\pi([u_1, \dots, u_n]) = 0$

We call such a ∂_v^π the *residue homomorphism*. It turns out that the so-defined homomorphism depends on the chosen valuation π , that is, for an element $\alpha \in K_*^{\text{MW}}(F)$ and $u \in \mathcal{O}_v^\times$, we have that

$$\partial_v^\pi(\langle u \rangle \alpha) = \langle \bar{u} \rangle \partial_v^\pi(\alpha).$$

This is the reason for [Mor12] to introduce the theory of *twisted Milnor Witt K -theory*. The twists are considered with respect to an invertible sheaf, equivalently, a line bundle³. Let V be an F -vector space of rank 1 with $V^\times$ comprising the non-zero elements of V . The group $V^\times$ acts transitively on $F^\times$, and this gives the free Abelian group $\mathbb{Z}[V^\times]$ a structure of a $\mathbb{Z}[F^\times]$ -algebra. On the other hand, we know that $K_*^{\text{MW}}(F)$ is a $K_0^{\text{MW}}(F)$ -algebra and there is a canonical map $F^\times \rightarrow K_0^{\text{MW}}(F)$ define as $u \mapsto \langle u \rangle \in K_0^{\text{MW}}(F)$. Thus, all the groups $K_n^{\text{MW}}(F)$ also inherits the structure of $\mathbb{Z}[F^\times]$ -algebra. Now, we can conveniently set the *Milnor-Witt K -theory twisted by V* as

$$K_n^{\text{MW}}(F, V) := K_n^{\text{MW}}(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[V^\times]$$

With this definition, we can now consider the twisted residue homomorphism with respect to a line bundle L . The homomorphism

$$\partial_v : K_*^{\text{MW}}(F, L) \rightarrow K_{*-1}^{\text{MW}}(\kappa(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^* \otimes L)$$

is defined by $\partial_v(\alpha \otimes l) = \partial_v^\pi(\alpha) \otimes \bar{\pi}^* \otimes l$, for $l \in L$. Here, $(\mathfrak{m}_v/\mathfrak{m}_v^2)^*$ represents the relative tangent space which, by definition, is the dual of the relative cotangent space $\mathfrak{m}_v/\mathfrak{m}_v^2$ (seen as $\kappa(v)$ -vector spaces).

Lemma 3.23. The homomorphism ∂_v is well-defined and is independent of the uniformizer π .

Proof. See [Fas20, Lemma 1.19]. □

Motivic Brouwer degree: We have yet another fundamental tool in the Milnor-Witt K -theory which gives the description of the non-trivial $\mathbb{A}^1$ -homotopy sheaf of the motivic sphere $\mathbb{A}^n \setminus \{0\}$, for $n \geq 2$.

Theorem 3.24. ([Mor12, Theorem 6.40]) For $n \geq 2$, we have a canonical isomorphism of strictly $\mathbb{A}^1$ -invariant sheaves

$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong \pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n}) \cong K_n^{\text{MW}}$$

This theorem helps one to understand the endomorphism ring $[\mathbb{A}^n \setminus \{0\}, \mathbb{A}^n \setminus \{0\}]_{\mathbb{A}^1}$, for all $n \geq 2$. In particular, [Mor12, Corollary 6.43] helps us to identify the zeroth part of Milnor-Witt sheaves with the Grothendieck-Witt ring of quadratic forms over a field k :

$$(3.25) \quad [\mathbb{A}_k^n \setminus \{0\}, \mathbb{A}_k^n \setminus \{0\}]_{\mathbb{A}^1} \cong K_0^{\text{MW}}(k) \cong \text{GW}(k).$$

If $f : \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{A}^n \setminus \{0\}$ is a morphism of spaces, then its class of $[f]$ in $\mathcal{H}(k)$ is called the *motivic Brouwer degree* of f . To compute this Brouwer degree, one uses the following cohomological technique.

Lemma 3.26. Let $f : \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{A}^n \setminus \{0\}$ be a morphism in $\mathcal{H}(k)$. Then f is an isomorphism if and only if

$$f^* : H^{n-1}(\mathbb{A}^n \setminus \{0\}, K_n^{\text{MW}}) \rightarrow H^{n-1}(\mathbb{A}^n \setminus \{0\}, K_n^{\text{MW}})$$

is an isomorphism of Milnor-Witt K -theory groups. In other words, the degree of f is 1.

Proof. See [DF18, Lemma 2.2]. □

³in fact, one uses the graded line bundles in more generality, see [Fas20, §1.3]

4. RELATIVE $\mathbb{A}^1$ -CONTRACTIBILITY OF KORAS-RUSSELL THREEFOLDS

Definition 4.1. For a perfect field k , the *Koras-Russell threefolds of first kind* are defined by the explicit polynomial equation

$$(4.2) \quad \mathcal{K} := \{x^n z - y^r - t^s - x = 0\} \subset \mathbb{A}_k^4$$

where the integers $n \geq 2$ and $r, s \geq 1$ such that $\gcd(r, s) = 1$.

Traditionally, Koras-Russell threefolds are studied extensively over a field of characteristic zero. But in the light of coprimality of the integers r, s and the usefulness of the coordinate ' x ' to make the Jacobian criterion for smoothness work, we here record that it is also well-defined over any perfect field (in fact, k can be arbitrary as well) and more so over arithmetic curves such as the integers $\text{Spec } \mathbb{Z}$. Note that the induced projection map $f : \mathcal{K} \rightarrow \mathbb{A}_x^1$ has all fibers except 0 isomorphic to $\mathbb{A}^2$ and the fiber at the origin isomorphic to a cylinder on the cuspidal curve $\mathbb{A}_x^1 \times C_{r,s}$ (see Fig. 1). More generally, one studies a deformed version of the Koras-Russell threefolds given by $\{x^m z = xq(x) + y^r + t^s\}$, for any polynomial $q(x)$ such that $q(0) \in k^\times$ (see e.g., [DMP11]). In [DF18], the authors prove the $\mathbb{A}^1$ -contractibility of the Koras-Russell threefold of first kind over a field of characteristic zero. In the first part of this article, we shall prove that this indeed can be extended over any field and, as a consequence, over general base schemes. To put the strategy into context, we shall explain some elements from [DF18]. The basic ingredient is a version of the five lemma (cf.

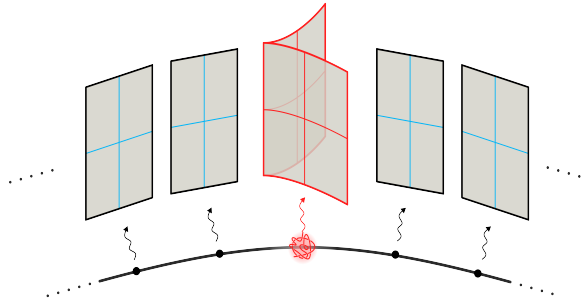


FIGURE 1. Fibers of the Koras-Russell threefolds projected to $\mathbb{A}_x^1$ with the generic point indicated in red.

Lemma A.1). To adapt this lemma to our situation, we note that $\mathcal{K}$ contains a hypersurface given by $\{z = 0\} := P \subset \mathcal{K}$ which can be shown to be isomorphic to $\mathbb{A}^2$ with co-ordinates y and t , i.e., $P \cong \text{Spec } k[y, t]$. Moreover, $\mathcal{K}$ also contains an affine line given by $L := \{x = y = t = 0\} \cong \mathbb{A}^1$, i.e., $L \cong \text{Spec}(k[z])$, which gives us $P \setminus L \simeq \mathbb{A}^2 \setminus \{0\}$. By definition, the varieties L and P intersect transversally in $\mathbb{A}_k^4$ at the point $(0, 0, 0, 0)$ giving us the following pullback diagram:

$$(4.3) \quad \begin{array}{ccc} P \setminus L \simeq \mathbb{A}^2 \setminus \{0\} & \longrightarrow & P \simeq \mathbb{A}^2 \\ \downarrow & & \downarrow \\ \mathcal{K} \setminus L & \longrightarrow & \mathcal{K} \end{array}$$

Consider the associated cofiber sequence given by the following commutative diagram

$$(4.4) \quad \begin{array}{ccccc} \mathbb{A}^2 \setminus \{0\} & \longrightarrow & \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 / \mathbb{A}^2 \setminus \{0\} \\ \downarrow i & & \downarrow \phi & & \downarrow j \\ \mathcal{K} \setminus L & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{K} / \mathcal{K} \setminus L \end{array}$$

In the light of Lemma A.1, it suffices to show that the morphisms i and j are $\mathbb{A}^1$ -weak equivalences in Spc_k .

4.1. $\mathbb{A}^1$ -weak equivalence of j : The arguments to prove the $\mathbb{A}^1$ -weak equivalence of j are exactly as those of [DF18], which is derived from the purity isomorphism ([MV99, Theorem 2.23]). In fact, the defining equation of $\mathcal{K}$ gives us a description of the normal cone of L inside $\mathcal{K}$. Namely, by rewriting $\mathcal{K}$ as $\{x(x^{m-1}z - 1) = y^r + t^s\}$, one observes that the regular function $(x^{m-1}z - 1)$ invertible in L (as it is equal to -1 in L) and that the normal cone $N_{\mathcal{K}}(L)$ is generated by y and t . Coupling this data with the homotopy purity, we have that

$$X/(X \setminus L) \simeq \mathrm{Th}(N_{\mathcal{K}}(L)) \simeq L_+ \wedge (\mathbb{P}^1)^{\wedge 2}.$$

Again by purity, the quotient space $\mathbb{A}^2 / \mathbb{A}^2 \setminus \{0\}$ is identified with $(\mathbb{P}^1)^{\wedge 2}$ (cf. [AE17, §4.6] or [ADF17, §2.1]). The inclusion $\mathrm{Spec} k \simeq \{0\} \hookrightarrow L \simeq \mathbb{A}^1$ induces a the composite

$$\mathrm{Spec} k \wedge (\mathbb{P}^1)^{\wedge 2} \simeq \mathbb{A}^2 / \mathbb{A}^2 \setminus \{0\} \xrightarrow{j} \mathcal{K} / \mathcal{K} \setminus L \simeq L_+ \wedge (\mathbb{P}^1)^{\wedge 2}$$

which is an $\mathbb{A}^1$ -weak equivalence (cf. [Voe03, Lemma 2.1]) and so is j .

4.2. $\mathbb{A}^1$ -weak equivalence of i . To prove the $\mathbb{A}^1$ -weak equivalence of $i : \mathbb{A}^2 \setminus \{0\} \rightarrow \mathcal{K} \setminus L$, the strategy is to first produce an explicit $\mathbb{A}^1$ -weak equivalence $g : \mathcal{K} \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$. Then, using a key property of motivic Brouwer degree as in Lemma 3.26, we show that the composite $g \circ i : \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k .

The explicit $\mathbb{A}^1$ -weak equivalence via Zariski local triviality. In what follows, we shall explain the strategy to produce the explicit $\mathbb{A}^1$ -weak equivalence $g : \mathcal{K} \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$ that has the structure of a Zariski locally trivial $\mathbb{A}^1$ -bundle over $\mathbb{A}^2 \setminus \{0\}$ and the corresponding adaptations when k is perfect. Consider the affine scheme $\mathcal{K}(s) := \{x^m z = x + y^r + t^s\}$ along with the strictly quasi-affine scheme $\mathcal{K}(s) \setminus \{0\}$. Note that for $s = 1$, we have $\mathcal{K}(1) := \{x^m z = x + y^r + t\} \cong \mathbb{A}_k^3$, by the change of co-ordinates and as a result, $\mathcal{K}(1) \setminus L \simeq \mathbb{A}^2 \setminus \{0\}$. Now the strategy is to prove the existence of a quasi-affine fourfold W that is the total space of a Zariski locally trivial $\mathbb{A}^1$ -bundle over $\mathcal{K}(s) \setminus L$, for all $s \geq 1$. This fourfold W is produced by exploiting the Danielewski fiber product trick that was initially constructed to produce a counterexample to the cancellation problem for surfaces ([Dan89], see also [Dub07]). Here, we show that the same strategy can be extended over all perfect fields with suitable modifications.

Proposition 4.5. Let k be any perfect field. Then for every $s \geq 1$, there exists a smooth strictly quasi-affine fourfold W which is simultaneously the total space of a Zariski locally trivial $\mathbb{A}^1$ -bundles over $\mathcal{K}(s) \setminus L$.

Proof. The desired fourfold W is constructed as the pullback of two strictly quasi-affine schemes over an algebraic space $\mathfrak{S}$

$$\begin{array}{ccc} W & \xrightarrow{\text{pr}_s} & \mathcal{K}(s) \setminus L \\ \text{pr}_1 \downarrow & & \downarrow \\ \mathcal{K}(1) \setminus L & \longrightarrow & \mathfrak{S} \end{array}$$

whose existence is proven in Lemma 4.7 where we also show that for all $s \geq 1$, the quasi-affine threefolds $\mathcal{K}(s)$ all have a structure of an étale locally trivial $\mathbb{A}^1$ -bundle $\rho : \mathcal{K}(s) \rightarrow \mathfrak{S}$ fibered over the same algebraic space $\mathfrak{S}$. Apriori, by taking the fiber product $W := (\mathcal{K}(1) \setminus L) \times_{\mathfrak{S}} (\mathcal{K}(s) \setminus L)$, we see that W only has the structure of an algebraic space. But by noticing that the natural projections $\text{pr}_s : W \rightarrow \mathcal{K}(s) \setminus L$ is an affine morphism and $\mathcal{K} \setminus L$ is a strictly quasi-affine scheme, which consequently implies that W also has to be a strictly quasi-affine scheme. The fact that this étale equivalence descends also to the equivalence in the Zariski topology is due to the deep fact that the transition group $\text{Aut}(\mathbb{A}_k^1) = \mathbb{G}_{m,k} \ltimes \mathbb{G}_a$ is special, in the sense of [Gro58, §5]. $\square$

As a direct consequence, we obtain our desired result.

Corollary 4.6. The map $g : \mathcal{K}(s) \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$ is an $\mathbb{A}^1$ -weak equivalence for all $s \geq 1$.

Proof. This is due to the observation that Zariski locally trivial $\mathbb{A}^n$ -bundles preserve $\mathbb{A}^1$ -weak equivalences (see, for example [AØ21, Lemma 3.1.3]). $\square$

Strategy to construct $\mathfrak{S}$ over perfect fields: The existence of the claimed algebraic space $\mathfrak{S}$ is due to the fundamental fact in moduli theory that if an algebraic group G acts freely on a scheme X , then the quotient X/G exists in the category of algebraic spaces ([Stacks, Tag 071R]). When the char $k = 0$, producing a $\mathbb{G}_a$ -action on an affine variety X is equivalent to providing a locally nilpotent derivation D on its coordinate ring $\Gamma(X)$, which follows due to the bijection of the exponential map $t \mapsto \exp(tD)$ (cf. [Fre06, Chapter 1, §1.5]). However, if the char $k = p > 0$, this simple correspondence does not a priori apply, for example, $1/p!$ in the exponential function is not well-defined. Nevertheless, one still gets a correspondence by considering a locally finite iterative higher derivations ∂ instead of a single locally nilpotent derivation. The correspondence is obtained by considering the truncated version of $\exp(t\partial)$ (see [Miy78, Part I, §1], [Oku04, §5]). Given the existence of such a family of locally finite iterative higher derivations, we have that for a $\mathbb{G}_a$ -action on $\mathcal{K}(s)$ its algebraic quotient coincides with the projection $\text{pr}_{x,t} : \mathbb{A}^4 \rightarrow \mathbb{A}^2$ restricted to $q : \mathcal{K}(s) \rightarrow \mathbb{A}^2$ with the fixed point locus L . Hence, this $\mathbb{G}_a$ -action induces a fixed point free $\mathbb{G}_a$ -action on $\mathcal{K}(s) \setminus L$

$$\begin{array}{ccc} \mathcal{K}(s) \setminus L & \xrightarrow{q} & \mathbb{A}^2 \setminus \{0\} \\ & \searrow & \nearrow \\ & (\mathcal{K}(s) \setminus L) / \mathbb{G}_a & \end{array}$$

such that it factors through an étale $\mathbb{G}_a$ -torsor over its geometric quotient $(\mathcal{K}(s) \setminus L) / \mathbb{G}_a$ as above. The desired algebraic space $\mathfrak{S}$ is then constructed by an étale equivalence relation. Observe that for the morphism $q : \mathcal{K} \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$, the fibers over points $x \neq 0$ forms a Zariski locally trivial $\mathbb{A}^1$ -bundle, which is actually a trivial $\mathbb{A}^1$ -bundle; this is because if $x \neq 0$, one has a slice (i.e., $\partial a = 1$) of the form $a = f(y)/x^m$, for some suitable polynomial $f(y) \in k[y]$.

On the other hand, the fiber over the punctured affine line $C_0 := \{x = 0\} = \text{Spec } k[t^{\pm 1}]$ is given by $C \times \text{Spec } k[z]$ where $C := \text{Spec } k[t^{\pm 1}, y] / \langle y^r + t^s \rangle$. The curve $C \rightarrow C_0$ can be identified with a

finite étale cover of degree r with respect to the projection $(y, t) \mapsto t$. This is where the case over fields of positive characteristics crucially differs from that of characteristic zero. If $\text{char } k = p \mid r$, then one cannot form a finite étale cover as claimed above. But since we have that $\gcd(r, s) = 1$, we can rework the proof as follows. If $p \nmid r$, then we can then consider the finite étale cover

$$C := \text{Spec } k[t^{\pm 1}, y] / \langle y^r + t^s \rangle \rightarrow \text{Spec } k[t^{\pm 1}] := C_0$$

$$(y, t) \mapsto y$$

of degree s with respect to the other coordinate, where we can still extract the roots (i.e., $(y^r)^{1/s}$) since in this case $p \nmid s$ by the coprimality.

Lemma 4.7. For any (perfect) field k , there exists a smooth algebraic space $\delta : \mathfrak{S} \rightarrow \mathbb{A}_k^2 \setminus \{0\}$ such that for every $s \geq 1$, the map $q : \mathcal{K}(s) \setminus L \rightarrow \mathbb{A}_k^2 \setminus \{0\}$ factors through an étale-locally trivial $\mathbb{A}^1$ -bundle $\rho : \mathcal{K}(s) \setminus L \rightarrow \mathfrak{S}$.

The strategy of the proof follows from a more general principle involved in constructing an algebraic space from a surface by replacing a desired curve by its finite étale covering ([DF14, §1.0.1]). Here we will brief on the working strategy and ideas of the proof following [DF18].

Proof. Without loss of generality, let us assume that $\text{char } k = p \nmid r$. Observe that the strictly quasi-affine threefolds $\mathcal{K} \setminus L$ are covered by two principal affine opens, (say) $V_x = \{x \neq 0\}$ where it is a trivial $\mathbb{A}^1$ -bundle as observed above, and $V_t = \{t \neq 0\}$ where the suitable modifications discussed above have to be implemented. As a consequence, it suffices to build an algebraic space $\mathfrak{S}_t$ locally on V_t and prove that it compatibly extends to the desired algebraic space globally. To this end, we need to prove that there (locally) exists an algebraic space $\delta_t : \mathfrak{S}_t \rightarrow U_t$ such that $q|_{V_t} : V_t \rightarrow U_t$ factors through an étale locally trivial $\mathbb{A}^1$ -bundle $\rho_t : V_t \rightarrow \mathfrak{S}_t$. Moreover, such a factorization via δ_t should be an isomorphism when restricted to the overlap $U_{xt} := U_x \cap U_t$. Here, $U_x = \text{Spec } k[x^{\pm 1}, t]$ and $U_t = \text{Spec } k[x, t^{\pm 1}]$ are corresponding principal affine open subsets in $\mathbb{A}^2 \setminus \{0\}$. The desired (globally) algebraic space $\delta : \mathfrak{S} \rightarrow \mathbb{A}^2 \setminus \{0\}$ is then obtained by gluing U_x and $\mathfrak{S}_t$ by the identity along U_{xt} and such that $\delta_t^{-1}(U_{xt}) \cong U_{xt}$.

$$\begin{array}{ccc} \mathcal{K}(s) \setminus L & \xrightarrow{q} & \mathbb{A}^2 \setminus \{0\} \\ \rho \downarrow & \nearrow \delta & \\ \mathfrak{S} & & \end{array} \quad \begin{array}{ccc} V_t & \xrightarrow{q|_{V_t}} & U_t \\ \rho_t \downarrow & \nearrow \delta_t & \\ \mathfrak{S}_t & & \end{array}$$

Let $h : \text{Spec } R =: C \rightarrow C_0$ be the Galois closure of the finite étale map $h_0 : C_1 \rightarrow C_0$ with $C_1 := \text{Spec } k[t^{\pm 1}][y] / (y^r + t^s)$. In other words, C gives the normalization of C_1 in the Galois closure κ with respect to the field extension

$$k(t) \hookrightarrow k(t)[y] / (y^r + t^s).$$

The above-mentioned Galois closure κ is precisely obtained by extracting the r th-roots from t^s , which is valid since $p \nmid r$ by assumption. However, by the coprimality of r and s , this is equivalent to extracting the r th-roots of t along with all the r th-roots of -1 . Thus, neither κ nor the curves C_1 or C depend on s . By construction, this gives us an étale G -torsor $h : C \rightarrow C_0$ with G being the Galois group $\text{Gal}(\kappa/k(t))$ and a factorization

$$h : C \xrightarrow{h_1} C_1 \xrightarrow{h_0} C_0$$

where $h_1 : C \rightarrow C_1$ is an étale torsor under some subgroup $H \leq G$ of index r equal to the degree of h_1 , corresponding to the fact that we have made an r -fold Galois cover of C_0 . Now to complete the proof, one can follow the same strategy performed by the authors in [DF18, Lemma 3.2]. Indeed, they show that there exists a scheme S obtained by gluing r -copies of $S_{\bar{g}}$, $\bar{g} \in G/H$, by the identity along curves $C_0 \times C$ such that the group G acts freely on $S_{\bar{g}}$ giving rise to a geometric quotient $\mathfrak{S}_t := S/G$ in the category of algebraic spaces in the form of an étale G -torsor over $\mathfrak{S}$. Moreover, this local morphism glues to a global G -invariant morphism which then descends to the corresponding (global) quotient. We present their proof here for completeness and the reader's clarity.

By definition, the polynomial $y^r + t^s \in R[y]$ splits over κ as $\prod_{\bar{g} \in G/H} (y - \lambda_{\bar{g}})$, for some elements $\lambda_{\bar{g}} \in R$ and $\bar{g} \in G/H$. The Galois group G then acts transitively on the quotient G/H via $g' \cdot \lambda_{\bar{g}} = \lambda_{(g')^{-1} \cdot \bar{g}}$. Since $h : C_1 \rightarrow C_0$ is étale, we have that for distinct elements $\bar{g}, \bar{g}' \in G/H$, the element $\lambda_{\bar{g}} - \lambda_{\bar{g}'}$ is an invertible regular function on C . Consequently, we can find a collection of elements $\sigma_{\bar{g}}(x) \in B =: R[x]$ with corresponding residue classes $\lambda_{\bar{g}} \in R = B/xB$ modulo x on which G acts via

$$g' \cdot \sigma_{\bar{g}}(x) \mapsto \sigma_{(g')^{-1} \cdot \bar{g}}(x)$$

along with a G -invariant polynomial $s(x, y) \in B[y]$ such that in the ring $B[y]$, we have

$$(4.8) \quad y^r + t^s + x = \prod_{\bar{g} \in G/H} (y - \sigma_{\bar{g}}(x)) + x^m s(x, y).$$

Now observe that the principal open subset V_t under consideration when base changed along B , (say) $\hat{V}_t := V_t \times_{U_t} \text{Spec}(B)$ is isomorphic to the closed subvariety of $\text{Spec}(B[y, z_1])$ defined by the equation

$$(4.9) \quad \hat{V}_t \cong \{x^m z_1 = \prod_{\bar{g} \in G/H} (y - \sigma_{\bar{g}}(x))\} \subset \text{Spec}(B[y, z_1])$$

where $z_1 = z - s(x, y)$. As mentioned before, the regular function $\lambda_{\bar{g}} - \lambda_{\bar{g}'}$ is invertible whenever $\bar{g} \neq \bar{g}'$, the closed subscheme $\{x = 0\} \subset \hat{V}_t$ is the disjoint union of r -closed subschemes $D_{\bar{g}} \cong \text{Spec}(R[z_1])$ with corresponding defining ideals $(x, y - \sigma_{\bar{g}}(x)) \in \Gamma(\hat{V}_t, \mathcal{O}_{\hat{V}_t})$ on which G acts by permutation. Furthermore, the described affine variety $\hat{V}_t$ is covered by affine open subsets of the form $\hat{V}_{t, \bar{g}}$, where

$$\hat{V}_{t, \bar{g}} = \hat{V}_t \setminus \bigcup_{\bar{g}' \in (G/H) \setminus \{\bar{g}\}} D_{\bar{g}'}$$

for $\bar{g} \in G/H$. Now, due to the extraction of roots as in Eq. (4.8), we have a well-defined rational map

$$\hat{V}_{t, \bar{g}} \dashrightarrow \text{Spec } B[u_{\bar{g}}], \quad (x, y, z_1) \mapsto \left(x, \frac{y - \sigma_{\bar{g}}(x)}{x^m} = \frac{z}{\prod_{\bar{g}' \in (G/H) \setminus \{\bar{g}\}} (y - \sigma_{\bar{g}'}(x))} \right)$$

which is in fact an isomorphism of schemes over $\text{Spec } B$. As a consequence, one verifies that the morphism $\hat{q} : \hat{V}_t = V_t \times_{U_t} \text{Spec } B \rightarrow \text{Spec } B$ is faithfully flat and that $\hat{q}$ factors through a Zariski locally trivial $\mathbb{A}^1$ -bundle $\hat{\rho} : \hat{V}_t \rightarrow S$ over the scheme $\hat{\delta} : S \rightarrow \text{Spec } B$. The latter scheme S is obtained by gluing r copies of $S_{\bar{g}}$, for $\bar{g} \in G/H$, of $\text{Spec } B \cong C \times \mathbb{A}^1$ by identity along the principal open subsets $\text{Spec } B_x \cong C \times \text{Spec}(k[x^{\pm 1}]) \subset S_{\bar{g}}$. Put differently, the morphism $\hat{\rho}_t : \hat{V}_t \rightarrow S$ is a Zariski-locally trivial $\mathbb{A}^1$ -bundle with local trivializations $\hat{V}_t|_{S_{\bar{g}}} \cong \text{Spec } B[u_{\bar{g}}]$ and transition functions over

$S_{\bar{g}} \cap S_{\bar{g}'} \cong \text{Spec } B_x$ of the form

$$u_{\bar{g}} \mapsto u_{\bar{g}'} + x^{-m}(\sigma_{\bar{g}}(x) - \sigma_{\bar{g}'}(x)).$$

The action of G on $\hat{V}_t$ descends to a fixed point-free action on S defined locally by

$$S_{\bar{g}} \ni (c, x) \mapsto (g' \cdot c, x) \in S_{(\bar{g}')^{-1} \cdot \bar{g}}.$$

This G -action produces a quotient which is a scheme only if $h_0 : C_1 \rightarrow C_0$ is an isomorphism, but always exists as an algebraic space in the form of an étale G -torsor $S \rightarrow \mathfrak{S}_t := S/G$. By construction of $\mathfrak{S}_t$, we get the following cartesian square

$$\begin{array}{ccc} \hat{V}_t & \longrightarrow & V_t \cong \hat{V}_t/G \\ \hat{\rho}_t \downarrow & & \downarrow \rho_t \\ S & \longrightarrow & \mathfrak{S}_t := S/G \end{array}$$

with the horizontal maps being étale G -torsors. Consequently, the induced morphism $\rho_t : V_t \rightarrow \mathfrak{S}_t$ is an étale locally trivial $\mathbb{A}^1$ -bundle. It remains to observe that the G -invariant morphism $\text{pr}_1 \circ \hat{\delta} : S \rightarrow \text{Spec } B \cong U_t \times_{C_0} C \rightarrow U_t$ descends to a morphism $\delta_t : \mathfrak{S}_t \rightarrow U_t$ and $\hat{\delta}$ restricts to an isomorphism outside $\{x = 0\} \subset U_t$. In contrast, over the punctured fiber, we have that the $\delta^{-1}(\{x = 0\})$ is isomorphic to the quotient of $C \times G/H$ by the diagonal action of G , and hence to $C/H \cong C_1$. Thus, the algebraic space $\mathfrak{S}_t$ extends to globally $\delta : \mathfrak{S} \rightarrow \mathbb{A}^2 \setminus \{0\}$. $\square$

Remark 4.10. The perfectness assumption on k is not strictly necessary in Lemma 4.7. Indeed, any field k facilitating the extraction of the r -th roots of t (equivalently, the s -th roots of y) should do the job.

Motivic Brouwer degree. Let $g : \mathcal{K} \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$ be the $\mathbb{A}^1$ -weak equivalence from Corollary 4.6. Recall that our aim is to show that the morphism $i : \mathbb{A}^2 \setminus \{0\} \rightarrow \mathcal{K} \setminus L$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k . To this end, it is enough to show that the composite $i \circ g : \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{A}^2 \setminus \{0\}$

$$\mathbb{A}^2 \setminus \{0\} \xrightarrow{i} \mathcal{K} \setminus L \xrightarrow{g} \mathbb{A}^2 \setminus \{0\}$$

is an $\mathbb{A}^1$ -weak equivalence in Spc_k as it then follows due to the two-out-of-three property of model categories (for e.g., [Qui67, §1.1]). To show that $i \circ g$ is an $\mathbb{A}^1$ -weak equivalence, we use the associated $\mathbb{A}^1$ -Brouwer degree map from the motivic homotopy theory (cf. Theorem 3.24). The morphism g being an $\mathbb{A}^1$ -weak equivalence, we can identify $\mathcal{K} \setminus L$ in the Milnor-Witt K -theory as $H^1(\mathcal{K} \setminus L, K_2^{\text{MW}}) = K_0^{\text{MW}}(k) \cdot \mu$, where $\mu = g^*(\xi)$ is the pullback of the explicit generator. In the view of Lemma 3.26, it is enough to show that the induced map on the Milnor-Witt K -theory sheaves

$$i^* : H^1(\mathcal{K} \setminus L, K_2^{\text{MW}}) \rightarrow H^1(\mathbb{A}^2 \setminus \{0\}, K_2^{\text{MW}})$$

is an isomorphism. The Eq. (4.4) induces the following commutative diagram in Milnor-Witt K -theory

$$(4.11) \quad \begin{array}{ccccccc} H^1(\mathcal{K} \setminus L, K_2^{\text{MW}}) & \xrightarrow{\partial} & H^2(\mathcal{K}/(\mathcal{K} \setminus L), K_2^{\text{MW}}) & \longrightarrow & H^2(\mathcal{K}, K_2^{\text{MW}}) & \longrightarrow & H^2(\mathcal{K} \setminus L, K_2^{\text{MW}}) \\ \downarrow i^* & & \downarrow j^* & & \downarrow & & \downarrow i^* \\ H^1(\mathbb{A}^2 \setminus \{0\}, K_2^{\text{MW}}) & \xrightarrow{\partial'} & H^2((\mathbb{P}^1)^{\wedge 2}, K_2^{\text{MW}}) & \longrightarrow & H^2(\mathbb{A}^2, K_2^{\text{MW}}) & \longrightarrow & H^2(\mathbb{A}^2 \setminus \{0\}, K_2^{\text{MW}}) \end{array}$$

Since j is an $\mathbb{A}^1$ -weak equivalence, this implies that j^* is an isomorphism. By $\mathbb{A}^1$ -homotopy invariance of Milnor-Witt K -theory, we have that $H^*(\mathbb{A}^2, K_2^{\text{MW}}) = 0$, which due to the exactness renders that ∂' is also an isomorphism. Observe that the groups $H^1(\mathcal{K} \setminus L, K_2^{\text{MW}})$ and $H^2(\mathcal{K}/(\mathcal{K} \setminus L), K_2^{\text{MW}})$ are actually free K_0^{MW} -modules of rank 1 due to [AF14, Lemma 4.5]. This implies that ∂ is $K_0^{\text{MW}}(k)$ -linear and, in particular, an injective group homomorphism. Thus, it remains to only verify that ∂ is surjective. This is proven in Proposition 4.12 by showing that every cocycle of $H^2(\mathcal{K}/(\mathcal{K} \setminus L), K_2^{\text{MW}})$ is a boundary of some cocycle of $H^1(\mathcal{K} \setminus L, K_2^{\text{MW}})$.

Proposition 4.12. For any perfect field k , the connecting homomorphism

$$\partial : H^1(\mathcal{K} \setminus L, K_2^{\text{MW}}) \rightarrow H^2(\mathcal{K}/(\mathcal{K} \setminus L), K_2^{\text{MW}})$$

is surjective.

Except for our style of presentation, the contents of the proof are essentially the same as in [DF18, Proposition 3.3]. We will include it here for the sake of concreteness.

Proof. Due to [Fas20, §2.2], we have the relation $H^n(\mathcal{K}/(\mathcal{K} \setminus L), K_*^{\text{MW}}) \cong H_L^n(\mathcal{K}, K_*^{\text{MW}})$, for all $\forall n \in \mathbb{Z}$. In particular, we have that $H^2(\mathcal{K}/(\mathcal{K} \setminus L), K_2^{\text{MW}}) \cong H_L^2(\mathcal{K}, K_2^{\text{MW}})$. Recall from Section 4.1 that the normal bundle of L in $\mathcal{K}$ is explicitly generated by the sections y and t . Using the localization exact sequence of Chow-Witt theory ([Fas08, §10.4]), this provides us with a long exact sequence and, as a result, also the generators for $H_L^2(\mathcal{K}, K_2^{\text{MW}})$ explicitly as the following class of cocycle:

$$(4.13) \quad \beta := \langle 1 \rangle \otimes \bar{t} \otimes \bar{y} \in K_0^{\text{MW}}(\kappa(L), \wedge^2 \mathfrak{m}_L / \mathfrak{m}_L^2).$$

To prove our claim, it is enough to show that this cocycle is indeed a boundary of a cocycle $\alpha \in H^1(\mathcal{K} \setminus L, K_2^{\text{MW}})$. The proof exploits several machinery developed in [Mor12]. Consider the integral subvarieties of $\mathcal{K}$ as follows:

$$M := \{y = 0\} \subset \mathcal{K} \quad N := \{t = 0\} \subset \mathcal{K} \quad L' := \{y = t = 0, x^{m-1}z = 1\} \subset \mathcal{K}$$

By the construction, we have $M \cap N = L \sqcup L'$. By the definition of M and N , we have that the element $\gamma := [y] \otimes \bar{t} \in K_1^{\text{MW}}(\kappa(N), \mathfrak{m}_N / \mathfrak{m}_N^2)$ has a non-trivial boundary only on L and L' with same values of the boundaries in respective residue fields; the boundary of γ on L is given by $\langle 1 \rangle \otimes \bar{t} \wedge \bar{y} \in K_0^{\text{MW}}(\kappa(L), \wedge^2(\mathfrak{m}_L / \mathfrak{m}_L^2))$ and on L' given by $\langle 1 \rangle \otimes \bar{t} \wedge \bar{y} \in K_0^{\text{MW}}(\kappa(L'), \wedge^2(\mathfrak{m}_{L'} / \mathfrak{m}_{L'}^2))$. Eventually, it turns out that $\gamma \in K_1^{\text{MW}}(\kappa(N), \mathfrak{m}_N / \mathfrak{m}_N^2)$ is not a cocycle on $\mathcal{K} \setminus L$, but we can produce one by doing the following modification. Note that the symbol

$$\alpha' := [x^{m-1}z - 1] \otimes \bar{y} \in K_1^{\text{MW}}(\kappa(M), \mathfrak{m}_M / \mathfrak{m}_M^2)$$

has a non-trivial boundary only on L' . Indeed from the equation of $\mathcal{K}$, we have that $x(x^{m-1}z - 1) = y^r + t^s$ and again since r, s are coprime, we have that any prime containing $x^{m-1}z - 1$ and y must also contain the t and in turn the prime ideal corresponding to L' (from the definition of L'). We now compute the boundary of α' on L' , taking advantage of the above-mentioned primality condition of the ideals. Since x is invertible in L' , we can write

$$(x^{m-1}z - 1) = (y^r + t^s)/x = x^{-1}y^r + x^{-1}t^s \in \mathcal{O}_{\mathfrak{m}_{L'}}.$$

Therefore, the boundary of α' is the same as the boundary of $[x^{-1}t^s] \otimes \bar{y}$ on L' . The latter can be computed using the explicit relations given in [Mor12, Lemma 3.5]. In particular, we have that

$$[x^{-1}t^s] = [x^{-1}] + \langle x^{-1} \rangle [t^s] = [x^{-1}] + \langle x \rangle [t^s].$$

Now due to Lemma 3.21, we have that $[t^s] = s_\epsilon[t]$. Finally, using the boundary formulas as stated in [Mor12, Proposition 3.17], we obtain the boundary of α' as

$$d(\alpha') = \langle x \rangle s_\epsilon \otimes \bar{y} \wedge \bar{t} = \langle -x \rangle s_\epsilon \otimes \bar{t} \wedge \bar{y}.$$

Set $S := \{x^{m-1}z = 1\} \subset \mathcal{K}$ as a codimension 1 subvariety of $\mathcal{K}$. Since M and S are different codimension 1 subvarieties, we compute the boundary as follows:

$$d([y, x^{m-1}z - 1]) = [x^{m-1}z - 1] \otimes \bar{y} + \epsilon[y] \otimes \overline{x^{m-1}z - 1}$$

As $d^2 = 0$ and $\epsilon = -\langle -1 \rangle$, we have

$$d([y] \otimes \overline{x^{m-1}z - 1}) = \langle -1 \rangle d([x^{m-1}z - 1] \otimes \bar{y}) = \langle x \rangle s_\epsilon \otimes \bar{t} \wedge \bar{y}.$$

Now x is an unit in S , we have that

$$d(\langle x \rangle [y] \otimes \overline{x^{m-1}z - 1}) = s_\epsilon \otimes \bar{t} \wedge \bar{y}.$$

Similarly,

$$d([x^{m-1}z - 1] \otimes \bar{t}) = \langle x \rangle r_\epsilon \otimes \bar{t} \wedge \bar{y}$$

and as a consequence, we have

$$d([t] \otimes \overline{x^{m-1}z - 1}) = \langle -1 \rangle d([x^{m-1}z - 1] \otimes \bar{t}) = \langle -x \rangle r_\epsilon \otimes \bar{t} \otimes \bar{y}.$$

Thus,

$$d(\langle -x \rangle [t] \otimes \overline{x^{m-1}z - 1}) = r_\epsilon \otimes \bar{t} \wedge \bar{y}.$$

As $\gcd(r, s) = 1$, we can choose $g, h \in \mathbb{N}$ such that $gr - hs = 1$. For any integers $p, q \in \mathbb{Z}$, we have $(pq)_\epsilon = p_\epsilon q_\epsilon$, and so we have $g_\epsilon r_\epsilon - h_\epsilon s_\epsilon = \langle \pm 1 \rangle$ ⁴. Altogether, we have obtained that

$$d(g_\epsilon \langle -x \rangle [t] \otimes \overline{x^{m-1}z - 1} - h_\epsilon \langle x \rangle [y] \otimes \overline{x^{m-1}z - 1}) = \langle \pm 1 \rangle \otimes \bar{t} \wedge \bar{y}.$$

This gives us the desired cocycle α by modifying α' as

$$\alpha := [y] \otimes \bar{t} - \langle \pm 1 \rangle \left(g_\epsilon \langle -x \rangle [t] \otimes \overline{x^{m-1}z - 1} - h_\epsilon \langle x \rangle [y] \otimes \overline{x^{m-1}z - 1} \right)$$

that maps to the generator of $H_L^2(\mathcal{K}, K_2^{\text{MW}})$ under ∂ . □

Remark 4.14. When $\text{char } k = 0$, the authors in [DF18] also present another proof, which they call the *lazy proof*, exploiting the stable $\mathbb{A}^1$ -contractibility of $\mathcal{K}$ proven in [HKØ16, Theorem 4.2]. In brief, the proof in the stable $\mathbb{A}^1$ -homotopy category heavily relies on the fact that every smooth quasi-projective scheme is dualizable in $\text{SH}(k)$. This is, in turn, proven by exploiting the existence of resolution of singularities in characteristic zero. Hence, assuming the existence of resolution of singularities over perfect fields, the lazy proof works verbatim in our context.

⁴it is odd if gr is odd and $\langle -1 \rangle$ otherwise

4.3. Relative $\mathbb{A}^1$ -contractibility of $\mathcal{K}$. All these results allow us to extend the $\mathbb{A}^1$ -contractibility of $\mathcal{K}$ in the unstable $\mathbb{A}^1$ -homotopy category over generalized base schemes.

Proposition 4.15. For a perfect field k , the canonical morphism of the Koras-Russell threefolds of first kind as in Eq. (4.2) $f : \mathcal{K} \rightarrow \text{Spec } k$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k .

Proof. From the arguments of Section 4.1 and Section 4.2, we have that the morphisms i and j are $\mathbb{A}^1$ -weak equivalences in Spc_k . Consequently, we get that $\phi : \mathbb{A}^2 \rightarrow \mathcal{K}$ is an $\mathbb{A}^1$ -weak equivalence in Eq. (4.4); the conclusion follows as $\mathbb{A}^2$ is clearly $\mathbb{A}^1$ -contractible over k . $\square$

Theorem 4.16. The family of Koras-Russell threefolds of the first kind, as in Eq. (4.2) $f : \mathcal{K} \rightarrow \text{Spec } \mathbb{Z}$ is relatively $\mathbb{A}^1$ -contractible in $\mathcal{H}(\mathbb{Z})$. In particular, $\mathcal{K}$ is an $\mathbb{A}^1$ -contractible scheme in $\mathcal{H}(\mathbb{Z})$ that is not isomorphic to $\mathbb{A}_{\mathbb{Z}}^3$.

Proof. The fiber of f over the generic point $o \in \text{Spec } \mathbb{Z}$ is $\mathbb{A}^1$ -contractible, as the quotient field $\text{Spec } \mathbb{Q}$ has characteristic zero, for which the proof follows from [DF18]. As a result, we have that the structure morphism $f_o : \mathcal{K}_o \rightarrow \text{Spec } \mathbb{Q}$ is an $\mathbb{A}^1$ -weak equivalence in $\text{Spc}_{\mathbb{Q}}$. On the other hand, for all closed points $q \in \text{Spec } \mathbb{Z}$, the corresponding residue fields being isomorphic to $\text{Spec } \mathbb{F}_q$ (which, in particular, is perfect), the $\mathbb{A}^1$ -contractibility of the closed fibers $f_q : \mathcal{K}_q \rightarrow \text{Spec } \mathbb{F}_q$ follows from Proposition 4.15.

$$\begin{array}{ccccc}
 \mathcal{K}_o & \longrightarrow & \mathcal{K} & \longleftarrow & \mathcal{K}_q \\
 \searrow f_o & & \downarrow f & & \swarrow f_q \\
 \text{Spec } \mathbb{Q} \cong \text{Spec } \kappa_o & \longrightarrow & \text{Spec } \mathbb{Z} & \longleftarrow & \text{Spec } \kappa_q \cong \text{Spec } \mathbb{F}_q
 \end{array}$$

Altogether, $f : \mathcal{K} \rightarrow \text{Spec } \mathbb{Z}$ is a morphism of motivic spaces such that all its fibers are relatively $\mathbb{A}^1$ -contractible over the corresponding (residue) fields. The conclusion is a consequence of Proposition 3.19. $\square$

The above theorem generalizes, giving us the following consequence.

Corollary 4.17. Let S be any Noetherian scheme with perfect residue fields. Let $\mathcal{K}$ be the family of Koras-Russell threefolds of the first kind as in Eq. (4.2) of relative dimension 3 over S . Then $f : \mathcal{K} \rightarrow S$ is an $\mathbb{A}^1$ -weak equivalence in Spc_S .

Proof. Let us first observe that the canonical map $f : \mathcal{K} \rightarrow S$ is well-defined and smooth, as all the fibers of f either over the quotient field or the residue fields are smooth schemes (cf. Lemma A.3). Choose a point $s \in S$ with residue fields κ_s . Now, if κ_s has characteristic zero, then the $\mathbb{A}^1$ -contractibility of $\mathcal{K} \rightarrow \text{Spec } \kappa_s$ follows from [DF18]. If, on the other hand, $\text{char } \kappa_s = q > 0$, it follows from Theorem 4.16 that $\mathcal{K} \rightarrow \text{Spec } \kappa_s$ is $\mathbb{A}^1$ -contractible since by assumption we have that κ_s is a perfect field. In all, we have that all the fibers of the morphism $f : \mathcal{K} \rightarrow S$ are $\mathbb{A}^1$ -contractible, and the proof is once again a consequence of Proposition 3.19. $\square$

Remark 4.18. Since the arguments of Proposition 4.12 follow from [Mor12, §3, Lemma 5.10], which in turn is valid over arbitrary base fields, Proposition 4.15 also remains true over any field k . Hence, more generally, the hypothesis on S in Corollary 4.17 can be promoted to a Noetherian base scheme, whence the relative $\mathbb{A}^1$ -contractibility of $\mathcal{K}$.

5. RELATIVE $\mathbb{A}^1$ -CONTRACTIBILITY OF KORAS-RUSSELL PROTOTYPES OVER A BASE SCHEME

Following the relative $\mathbb{A}^1$ -contractibility of Koras-Russell threefolds, we now extend the relative $\mathbb{A}^1$ -contractibility of the generalized Koras-Russell varieties introduced and studied by the authors in [DG25]. Let us begin with the definition of these varieties. As for the notation, k will denote a perfect field and $k^{[n]}$ will denote the polynomial ring in n -variables.

Definition 5.1. Let $r, s \geq 2$ be any coprime integers. Consider the triple $(m, \underline{n}, \psi)$ with integers $m \geq 0$ and a $(m+1)$ -tuple $\underline{n} = (n_0, n_1, \dots, n_m)$ of multi-index with $n_i > 1$ and a polynomial $\psi \in k^{[1]}$ such that $\psi(0) \neq 0$. Then consider the coordinate ring

$$R_m(\underline{n}, \psi) := k[x_0, x_1, \dots, x_m, y, z, t] / (\underline{x}^{\underline{n}} z + y^r + t^s + x_0 \psi(\underline{x}))$$

where $\underline{x} = \prod_{i=0}^m x_i$ and $\underline{x}^{\underline{n}} = \prod_{i=0}^m x_i^{n_i}$. The *generalized Koras-Russell varieties* are defined as the smooth affine variety associated to this coordinate ring, i.e.,

$$(5.2) \quad \mathcal{X}_m(\underline{n}, \psi) := \text{Spec } R_m(\underline{n}, \psi)$$

For every $m \geq 0$, the so-obtained affine variety $\mathcal{X}_m(\underline{n}, \psi)$ is of dimension $(m+3)$ and its smoothness is a consequence of the Jacobian criterion owing to the assumption that $\psi(0) \neq 0$. Let us now look at a concise example of one such variety in relative dimension $(m+3)$ constructed over the affine space $\mathbb{A}_k^{n-3}$, for $n \geq 4$. Putting $\psi(\underline{x}) = 1 + \underline{x} + \sum_{i=2}^{n-2} a_i \underline{x}^i$, which is clearly non-vanishing at the origin, we get that

$$(5.3) \quad \mathcal{X}_m(n, \psi) := \{\underline{x}^{\underline{n}} z = y^r + t^s + x_0(1 + \underline{x} + \sum_{i=2}^{n-2} a_i \underline{x}^i)\} \subseteq \mathbb{A}_k^{n-3} \times \mathbb{A}_k^{m+3}$$

with coordinates given by $\text{Spec}(k[a_2, \dots, a_{n-2}][x_0, \dots, x_m, y, z, t])$. This makes it a smooth affine scheme under the projection map $\mathcal{X}_m(n, \psi) \rightarrow \mathbb{A}_k^{n-3}$ given by projecting onto the coefficients $\{a_i\}$.

Theorem 5.4. Let k be any perfect field. Then under the notations and assumptions as in Eq. (5.2), the canonical morphism $\mathcal{X}_m(\underline{n}, \psi) \rightarrow \text{Spec } k$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k .

The authors in [DG25] have proven the $\mathbb{A}^1$ -contractibility of these varieties over fields of characteristic zero. We can now promote this result over perfect fields due to Proposition 4.15 following the original proof. Let us note that all these varieties are all stably k -isomorphic ([DG25, Proposition 3]), that is, $\mathcal{X}_m(\underline{n}, \psi) \times \mathbb{A}_k^1 \cong \mathcal{X}_m(\underline{n}, 1) \times \mathbb{A}_k^1$, for any polynomial $\psi \in k^{[1]}$ as above. Hence, in the light of $\mathbb{A}^1$ -homotopy invariance, it is enough to show that $\mathcal{X}_m := \mathcal{X}_m(\underline{n}, 1)$ is $\mathbb{A}^1$ -contractible. Observe that when $m = 0$ and $\psi(\underline{x}) = 1$, $\mathcal{X}_0(n_0, 1)$ is precisely the family of Koras-Russell threefolds studied in the previous section, which we proved to be $\mathbb{A}^1$ -contractible (Proposition 4.15).

Proof of Theorem 5.4. Let us fix

$$A_{m-1} = k[x_1, \dots, x_{m-1}, y, t] \quad \text{and} \quad R_m = \frac{A_{m-1}[x_0, x_m, z]}{\prod_{i=0}^m x_i^{n_i} z + y^r + t^s + x_0}.$$

Then consider the following closed subschemes of $\mathcal{X}_m$

$$(5.5) \quad W_m = \{x_m = 0\} \cong \text{Spec}(R_m/x_m R_m) \cong \text{Spec}(A_{m-1}[z]) \cong \mathbb{A}_k^{m+2}$$

$$(5.6) \quad H_m = \{z = 0\} \cong \text{Spec}(R_m/z R_m) \cong \text{Spec}(A_{m-1}[x_m]) \cong \mathbb{A}_k^{m+2}$$

$$(5.7) \quad P_m = W_m \cap H_m \cong \text{Spec}(R_m/(x_m, z) R_m) \cong \text{Spec}(A_{m-1}) \cong \mathbb{A}_k^{m+1}$$

Let $j_m : H_m \rightarrow \mathcal{X}_m$ and $i_m : H_m \setminus P_m \rightarrow \mathcal{X}_m \setminus W_m$ be the canonical closed immersions. By setting $Z := x_m z$, we have an isomorphism

$$\mathcal{X}_m \setminus W_m \cong \operatorname{Spec} \left(A_{m-2}[x_0, x_{m-1}, Z][x_m^{-1}] / (x_0^{n_0} \prod_{i \neq m} x_i^{n_i} Z + y^r + t^s + x_0) \right) \cong \mathcal{X}_{m-1} \times \mathbb{G}_{m,k}$$

for which i_m equals the product

$$j_{m-1} \times \operatorname{id}_{\mathbb{G}_{m,k}} : H_m \setminus P_m \cong H_{m-1} \times_k \mathbb{G}_{m,k} = \operatorname{Spec} (A_{m-2}[x_{m-1}][x_m^{\pm 1}]) \rightarrow \mathcal{X}_{m-1} \times_k \mathbb{G}_{m,k}.$$

By setting up the cofiber sequence as in Eq. (4.4), we get the following diagram

$$(5.8) \quad \begin{array}{ccccc} H_m \setminus P_m & \longrightarrow & H_m & \longrightarrow & H_m / (H_m \setminus P_m) \\ \downarrow i_m & & \downarrow j_m & & \downarrow l_m \\ \mathcal{X}_m \setminus W_m & \longrightarrow & \mathcal{X}_m & \longrightarrow & \mathcal{X}_m / (\mathcal{X}_m \setminus W_m) \end{array}$$

associated to the open immersions $H_m \setminus P_m \rightarrow H_m$ and $\mathcal{X}_m \setminus W_m \rightarrow \mathcal{X}_m$. Observe that H_m is $\mathbb{A}^1$ -contractible by Eq. (5.6) and so $\mathcal{X}_m$ is $\mathbb{A}^1$ -contractible provided j_m is an $\mathbb{A}^1$ -weak equivalence. For $m = 0$, the morphism $j_0 : H_0 \rightarrow \mathcal{X}_0$ is an $\mathbb{A}^1$ -weak equivalence due to Proposition 4.15. We now establish this property for $m \geq 1$ by induction on m .

To show that j_m is an $\mathbb{A}^1$ -weak equivalence, it suffices to show that i_m and l_m are $\mathbb{A}^1$ -weak equivalences in Spc_k in the light Lemma A.1. The $\mathbb{A}^1$ -weak equivalence of i_m follows from the fact that $i_m \simeq j_{m-1} \times \operatorname{id}_{\mathbb{G}_{m,k}}$ and by the induction hypothesis, j_{m-1} is an $\mathbb{A}^1$ -weak equivalence. On the other hand, l_m is also an $\mathbb{A}^1$ -weak equivalence. Indeed, as P_m is obtained as the transversal intersection of two smooth subvarieties inside $\mathcal{X}_m$, the normal cone $N_{P_m} H_m$ of P_m in H_m is in fact the restriction of the normal cone $N_{W_m} \mathcal{X}_m$ of W_m in $\mathcal{X}_m$ to P_m . But the normal cone of $N_{W_m} \mathcal{X}_m$ is trivial, attributed to the analogous reason as in Section 4.1, whence the triviality of $N_{P_m} H_m$. The proof now follows from the homotopy purity [MV99, Theorem 2.23] as all these varieties are smooth. In particular,

$$H_m / (H_m \setminus P_m) \simeq \operatorname{Th}(N_{P_m} H_m) \simeq P_m \wedge \mathbb{P}^1$$

and similarly,

$$\mathcal{X}_m / (\mathcal{X}_m \setminus W_m) \simeq \operatorname{Th}(N_{W_m} \mathcal{X}_m) \simeq W_m \wedge \mathbb{P}^1.$$

The map l_m coincides with the following map observed under the above-mentioned isomorphisms

$$\operatorname{Th}(N_{P_m} H_m) \simeq P_m \wedge \mathbb{P}^1 \xrightarrow{\iota \wedge \mathbb{P}^1} W_m \wedge \mathbb{P}^1 \simeq \operatorname{Th}(N_{W_m} \mathcal{X}_m).$$

obtained as the $\mathbb{P}^1$ -suspension of the closed immersion $\iota : P_m \hookrightarrow W_m$. Hence, the $\mathbb{A}^1$ -weak equivalence of ι implies that of l_m . $\square$

Corollary 5.9. Let S be any Noetherian scheme with perfect residue fields. Then under the assumptions as in Eq. (5.2), the canonical morphism $\varphi : \mathcal{X}_m(\underline{n}, \psi) \rightarrow S$ is an $\mathbb{A}^1$ -weak equivalence in Spc_S .

Proof. The strategy of proof is precisely as that of Corollary 4.17; choose any point $s \in S$ and consider the induced morphism on the corresponding fiber $\varphi_s : \mathcal{X}_m(\underline{n}, \psi)_s \rightarrow \operatorname{Spec} \kappa_s$, where

$$\mathcal{X}_m(\underline{n}, \psi)_s := \mathcal{X}_m(\underline{n}, \psi) \times_{\mathrm{Spec} \kappa_s} \mathrm{Spec} \kappa_s.$$

$$\begin{array}{ccc} \mathcal{X}_m(\underline{n}, \psi)_s & \longrightarrow & \mathcal{X}_m(\underline{n}, \psi) \\ \downarrow \varphi_s & & \downarrow \varphi \\ \mathrm{Spec} \kappa_s & \longrightarrow & S \end{array}$$

Since by the hypothesis all such fields κ_s are perfect, φ_s is an $\mathbb{A}^1$ -weak equivalence in Spc_{κ_s} owing to Theorem 5.4. The proof now follows from Proposition 3.19. $\square$

Remark 5.10. The theorems 4.16 and 5.4 enable us to produce exotic families of smooth affine varieties over fields of positive characteristic. For a comparison, let us now quickly recall the Asanuma-Gupta varieties for the readers convenience: let k be a field of characteristic $p > 0$ and let $f(z, t) := z^{p^e} + t + t^{sp}$ be the non-trivial line embedded into $\mathbb{A}_k^2$ such that the integers $e, s \geq 1$ with $p^e \nmid sp$ and $sp \nmid p^e$. Then the *Asanuma-Gupta varieties* are defined by the explicit polynomial equations

$$\begin{aligned} \mathcal{AG}_3 &:= \mathrm{Spec} \left(\frac{k[x, y, z, t]}{x^m y - f(z, t)} \right) \subset \mathbb{A}_k^4 \quad \text{where integers } m \geq 2 \\ \mathcal{AG}_{m+2} &:= \mathrm{Spec} \left(\frac{k[x_1, \dots, x_m, y, z, t]}{x_1^{r_1} \dots x_m^{r_m} y - f(z, t)} \right) \subset \mathbb{A}_k^{m+3} \quad \text{where integers } m, r_i \geq 2 \end{aligned}$$

Observe that due to [Asa87, Theorem 5.1], we have that the projection map $\tilde{\mathrm{pr}}_x : \mathcal{AG}_3 \rightarrow \mathbb{A}_x^1$ onto the x -axis is an $\mathbb{A}^2$ -fiber space, which, in particular, says that all the fibers of $\tilde{\mathrm{pr}}_x$ are smooth. In contrast, recall from Section 4 that for the projection $\mathrm{pr}_x : \mathcal{K} \rightarrow \mathbb{A}_x^1$, the generic fiber is isomorphic to $C_{r,s} \times \mathbb{A}_k^1$, which is clearly singular. Similarly, one can show that the $\tilde{\mathrm{pr}}_{x_1} : \mathcal{AG}_{m+2} \rightarrow \mathbb{A}_{x_1}^1$ is an $\mathbb{A}_k^{m+3}$ -fiber space while $\mathrm{pr}_{x_0} : \mathcal{X}_m(\underline{n}, \psi) \rightarrow \mathbb{A}_{x_0}^1$ is isomorphic to the singular variety $C_{r,s} \times \mathbb{A}_k^{m+1}$ over its generic point $x_0 = 0$. Thus, from these one infers that these results provide us with a distinct class of ‘potential’ counter-examples to the Zariski Cancellation Problem in all dimensions ≥ 3 in positive characteristics.

Moduli of Arbitrary Smooth Exotic Affine Varieties. In [AD07, Theorem 5.1], the authors constructed an infinite collection of pairwise non-isomorphic exotic $\mathbb{A}^1$ -contractible strictly quasi-affine varieties of relative dimensions ≥ 4 . They also showed that such a situation cannot occur in dimensions 1 and 2 over fields of characteristic zero ([AD07, Claims 5.7 and 5.8]). The missing gaps in this picture were whether there exists an analogous collection for threefolds and if there exists a collection consisting solely of *affine* varieties in dimension ≥ 4 . The first gap was answered positively by the authors in [DF18, Corollary 1.3] by devising a general method to produce a moduli of arbitrary positive dimension of pairwise non-isomorphic, stably isomorphic, $\mathbb{A}^1$ -contractible smooth affine threefolds inspired from [DMP11]. The second gap is also answered positively due to the recent work of [DG25] exploiting the $\mathbb{A}^1$ -contractibility of the generalized Koras-Russell varieties. Our results (Corollary 4.17 and Corollary 5.9) add significance to this overall picture by producing a moduli of arbitrary positive dimension of pairwise non-isomorphic, stably isomorphic, $\mathbb{A}^1$ -contractible smooth *affine* schemes over arbitrary base schemes in *relative dimension* ≥ 3 .

6. EXOTIC MOTIVIC SPHERES

Recall that in motivic homotopy theory, we have two types of sphere objects: the *simplicial circle* from the algebraic topology $S^1 := \Delta^1/\partial\Delta$, which exists as a singular scheme. It is pointed by the image $\partial\Delta \rightarrow \Delta$. The other circle comes from algebraic geometry called the *Tate circle* $\mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$ which exists as a smooth scheme. It is pointed by 1. As the unstable $\mathbb{A}^1$ -homotopy category is symmetric monoidal with respect to the smash product, one forms *generalized motivic spheres* via

$$\mathbb{S}^{p,q} := (S^1)^{\wedge(p-q)} \wedge \mathbb{G}_m^{\wedge q}$$

for all $p \geq q \geq 0$. For notational convenience, we will adapt to write S^{p-q} for $(S^1)^{\wedge(p-q)}$ and $\mathbb{G}_m^q$ for $(\mathbb{G}_m)^{\wedge q}$.

Example 6.1. The following are some basic examples of motivic spheres.

- (1) By definition, we have $\mathbb{S}^{1,0} = S^1$ and $\mathbb{S}^{1,1} = \mathbb{G}_m$. More generally, for every $n \geq 1$, we have

$$\mathbb{S}^{2n-1,n} \simeq \mathbb{A}^n \setminus \{0\}.$$

- (2) For every $n \geq 1$, we have that

$$\mathbb{S}^{2n,n} = \Sigma_{S^1} \mathbb{S}^{2n-1,n} \simeq \mathbb{A}^n / (\mathbb{A}^n \setminus \{0\}) \simeq \mathbb{P}^n / (\mathbb{P}^n \setminus \{0\}) \simeq \mathbb{P}^n / \mathbb{P}^{n-1}.$$

In particular, for $n = 1$, we get $\mathbb{S}^{2,1} \simeq \mathbb{P}^1$,

- (3) The authors in [ADF17] study a certain family of smooth affine quadrics given by

$$Q_{2n} := \text{Spec} \left(k[x_1, \dots, x_m, y_1, \dots, y_m] / \left\langle \sum_i x_i y_i - 1 \right\rangle \right)$$

$$Q_{2n-1} := \text{Spec} \left(k[x_1, \dots, x_m, y_1, \dots, y_m, z] / \left\langle \sum_i x_i y_i - z(1+z) \right\rangle \right)$$

and show that these quadrics indeed form smooth models for the motivic spheres

$$Q_{2n-1} \simeq \mathbb{S}^{2n-1,n} \quad \text{and} \quad Q_{2n} \simeq \mathbb{S}^{2n,n} \simeq (\mathbb{P}^1)^{\wedge n}.$$

Coupling with previous examples, one sees that $\Sigma_{S^1} \mathbb{A}^n \setminus \{0\} \simeq (\mathbb{P}^1)^{\wedge n}$, for all $n \geq 1$,

- (4) For all integers $p > 2q$, the motivic spheres $\mathbb{S}^{p,q}$ cannot be represented by a smooth scheme [ADF17, Proposition 2.3.1].

The main objective of this section is to prove the existence of exotic motivic spheres in higher dimensions. This study, in contrast, can be seen as a *compact analog* to that of exotic affine varieties as discussed in Section 4 and Section 5. We begin with the following definition.

Definition 6.2. Let S be a base scheme. A smooth scheme $X \rightarrow S$ of finite type and of relative dimension $n \geq 0$ is called an *exotic motivic sphere* if it is $\mathbb{A}^1$ -homotopic to the scheme $\mathbb{A}_S^n \setminus \{0\}$ without being isomorphic as S -schemes, i.e.,

$$X \simeq \mathbb{A}_S^n \setminus \{0\} \quad \text{and} \quad X \not\cong \mathbb{A}_S^n \setminus \{0\}.$$

Question 6.3. Let S be a reasonably arbitrary base scheme. Then for a smooth scheme $X \rightarrow S$, does $X \simeq_{\mathbb{A}^1} \mathbb{A}_S^n \setminus \{0\}$ imply the isomorphism $X \cong \mathbb{A}_S^n \setminus \{0\}$ of S -schemes?

6.1. Overview of Question 6.3. In dimension 1, $\mathbb{S}^{0,1} := \mathbb{G}_m$ is the unique motivic sphere up to isomorphism over any reduced base scheme.

Proposition 6.4. Over a reduced scheme S , if a smooth scheme $X \rightarrow S$ is $\mathbb{A}^1$ -homotopic to $\mathbb{G}_{m,S}$, then it is isomorphic to $\mathbb{G}_{m,S}$.

Proof. Let us begin with the case of fields ⁵. Let X be a smooth curve that is $\mathbb{A}^1$ -homotopic to $\mathbb{G}_m$. Then we have that $\pi_0^{\mathbb{A}^1}(X) \simeq \pi_0^{\mathbb{A}^1}(\mathbb{G}_m) \simeq \mathbb{G}_m$ which leaves us to show that X is $\mathbb{A}^1$ -rigid. Suppose the contrary, then recall that we have a canonical surjection of Nisnevich sheaves $X \rightarrow \pi_0^{\mathbb{A}^1}(X)$ ([MV99, §2, Corollary 3.22]) which gives us a dominant morphism $\varphi : X \rightarrow \mathbb{G}_m$. Since by supposition X is not $\mathbb{A}^1$ -rigid, then there is a finite separable extension L/k and a $\mathbb{A}^1$ -homotopy $H : \mathbb{A}_L^1 \rightarrow X$ such that the induced maps on the 0-section and 1-section of $\text{Spec } L \rightarrow X$ do not agree $H(0) \neq H(1)$ (cf. [AM11, Lemma 2.1.11]). Hence, H is a dominant map which makes the composition $\varphi \circ H : \mathbb{A}_L^1 \rightarrow X \rightarrow \mathbb{G}_m$ constant (since $\mathbb{G}_m$ is $\mathbb{A}^1$ -rigid). In particular, φ is constant, which contradicts the fact that φ is dominant. For any reduced scheme S , it suffices to check locally on affine charts. But since for any reduced ring R , $\mathbb{G}_{m,R}$ remains $\mathbb{A}^1$ -rigid, the proof follows. $\square$

For a field k of characteristic zero, it has been recently shown by [CR24b, Theorem 3.1] that in dimension 2 if an open subscheme U of an affine surface X is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{A}^2 \setminus \{0\}$, then $U \cong \mathbb{A}^2 \setminus \{0\}$ as k -schemes. This question is answered negatively in dimension 3 via the Koras-Russell threefolds of the first kind (cf. [CR24b, §3]). Thus, the question remains widely open in dimensions ≥ 4 over fields and consequently, over a base scheme. In this article, we close this question in all dimensions ≥ 4 over fields using the generalized Koras-Russell varieties.

6.2. Existence of exotic motivic spheres in higher dimensions. We show that the prototypes studied in Section 5 provide counterexamples to the Question 6.3 in every dimension ≥ 4 . Due to the previously mentioned fact that each of these varieties $\mathcal{X}_m(\underline{n}, \psi)$ is stably isomorphic, it is enough to prove this fact for the varieties $\mathcal{X}_m := \mathcal{X}_m(n, 1)$, noting, nevertheless, that any variety in this family indeed provides a counter-example. The strategy of the proof that we present below is inspired by that of the classical Koras-Russell threefolds of [CR24b]. For simplicity, let us consider the family of smooth affine varieties defined as in Eq. (5.3)

$$(6.5) \quad \mathcal{X}_m := \{\underline{x}^n z = y^r + t^s + x_0\} \subset \mathbb{A}_k^{m+4} \cong \text{Spec}(k[x_0, \dots, x_m, y, z, t]).$$

The canonical morphism $\mathcal{X}_m \rightarrow \text{Spec } k$ makes it into a smooth affine scheme of dimension $(m+3)$ for all $m \geq 0$ and $n \geq 2$ and $\underline{x} := \prod_{i=0}^m x_i$. The variety $\mathcal{X}_m$ is obtained by setting $\psi = 1$ and the index $\underline{n} = n$ in Eq. (5.2). Observe that $p = (1, 1, \dots, 1, 0, 1, 0) \in \mathcal{X}_m$ is a rational point. Define the map

$$\begin{aligned} \phi : \mathcal{X}_m &\rightarrow \mathbb{A}_k^{m+3} \\ (x_0, \dots, x_m, y, z, t) &\mapsto (x_0, \dots, x_m, y, t) \end{aligned}$$

(forgetting ' z '). Then the image $\phi(p) = (1, \dots, 1, 0, 0) =: q \in \mathbb{A}_k^{m+3}$.

Lemma 6.6. Let k be a field. For all $m \geq 0$, the induced map $d\phi_p : T_p \mathcal{X}_m \rightarrow T_q \mathbb{A}_k^{m+3}$ is an isomorphism of tangent spaces.

⁵The author thanks Biman Roy for his generous explanation of the proof over fields.

Proof. Let $f := f(x_0, \dots, x_m, y, z, t) = \underline{x}^n z - y^r - t^s - x_0$. We now compute the partial derivatives of f

$$\begin{aligned} \frac{\partial f}{\partial x_0} &= nx_0^{n-1} x_1^n \dots x_m^n z - 1 & \frac{\partial f}{\partial x_j} &= nx_0^n x_1^n \dots x_j^{n-1} \dots x_m^n z, \quad \text{for all } 0 < j \leq m, \\ \frac{\partial f}{\partial y} &= -ry^{r-1} & \frac{\partial f}{\partial z} &= \underline{x}^n & \frac{\partial f}{\partial t} &= -st^{s-1} \end{aligned}$$

The total derivative $\nabla(f)$ of f can be computed as

$$\nabla f = \left((nx_0^{n-1} x_1^n \dots x_m^n z - 1), \dots, (nx_0^n x_1^n \dots x_j^{n-1} \dots x_m^n z), \dots, -ry^{r-1}, \underline{x}^n, -st^{s-1} \right)$$

which at the point p is given by $\nabla f|_p = (n-1, n, \dots, n, 0, 1, 0)$. Recall that the tangent space of $\mathcal{X}_m$ at p is the kernel of the total derivative at p . And hence, depending on the parity of m , we find that the tangent space at p is given as follows: If m is even, we have

$$T_p \mathcal{X}_m = \{(a, b, -b, \dots, b, -b, c, -a(n-1), d) \mid a, b, c, d \in k\} \subset \mathbb{A}_k^{m+4}$$

If m is odd, we (respectively) have

$$T_p \mathcal{X}_m = \{(a, b, -b, \dots, 0, c, -a(n-1), d) \mid a, b, c, d \in k\} \subset \mathbb{A}_k^{m+4}$$

Nevertheless, in both cases, the induced map $d\phi_p$ is given by the

$$(a, b, -b, \dots, b, -b, c, -a(n-1), d) \mapsto (a, b, -b, \dots, b, -b, c, d)$$

or (respectively)

$$(a, b, -b, \dots, 0, c, -a(n-1), d) \mapsto (a, b, -b, \dots, 0, c, d)$$

which are clearly isomorphic. This, in particular, implies that the normal bundle of $\mathcal{X}_m$ at p is isomorphic to that of $N_q \mathbb{A}_k^{m+3}$, which is trivial. $\square$

Lemma 6.7. Let k be any infinite perfect field. Then the strictly quasi-affine variety $\mathcal{X}_m \setminus \{p\}$ is $\mathbb{A}^1$ -chain connected.

Proof. The strategy of the proof takes into account the fact that the classical Koras-Russell threefolds $\mathcal{K}$ are $\mathbb{A}^1$ -chain connected ([DPØ19, Example 2.28]). Let L/k be any finitely generated field extension. Recall from Definition 3.5 that we need to show that all the fibers and any points between fibers can be connected by elementary $\mathbb{A}^1$ -homotopies. Consider the projection

$$\begin{aligned} \text{pr} : \mathcal{X}_m &\rightarrow \mathbb{A}_k^1 \quad \text{defined by} \\ (x_0, \dots, x_m, y, z, t) &\mapsto x_0. \end{aligned}$$

First, let us observe that the fibers over points of $\mathbb{G}_m$ are $\mathbb{A}^1$ -chain connected. Indeed, choose a point $\alpha \in \mathbb{G}_m$. If $\alpha \neq 1$, then $\text{pr}^{-1}(\alpha) \simeq \mathbb{A}_k^{m+2}$ which is clearly $\mathbb{A}^1$ -chain connected and if $\alpha = 1$, we have $\text{pr}^{-1}(\alpha) \simeq \mathbb{A}_k^{m+2} \setminus \{0\}$ which is again $\mathbb{A}^1$ -chain connected. In all, for every point $\alpha \in \mathbb{G}_m$, any two points L -points in $\text{pr}^{-1}(\alpha)$ can be connected via chains of $\mathbb{A}_L^1$'s. The fiber over the point $\{x_0 = 0\}$ is given by $\text{pr}^{-1}(0) \simeq \mathbb{A}_k^{m+1} \times_k C_{r,s}$, where $C_{r,s} := \{y^r - t^s = 0\}$ is the cuspidal curve. The cuspidal curve is $\mathbb{A}^1$ -chain connected as witnessed by Example 3.8. Hence, the fiber $\text{pr}^{-1}(0)$ is $\mathbb{A}^1$ -chain connected given by the naïve $\mathbb{A}^1$ -homotopy

$$\begin{aligned} \mathbb{A}_L^1 &\rightarrow \mathbb{A}^{m+1} \times \Gamma_{r,s} \\ \gamma &\mapsto (a_0 \gamma, \dots, a_m \gamma, b \gamma^s, c \gamma^r) \end{aligned}$$

joining $(0, \dots, 0)$ with the points $(a_0, \dots, a_m, b, c)$. Now, to conclude, we only have to show that points between different fibers can be joined via chains of $\mathbb{A}_L^1$'s. Consider the $\mathbb{A}^1$ -homotopy given as follows; let $f(w)$ and $g(w)$ be two polynomials in $k[w]$ such that w^n divides $f(w)^r + g(w)^s + w$. The existence of such polynomials can be demonstrated by lifting the polynomials mod w^n by inducting on $n \geq 0$. The case $n = 0$ is trivial and for the case $n = 1$, choose $f(\alpha w) = 1$ and $g(\alpha w) = -1$. For the case $n = 2$, choose $f(\alpha w) = 1 + w$ and $g(\alpha w) = w - 1$. By proceeding in the same fashion, we can obtain a general strategy by setting:

$$f(\alpha w) := 1 + a_0 w + a_1 w^2 + \dots + a_{n-2} w^{n-1}$$

and

$$g(\alpha w) := 1 - w - \dots - w^{n-1}$$

for some coefficients $a_i \in k$. One then has to choose $\{a_i\}$'s such that the coefficients of w_i vanish for all $i \leq n - 1$. Now, define a map $\theta : \mathbb{A}_L^1 \rightarrow \mathcal{X}_m \setminus \{p\}$ by

$$w \mapsto \left(\alpha w, 1, \dots, 1, \frac{f(\alpha w)^r + g(\alpha w)^s + \alpha w - 1}{(\alpha w)^n}, f(\alpha w), g(\alpha w) \right).$$

This connects points from the fibre $\text{pr}^{-1}(0)$ with points of the fiber $\text{pr}^{-1}(\alpha)$, for $\alpha \in \mathbb{G}_m$. Crucially, note that our chosen point $p = (1, \dots, 1, 0, 1, 0)$ does not lie in the image of θ . Suppose on contrary, if $p \in \text{Im}(\theta)$, then we have that $\theta(a) = p = (1, \dots, 1, 0, 1, 0)$, for some $a \in \mathbb{A}_L^1$. This gives us $\alpha w = 1$, $f(\alpha w) = 1$, and $g(\alpha w) = 0$. Upon direct substitution and comparison with the point p , we see that

$$\frac{f(\alpha w)^r + g(\alpha w)^s + \alpha w - 1}{(\alpha w)^n} = 1 \neq 0.$$

Hence, the quasi-affine variety $\mathcal{X}_m \setminus \{p\}$ is $\mathbb{A}^1$ -chain connected. $\square$

We now have all the tools to prove our key result that answers Question 6.3.

Theorem 6.8. Let k be an infinite perfect field. Then for every $m \geq 0$, the strictly quasi-affine variety $\mathcal{X}_m \setminus \{p\}$ of dimension $(m + 3)$ is $\mathbb{A}^1$ -homotopic to $\mathbb{A}_k^{m+3} \setminus \{q\}$ but is not isomorphic to $\mathbb{A}_k^{m+3} \setminus \{q\}$ as a k -scheme.

Proof. Let $\mathcal{X}_m$ be the smooth affine variety as described in Eq. (6.5) with the rational point $p = (1, \dots, 1, 0, 1, 0) \in \mathcal{X}_m$ and with its image $q = (1, \dots, 1, 0, 0) \in \mathbb{A}_k^{m+3}$. We first show that $\mathcal{X}_m \setminus \{p\}$ is $\mathbb{A}^1$ -homotopic to $\mathbb{A}_k^{m+3} \setminus \{q\}$. For this, let us set up the following commutative diagram whose rows are cofiber sequences

$$(6.9) \quad \begin{array}{ccccc} \mathcal{X}_m \setminus \{p\} & \longrightarrow & \mathcal{X}_m & \longrightarrow & \mathcal{X}_m / (\mathcal{X}_m \setminus \{p\}) \\ \downarrow \phi & & \downarrow \psi & & \downarrow \rho \\ \mathbb{A}_k^{m+3} \setminus \{q\} & \longrightarrow & \mathbb{A}_k^{m+3} & \longrightarrow & \mathbb{A}_k^{m+3} / (\mathbb{A}_k^{m+3} \setminus \{q\}) \end{array}$$

Due to Theorem 5.4, we have that $\psi : \mathcal{X}_m \rightarrow \mathbb{A}_k^{m+3}$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k . Next, we observe that ρ is also an $\mathbb{A}^1$ -weak equivalence. Indeed, since the normal bundle associated to the closed immersion $\{q\} \hookrightarrow \mathbb{A}_k^{m+3}$ is trivial, by the $\mathbb{A}^1$ -homotopy purity ([MV99, Theorem 2.23]), we have that

$$\mathbb{A}_k^{m+3} / (\mathbb{A}_k^{m+3} \setminus \{q\}) \simeq \text{Spec } k \wedge (\mathbb{P}^1)^{\wedge(m+3)} \simeq (\mathbb{P}^1)^{\wedge(m+3)}.$$

Since $\{p\} \in \mathcal{X}_m$ and $\mathcal{X}_m$ are smooth subvarieties, due to the aforementioned purity, we have

$$\mathcal{X}_m/(\mathcal{X}_m \setminus \{p\}) \simeq \mathrm{Th}(N_p \mathcal{X}_m).$$

By Lemma 6.6, we have that the normal bundle $N_p \mathcal{X}_m$ is isomorphic to a trivial bundle and hence that

$$\mathrm{Th}(N_p \mathcal{X}_m) \simeq \{p\} \wedge (\mathbb{P}^1)^{\wedge(m+3)} \simeq (\mathbb{P}^1)^{\wedge(m+3)}.$$

Hence, we have that

$$(\mathbb{P}^1)^{\wedge(m+3)} \simeq \mathcal{X}_m/(\mathcal{X}_m \setminus \{p\}) \xrightarrow{\rho} \mathbb{A}_k^{m+3}/(\mathbb{A}_k^{m+3} \setminus \{q\}) \simeq (\mathbb{P}^1)^{\wedge(m+3)}$$

which due to the fact that the map ρ is induced from the map $d\phi_p$ becomes an $\mathbb{A}^1$ -weak equivalence in the light of [Voe03, Lemma 2.1]. The $\mathbb{A}^1$ -weak equivalence of ρ implies, by definition, that the simplicial suspension $\Sigma_{S^1} \phi : \Sigma_{S^1} \mathcal{X}_m \setminus \{p\} \rightarrow \Sigma_{S^1} \mathbb{A}_k^{m+3} \setminus \{q\}$ is an $\mathbb{A}^1$ -weak equivalence of motivic spaces. Now observe that $\mathcal{X}_m \setminus \{p\}$ is $\mathbb{A}^1$ -chain connected from Lemma 6.7 and whence by [AM11, Proposition 2.2.7] is thereby $\mathbb{A}^1$ -connected as well. Moreover, it is of codimension $d = m + 3 \geq 2$ in $\mathcal{X}_m$, for $m \geq 0$. Thus, due to [AD09, Theorem 4.1], this implies that the open subschemes $\mathcal{X}_m \setminus \{p\} \hookrightarrow \mathcal{X}_m$ have the same $\mathbb{A}^1$ -homotopy sheaves in a range

$$\pi_i^{\mathbb{A}^1}(\mathcal{X}_m \setminus \{p\}) \xrightarrow{\sim} \pi_i^{\mathbb{A}^1}(\mathcal{X}_m) \quad \text{for all } 0 \leq i \leq d - 2$$

and in particular, $\pi_1^{\mathbb{A}^1}(\mathcal{X}_m \setminus \{p\})$ is trivial, which by the $\mathbb{A}^1$ -Hurewicz theorem [Mor12, Theorem 6.35, Theorem 6.37] imply that ϕ is an $\mathbb{A}^1$ -homology equivalence. To conclude, observe that due to [Shi22, Theorem 1.1], we have that $\mathcal{X}_m \setminus \{p\} \xrightarrow{\sim} \mathbb{A}_k^{m+3} \setminus \{q\}$ is an $\mathbb{A}^1$ -weak equivalence in Spc_k .

Now, we show that these quasi-affine varieties cannot be isomorphic as k -schemes. Suppose, on the contrary, assume that there is an isomorphism $\sigma : \mathcal{X}_m \setminus \{p\} \rightarrow \mathbb{A}_k^{m+3} \setminus \{q\}$ with an inverse $\tau : \mathbb{A}_k^{m+3} \setminus \{q\} \rightarrow \mathcal{X}_m \setminus \{p\}$. Now since the point $\{p\}$ (resp. $\{q\}$) is of codimension at least 2 in $\mathcal{X}_m$ (resp. in $\mathbb{A}_k^{m+3}$) respectively, the morphism σ and τ can be continuously extended to a smooth morphism $\sigma' : \mathcal{X}_m \rightarrow \mathbb{A}_k^{m+3}$ and $\tau' : \mathbb{A}_k^{m+3} \rightarrow \mathcal{X}_m$. Both the maps σ' and τ' take the same values with the identity maps away from the complement of a rational point, which implies that both σ' and τ' are isomorphisms. But due to the fact that $\mathrm{ML}(\mathcal{X}_m) \not\cong \mathrm{ML}(\mathbb{A}_k^{m+3})$ (cf. [GG23, Proposition 3.4]), this is absurd. Thus, the quasi-affine varieties $\mathcal{X}_m \setminus \{p\}$ and $\mathbb{A}_k^{m+3} \setminus \{q\}$ cannot be isomorphic as k -schemes. $\square$

Remark 6.10. The assumption on the base field to be infinite in Theorem 6.8 is due to the limitation as expressed in [AD09, Remark 2.16]. In essence, for a smooth scheme X with an open subscheme $U \hookrightarrow X$ and its closed complement $Z := X \setminus U$ of codimension ≥ 2 , if $X(k) \subseteq Z$, then clearly U has no rational point, whence it cannot be $\mathbb{A}^1$ -connected. However, for our contexts, we expect that this explicit quasi-affine variety $\mathcal{X}_m \setminus \{p\}$ should still be $\mathbb{A}^1$ -connected over a perfect base field and consequently, we should be able to apply the $\mathbb{A}^1$ -excision statement to promote the Theorem 6.8 to perfect fields.

Alternatively if one knows the $\mathbb{A}^1$ -simply connectedness of $\mathcal{X}_m \setminus \{p\}$, the infinite assumption in Theorem 6.8 can be lifted in the light of [Shi22, Corollary 3.6]. This raises the following question.

Question 6.11. For a smooth (affine) k -variety, does $\mathbb{A}^1$ -chain connectedness imply $\mathbb{A}^1$ -simply connectedness?

A note on base change for motivic spheres. In this short section, we will comment on the base change properties of motivic spheres. In anticipation of the existence of exotic motivic spheres over fields (Theorem 6.8), it is natural to systematically ask for the verity over a base scheme S . The question of interest can be framed as follows:

Question 6.12. Let S be any "reasonable" base scheme. Then for every $m \geq 0$, are the smooth quasi-affine varieties $\mathcal{X}_m \setminus \{p\}$ and $\mathbb{A}_S^{m+3} \setminus \{0\}$ necessarily $\mathbb{A}^1$ -homotopic. If so, are they isomorphic as S -schemes?

We intuitively anticipate that they should *not* be isomorphic as S -schemes owing to the failure of relative Makar-Limanov invariants as in the case of fields. To show that they are $\mathbb{A}^1$ -homotopic, one might hope for the following approach.

Gluing motivic spheres. One may choose to work fiberwise as in the case of $\mathbb{A}^1$ -contractibility. For concreteness, fix S to be a base scheme with infinite perfect residue fields. Let us now decipher the obstruction to taking up this approach. For every point $s \in S$, let us consider the base change $f_s : (\mathcal{X}_m \setminus \{p\})_s \rightarrow \text{Spec } \kappa_s$, where κ_s is the corresponding residue field at $s \in S$. Thus, by Theorem 6.8, we have that for every $s \in S$, the quasi-affine schemes $(\mathcal{X}_m \setminus \{p\})_s$ are $\mathbb{A}^1$ -homotopic to $\mathbb{A}_{\kappa_s}^{m+3} \setminus \{q\}$ in $\text{Spec } \kappa_s$.

$$\begin{array}{ccc} \mathbb{A}_{\kappa_s}^{m+3} \setminus \{0\} \simeq (\mathcal{X}_m \setminus \{p\})_s & \longrightarrow & \mathcal{X}_m \setminus \{p\} \\ f_s \downarrow & & \downarrow f \\ \text{Spec } \kappa_s & \longrightarrow & S \end{array}$$

But since in general the $\mathbb{A}^1$ -homotopy classes of motivic spheres are not trivial, owing to the non-triviality the motivic Brouwer degree (cf. [Mor12, Corollary 6.43]), knowing that the fibers $(\mathcal{X}_m \setminus \{p\})_s$ are $\mathbb{A}^1$ -homotopic to $\mathbb{A}_{\kappa_s}^{m+3} \setminus \{q\}$ does not allow us conclude that the original space $\mathcal{X}_m \setminus \{p\}$ is $\mathbb{A}^1$ -homotopic to $\mathbb{A}_S^{m+3} \setminus \{q\}$. This can be explained by the lack of an analogous *gluing lemma* for motivic spheres in comparison to that of contractible objects as provided by Proposition 3.19. Anticipating such a gluing lemma for motivic spheres, it is straightforward to verify that there are exotic motivic spheres in relative dimensions ≥ 3 over an appropriate base scheme S .

The relative approach. An alternative approach would be to directly show that the varieties in Question 6.12 are $\mathbb{A}^1$ -homotopic over S itself. However, this requires one to extend the tools exploited in the proof of Theorem 6.8 to the relative setting, which is, to the author's knowledge, yet to be fully understood.

Uniqueness of motivic spheres in relative dimension 2. In anticipation of the analogous gluing lemma, one could attempt to establish that there are no exotic motivic spheres in relative dimensions 2. Here is one possible formulation.

Conjecture 6.13. Let S be any Noetherian scheme with characteristic zero residue fields. Let $f : X \rightarrow S$ be a smooth scheme of relative dimension 2. Then if X is $\mathbb{A}^1$ -homotopic to $\mathbb{A}_S^2 \setminus \{0\}$, then X is isomorphic to $\mathbb{A}_S^2 \setminus \{0\}$ as a S -scheme.

Due to [CR24b, Theorem 3.1], we have that there are no exotic motivic spheres over a field of characteristic zero. If S is any Noetherian scheme with characteristic zero residue fields, then by the "anticipated" gluing lemma, it is verbatim to verify the nonexistence of exotic motivic spheres over S in relative dimension 2 as well.

APPENDIX A.

Lemma A.1. Let $\mathcal{C}$ be any pointed model category and consider the following commutative diagram in $\mathcal{C}$ where the rows are cofiber sequences.

$$\begin{array}{ccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C \\ f \downarrow & & g \downarrow & & \downarrow h \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' \end{array}$$

If f and g are weak equivalences, then so is h . In addition, assume that f and h are weak equivalences and B is contractible in $\mathcal{C}$, then g is a weak equivalence in $\mathcal{C}$ and consequently, B' is contractible in $\mathcal{C}$.

Proof. A proof is provided in [Hov07, Proposition 6.5.3 (b)] and is famously known as the *very weak five lemma*. $\square$

The following lemma, due to Ken Brown, is extremely useful in model categories.

Lemma A.2. Suppose $\mathcal{C}$ is a model category and $\mathcal{D}$ is a category with a subcategory of weak equivalences that satisfies the two out of three axiom. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor which takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.

Proof. This is due to [Hov07, Lemma 1.1.12]. $\square$

Lemma A.3. Let S be any separated scheme of finite type. Then for any affine S -scheme X , we have that the canonical morphism $f : X \rightarrow S$ is an affine morphism which is locally of finite presentation. Moreover, we have that f is smooth if all the fibres of $f : X \rightarrow S$ are smooth.

Proof. This follows from [Stacks, Lemma 01SG] and [Stacks, Lemma 01V8]. $\square$

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